Problem Set 8

Disclaimer For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail.
Late homework policy. Late work will be accepted only with a medical note or for another approved reason.

Cooperation policy. You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

Part I. These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

Part II. These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

Part I(25 points)

(a)	(5 points)	p. 146,	Section II.6,	Problem 6.2(c)
(b)	(10 points)	p. 149,	Section II.6,	Problem 6.7
(c)	(10 points)	p. 150,	Section II.6,	Problem 6.11 (b),(c),(d)

Part II(25 points)

Problem 1(5 points) Let X be a scheme satisfying condition (*) on p. 130 of the textbook. For every invertible element f in $\mathcal{O}_X(X)$, prove that the Weil divisor (f) is the zero divisor.

Problem 2(10 points) Now assume further that X is normal, i.e., the stalk of X at every point is integrally closed in its fraction field. Let f be an element of the total ring of fractions $K(X) = \mathcal{O}_{X,\eta_X}$ such that (f) is the zero divisor. Prove that f is the image under restriction of an invertible element in $\mathcal{O}_X(X)$. (Hint: The scheme X has a covering by open affines U such that $\mathcal{O}_X(U)$ in integrally closed. Now use Proposition 6.3A on p. 132 to conclude that f is an invertible element in $\mathcal{O}_X(U)$ for each U.)

Problem 2(10 points) I misspoke in lecture when I asserted the previous result was true whenever X satisfies condition (*) on p. 130 of the textbook. Verify the following gives an example of such a scheme where the previous result fails. Let k be a field of $\mathrm{char}(k) \neq 2$. Let A be the subring of $k[x, y]$ of all elements $p(x, y)$ such that $p(1, 0)$ equals $p(0, 1)$. This is a finitely generated k-algebra by general principles, but here is an explicit argument. The subalgebra A contains the finitely generated k-algebra $B = k[x^2 - x, y^2 - y]$. The ring $k[x, y]$ is a finite B-module since x and y each satisfy obvious monic polynomials over B. Hence the B-submodule A is a finite B-module (by the Hilbert basis theorem). Thus A is a finitely generated k-algebra (use the generators of B together with generators of A as a B-module).

Let Y be the affine scheme Spec A. Let Z be the affine scheme Spec $k[x, y]$. Let $f : Z \to Y$ be the obvious morphism. Verify that f maps $(1, 0)$ and $(0, 1)$ to a single point p of Y. Verify that $f : Z - \{(1, 0), (0, 1)\} \to Y - \{p\}$ is an isomorphism. Conclude that Y satisfies condition (*) (since Z obviously does), $K(Y)$ equals $K(Z)$, the prime Weil divisors of Y are in bijection with the prime Weil divisors of Z, and the associated valuations of $K(Y) = K(Z)$ are equal.

Let X be the affine open subscheme $D((x - y)^2(x + y))$ of Y. Let f be $(x - y)/(x + y)$. Use the previous paragraph to conclude that the Weil divisor of (f) on X is the zero Weil divisor (of course the Weil divisor on Y is *not* the zero divisor, but X is precisely the complement of the support of this divisor). Since $f(1, 0)$ does not equal $f(0, 1)$, f is not in $\mathcal{O}_Y(X)$.

Extra credit(5 points) Compute generators and relations for the k-algebra A. You may make whatever hypothesis you like about the characteristic. This is a good opportunity to use your computer algebra skills.