Recap. For a scheme $X$ with $\mathcal{K}_X$ = sheaf of $\mathcal{O}_X$-algebras associated to presheaf $S(W) = \{s \in \mathcal{O}_X(W) \mid s: \mathcal{O}_W \to \mathcal{O}_U \text{ injective}\}$, $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ is the group of invertible $\mathcal{O}_X$-submodules of $\mathcal{K}_X$—Cartier divisors. If $X$ is integral and normal, then the induced map of sheaves, $(\nu_0)_0 : \mathcal{K}_X^*/\mathcal{O}_X^* \to \bigoplus_{\text{prime } \mathfrak{p}_0 \in \mathbb{Z}} \mathcal{O}_{X, \mathfrak{p}_0}$, is injective, where the direct sum ranges over all irreducible closed subsets $D \subseteq X$ whose prime ideal has height one, i.e., whose stalk $\mathcal{O}_{X, \mathfrak{p}_0}$ is a DVR with valuation $\nu_0$ (with convention that $\nu_0^{-1}(\mathbb{Z})$ equals $\mathcal{O}_{X, \mathfrak{p}_0} - \mathcal{O}_{X, \mathfrak{p}_0}^\times$).

The group of global sections of this sheaf is the group of Weil divisors, and the induced map of global sections is the natural map from the group of Cartier divisors to the group of Weil divisors.

**Fact.** A Cartier divisor is effective if and only if each $\nu_0$ is $\geq 0$.

**Proof.** This is the commutative algebra result that an integrally closed integral domain equals the intersection of all localizations at height-one prime ideals.

**Algebra Fact.** A Noetherian integral domain $R$ is a UFD if and only if the Weil divisor class group is zero.

**Corollary.** For a Noetherian UFD $R$, the Weil divisor class group of $R[\frac{1}{D}] = \text{Proj } R[\mathfrak{a}_0, \ldots, \mathfrak{a}_n]$ ($n > 0$) is $\mathbb{Z}$.

**Proof.** By Gauss, $R[\mathfrak{a}_0, \ldots, \mathfrak{a}_n]$ is a UFD. Thus, every homogeneous prime ideal $\mathfrak{p}_D$ of height one is principal with a homogeneous generator, $\mathfrak{p}_D = (F)$, $\deg (F) = e$. The isomorphism of graded modules, $R[\mathfrak{a}_0, \ldots, \mathfrak{a}_n] \cong \mathbb{Z}[F] \to \mathfrak{p}_D$, shows that the class of $D$ equals $(F)$. 

Proposition. For $X$ integral, normal, for $C \subset X$ closed with open complement $U$, there is a diagram of exact sequences

$$\begin{array}{c}
0 \to \bigoplus_{c_i \cup 1} \mathbb{Z} \cdot \langle c_i \rangle \to \text{Weil}(X) \xrightarrow{\text{incl}} \text{Weil}(U) \to 0 \\
\downarrow \quad \downarrow \\
\bigoplus_{c_i \cup 1} \mathbb{Z} \cdot \langle c_i \rangle \to \text{Cl}(X) \to \text{Cl}(U) \to 0
\end{array}$$

Proof. The splitting map $\text{Weil}(U) \to \text{Weil}(X)$ is given by closure.

Proposition. For $X$ integral, normal, $\text{Cl}(X) \to \text{Cl}(\mathbb{A}^n_X)$ is an isomorphism.

Proof. The restriction of each Weil divisor on $\mathbb{A}^n_X$ to the generic fiber $\mathbb{A}^n_{\text{Fr}_k(X)}$ is principal. After modifying by this principal divisor, the prime divisors in the support are disjoint from the generic fiber, i.e., they are inverse images of prime Weil divisors in $X$.

Definition. For a quasi-separated, finite type morphism $\pi : X \to Y$, an invertible $\mathcal{O}_X$-module $L$ is $\pi$-relatively very ample if there exists a dense open immersion of $B$-schemes, $X \subseteq \overline{	ext{Proj}}_B S$

with $L = \pi^* \mathcal{O}(1)$.

Fact. If $B$ affine, then $L$ is very ample $\Rightarrow$ locally closed immersion $\pi : X \subset \mathbb{P}^n_B$ pulling back $\mathcal{O}(1)$ to $L$. Also $f : X \to \mathbb{P}^n_B$ is a locally closed immersion of $B$-schemes $\Rightarrow$ true for each geometric fiber over $B$.

Criterion. For a field $k$, a morphism of finite type $k$-schemes $f : X \to \mathbb{P}^n_k$ is a locally closed immersion $\iff$ both

(i) $\mathcal{O}_X \to f^* \mathcal{O}(1)$ separates points, and

(ii) each $\mathcal{O}_X(1) \to f^* \mathcal{O}(1) \to \mathcal{O}(1)$ is surjective ("separates tangent vectors").
I. Let $X \to \text{Spec } A$ be a finite type, q-sep. morphism. Let $L$ be an invertible sheaf on $X$.

Theorem, Part I. TFAE.

1. $\exists n > 0$ s.t. $L^\otimes n$ is very ample.

2. $\exists n > 0$ and sections $s_0, \ldots, s_r \in \Gamma(X, L^\otimes n)$ s.t. each $D(s_i)$ is quasi-affine.

3. $\exists n > 0$ and sections $s_0, \ldots, s_r \in \Gamma(X, L^\otimes n)$ s.t. each $D(s_i)$ is affine.

Proof. $(1) \Rightarrow (2)$. Let $X \xrightarrow{i} X' \xrightarrow{j} \mathbb{P}^n_A$ be a comm. dis. wr $f^* \mathcal{O}(1) = L^\otimes n$.

Let $s_i = f^* x_i$. Then $D(s_i) = f^{-1}(D_+(x_i)) = i^{-1}(j^{-1}(D_+(x_i)))$.

Because $j$ is a closed immersion, $D_+(x_i)$ is affine, $j^{-1}(D_+(x_i))$ is affine. Because $i$ is open, $i^{-1}(j^{-1}(D_+(x_i)))$ is quasi-affine.

$(2) \Rightarrow (3)$. For every $p \in X$, $\exists \mathcal{F} \in \mathcal{V}$ and $f \in \mathcal{F}$ s.t. $p \in D(f)$ is an open aff. nbhd. Because $X$ is q-cpt, suffices to consider finitely many $f_i$. By basic prop, $\exists N$ s.t. each $s_i^N f_i$ extends to a section $t_i$ of $L^\otimes (N)$.

Form $s_i \cdot s_i^N f_i$ for safe measure. Then $D(s_i, s_i^N f_i) = D(s_i) \cap D(s_i^N f_i) = D(f_i) \cap D(s_i)$ is aff.

$(3) \Rightarrow (1)$. Each $D(s_i) \to \text{Spec } A$ is finite type.
Let $T$ be a section of $L$, $\text{deg}(T) = a$. Then exists a $D \in \text{Div}(X)$ so $T \sim D$.

Proof. (3) $\Rightarrow$ (4): Some $g$ is a generator of $\mathcal{O}(D)$. Hence $T \sim D$.

(5) $\Rightarrow$ (3): $T$ is ample.

(4) $\Rightarrow$ (5): Let $T \sim D + F$. By $\text{Div}(X)$, so $T \sim D$. Then $P_f$. II. Then $\mathcal{O}(D)$ is a closed immersion.

Therefore, $f = X \dashrightarrow Y$. $\mathcal{O}(D)$ is a closed immersion.

Thus, $f: (D, (x_0)) = D(x_0) = \mathcal{O}(D)$. Also, the pullback map $f^*: \mathcal{O}(D) \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$.

Thus, the section of $f^*$ is a unique morphism $f: X \rightarrow Y$. The sections to $\mathcal{O}(D)$ are

for $(f(x_0), (g(x_0)))$, $g \in \mathcal{O}(D)$. By basic facts.

So $f(1, 2, \ldots, n)$ defines a generator of $\mathcal{O}(D)$.
Definitions of relatively ample & relatively very ample. \( f : X \to Y \), \( L \) on \( X \).

1. \( f \) is \( \text{sep} \) \& \( q \)-cplt, \( L \) \( f \)-rel. very ample
   \( \Rightarrow \) \( L \) \( f \)-rel. ample.

2. \( f \) \( \text{f.p.} \) \& \( q \)-sep & \( q \)-cplt, \( L \) \( f \)-rel. ample
   \( \Rightarrow \) \( \exists N > 0 \) s.t. \( L^N \) is \( f \)-rel. very ample.

Relative Proj. P.F. Blowing up \( A = \mathbb{R}^n \).

\( \nu^* I : \mathcal{O}_X = \mathcal{O}_X(1) \). "Universal property."
Denote \( g : Y \to X \). \( (g^* I)_{\text{pure}} := \text{Image } (g^* I \to \text{Hom} (\text{Hom}(g^* I, \mathcal{O}_X), \mathcal{O}_X)) \).
If \( (g^* I)_{\text{pure}} \to \mathcal{O}_Y \) is injective, then \( g \) factors through \( \nu \) if and only if \( g^* I \cdot \mathcal{O}_Y \) is invertible, in which case the factorization is unique.

Local description. \( X = \text{Spec } A, \ I = \langle f_1, \ldots, f_r \rangle \).
Form \( \mathbb{P}^r_A \text{ and } \mathbb{V}(x_1 f_i - x_i f_1) \text{ associated.} \)
If \( X \) is reduced, then \( X = \text{closure of } \mathbb{V} \cap \pi^{-1} (\text{Spec } A - \mathbb{V}(I)) \).
In general \( X = \mathbb{V}(P(x_i) |_P A = 0 \text{ in } A) \).

Thm 7.7. \( f : X \to Y \) birel & proj, \( Y \) q-proj, then \( f \) is isomorphic to \( \text{Bl}_{\nu} Y \) for some (non-unique) \( \nu \).

\[ \text{II. Differentials.} \]
Definition. Let \( f : X \to Y \) be a morphism. Let \( \Delta_{X,Y} : X \to X \times_Y X \) be the diagonal. It is a locally closed immersion. Let \( I \) denote the ideal sheaf.
Then \( \Omega_{X,Y} := \Delta^* I = I/I^2 \).
Denote by \( j : \mathcal{O}_X \to \Omega_{X,Y} \) the morphism \( p_1 \to p_2, p_3 \to p_4 \).
Then $d$ is an $f^*\Omega_Y$-derivation. Moreover, it is the universal $f^*\Omega_Y$-derivation.

Local. $B \cong A$. $\Omega_{B/A} = \mathcal{I}/\mathcal{I}^2$, $\mathcal{I} := \ker(B \otimes_k B \to B)$

$d: B \to \Omega_{B/A}$, $b \mapsto b \partial_1 - 1 \partial_0 b$.

$\phi: B \to M$, $\phi(b \partial_1 - 1 \partial_0 b) = b$. $\phi: B \otimes_k B \to M$

$\phi(b \otimes b \partial_1 - 1 \partial_0 b_2) = b$, $\phi(b \partial_2) = 0, \ldots$

**Fundamental exact sequences**

1. $\xymatrix{ X \ar[r]^f & Y \ar[r]^-{\phi} & Z, \quad f^*\Omega_{Y/Z} \ar[r] & \Omega_{X/Z} \ar[r] & \Omega_{X/Y} \ar[r] & 0}$

2. $\xymatrix{ X \ar[r]^i & Y, \quad \mathcal{O}_X \ar[r]^-{\phi} & \mathcal{O}_Y \ar[r] & \mathcal{O}_{X/Y} \ar[r] & 0}$

**Example.** $A^n_Y \to Y$. $\Omega_{A^n_Y/Y} = \mathcal{O}_{A^n_Y} \{\frac{\partial x_1}{\partial y_1}, \ldots, \frac{\partial x_n}{\partial y_n}\}$

$X \to A^n_Y$, $X = \{x_1, \ldots, x_l\}$. $\Omega_{X/Y} = \mathcal{O}_X \{\frac{\partial x_1}{\partial y_1}, \ldots, \frac{\partial x_n}{\partial y_n}\}$

**Euler sequence** $U \to \mathbb{P}^n_Y$

$0 \to \pi^*\Omega_{\mathbb{P}^n/Y} \ar[r] & \Omega_{U/Y} \ar[r] & \Omega_{U/\mathbb{P}^n} \ar[r] & 0$

$\mathcal{O}_{\mathbb{P}^n} \otimes_{\mathcal{O}_{X/Y}} \mathcal{O}_U \ar[r] & \mathcal{O}_U$

$dx: 1 \to x: \quad \mathbb{P}^n_Y \to \mathbb{P}^n_Y \mathcal{O}_{\mathbb{P}^n} \mathcal{O}_{\mathbb{P}^n}(-1) \ar[r] & \mathcal{O}_{\mathbb{P}^n} \ar[r] & 0$
Def. The morphism $f : X \to Y$ is smooth if
(1) $f$ is flat of relative dim $d$.
(2) $f$ is locally finitely presented.
(3) $\Omega_{X/Y}$ is locally free of rank $d$.

Jacobian Criterion
\[ X = \text{Spec } k, \quad k = \mathbb{K}. \] Then $X$ is smooth iff $\mathcal{O}_X$ is locally f. type & $\mathcal{A}$ (closed point $x$ of $X$), $\mathcal{O}_{x,x}$ is a regular local ring.

Non-example, $\text{char}(k) = p$, $\text{Spec } k \to X$, $\mathcal{O} \leftarrow k$. If $k$ is not a finite field, then
\[ k = \mathbb{F}_p(+) \], $K = k[u]/u^p$. $\text{Spec } K \to \text{Spec } k$

is not smooth. $K_{\text{et}} K = k[u, u^p]/(u^p - 1, u^{p^2} - 1)$

$u, u^2$ generate $\Omega_{K_{\text{et}} K}$, $\Omega_{K_{\text{et}} K} \cong K\langle du \rangle$

Bertini's theorem

- formally smooth: $f'$ exists
- formally unramified: $f'$ is unique if it exists
- formally étale: $f'$ exists and is unique.