Def. For every morphism of locally ringed spaces, $f: X \to S$, the diagonal ideal sheaf $\Delta_{X/S}$ is the kernel of the surjective morphism of sheaves of commutative unital rings $\Delta^+_X: \Delta^+_X(\mathcal{O}_{X,x}) \to \mathcal{O}_X$.

The two splittings, $\Delta^-_{X/S}(\mathcal{O}_{X,x}) \cong \mathcal{O}_X = \Delta^+_{X/S}(\mathcal{O}_{X,x}) \cong \mathcal{O}_X$, give $\Delta^+_{X/S}(\mathcal{O}_{X,x})$ a structure of $\mathcal{O}_X$-free $\mathcal{O}_X$ algebra that is an isomorphism.

For every integer $l \geq 0$, the functor $\mathcal{O}_{X/S}: \mathcal{O}_X$-mod $\to$ $\mathcal{O}_X$-mod associates to every left $\mathcal{O}_X$-module $E$ the left $\mathcal{O}_X$-module,

$$\mathcal{O}_{X/S}(E) := (\Delta^+_{X/S}(\mathcal{O}_{X,x})/\Delta^+_{X/S}(\mathcal{O}_{X,x})) \otimes_{\mathcal{O}_X} E.$$ 

This is functor contravariant in the $S$-scheme $X$, and compatible with arbitrary base change of $S$.

In particular, the **Atiyah sequence** is the short exact sequence (locally split), $0 \to \Omega_{X/S} \otimes_{\mathcal{O}_X} E \to \mathcal{O}_{X/S}(E) \to E \to 0$.

For a quasi-coherent $\mathcal{O}_S$-module $F$, for $X = \text{Proj}(\mathcal{F})$, the **Atiyah** sequence of $\mathcal{O}_{\text{Proj}(\mathcal{F})}(1)$ is the Euler sequence twisted by $\mathcal{O}_{\text{Proj}(\mathcal{F})}(1)$, i.e., $\mathcal{O}_{\text{Proj}(\mathcal{F})}(1)$ equals $f^*F$ and the Atiyah sequence is

$$0 \to \Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}(1) \to f^*F \to \mathcal{O}_{\text{Proj}(\mathcal{F})}(1) \to 0$$

For every $S$-scheme $g: Y \to S$ and $S$-morphism $h: Y \to \text{Proj}(\mathcal{F})$, the Atiyah sequence of $h^*\mathcal{O}_{\text{Proj}(\mathcal{F})}(1)$ fits into a push-out diagram:

$$0 \to \Omega_{Y/S} \to \mathcal{O}_{Y/S}(f^*(\mathcal{O}(1))) \to h^*\mathcal{O}(1) \to 0$$

If $g$ is smooth, then $\Omega_{Y/S}$ is locally free of finite rank, and the transpose is

$$0 \to f^*\mathcal{O}(-1) \to \mathcal{O}_{Y/S}(f^*(\mathcal{O}(1))) \to \Omega_{Y/S} \otimes_{\mathcal{O}_S} f^*\mathcal{O}(1(-1)) \to 0$$

$$0 \to f^*\mathcal{O}(-1) \to g^*f^*(\mathcal{O}(1)) \to \Omega_{Y/S} \otimes_{\mathcal{O}_S} f^*\mathcal{O}(1(-1)) \to 0$$
If \( F \) is locally free of finite rank and \( g \) is smooth, these are all locally free \( O_F \)-modules of finite rank. The transpose is
\[
0 \to h^* O_F \to g^* F \to h^* U(1) \to 0
\]
and
\[
0 \to \Omega_{Y/S} \otimes_{O_F} h^* O_F \to P_{Y/S}(h^* U(1)) \to h^* U(1) \to 0
\]
The morphism of projective space bundles over \( P_{S F} \),
\[
P_{P_{S F}}(\Omega_{P_{S F}} \otimes O_{P_{S F}}(-1)) \to P_{S F}(F^*) = P_{S F} \times_S P_{S F}
\]
is the universal hyperplane, i.e., the partial flag bundle \( \text{Flag}(1, n-1, S) \).
The pullback of this bundle over \( Y \) is the “incidence scheme” of hyperplanes intersecting \( Y \). The zero scheme of the composite,
\[
\pi^* (\Omega_{Y/S} \otimes_{O_F} h^* O_F \otimes O_{P_{S F}}(-1)) \to h^* (\Omega_{Y/S} \otimes_{O_F} O_{P_{S F}}(-1)) \to h^* O_{P_{S F}}(+1)
\]
is the scheme of “tangent hyperplanes” to \( Y \).

If \( g \) is unramified, the composite is surjective, and the zero scheme \( \text{Flag} \) is a projective space subbundle of \( C \) equal to the rank of \( \Omega_{Y/S} \otimes_{O_F} h^* O_F \otimes O_{P_{S F}}(-1) \), i.e., the relative dimension of \( Y/S \).

Thus, the codimension of the zero scheme in \( P_{Y}(g^* F^*) \) equals \( 1 + \dim(Y/S) \).

Assuming that \( h \) is quasi-compact and quasi-separated, the image in \( P_{S F}/S \) of this zero scheme is constructible and every irreducible, locally closed subset has fiber dimension over \( S \) strictly less than \( \dim(h) \).

Thus the open complement of the closure of the image is dense in every fiber of \( P_{S F}/S \). This is the maximal open subscheme over which \( Y_{x_{P_{S F}}} \text{Flag}(1, n-1, F) \) is smooth.
1. Criterion for a cohomological/homological $F$-functor to be universal: \((\{F^i\}, \{F^0\})\) is universal if \(\forall i \geq 0\), \(\mathbb{Q} F^i\) is effaceable/injective, i.e., \(\forall \text{ object } M, F\) injection \(M \xrightarrow{w} N\) s.t. \(F^i(w) = 0\). An object \(M\) is \(F\)-acyclic if \(\forall i > 0, F^i(M) = 0\). If every object \(M\) has an injection into an \(F\)-acyclic object then \(F\) is effaceable, thus universal.

Example. Let \(B\) be a flat \(A\)-algebra. Then every projective \(B\)-module is \(A\)-flat. Thus, for every \(A\)-module \(M\) and \(B\)-module \(N\), \(\text{Tor}_p^B(\mathbb{Q} \otimes_A M, N) \cong \text{Tor}_p^A(M, \mathbb{Q} \otimes_A N)\).

2. Left adjoint functors preserve right exactness, colimits & projective objects. Right adjoints preserve left exactness, limits (= inverse limits) and injective objects.

Applications.

1. \(R\text{-mod}\) has enough injective objects.
2. \(\mathbb{A}^n\text{-mod}\) \(\mathcal{O}_X\text{-mod}\) has enough injectives.

3. Injective objects of \(\mathcal{O}_X\text{-mod}\) are flasque.

Given \(v \subset u\), since \(i_v^! \mathcal{O}_V \to i_u^! \mathcal{O}_U\) is injective,

\[\text{Hom}(i_v^! \mathcal{O}_V, F) \to \text{Hom}(i_u^! \mathcal{O}_U, F)\] is surjective.
Prop. 2.5. Flasque sheaves are $\mathcal{O}_X$-cyclic.

Proof: Let $F$ be flasque. Let $0 \to F \to I \to G \to 0$ be a monomorphism to an injective $\mathcal{O}_X$-module.

By earlier argument, $G$ is flasque & $\mathcal{I}(X) \to G(X)$ is surjective. Since $H^i(X, I) = 0$, then $H^i(X, F) = 0$.

But then, since $G$ is flasque, $H^i(X, G) = 0$. ... \( \square \)

Consequence $\Rightarrow$ Prop. 2.6. $H^p(X, -): \mathcal{O}_X \text{-mod} \to \Gamma(X, \text{Ab}_X) \to \mathbb{Z}$ agrees with $\mathcal{O}_X \text{-mod} \to \text{Ab}_X \xrightarrow{H^p(X, -)} \mathbb{Z} \text{-mod}$.

Vanishing thm. of Groth. $X$ Noether of dim $n$ $\implies X$ has cd $\leq n$.

Lemma 2.8. $X$ Noether $\implies \text{colim}_t$ of flasques is flasque.

Prop. 2.9. $X$ Noether $\implies H^p(X, \lim_i F_i) = \lim_i H^p(X, F_i)$.

Lemma 2.10. In general, for a closed embedding $Y \subseteq X$, $H^p(X, i_* F) = H^p(Y, F)$.

Reason: it is exact & sends flasques to flasques.

Pf of thm 1. Reduction 1. Suffices to prove the result when $X$ is irreducible.
Base case. \( \dim X = 0 \), known.

Reduction 2. Every sheaf is a filtered colimit of ines of \( \Phi_i ! \mathbb{Z}_u \rightarrow \mathbb{F} \). So suffices to prove result for ines.

Reduction 3. Using L.E.S., reduce to \( \mathbb{F} = \text{et} \) of \( i_u ! \mathbb{Z}_u \). \( 0 \rightarrow \mathbb{L} \rightarrow i_u ! \mathbb{Z}_u \rightarrow \mathbb{F} \rightarrow 0 \).

Reduction 4. There exists \( \xi \mathbb{Z}_x \rightarrow \mathbb{L} \) inj.

whose cohom has support strictly \( \subset X \). Induction reduces to the case \( \mathbb{F} = i_u ! \mathbb{Z}_u \).

Final step. \( 0 \rightarrow i_u ! \mathbb{Z}_u \rightarrow \mathbb{L}_x \rightarrow i_u ^* \mathbb{Z}_x \rightarrow 0 \)

Since \( \dim Y < \dim X \), \( H^p (X, i_u ^* \mathbb{Z}_x) = 0 \) for \( p \geq n \) \( \Rightarrow \) \( H^q (X, i_u ! \mathbb{Z}_u) = H^q (X, \mathbb{Z}_x) \).

Since \( X \) is irre. \( \mathbb{L}_x \) is flasque. So \( H^q (X, i_u ! \mathbb{Z}_u) = 0 \) for \( q > n \).

Cohom on an affine scheme. B-1 Prop. 5.6, if \( \mathbb{A}^n X \) is affine \& \( \mathbb{F} \) is \( q \)-coh, then \( H^r (X, \mathbb{F}) = 0 \). What about higher cohomology? There is a bootstrap method induction argument.
using Čech cohomology. But it uses spectral sequences, which we will try to avoid. So we will stick to the Noetherian case and give a more direct argument.

Goal. Prove injective objects in the category of $q$-cohom sheaves are flasque, if $X$ is Noetherian.

Reason. Then the same sort of argument as before proves all $H^i(X, F) = 0$.

Before proving this:

Then 3.7. (Serre's criterion for affineness) Let $X$ be a $q$-cpct scheme. TEA

(i) $X$ is affine
(ii) $H^0(X, F) = 0$ for all $q$-coht $F$
(iii) $H^i > 0 (X, I) = 0$ for all $q$-coht $I$ in $\mathcal{O}_X$.

PF: $(i) \implies (ii) \implies (iii)$.

$(iii) \implies (i)$, $0 \to I_{Y_{up}} \to I_Y \to I(p) \to 0$ 

$\exists$ $f \in I_Y$ not in $I_{Y_{up}}$. So $p \in D(f) \subset X - Y = U$. So $X_f = U_f$ is affine. $q$-cpctness filters.
Rash idea: $I$ inj. $\Rightarrow I$ is divisible. $\Rightarrow I \rightarrow I$ soc.

Lem. 3.2: $I$ inj. $\Rightarrow \Gamma_n(I)$ inj.

$\exists J$ s.t. $A \rightarrow J$, by Noeth.

$\exists \psi : \mathcal{C} \rightarrow \mathcal{D}$ such that $\psi(b) = 0$.

Then $\forall \theta : \mathcal{C} \rightarrow \mathcal{D}$, $\psi(\theta(b)) = 0$.

$\exists N$ such that $\forall \theta : \mathcal{C} \rightarrow \mathcal{D}$, $\psi(\theta(a^N b)) = 0$.

$L \rightarrow L \cap 2^N \rightarrow J$.

$A \rightarrow A/2^N \rightarrow J$.

$J \rightarrow J/2^N \rightarrow \cdots$.

$L \rightarrow L/2^N \rightarrow L/2^N \rightarrow \cdots \rightarrow L$, $\cdots$. 