Problem Set 9

Disclaimer For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail.

Late homework policy. Late work will be accepted only with a medical note or for another approved reason.

Cooperation policy. You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

Part I. These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

Part II. These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

Part I(25 points)

(a)	(5 points)	p. 169,	Section II.7,	Problem 7.3(a)
(b)	(10 points)	p. 169,	Section II.7,	Problem 7.5(a),(e)
(c)	(10 points)	p. 171,	Section II.7,	Problem 7.12

Part II(25 points)

These exercises establish an assertion made about Exercise 1, Part II of Problem Set 2: There exist ring homomorphisms $A \to B$ such that Spec $B \to$ Spec A is an open immersion, yet B is not of the form $A[1/a]$, i.e., not every open affine in Spec A is a distinguished open affine. The same example can be used to produce a quasi-projective k-scheme X such that $\mathcal{O}_X(X)$ is not a finitely generated k-algebra.

Let k be a field and let X be the curve in $\mathbb{P}^2_k$ with equation $(x-y)^3 - xyz = 0$. This is a nodal plane cubic with node at $Z = [0,0,1]$. Let $\nu : \mathbb{P}^1_k \to \mathbb{P}^2_k$ be the morphism $[T_0, T_1] \mapsto [x,y,z] = [T_0^2 T_1, T_0 T_1^2, (T_0 - T_1)^3]$. As in the hint to Exercise (b), Part I of Problem Set 7, the map ν is the normalization, maps $\mathbb{G}_m$ isomorphically to $X - \{Z\}$ and maps 0 and ∞ to Z. Let $\Lambda : \mathrm{CaCl}^0(X) \to \mathbb{G}_m(k)$ be the isomorphism as defined in that hint.

Problem 1(5 points) Prove that $U := X - \{\nu(1)\}$ is an affine open subscheme of X. (**Hint.** What is the intersection of X with the vanishing locus of z?)

Problem 2(10 points) Let λ be an element of $\mathbb{G}_m(k)$ different from 1. Consider the elements in $\mathcal{O}_X(U)$,

$$f_1 = [(1-\lambda)^2(x-y) - \lambda z]/z, \quad f_2 = [\lambda x^2 - (1+\lambda)xy + y^2]/z^2.$$

Use Exercise II.2.17 to deduce that the open subset $V_\lambda := X - \{\nu(1), \nu(\lambda)\}$ is an affine open subscheme of X, thus an affine open subscheme of U. Later, Exercise III.4.2 will give a more general proof of this fact.

Problem 2(10 points) Assume there exists λ in k which is not a root of unity, i.e., k is not an algebraic extension of a finite field. Now show that V_λ is not a distinguished open affine subscheme of U as follows. If V_λ equals $D(g)$ for some element g of $\mathcal{O}_X(U)$, then the principal divisor of g on X has the form $-m[\nu(1) - \nu(\lambda)]$ for some positive integer m. By the hint for Exercise (b), Part I of Problem Set 7, Λ maps $\mathcal{O}_X(-m[\nu(1) - \nu(\lambda)])$ to λ^{-m} in $\mathbb{G}_m$. Thus, if λ is not a root of unity, then $-m[\nu(1) - \nu(\lambda)]$ is not a principal divisor.

Extra credit(5 points) Let $\mu : \mathbb{G}_m \times \mathbb{P}^1_k \to \mathbb{P}^1_k$ be the morphism sending $(\lambda, [T_0, T_1])$ to $[T_0, \lambda \cdot T_1]$. This is an action of $\mathbb{G}_m$ on $\mathbb{P}^1$. Prove there exists an action μ_X of $\mathbb{G}_m$ on X such that $\nu : \mathbb{P}^1 \to X$ is a $\mathbb{G}_m$-equivariant morphism (for instance, find an open affine covering of $\mathbb{G}_m \times X$ and describe the morphism μ_X on these opens by giving its homogeneous coordinates).

Deduce that for every $\lambda \in \mathbb{G}_m$, $X - \{\nu(\lambda)\}$ is isomorphic to U, and thus is affine. Of course also $X - \{Z\}$ equals $D_+(xy)$. Thus, the complement of every singleton in X is an open affine. Because X is separated, every intersection of open affine schemes is affine. Conclude that the complement of every finite subset of closed points in X is affine. Again, this will follow later from Exercise III.4.2.