

## Problem Set 14

**Disclaimer** For open-ended problems, part of the problem is to give a precise formulation. Especially for the problems in Part II, you should do as much of the problems as is useful to you. For each problem, it is important you understand how to verify all details. However, if you are pressed for time, you may write-up only the most important steps, instead of every detail. As an adult, you are expected to use your judgment in interpreting what this precisely means.

**Late homework policy.** Late work will be accepted only with a medical note or for another Institute-approved reason.

**Cooperation policy.** You are strongly encouraged to work with others, but the final write-up must be entirely your own and based on your own understanding.

**Part I.** These problems are from the textbook. You are expected to read *all* the problems from the sections of the textbook covered that week. You are asked to write-up and turn-in only the problems assigned below.

**Part II.** These problems are not necessarily from the textbook. Often they will be exercises in commutative algebra, category theory, homological algebra or sheaf theory.

**Part I**(30 = 25 + 5 points!) You can do any 25 out of 30 points, or do all 30 points to get 5 points extra credit.

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|-----|-------------|---------|----------------|----------------------------|-------------------|
| (a) | (10 points) | p. 217, | Section III.3, | Problem 3.7(a)             | See note below    |
| (b) | (15 points) | p. 188, | Section II.8,  | Problem 8.6(a),(b) and (c) | 5 points for each |
| (c) | (5 points)  | p. 188, | Section II.8,  | Problem 8.7                | See note below    |

**Note (a).** Here is one way to organize the problem. First do the case when  $\mathfrak{a}$  is principal. Then show that formation of the limit is compatible with inclusion of ideals, i.e., if  $\mathfrak{b} \subset \mathfrak{a}$  is an inclusion of ideals there is a natural map of  $A$ -modules

$$\varinjlim_n \mathrm{Hom}_A(\mathfrak{a}^n, M) \rightarrow \varinjlim_n \mathrm{Hom}_A(\mathfrak{b}^n, M).$$

Finally, let  $\mathfrak{a}$  be an arbitrary ideal and let  $f_1, \dots, f_r$  be a set of generators of  $\mathfrak{a}$ . Consider the sequence of  $A$ -modules,

$$\varinjlim_n \mathrm{Hom}_A(\mathfrak{a}^n, M) \rightarrow \prod_{1 \leq i \leq r} \varinjlim_n \mathrm{Hom}_A(f_i^n A, M) \Rightarrow \prod_{1 \leq i < j \leq r} \varinjlim_n \mathrm{Hom}_A((f_i f_j)^n A, M).$$

**Note (c).** First of all, what does Exercise II.8.6 say when  $f : A \rightarrow B$  is the identity homomorphism? Next, what can you get out of Proposition II.5.6?

**Part II**(25 points)

**Problem 1**(15 points) Do Exercise III.4.10 on p. 225. If you want, here is one way to think about it. First of all, use Prop. III.6.3 and Lemma III.6.6 to think of  $H^1(X, \mathcal{F} \otimes \mathcal{T})$  as  $\text{Ext}_{\mathcal{O}_X}^1(\Omega_{X/k}, \mathcal{F})$ , and then think of this as extension classes as in Exercise III.6.1. Given an extension of sheaves of rings (*not necessarily* of  $\mathcal{O}_X$ -modules!)

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_{X'} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

consider the sequence obtained from Proposition II.8.12,

$$\mathcal{F} \longrightarrow \Omega_{X'/k}/\mathcal{F} \cdot \Omega_{X'/k} \longrightarrow \Omega_{X/k} \longrightarrow 0.$$

What *kind* of Abelian sheaves are these (are they modules over some sheaf of rings)? Using Exercise II.8.7, what can you say about this sequence locally? What does this imply globally?

For the other direction, given an extension of  $\mathcal{O}_X$ -modules

$$0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \Omega_{X/k} \longrightarrow 0$$

consider the sheaf,

$$\mathcal{A}(U) := \{(a, e) \in \mathcal{O}_X(U) \oplus \mathcal{E}(U) \mid d(a) = \beta(e)\}.$$

What can you do with this sheaf?

For some real fun, now take a look at Exercise III.5.9.

**Problem 2**(5 points) Let  $A$  be a Noetherian ring. For every  $A$ -module  $M$  and every ideal  $\mathfrak{a} \subset A$ , the  $\mathfrak{a}$ -depth of  $M$ , denoted  $\mathfrak{a} - \text{depth}(M)$ , is the supremum over all nonnegative integers  $r$  for which there exists a sequence of elements  $a_1, \dots, a_r \in \mathfrak{a}$  that are regular for  $M$ , i.e.,  $a_1$  is a nonzero divisor on  $M$ ,  $a_2$  is a nonzerodivisor on  $M/a_1M$ ,  $a_3$  is a nonzerodivisor on  $M/\langle a_1, a_2 \rangle M$ , etc.

For every positive integer  $n$  show that  $\mathfrak{a}^n - \text{depth}(M) \geq \mathfrak{a} - \text{depth}(M)$ . Also, if  $\mathfrak{a} - \text{depth}(M) \geq 1$ , show that the map  $M \rightarrow \text{Hom}_A(\mathfrak{a}, M)$  is injective. Combine these two facts with Exercise III.3.7(a) to deduce that if  $\mathfrak{a} - \text{depth}(M) \geq 1$ , then the following map is injective,

$$\widetilde{M}(\text{Spec } A) \rightarrow \widetilde{M}(\text{Spec } A - \mathbb{V}(\mathfrak{a})).$$

By working with an open affine cover, this implies the following. Let  $X$  be a locally Noetherian scheme, let  $Y$  be a closed subset, and let  $\mathcal{F}$  be a quasi-coherent sheaf on  $X$ . If  $\mathfrak{m}_y - \text{depth}(\mathcal{F}_y) \geq 1$  for every  $y \in Y$ , the following map of  $\mathcal{O}_X(X)$ -modules is an isomorphism,

$$\mathcal{F}(X) \rightarrow \mathcal{F}(X - Y).$$

**Problem 3**(10 points) This problem justifies the connection between Serre's condition  $S_2$  and the algebraic Hartog's phenomenon discussed in lecture. Let  $A$  be a Noetherian ring and let  $M$  be a

finite  $A$ -module. Let  $a_1, a_2$  be a regular sequence for  $M$ . Using the fact that the Koszul complex of  $M$  with respect to  $a_1, a_2$  is acyclic, show that the induced map,

$$M \rightarrow \text{Hom}_A(\langle a_1, a_2 \rangle, M)$$

is an isomorphism. Now use that for every integer  $n$  the sequence  $a_1^n, a_2^n$  is  $M$ -regular to deduce that the following map is an isomorphism,

$$M \rightarrow \varinjlim_n \text{Hom}_A(\langle a_1, a_2 \rangle^n, M).$$

Combined with Exercise III.3.7(a), deduce that the following map is an isomorphism,

$$\widetilde{M}(\text{Spec } A) \rightarrow \widetilde{M}(\text{Spec } A - \mathbb{V}(\langle a_1, a_2 \rangle)).$$

Finally, by working with an open affine covering, using the previous exercise, and choosing length 2 regular sequences contained in  $\mathfrak{m}_y$ , deduce the following result.

Let  $X$  be a locally Noetherian scheme, let  $Y$  be a closed subset, and let  $\mathcal{F}$  be a coherent sheaf. If  $\mathfrak{m}_y - \text{depth}(\mathcal{F}_y) \geq 2$  for every  $y \in Y$ , then the following map of  $\mathcal{O}_X(X)$ -modules is an isomorphism,

$$\mathcal{F}(X) \rightarrow \mathcal{F}(X - Y).$$

**Extra credit**(10 points) Let  $X$  be a reduced, locally Noetherian scheme, let  $Y$  be a closed subset of  $X$ , let  $i : U \rightarrow X$  be the open complement of  $Y$ , and let  $\mathcal{F}$  be a locally free sheaf on  $U$  of finite rank. If  $\mathfrak{m}_y - \text{depth}(\mathcal{O}_{X,y}) \geq 2$  for every  $y \in Y$ , prove that  $i_*\mathcal{F}$  is a coherent sheaf.

**Hint.** First of all, this is local, so assume that  $X$  is affine,  $X = \text{Spec } A$ . For every coherent sheaf  $\mathcal{G}$  on  $U$ , use Exercise II.5.15 to find an integer  $r$  and a map of  $A$ -modules,

$$\phi : A^{\oplus r} \rightarrow \mathcal{G}(U)$$

for which the induced map of  $\mathcal{O}_U$ -modules,

$$\tilde{\phi} : \mathcal{O}_U^{\oplus r} \rightarrow \mathcal{G}$$

is surjective. Iterate this to find integers  $r$  and  $s$  and a sequence of  $A$ -modules,

$$A^{\oplus s} \xrightarrow{\psi} A^{\oplus r} \xrightarrow{\phi} \mathcal{G}(U)$$

such that the following sequence of  $\mathcal{O}_U$ -modules is exact,

$$\mathcal{O}_U^{\oplus s} \xrightarrow{\tilde{\psi}} \mathcal{O}_U^{\oplus r} \xrightarrow{\tilde{\phi}} \mathcal{G} \longrightarrow 0.$$

Now take  $\mathcal{G} = \mathcal{F}^\vee$ , and define  $M$  to be the kernel of the transpose map,

$$M := \text{Ker}(\psi^\dagger : A^{\oplus s} \rightarrow A^{\oplus r}).$$

What can you say about the following map of  $\mathcal{O}_X(X)$ -modules

$$\widetilde{M}(X) \rightarrow \widetilde{M}(X - Y) ?$$