18.725 PROBLEM SET 8

Due date: Wednesday, November 24 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 2, 3, and 4. There will be more problems posted soon, and you will be asked to do 1 more problem to a total of 5.

Required Problem 1: Recall from Definition 14.12 that a regular morphism of varieties $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *projective* if for every open affine $U \subset Y$ there exists a projective variety Z, and a closed immersion $i : F^{-1}(U) \to U \times Z$ such that the restriction morphism $F : F^{-1}(U) \to U$ equals $\mathrm{pr}_U \circ i$. To be precise, this is the definition of *weakly projective*. A regular morphism of varieties $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is *strongly projective* if there exists a projective variety Z and a closed immersion $i : X \to Y \times Z$ such that $F = \mathrm{pr}_Y \circ i$.

Let X be a quasi-projective variety and denote by $j : X \hookrightarrow \mathbb{P}^n_k$ a locally closed immersion. Let $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a regular morphism of algebraic varieties. Prove the following are equivalent,

(i) F is weakly projective,
(ii) F is proper,
(iii) the graph morphism $F \times j : X \to Y \times \mathbb{P}^n_k$ has closed image, and
(iv) F is strongly projective.

Solution:(i)⇒(ii) Corollary 24.17 proves that every weakly projective morphism is proper.

(ii)⇒(iii) The variety $\mathbb{P}^n_k$ is separated, i.e., the constant morphism $\mathbb{P}^n_k \to \mathbb{P}^0_k$ is separated. By Lemma 14.5, separated morphisms satisfy base-change, so $\mathrm{pr}_Y : Y \times \mathbb{P}^n_k \to Y$ is separated. The composition of $F \times j$ and pr_Y is F, which is proper by hypothesis. By Prop. 24.14, $F \times j$ is proper, in particular it is closed. Therefore $(F \times j)(X) \subset Y \times \mathbb{P}^n_k$ is closed.

(iii)⇒(iv) Here is one argument (not the shortest one). By hypothesis, $(F \times j)(X) \subset Y \times \mathbb{P}^n_k$ is a closed subset. To prove that $F \times j$ is a closed immersion, it suffices to prove that $F \times j : X \to (F \times j)(X)$ is an isomorphism. Consider the projection $\mathrm{pr}_{\mathbb{P}^n_k} : (F \times j)(X) \to \mathbb{P}^n_k$. The composition of $\mathrm{pr}_{\mathbb{P}^n_k}$ and $F \times j$ is j, so $\mathrm{pr}_{\mathbb{P}^n_k}((F \times j)(X)) \subset j(X)$. The induced set map $\mathrm{pr}_{\mathbb{P}^n_k} : (F \times j)(X) \to j(X)$ is a regular morphism by the universal property of the induced SWF structure. Because j is a locally closed immersion, $j : X \to j(X)$ is an isomorphism. Therefore $j^{-1} \circ \mathrm{pr}_{\mathbb{P}^n_k} : (F \times j)(X) \to X$ is a regular morphism. It is straightforward that this is an inverse of $F \times j : X \to (F \times j)(X)$, proving that $F \times j : X \to Y \times \mathbb{P}^n_k$ is a closed immersion. Because $\mathrm{pr}_Y \circ (F \times j) = F$, this factorization of F proves F is strongly projective.

(iv)⇒(i) This is obvious.

Required Problem 2 In each of the following cases, X is an irreducible affine variety and $L/k(X)$ is a finite algebraic field extension. In each case compute the associated normalization $F : Y \to X$, i.e., write down the equations defining F in some affine space and the coordinates of the morphism F. In all cases, $\mathrm{char}(k) = 0$.

(a) $X = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2_k$, $L = k(X)$.

Solution: Denote $A = k[X]$ and denote by B the integral closure of A in L. Let $b = y/x \in L$. Then $b^2 = y^2/x^2 = x^3/x^2 = x$. So b satisfies the monic polynomial $t^2 - x$, i.e., $b \in B$. Moreover $x = b^2$ and $y = bx = b^3$. So $k[X] = k[x, y] \subset k[b] \subset B$. Therefore the integral closure of $k[X]$ in L is the integral closure of $k[b]$ in L. But since $k[b] \cong k[t]$ is a UFD, it is already integrally closed by Gauss's Lemma. Thus $B = k[b]$. So $Y = \mathbb{A}^1_k$ and $F : \mathbb{A}^1_k \to X$ by $b \mapsto (b^2, b^3)$ is the normalization.

(b) $X = \mathbb{V}(y^p - x^q) \subset \mathbb{A}^2_k$, p and q are relatively prime positive integers, $L = k(X)$.

Solution: Denote $A = k[X]$ and denote by B the integral closure of A in L. Because p and q are relatively prime, by the division algorithm there exist integers r, s such that $rp + sq = 1$. Let $b = x^r y^s \in L$. Then $b^p = x^{pr} y^{ps}$. Because $y^p = x^q$, this is $b^p = x^{pr+qs} = x^1 = x$. Similarly, $b^q = x^{qr} y^{qs} = y^{pr+qs} = y^1 = y$. Since b satisfies the monic polynomial $t^p - x$, $b \in B$. And $x, y \in k[b]$, so $k[X] \subset k[b] \subset B$. Because $k[b] \cong k[t]$ is a UFD, $k[b]$ is integrally closed by Gauss's Lemma. Thus $B = k[b]$. So $Y = \mathbb{A}^1_k$ and $F : \mathbb{A}^1_k \to X$ by $b \mapsto (b^p, b^q)$ is the normalization.

(c) $X = \mathbb{A}^1_k$, $L = k(X)[t]/\langle t^2 + (1/x)t + 1\rangle$,

Solution: Denote $A = k[X] = k[x]$ and denote by $B \subset L$ the integral closure of A. Let $u = xt \in L$. Then,

$$u^2 = x^2t^2 = x^2(-(1/x)t - 1) = -xt - x^2 = -u - x^2.$$

Since u satisfies the monic polynomial $f(y) = y^2 + y + x^2$, u is in B. Of course $k[X][u] \subset B$ is isomorphic to $C = k[x, y]/\langle y^2 + y + x^2\rangle$. The claim is that C is integrally closed. To prove this, it suffices to prove that the Jacobian ideal of $y^2 + y + x^2$ is the unit ideal in C, because then $\mathbb{V}(y^2 + y + x^2)$ is even smooth. The Jacobian ideal is $\langle 2y + 1, 2x\rangle$. But,

$$1 = (2y + 1)(2y + 1) + (2x)(2x) - 4(y^2 + y + x^2),$$

so $\langle 2y + 1, 2x\rangle C$ is all of C. Therefore $Y = \mathbb{V}(y^2 + y + x^2) \subset \mathbb{A}^2_k$, and $F : Y \to X$ is $F(a, b) = a$.

(d) $X = \mathbb{V}(y^2 - x^2(x - z)) \subset \mathbb{A}^3_k$, $L = k(X)$,

Solution: Denote $A = k[X]$ and denote by B the integral closure of A in L. Let $b = y/x$. Then $b^2 = y^2/x^2 = (x - z)$. Since b satisfies the monic polynomial $t^2 - (x - z)$, $b \in B$. Moreover, $y = bx$ and $z = x - b^2$, so $k[X] \subset k[x, b] \subset B$. But $k[x, b] \cong k[x, y]$ is a UFD, hence integrally closed by Gauss's Lemma. Thus the integral closure of $k[X]$ is $B = k[x, b]$. Therefore $Y = \mathbb{A}^2_k$ and $F : Y \to X$ is $F(a, b) = (a, ab, a - b^2)$.

Required Problem 3 Let X be a variety. A *rank r subbundle of* $X \times \mathbb{A}^n_k$ is a pair (E, ϕ) of a rank r vector bundle E on X together with a morphism of Abelian cones on X, $\phi : E \to X \times \mathbb{A}^n_k$ such that for every point $p \in X$, the corresponding map $\phi_p : E_p \to \mathbb{A}^n_k$ is injective, where E_p denotes the fiber of E over p. An *equivalence of rank r subbundles*, $\psi : (E_1, \phi_1) \to (E_2, \phi_2)$ is a morphism of Abelian cones on X, $\psi : E_1 \to E_2$ such that $\phi_2 \circ \psi = \phi_1$. For every regular morphism $F : Y \to X$ and every rank r subbundle of $X \times \mathbb{A}_k^n$, (E, ϕ), the *pullback subbundle* is defined to be $(Y \times_X E, F^*\phi)$ where $F^*\phi : Y \times_X E \to Y \times \mathbb{A}_k^n$ is $\mathrm{pr}_Y \times (\mathrm{pr}_{\mathbb{A}_k^n} \circ \phi \circ \mathrm{pr}_E)$.

(i) Prove that $F^*\phi$ is injective on fibers.

Solution: For every $y \in Y$, the fiber of $Y \times_X E$ over y is the fiber of E over $x = F(y)$, and the fiber of $Y \times \mathbb{A}_k^n$ is just $\mathbb{A}_k^n$, which is the fiber of $X \times \mathbb{A}_k^n$ over x. The fiber of $F^*\phi$ over y, i.e., $F^*\phi : \{y\} \times_Y (Y \times_X E) \to \{y\} \times_Y (Y \times \mathbb{A}_k^n)$, is the fiber of ϕ over x, which is injective by hypothesis.

(ii) Prove that if (E_1, ϕ_1) and (E_2, ϕ_2) are equivalent rank r subbundles of $X \times \mathbb{A}_k^n$, then $(Y \times_X E_1, F^*\phi_1)$ and $(Y \times_X E_2, F^*\phi_2)$ are equivalent rank r subbundles of $Y \times \mathbb{A}_k^n$.

Solution: Let $\psi : E_1 \to E_2$ be a morphism of Abelian cones such that $\phi_2 \circ \psi = \phi_1$. Then $F^*\psi : Y \times_X E_1 \to Y \times_X E_2$ is a morphism of Abelian cones. Because F^* is a functor, $F^*\phi_2 \circ F^*\psi = F^*\phi_1$. Therefore $(Y \times_X E_1, F^*\phi_1)$ is equivalent to $(Y \times_X E_2, F^*\phi_2)$.

(iii) Let $G : Z \to Y$ be a regular morphism. For every rank r subbundle of $X \times \mathbb{A}_k^n$, (E, ϕ), prove that $(Z \times_X E, (F \circ G)^*\phi)$ is equivalent to $(Z \times_Y (Y \times_X E), G^*(F^*\phi))$.

Solution: The point is that the canonical isomorphism $Z \times_X E \to Z \times_Y (Y \times_X E)$ is an isomorphism of vector bundles over Z. This is straightforward and left to the reader.

Together, (i)–(iii) prove the existence of a contravariant functor,

$$\underline{\mathrm{Grass}(r,n)} : k - \mathrm{Varieties} \to \mathrm{Sets},$$

where $\underline{\mathrm{Grass}(r,n)}(X)$ is the set of equivalence classes of rank r subbundles of $X \times \mathbb{A}_k^n$, and where $\underline{\mathrm{Grass}(r,n)}(F) : \underline{\mathrm{Grass}(r,n)}(X) \to \underline{\mathrm{Grass}(r,n)}(Y)$ is the set map that sends the equivalence class $[(E, \phi)]$ to the equivalence class $[(Y \times_X E, F^*\phi)]$. This functor is called the *Grassmann functor*.

Required Problem 4: This problem proves the existence of a universal object for the Grassmann functor, i.e., a k-variety $\mathrm{Grass}(r,n)$ together with a rank r subbundle of $\mathrm{Grass}(r,n) \times \mathbb{A}_k^n$, (E, ϕ), such that for every variety X and every rank r subbundle (E', ϕ'), there is a unique morphism $F : X \to \mathrm{Grass}(r,n)$ such that $F^*(E, \phi)$ is equivalent to (E', ϕ').

(i) For every r-tuple $\underline{i} = (i_1, \ldots, i_r)$ of integers satisfying $1 \leq i_1 < \cdots < i_r \leq n$, define $U_{\underline{i}} \subset \mathrm{Hom}(\mathbb{A}_k^r, \mathbb{A}_k^n)$ to be the closed subvariety of $n \times r$ matrices such that for every $k, l = 1, \ldots, r$,

$$A_{i_k, l} = \begin{cases} 1, & k = l, \\ 0, & k \neq l \end{cases}$$

Denote by $\phi_{\underline{i}} : U_{\underline{i}} \times \mathbb{A}_k^r \to U_{\underline{i}} \times \mathbb{A}_k^n$ the morphism given by the matrix A. Prove that $(U_{\underline{i}} \times \mathbb{A}_k^r, \phi_{\underline{i}})$ is a rank r subbundle.

Solution: It is clear that this morphism is linear on fibers, thus it is a morphism of Abelian cones. Let $\mathrm{Id}_{U_{\underline{i}}} : \chi_{\underline{i}} : U_{\underline{i}} \times \mathbb{A}_k^n \to U_{\underline{i}} \times \mathbb{A}_k^r$ be the morphism defined below. The composition $(\mathrm{Id}_{U_{\underline{i}}} \times \chi_{\underline{i}}) \circ \phi_{\underline{i}}$ is the identity morphism. Therefore $\phi_{\underline{i}}$ is injective on fibers.

(ii) Let $\underline{i}$ be an r-tuple as above. Denote by $\chi_{\underline{i}} : \mathbb{A}_k^n \to \mathbb{A}_k^r$ the projection of $\mathbb{A}_k^n$ onto the coordinates x_{i_k}, $k = 1, \ldots, r$. Let X be a variety and let (E, ϕ) be a rank r

subbundle of $X \times \mathbb{A}_k^n$ such that composition of ϕ with $\mathrm{Id}_X \times \chi_{\underline{i}} : X \times \mathbb{A}_k^n \to X \times \mathbb{A}_k^r$ is an isomorphism. Prove there exists a unique morphism $F : X \to U_{\underline{i}}$ such that $F^*(U_{\underline{i}} \times \mathbb{A}_k^r, \phi_{\underline{i}})$ is equivalent to (E, ϕ).

Solution: There is only one idea in this solution, which is to convert the morphism ϕ into an $n \times r$ matrix whose entries are elements of $\mathcal{O}_X(X)$, and then use this matrix to define a morphism $X \to \mathrm{Hom}_k(\mathbb{A}_k^r, \mathbb{A}_k^n)$ whose image is contained in $U_{\underline{i}}$. However, the details are a bit tedious. Lemmas are used to organize the details.

Lemma 0.1. *Let X and Y be abstract algebraic varieties and let $(X \times Y, pr_X, pr_Y)$ be a fiber product. The induced k-algebra homomorphism, $pr_X^{\#} \otimes pr_Y^{\#} : \mathcal{O}_X(X) \otimes_k \mathcal{O}_Y(Y) \to \mathcal{O}_{X\times Y}(X \times Y)$, is an isomorphism.*

Proof. The first case is when X and Y are affine algebraic varieties. Then this follows from Cor. 13.9.

The second case is where X is general and Y is affine. Let $(X_\alpha)_{\alpha\in A}$ be an open affine covering of X, and for every pair $\alpha, \alpha' \in A$, let $(X_{\alpha,\alpha',\gamma})_{\gamma\in A_{\alpha,\alpha'}}$ be an open affine covering of $X_\alpha \cap X_{\alpha'}$. By the gluing lemma, there is an exact sequence,

$$0 \to \mathcal{O}_X(X) \to \prod_{\alpha\in A} \mathcal{O}_X(X_\alpha) \to \prod_{(\alpha,\alpha')\in A\times A, \gamma\in A_{\alpha,\alpha'}} \mathcal{O}_X(X_{\alpha,\alpha',\gamma}).$$

Because tensor product of k-vector spaces preserves exact sequences, there is an exact sequence,

$$0 \to \mathcal{O}_X(X)\otimes_k\mathcal{O}_Y(Y) \to \prod_{\alpha\in A} \mathcal{O}_X(X_\alpha)\otimes_k\mathcal{O}_Y(Y) \to \prod_{(\alpha,\alpha')\in A\times A, \gamma\in A_{\alpha,\alpha'}} \mathcal{O}_X(X_{\alpha,\alpha',\gamma})\otimes_k\mathcal{O}_Y(Y).$$

But also $(X_\alpha \times Y)_{\alpha\in A}$ is an open affine covering of $X \times Y$. Using the first case, the sequence above is the exact sequence from the gluing lemma, i.e., $\mathcal{O}_X(X) \otimes \mathcal{O}_Y(Y) \to \mathcal{O}_{X\times Y}(X \times Y)$ is an isomorphism.

The final case where X is arbitrary and Y is arbitrary is proved by precisely the same argument as above, where now the second case is used in place of the first case. $\square$

Corollary 0.2. *For every variety X and every finite-dimensional k-vector space V, the natural k-algebra homomorphism $\mathcal{O}_X(X) \otimes_k Sym^*(V^\vee) \to \mathcal{O}_{X\times\mathbb{A}V}(X \times \mathbb{A}V)$ is an isomorphism.*

For the next lemma, let V and W be finite-dimensional k-vector spaces and let $\mathrm{Hom}_k(V, W)$ be the associated k-vector space of linear transformations. Denote by $\theta_{V,W} : \mathrm{Hom}_k(V, W) \times V \to W$ the unique set map $(T, v) \mapsto T(v)$. For every linear functional x on W, $x \circ \theta_{V,W}$ is a polynomial in linear functions on $\mathrm{Hom}_k(V, W) \times V$, namely,

$$x \circ \theta_{V,W} = \sum_{i=1}^{r} T_{x,\mathbf{v}_i} \circ \mathrm{pr}_{\mathrm{Hom}_k(V,W)} \cdot y_i \circ \mathrm{pr}_V,$$

where $(y_1, \dots, y_r)$ is any basis for $V^\vee$ with dual basis $(\mathbf{v}_1, \dots, \mathbf{v}_r)$, and where $T_{x,\mathbf{v}_i} : \mathrm{Hom}_k(V, W) \to k$ is $T \mapsto y(T(\mathbf{v}_i))$. Because $x \circ \theta_{V,W}$ is always a polynomial function, by the universal property of affine varieties, $\theta_{V,W} : \mathbb{A}\mathrm{Hom}_k(V, W)\times\mathbb{A}V \to \mathbb{A}W$ is a regular morphism. There is an induced map of vector bundles on $\mathbb{A}\mathrm{Hom}_k(V, W)$,

$$\widetilde{\theta}_{V,W} := \mathrm{pr}_{\mathrm{Hom}(V,W)} \times \theta_{V,W} : \mathbb{A}\mathrm{Hom}_k(V, W) \times \mathbb{A}V \to \mathrm{Hom}_k(V, W) \times \mathbb{A}W.$$

Lemma 0.3. *For every variety X and every map of vector bundles $\phi : X \times \mathbb{A}V \to X \times \mathbb{A}W$, there is a unique morphism $F : X \to \mathbb{A}Hom_k(V, W)$ such that $F^*\widetilde{\theta}_{V,W}$ equals ϕ.*

Proof. Consider $\mathrm{pr}_{\mathbb{A}W} \circ \phi : X \times \mathbb{A}V \to \mathbb{A}W$. By the universal property of affine varieties, this is equivalent to the k-algebra homomorphism $k[\mathbb{A}W] \to \mathcal{O}_{X\times\mathbb{A}V}(X \times \mathbb{A}V)$. By Lemma 0.1, $\mathcal{O}_{X\times\mathbb{A}V}(X \times \mathbb{A}V)$ is canonically isomorphic to $\mathcal{O}_X(X) \otimes_k k[\mathbb{A}V]$. Let $(\mathbf{v}_1, \dots, \mathbf{v}_r)$ be a basis for V with dual basis $(y_1, \dots, y_r)$ and let $(\mathbf{w}_1, \dots, \mathbf{w}_n)$ be a basis for W with dual basis $(x_1, \dots, x_n)$. Because ϕ is linear on fibers, for every $i = 1, \dots, n$, $(\mathrm{pr}_{\mathring{A}W} \circ \phi)^\#(x_i) = \sum_{j=1}^r a_{i,j} y_j$, for elements $a_{i,j} \in \mathcal{O}_X(X)$. By the universal property of affine varieties, there is a unique morphism $F : X \to \mathbb{A}\mathrm{Hom}_k(V, W)$ such that for every $1 \le i \le n$ and $1 \le j \le r$, $F^\#(T_{x_i,\mathbf{v}_j}) = a_{i,j}$. It is straightforward to check this is the unique regular morphism such that $F^*\widetilde{\theta}_{V,W}$ equals ϕ. □

Lemma 0.3 solves the problem, after a simple reduction of the original problem about rank r subbundles up to equivalence to a problem about morphisms $X\times\mathbb{A}^r_k \to X\times\mathbb{A}^n_k$ up to *equality.* As used in (i), observe that the composition of $\phi_{\underline{i}}$ and $\mathrm{Id}_{U_{\underline{i}}}\times\chi_{\underline{i}}$ is the identity morphism. For every morphism $F : X \to U_{\underline{i}}$, denote by

$$\alpha_F : X \times \mathbb{A}^r_k \to X \times_{U_{\underline{i}}} (U_{\underline{i}} \times \mathbb{A}^r_k),$$

the canonical isomorphism. Then for every morphism $F : X \to U_{\underline{i}}$, $(X \times \mathbb{A}^r_k, F^*\phi_{\underline{i}} \circ \alpha_F)$ is a rank r subbundle of $X \times \mathbb{A}^n_k$ with the additional property that $(\mathrm{Id}_X \times \chi_{\underline{i}}) \circ (F^*\phi \circ \alpha_F)$ is the identity morphism.

Denote by $\xi : X\times\mathbb{A}^r_k \to E$ the inverse of $(\mathrm{Id}_X\times\chi_{\underline{i}})\circ\phi$. Then $\phi\circ\xi : X\times\mathbb{A}^r_k \to X\times\mathbb{A}^n_k$ is the unique morphism such that both

(i) $(X \times \mathbb{A}^r_k, \phi \circ \xi)$ is a rank r subbundle equivalent to (E, ϕ), and
(ii) $(\mathrm{Id}_X \times \chi_{\underline{i}}) \circ (\phi \circ \xi)$ is the identity morphism $X \times \mathbb{A}^r_k \to X \times \mathbb{A}^r_k$.

By Lemma 0.3, there is a unique morphism $F : X \to \mathrm{Hom}_k(\mathbb{A}^r_k, \mathbb{A}^n_k)$ such that $F^*\phi_{\underline{i}} \circ \alpha_F$ equals $\phi \circ \xi$. Because $(\mathrm{Id} \times \chi_{\underline{i}}) \circ \phi \circ \xi$ is the identity, the image of F is contained in $U_{\underline{i}}$. Therefore $F : X \to U_{\underline{i}}$ is the unique morphism such that $F^*\phi_{\underline{i}}\circ\alpha_F$ equals $\phi \circ \xi$. By the previous paragraph, $F : X \to U_{\underline{i}}$ is the unique morphism such that $(X \times \mathbb{A}^r_k, F^*\phi_{\underline{i}} \circ \alpha_F)$ is equivalent to (E, ϕ).

(iii) For every pair of r-tuples $(\underline{i}, \underline{j})$, define $U_{\underline{i},\underline{j}} \subset U_{\underline{i}}$ to be the open set where the $r \times r$ submatrix $(A_{j_k,l})$ is invertible, i.e., the distinguished open affine of the determinant of this $r\times r$ matrix. Restricting $(U_{\underline{i}}, \phi_{\underline{i}})$ to $U_{\underline{i},\underline{j}}$, prove the composition of $\phi_{\underline{i}}$ with $\mathrm{Id}\times\chi_{\underline{j}}$ is an isomorphism. Deduce existence of a morphism $u_{\underline{i},\underline{j}} : U_{\underline{i},\underline{j}} \to U_{\underline{j},\underline{i}}$.

Solution: Denote by $D \in k[A_{i,j}|1 \le i \le n, 1 \le j \le r]$ the determinant of the $r\times r$ matrix $(A_{j_k,l}|1 \le k, l \le r)$. Then

$$\mathcal{O}_{U_{\underline{i}}}(U_{\underline{i},\underline{j}}) = k[A_{i,j}|1 \le i \le n, 1 \le j \le n][1/D]/\langle A_{i_k,l} - \delta_{k,l}\rangle.$$

By Cramer's rule, for every $1 \le k, l \le r$ there exists $B_{k,l} \in \mathcal{O}_{U_{\underline{i}}}(U_{\underline{i},\underline{j}})$ such that the matrix $(B_{k,l})$ is an inverse of the matrix $(A_{j_k,l})$. By the universal property of affine varieties, there exists a unique regular morphism, $\mathrm{pr}_{\mathbb{A}^r_k} \circ \widetilde{B}_{\underline{i},\underline{j}} : U_{\underline{i},\underline{j}} \times \mathbb{A}^r_k \to \mathbb{A}^r_k$, such that for every $1 \le k \le r$, $(\mathrm{pr}_{\mathbb{A}^r_k} \circ \widetilde{B}_{\underline{i},\underline{j}})^\#(y_k) = \sum_{l=1}^r B_{k,l} y_l$. Denote by $\widetilde{B}_{\underline{i},\underline{j}}$ the morphism $\mathrm{pr}_{U_{\underline{i},\underline{j}}} \times (\mathrm{pr}_{\mathbb{A}^r_k} \circ \widetilde{B}_{\underline{i},\underline{j}}) : U_{\underline{i},\underline{j}} \times \mathbb{A}^r_k \to U_{\underline{i},\underline{j}} \times \mathbb{A}^r_k$. This is a morphism of

Abelian cones and is the inverse of $(\mathrm{Id} \times \chi_{\underline{j}}) \circ \phi_{\underline{i}}$ (restricted to $U_{\underline{i},\underline{j}}$). By (ii), there exists a unique regular morphism $u_{\underline{i},\underline{j}} : U_{\underline{i},\underline{j}} \to U_{\underline{j}}$ such that

$$\widetilde{A}_{\underline{j}} \circ ((u_{\underline{i},\underline{j}} \circ \mathrm{pr}_{U_{\underline{i},\underline{j}}}) \times \mathrm{pr}_{\mathbb{A}^r_k}) = \widetilde{A}_{\underline{i}}|_{U_{\underline{i},\underline{j}}} \circ \widetilde{B}_{\underline{i},\underline{j}}$$

In other words,

$$u^*_{\underline{i},\underline{j}}\phi_{\underline{j}} \circ \alpha_{u_{\underline{i},\underline{j}}} = \phi_{\underline{i}}|_{U_{\underline{i},\underline{j}}} \circ \widetilde{B}_{\underline{i},\underline{j}}.$$

(iv) Prove the image of $u_{\underline{i},\underline{j}}$ is contained in $U_{\underline{j},\underline{i}}$ and that $u_{\underline{i},\underline{j}}$ and $u_{\underline{j},\underline{i}}$ are inverse isomorphisms.

Solution: The open subscheme $U_{\underline{j},\underline{i}} \subset U_{\underline{j}}$ is the largest open subset over which $(\mathrm{Id} \times \chi_{\underline{i}}) \circ \phi_{\underline{j}}$ is an isomorphism. So to prove the image of $u_{\underline{i},\underline{j}}$ is contained in $U_{\underline{j},\underline{i}}$, it suffices to prove the following is an isomorphism,

$$(\mathrm{Id} \times \chi_{\underline{i}}) \circ u^*_{\underline{i},\underline{j}}\phi_{\underline{j}} \circ \alpha_{u_{\underline{i},\underline{j}}}.$$

By (iii), this equals

$$(\mathrm{Id} \times \chi_{\underline{i}}) \circ \phi_{\underline{i}}|_{U_{\underline{i},\underline{j}}} \circ \widetilde{B}_{\underline{i},\underline{j}}.$$

By (i), $(\mathrm{Id} \times \chi_{\underline{i}}) \circ \phi_{\underline{i}}$ is the identity morphism. Therefore the morphism above is $\widetilde{B}_{\underline{i},\underline{j}}$, which is an isomorphism by construction. So the image of $u_{\underline{i},\underline{j}}$ is contained in $U_{\underline{j},\underline{i}}$.

Moreover, $u^*_{\underline{i},\underline{j}}\phi_{\underline{j}}$ is equivalent to $\phi_{\underline{i}}|_{U_{\underline{i},\underline{j}}}$ and $u^*_{\underline{j},\underline{i}}\phi_{\underline{i}}$ is equivalent to $\phi_{\underline{j}}|_{U_{\underline{j},\underline{i}}}$. By Problem 3(ii) and (iii), $u^*_{\underline{i},\underline{j}}u^*_{\underline{j},\underline{i}}\phi_{\underline{i}}|_{U_{\underline{i},\underline{j}}}$ is equivalent to $\phi_{\underline{i}}|_{U_{\underline{i},\underline{j}}}$. By the uniqueness in (i), $u_{\underline{i},\underline{j}} \circ u_{\underline{j},\underline{i}} = \mathrm{Id}_{U_{\underline{i},\underline{j}}}$. By symmetry, also $u_{\underline{j},\underline{i}} \circ u_{\underline{i},\underline{j}} = \mathrm{Id}_{U_{\underline{j},\underline{i}}}$. So these are inverse isomorphisms.

(v) Prove the collection $((U_{\underline{i}}), (U_{\underline{i},\underline{j}}), (u_{\underline{i},\underline{j}}))$ satisfies the gluing lemma for varieties. Denote the associated variety by $\iota_{\underline{i}} : U_{\underline{i}} \hookrightarrow \mathrm{Grass}(r, n)$.

Solution: Given $\underline{i}, \underline{j}$ and $\underline{k}$, $U_{\underline{j},\underline{i}} \cap U_{\underline{j},\underline{k}} \subset U_{\underline{j},\underline{i}}$ is the largest open subset where $(\mathrm{Id} \times \chi_{\underline{k}}) \circ \phi_{\underline{j}}$ is an isomorphism. By the same sort of argument as in (iii), $u^{-1}_{\underline{i},\underline{j}}(U_{\underline{j},\underline{i}} \cap U_{\underline{j},\underline{k}})$ is the largest open subset of $U_{\underline{i},\underline{j}}$ where $(\mathrm{Id} \times \chi_{\underline{k}}) \circ \phi_{\underline{i}}$ is an isomorphism, i.e.,

$$u^{-1}_{\underline{i},\underline{j}}(U_{\underline{j},\underline{i}} \cap U_{\underline{j},\underline{k}}) = U_{\underline{i},\underline{j}} \cap U_{\underline{i},\underline{k}}.$$

Moreover, the restriction to $U_{\underline{i},\underline{j}} \cap U_{\underline{i},\underline{k}}$ of both $u_{\underline{j},\underline{k}} \circ u_{\underline{i},\underline{j}}$ and $u_{\underline{i},\underline{k}}$ are morphisms that pullback $\phi_{\underline{k}}$ to a rank r bundle equivalent to $\phi_{\underline{i}}$. Therefore by the uniqueness in (i), these morphisms are equal. So the datum satisfies the hypothesis for the gluing lemma for morphisms.

(vi) Prove there exists a unique rank r subbundle of $\mathrm{Grass}(r, n) \times \mathbb{A}^n_k$, (E, ϕ), such that for every $\underline{i}$, $(\iota_{\underline{i}})^*(E, \phi)$ is equivalent to $(U_{\underline{i}} \times \mathbb{A}^r_k, \phi_{\underline{i}})$.

Solution: For every $\underline{i}$, define $E_{\underline{i}} \subset \iota_{\underline{i}}(U_{\underline{i}}) \times \mathbb{A}^r_k$ to be the image of $((\iota \circ \mathrm{pr}_{U_{\underline{i}}}) \times \mathrm{pr}_{\mathbb{A}^r_k}) \circ \phi_{\underline{i}}$. This is the closed subvariety $\mathbb{V}(y_i - \sum_{k=1}^r A_{i_k,j} y_j, 1 \leq i \leq n)$, and the restriction of $\mathrm{Id} \times \chi_{\underline{i}}$ is an isomorphism $E_{\underline{i}} \to \iota_{\underline{i}}(U_{\underline{i}}) \times \mathbb{A}^r_k$. Because of (ii), the restrictions of $E_{\underline{i}}$ and $E_{\underline{j}}$ to $\iota_{\underline{i}}(U_{\underline{i},\underline{j}}) = \iota_{\underline{j}}(U_{\underline{j},\underline{i}})$ are equal as subvarieties of $U \times \mathbb{A}^n_k$. Therefore there is a unique closed subvariety $E \subset \mathrm{Grass}(r, n) \times \mathbb{A}^n_k$ whose restriction to every $\iota_{\underline{i}}(U_{\underline{i}}) \times \mathbb{A}^n_k$ is $E_{\underline{i}}$. Denote by $\phi : E \to \mathrm{Grass}(r, n) \times \mathbb{A}^n_k$ the inclusion morphism. By construction, this is a subbundle such that for every $\underline{i}$, $(\iota_{\underline{i}})^*(E, \phi)$ is equivalent to $(U_{\underline{i}} \times \mathbb{A}^r_k, \phi_{\underline{i}})$.

(vii) Use (ii) to prove that $\mathrm{Grass}(r,n)$ and (E,ϕ) have the universal property.

Solution, Uniqueness: Let X be a variety, let (E_X,ϕ_X) be a rank r subbundle of $X\times\mathbb{A}^n_k$, and let $F_1,F_2 : X\to\mathrm{Grass}(r,n)$ be morphisms such that the pullbacks by F_1 and F_2 of (E,ϕ) are both equivalent to (E_X,ϕ_X). By construction $\iota_{\underline{i}}(U_{\underline{i}})\subset\mathrm{Grass}(r,n)$ is the largest open subset over which $(\mathrm{Id}\times\chi_{\underline{i}})\circ\phi$ is an isomorphism. Therefore, both $F_1^{-1}(U_{\underline{i}})$ and $F_2^{-1}(U_{\underline{i}})$ are equal to the largest open subset of X over which $(\mathrm{Id}\times\chi_{\underline{i}})\circ\phi_X$ is an isomorphism. Denote this open subset by $X_{\underline{i}}$. Then $\iota_{\underline{i}}^{-1}\circ F_1|_{X_{\underline{i}}}$ and $\iota_{\underline{i}}^{-1}\circ F_2|_{X_{\underline{i}}}$ are both morphisms such that the pullback of $(U_{\underline{i}}\times\mathbb{A}^r_k,\phi_{\underline{i}})$ are equivalent to the restriction to $X_{\underline{i}}$ of (E_X,ϕ_X). By the uniqueness in (ii), these two morphisms are equal. Therefore the restriction of F_1 and F_2 to $X_{\underline{i}}$ are equal for every $\underline{i}$, i.e., $F_1=F_2$.

Existence: Suppose that there exists an open covering (X_α) of X, and for every α there exists a morphism $F_\alpha : X_\alpha\to\mathrm{Grass}(r,n)$ with the property. By the uniqueness above, the datum (X_α,F_α) satisfies the hypotheses for the gluing lemma for morphisms, and thus there exists a morphism $F : X\to\mathrm{Grass}(r,n)$ such that for every α, $F|_{X_\alpha}=F_\alpha$. Then, to construct an equivalence $\psi : X\times_{\mathrm{Grass(r,n)}}E\to E_X$, again by the gluing lemma it suffices to construct an equivalent ψ_α over X_α for every α, which follows from the property of F_α. Therefore it suffices to prove there exists an open covering (X_α) of X, and for every α prove there exists a morphism $F_\alpha : X_\alpha\to\mathrm{Grass}(r,n)$ with the property for the restriction to X_α of (E_X,ϕ_X).

In particular, X is covered by open subsets over which E_X is trivial. Therefore, by the previous paragraph, it suffices to consider the case when $E_X=X\times\mathbb{A}^r_k$.

For every $\underline{i}$, the morphism $(\mathrm{Id}\times\chi_{\underline{i}})\circ\phi_X : X\times\mathbb{A}^r_k\to X\times\mathbb{A}^r_k$ is equivalent to an $r\times r$ matrix whose entries are elements of $\mathcal{O}_X(X)$. The determinant is an element $D_{\underline{i}}\subset\mathcal{O}_X(X)$. Define $X_{\underline{i}}\subset X$ to be the open subset where $D_{\underline{i}}$ is nonzero. For every $\underline{i}$, there is a morphism $F'_{\underline{i}} : X_{\underline{i}}\to U_{\underline{i}}$ as in (ii), and $F_{\underline{i}}=\iota_{\underline{i}}\circ F'_{\underline{i}} : X_{\underline{i}}\to\mathrm{Grass}(r,n)$ satisfies the property for the restriction of (E_X,ϕ_X) to $X_{\underline{i}}$. So, by the same argument as in the last paragraph, it suffices to prove that the open subsets $(X_{\underline{i}}|\underline{i})$ cover X.

For every $p\in X$, the fiber $\phi_X : \{p\}\times_X(X\times\mathbb{A}^r_k)\to\{p\}\times_X(X\times\mathbb{A}^n_k)$ is an injective map of vector spaces. In other words, the matrix of the induced linear transformation $\mathbb{A}^r_k\to\mathbb{A}^n_k$ has rank r. Thus some $r\times r$ minor is nonzero, i.e., there exists $\underline{i}$ such that $D_{\underline{i}}$ is nonzero at p. Therefore $p\in X_{\underline{i}}$, i.e., $(X_{\underline{i}}|\underline{i})$ is an open covering of X.

Problem 5: In this problem, do at least 2 of the parts (but you don't have to do all the parts). Recall for every integer $r\geq 0$, every vector space V and every vector space W, an *alternating, r-multilinear map* is a map $T : V^r\to W$ such that,

(i) for every $i=1,\dots,r$, and for every $(r-1)$-tuple $\underline{\mathbf{v}}=(\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{v}_{i+1},\mathbf{v}_r)\in V^{r-1}$, the map $T_{\underline{\mathbf{v}}} : V\to W$, $\mathbf{v}\mapsto T(\mathbf{v}_1,\dots,\mathbf{v}_{i-1},\mathbf{v},\mathbf{v}_{i+1},\dots,\mathbf{v}_r)$, is a k-linear map, and

(ii) for every $1\leq i<j\leq r$, for every r-tuple $\underline{\mathbf{v}}=(\mathbf{v}_1,\dots,\mathbf{v}_r)\in V^r$, $T(\underline{\mathbf{v}})=\mathbf{0}$ if $\mathbf{v}_i=\mathbf{v}_j$.

A pair $(\bigwedge^r(V),\tau)$ of a k-vector space $\bigwedge^r(V)$ and an alternating, r-multilinear map $\tau : V^r\to\bigwedge^r(V)$ is an r^{th} *exterior power of* V if for every alterating, r-multilinear map $T : V^r\to W$, there exists a unique k-linear map $L : \bigwedge^r(V)\to W$ such that

$T = L \circ \tau$. If the r^{th} exterior power of V exists (which it does!), it is unique up to unique isomorphism.

Let V be a finite-dimensional k-vector space and let $\mathcal{B} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an ordered basis for V. Define $\bigwedge^r(V)$ to be the free k-vector space with finite basis denoted $\mathcal{B}^{(r)} = (\mathbf{v}_{\underline{i}} | \underline{i} \in \Sigma_{n,r})$ where $\Sigma_{n,r}$ is the finite set,

$$\Sigma_{n,r} = \{\underline{i} = (i_1, \dots, i_r) | 1 \leq i_1 < \dots < i_r \leq n\}.$$

Define $\tau : V^r \to \bigwedge^r(V)$ to be the unique alternating, r-multilinear map such that for every $\underline{i} \in \Sigma_{n,r}$, $\tau(\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_r}) = \mathbf{v}_{\underline{i}}$.

(i) Prove that $(\bigwedge^r(V), \tau)$ is an r^{th} exterior power of V.

Solution: This is a standard result of multilinear algebra.

(ii) Let $L : V_1 \to V_2$ be a k-linear map of vector spaces, let $(\bigwedge^r(V_1), \tau_1)$ be an r^{th} exterior power of V_1 and let $(\bigwedge^r(V_1), \tau_2)$ be an r^{th} exterior power of V_2. Prove there exists a unique k-linear map $\bigwedge^r(L) : \bigwedge^r(V_1) \to \bigwedge^r(V_2)$ such that $\bigwedge^r(L) \circ \tau_1 = \tau_2 \circ (L^r)$.

Solution: The map $\tau_2 \circ (L^r) : V_1^r \to \bigwedge^r(V_2)$ is r-multilinear and alternating. By the universal property, there exists a unique k-linear map $\bigwedge^r(L) : \bigwedge^r(V_1) \to \bigwedge^r(V_2)$ such that $\bigwedge^r(L) \circ \tau_1 = \tau_2 \circ (L^r)$.

(iii) Let $\bigwedge^r$ be a rule that assigns to every k-vector space V an r^{th} exterior power $(\bigwedge^r(V), \tau)$. Prove there exists an associated covariant functor $\bigwedge^r : k-$ Vector spaces $\to k-$ Vector spaces which associates to every vector space V the vector space $\bigwedge^r(V)$ and which associates to every k-linear map $L : V_1 \to V_2$ the k-linear map $\bigwedge^r(L)$, i.e., check this rule respects identity morphisms and composition of k-linear maps. **Remark:** The only issue in defining such a functor is that the r^{th} exterior power is not unique – it is only unique up to unique isomorphism. This is not a serious issue (there is a canonical choice which is a quotient vector space of the free vector space with basis V^r).

(iv) In the same manner as Problem 8 from Problem Set 5, extend the notion of exterior power to vector bundles.

Solution: There are different solutions to this problem: each produces the same answer, but each emphasizes a different property of the answer. Here is one solution.

Let E, E' be Abelian cones over X. Denote by $E^{(r)}$ the r-fold fiber product $E^{(r)} = E \times_X E \times_X \cdots \times_X E$.

Definition 0.4. An *alternating, r-multilinear morphism of Abelian cones from E to E'* is a regular morphism $T : E^{(r)} \to E'$ such that,

(i) $\text{pr}_X \circ T : E^{(r)} \to X$ equals $\text{pr}_X : E^{(r)} \to X$, i.e., T is compatible with projection to X, and

(ii) for every $x \in X$, denoting by $E|_x = \{x\} \times_X E$ and $E'|_x = \{x\} \times_X E'$ the induced k-vector spaces, $T|_x : (E|_x)^r \to E'|_x$ is an alternating, r-multilinear map of k-vector spaces.

Definition 0.5. Let E be a vector bundle over X. An *r^{th} exterior power of E* is a pair $(\bigwedge^r(E), \tau)$ of a vector bundle $\bigwedge^r(E)$ over X together with an alternating, r-multilinear morphim of Abelian cones, $\tau : E^{(r)} \to \bigwedge^r(E)$, such that for every element $x \in X$, the restriction $\tau|_x$ is an r^{th} exterior power of $E|_x$.

Lemma 0.6. *Let E be a vector bundle over X, and let $(\bigwedge^r(E), \tau)$ be an r^{th} exterior power of E. For every morphism $F : Y \to X$, $(Y \times_X \bigwedge^r(E), F^*\tau \circ \alpha_F)$ is an r^{th} exterior power of $Y \times_X E$, where $\alpha_F : (Y \times_X E)^{(r)} \to Y \times_X E^{(r)}$ is the canonical isomorphism.*

Proof. This is straightforward. □

In particular, if V is a k-vector space and $(\bigwedge^r(V), \tau)$ is an r^{th} exterior power, then for $X = \mathbb{A}_k^0$ and $E = \mathbb{A}V$, the pair $(\mathbb{A}(\bigwedge^r(V)), \mathbb{A}\tau)$ is an r^{th} exterior power of E. By the lemma, for every variety X, $(X \times \mathbb{A}(\bigwedge^r(V)), \text{Id}_X \times \mathbb{A}\tau)$ is an r^{th} exterior power of $X \times \mathbb{A}V$ over X.

Let E be an Abelian cone over X with corresponding sheaf of sections $\mathcal{E}_{\text{sec}}$. In particular, $\mathcal{E}_{\text{sec}}(X)$ is an $\mathcal{O}_X(X)$-module in a natural manner. For every finite-dimensional vector space V, setting $E = X \times \mathbb{A}V$, the $\mathcal{O}_X(X)$-module $\mathcal{E}_{\text{sec}}(X)$ is canonically isomorphic to $\mathcal{O}_X(X) \otimes_k V$.

Let V be a finite-dimensional k-vector space and let E be an Abelian cone over X. For every morphism of Abelian cones, $\psi : X \times \mathbb{A}V \to E$, there is an induced map of $\mathcal{O}_X(X)$-vector spaces $\psi_* : \mathcal{O}_X(X) \otimes_k V \to \mathcal{E}_{\text{sec}}(X)$. By adjointness, this is equivalent to a map of k-vector spaces, $\psi_* : V \to \mathcal{E}_{\text{sec}}(X)$.

Lemma 0.7. *For every finite-dimensional k-vector space, V, and every Abelian cone over X, E, the following induced map is an isomorphism,*

$$Hom_{Ab.\ cone}(X \times \mathbb{A}V, E) \to Hom_{k-Vect.\ sp.}(V, \mathcal{E}_{sec}(X)).$$

Proof. **Injectivity:** Let $\psi : X \times \mathbb{A}V \to E$ be a morphism of Abelian cones such that ψ_* is the zero map. For every $v \in V$, there is a unique global section $s_v : X \to \mathbb{A}V$ whose projection to $\mathbb{A}V$ is the constant morphism with image v. By hypothesis, $\psi \circ s_v$ is the zero map. Therefore, for every $x \in X$, for every $v \in V$, $\psi(x, v) = \mathbf{0}_x \in \{x\} \times E$, i.e., ψ is the zero morphism.

Surjectivity: Let $L : V \to \mathcal{E}_{\text{sec}}(X)$ be a map of k-vector spaces. Let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an ordered basis of V. For every $i = 1, \dots, n$, let $s_i : X \to E$ be the global section which is $L(\mathbf{v}_i)$. Because E is an Abelian cone, there is a multiplication morphism $m : \mathbb{A}_k^1 \times E \to E$. Composing this with the morphism $\text{pr}_{\mathbb{A}^1} \times (s_i \circ \text{pr}_X) : \mathbb{A}_k^1 \times X \to \mathbb{A}_k^1 \times E$, and using the canonical isomorphism $\mathbb{A}_k^1 \times X \cong X \times \mathbb{A}_k^1$ gives a morphism $X \times \mathbb{A}_k^1 \to E$, denoted $\widetilde{s}_i$. The n-fold fiber product of $X \times \mathbb{A}_k^1$ over X is canonically isomorphic to $X \times \mathbb{A}_k^n$. Via the basis $(\mathbf{v}_1, \dots, \mathbf{v}_n)$, $X \times \mathbb{A}_k^n$ is canonically isomorphic to $X \times \mathbb{A}V$. Therefore there is an induced morphism

$$\widetilde{s}_1 \times \cdots \times \widetilde{s}_n : X \times \mathbb{A}V \to E \times_X \cdots \times_X E.$$

Composing with the addition map $E \times_X \cdots \times_X E \to E$ gives a morphism $\psi : X \times \mathbb{A}V \to E$. It is straightforward to check this is a morphism of Abelian cones and $\psi_* = L$. □

Lemma 0.8. *For every X and every finite-dimensional vector space V, the r^{th} exterior power of $X \times \mathbb{A}V$, $(X \times \mathbb{A}\bigwedge^r(V), \mathbb{A}\tau)$, has the universal property: For every Abelian cone E and every r-multilinear, alternating morphism of Abelian cones $T : (X \times \mathbb{A}V)^{(r)} \to E$, there is a unique morphism of Abelian cones $L : X \times \mathbb{A}\bigwedge^r(V) \to E$ such that $L \circ \tau = T$.*

Proof. **Uniqueness:** Let $L : X \times \mathbb{A}\bigwedge^r(V) \to E$ be a morphism of Abelian cones such that $L \circ \tau = T$. For every $x \in X$, conside the morphism $T|_x : V^r \cong (X \times \mathbb{A}V)^{(r)}|_x \to E|_x$. This is an r-multilinear, alternating map of k-vector spaces. And the induced map of k-vector spaces $L|_x : \bigwedge^r(V) \cong (X \times \mathbb{A}\bigwedge^r(v))|_x \to E|_x$ is a map of k-vector spaces such that $L|_x \circ \tau|_x = T|_x$. Because $\bigwedge^r(V)$ is an r^{th} exterior power of V, there is a unique such map $L|_x$. Since this holds for every $x \in X$, there is at most one morphism of Abelian cones $L : X \times \mathbb{A}\bigwedge^r(V) \to E$ such that $L \circ \tau = T$.

Existence: Let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an ordered basis for V. For every $\underline{i} \in \Sigma_{n,r}$, let $T \circ s_{\underline{i}} : X \to E$ be the global section obtained by composing T with the global section $s_{\underline{i}} = s_{\mathbf{v}_{i_1}} \times \cdots \times s_{\mathbf{v}_{i_r}} : X \to (X \times \mathbb{A}V)^{(r)}$. There is a unique map of k-vector space $\bigwedge^r(V) \to \mathcal{E}_{\text{sec}}(X)$ such that for every $\underline{i} \in \Sigma_{n,r}$, $\mathbf{v}_{\underline{i}} \mapsto T \circ s_{\underline{i}}$. By Lemma 0.7, there is a unique morphism $L : X \times \mathbb{A}\bigwedge^r(V) \to E$ such that L_* is this map of k-vector spaces. By construction, for every $\underline{i} \in \Sigma_{n,r}$, $L \circ \tau \circ s_{\underline{i}} = L \circ s_{\mathbf{v}_{\underline{i}}} = T \circ s_{\underline{i}}$. So for every $x \in X$, the induced map $L|_x : \bigwedge^r(V) \to E|_x$ is the unique map of k-vector spaces such that $L|_x \circ \tau|_x = T|_x$. Since this holds for every $x \in X$, $L \circ \tau = T$. □

Lemma 0.9. (i) *For every finite-dimensional vector space V and every r^{th} exterior power (E, T) of $X \times \mathbb{A}V$, there is a unique isomorphism of Abelian cones $L : X \times \mathbb{A}\bigwedge^r(V) \to E$ such that $L \circ \tau = T$.*

(ii) *For every vector bundle E on X, every r^{th} exterior power $(\bigwedge^r(E), \tau)$ of E satisfies the universal property from Lemma 0.8.*

(iii) *For every vector bundle E on X, there exists an r^{th} exterior power $(\bigwedge^r(E), \tau)$.*

Proof. **(i):** By Lemma 0.8, there is a unique morphism of Abelian cones $L : X \times \mathbb{A}\bigwedge^r(V) \to E$ such that $L \circ \tau = T$. At issue is whether L is an isomorphism. This can be checked locally. For every $x \in X$, there is an open neighborhood of x over which E is trivial. Thus, without loss of generality, assume $E = X \times \mathbb{A}_k^N$ for some integer N. Choosing an ordered basis for V, also $X \times \bigwedge^r(V) \cong X \times \mathbb{A}_k^M$ for some integer M. By Lemma 0.3, the morphism L is equivalent to a morphism $F : X \to \mathbb{A}\text{Hom}_k(k^M, k^N)$. Using Cramer's rule, etc., the morphism L is an isomorphism near x iff the image $F(x)$ is contained in the open subset (possibly empty) of isomorphisms. Thus L is an isomorphism near x iff $L|_x$ is an isomorphism.

By hypothesis, $T|_x : V^r \to E|_x$ is an r^{th} exterior power of V, therefore $L|_x$ is an isomorphism since exterior powers are unique up to unique isomorphism.

(ii): By the gluing lemma for morphisms, this can be proved locally on X. Locally on X, E is isomorphic to $X \times \mathbb{A}V$. By Lemma 0.8 and (i), every r^{th} exterior power of $X \times \mathbb{A}V$ satisfies the universal property.

(iii): Because of (ii), r^{th} exterior powers are unique up to unique isomorphism. Therefore, by the gluing lemma for varieties, it suffices to prove there exists an r^{th} exterior power locally on X. Locally on X, E is isomorphic to $X \times \mathbb{A}V$ and $(X \times \mathbb{A}\bigwedge^r(V), \mathbb{A}\tau)$ is an r^{th} exterior power of $X \times \mathbb{A}V$. □

Problem 6: Let $n, r \geq 0$ be integers. Define $N = \binom{n}{r}$. Let X be a variety.

(i) Using Problem 5(i) and (iv), give an isomorphism of the r^{th} exterior power of $X \times \mathbb{A}_k^n$ with $X \times \mathbb{A}_k^N$.

(ii) Applying (i) and Problem 5(iii) to the Grassmannian $\text{Grass}(r,n)$, define a tautological rank 1 subbundle of $\text{Grass}(r,n) \times \mathbb{A}_k^N$, $\bigwedge^r(\phi) : \bigwedge^r(S) \to \text{Gras}(r,n) \times \mathbb{A}_k^N$. Combined with Problem 7 from Problem Set 5, deduce existence of a regular morphism $F : \text{Grass}(r,n) \to \mathbb{P}^{N-1}$. This is the *Plücker embedding.*

(iii) For every $\underline{i} \in \Sigma_{n,r}$, denote by $x_{\underline{i}}$ the corresponding coordinate on $\mathbb{A}_k^N$. Prove that $F^{-1}(D_+(x_{\underline{i}}))$ equals $\iota(U_{\underline{i}})$. Conclude that F is an affine morphism.

Solution: By construction, $D_+(x_{\underline{i}}) \subset \mathbb{P}_k^{N-1}$ is the maximal open subset over which the composition of the tautological rank 1 subbundle $E_{1,N} \hookrightarrow \mathbb{P}_k^{N-1} \times \mathbb{A}_k^N$ with the projection to the $x_{\underline{i}}$ coordinate, $\mathbb{P}_k^{N-1} \times \mathbb{A}_k^N \to \mathbb{P}_k^{N-1} \times \mathbb{A}_k^1$ is an isomorphism. By construction, $U_{\underline{i}} \subset \text{Grass}(r,n)$ is the maximal open subset over which the composition of the tautological rank r subbundle $E_{r,n} \hookrightarrow \text{Grass}(r,n) \times \mathbb{A}_k^n$ with the projection $\text{Id} \times \chi_{\underline{i}} : \text{Grass}(r,n) \times \mathbb{A}_k^n \to \text{Grass}(r,n) \times \mathbb{A}_k^r$ is an isomorphism.

For a map of k-vector spaces, $L : k^r \to k^n$, the composition $\chi_{\underline{i}} \circ L : k^r \to k^r$ is an isomorphism iff the determinant of the matrix is nonzero, i.e., iff for the induced map of k-vector spaces $\bigwedge^r L : k \to k^N$, composition with the coordinate $x_{\underline{i}} : k^N \to k$ is an isomorphism. Therefore $F^{-1}(D_+(x_{\underline{i}})) = U_{\underline{i}}$.

Because $(D_+(x_{\underline{i}}) | \underline{i} \in \Sigma_{n,r})$ is an open affine covering of $\mathbb{P}_k^{N-1}$, and because every $F^{-1}(D_+(x_{\underline{i}}))$ is an affine variety, F is an affine morphism.

Problem 7: This problem continues the previous problem, proving the Plücker embedding is a closed immersion.

(i) Assume $n \geq r$. Let $\underline{i} = (1, \dots, r)$. The variety $U_{\underline{i}}$ is the closed subvariety of affine space $\mathbb{A}_k^{nr}$ of $n \times r$ matrices such that the first $r \times r$ rows form the identity matrix. Identify $U_{\underline{i}}$ with the affine space $\mathbb{A}_k^{(n-r)r}$ of $(n-r) \times r$ matrices A via the rule,

$$A \leftrightarrow \left(\frac{I_{r \times r}}{A} \right).$$

Denote the entries of A by $(a_{i,j} | 1 \leq i \leq n-r, 1 \leq j \leq r)$. These are coordinates on the affine space $U_{\underline{i}}$. For every $1 \leq i \leq n-r$ and $1 \leq j \leq r$, denote by $\underline{k} \in \Sigma_{n,r}$ the r-tuple,

$$\underline{k} = (1, \dots, j-1, j+1, \dots, r, r+i).$$

On the affine space $D_+(x_{\underline{i}})$, the rational function $x_{\underline{k}}/x_{\underline{i}}$ is a coordinate. Prove that $F^{\#}(x_{\underline{k}}/x_{\underline{i}}) = a_{i,j}$.

Solution: This is actually correct only up to a minus sign. The point is that the composition of $\phi_{\underline{i}}$ with the morphism $\text{Id} \times \chi_{\underline{k}}$ is given by an $r \times r$ matrix whose first $r-1$ rows are the coordinate vectors of the standard basis elements $\mathbf{e}_1, \dots, \mathbf{e}_{j-1}, \mathbf{e}_{j+1}, \dots, \mathbf{e}_r$ and whose final row is the $(r+i)^{\text{th}}$ row of $\phi_{\underline{i}}$, namely $(a_{i,1}, \dots, a_{i,r})$. Computing the determinant of this matrix by cofactor expansion gives $\pm a_{i,j}$.

(ii) Deduce that $F^{\#} : k[D_+(x_{\underline{i}})] \to k[U_{\underline{i}}]$ is surjective. Therefore $F : U_{\underline{i}} \to D_+(x_{\underline{i}})$ is a closed immersion. Argue this is true for every $\underline{i} \in \Sigma_{n,r}$, therefore $F : \text{Grass}(r,n) \to \mathbb{P}_k^{N-1}$ is a closed immersion.

Solution: The coordinate ring $k[U_{\underline{i}}]$ is the polynomial ring in the variables $a_{i,j}$, $1 \leq i \leq n-r$, $1 \leq j \leq r$. By (i), every variable is contained in the image of F. Therefore the image of F is all of $k[U_{\underline{i}}]$.

Clearly, using the action of the symmetric group on n letters on $\mathbb{P}^{N-1}$ and $\text{Grass}(r,n)$, the same result holds for every $\underline{i} \in \Sigma_{n,r}$.

Problem 8: Remark: The original formulation of this problem was wrong. Below is the correct formulation.

Here is a way to find generators for the homogeneous ideal of the projective variety $F(\text{Grass}(r,n)) \subset \mathbb{P}_k^{N-1}$. Denote by V the vector space $\mathbb{A}_k^n$ so that $\mathbb{A}_k^N$ equals $\bigwedge^r(V)$. Let $\tau_r : V^r \to \bigwedge^r(V)$ be the universal alternating r-linear map. Denote by $(\bigwedge^{r+1}(V), \tau_{r+1})$ an $(r+1)^{\text{st}}$ exterior power of V.

(i) Prove there is a unique 2-multilinear map $L : \bigwedge^r(V) \times V \to \bigwedge^{r+1}(V)$ such that $\tau_{r+1} = L \circ ((\tau_r \circ \text{pr}_{1,\dots,r}) \times \text{pr}_{r+1})$. Using adjointness, deduce existence of a map $\widetilde{L} : \bigwedge^r(V) \to \text{Hom}_k(V, \bigwedge^{r+1}(V))$ such that for every $\mathbf{w} \in \bigwedge^r(V)$ and every $\mathbf{v} \in V$, $\widetilde{L}(\mathbf{w})(\mathbf{v}) = L(\mathbf{w}, \mathbf{v})$.

(ii) Let $\mathbf{w}$ be an element of $\bigwedge^r(V) - \{\mathbf{0}\}$. Prove the image $[\mathbf{w}] \in \mathbb{P}(\bigwedge^r(V)) = \mathbb{P}_k^{N-1}$ is in $F(\text{Grass}(r,n))$ iff $\widetilde{L}(\mathbf{w})$ has rank at most $n-r$, i.e., iff the $(n-r+1)\times(n-r+1)$ minors of the matrix are all zero.

Solution: First of all, suppose $[\mathbf{w}] \in F(\text{Grass}(r,n))$. Then there exist linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_r \in k^n$ such that $\mathbf{w} = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$. There exists an ordered basis for k^n, $(\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n)$, and,

$$\widetilde{L}(\mathbf{w})(\mathbf{v}_i) = \begin{cases} \mathbf{v}_{\underline{1,\dots,r,i}}, & i = r+1, \dots, n, \\ \mathbf{0}, & i = 1, \dots, r \end{cases}$$

Therefore $\widetilde{L}(\mathbf{w})$ has rank $n-r$.

Conversely, suppose that $\widetilde{L}(\mathbf{w})$ has rank $\le n-r$. Let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be linearly independent elements in the kernel. There exists an ordered basis for k^n, $(\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_n)$. For every $\underline{i} \in \Sigma_{n,r}$ there exists an element $c_{\underline{i}} \in k$ such that,

$$\mathbf{w} = \sum_{\underline{i} \in \Sigma_{n,r}} c_{\underline{i}} \mathbf{v}_{\underline{i}}.$$

For every $\underline{i} \in \Sigma_{n,r}$, denote by $|\underline{i}|$ the set $\{i_1, \dots, i_r\}$. Clearly,

$$\mathbf{w} \wedge \mathbf{v}_j = \sum_{\underline{i} \in \Sigma_{n,r}, j \notin |\underline{i}|} \pm c_{\underline{i}} \mathbf{v} \underline{i}',$$

where $\underline{i}' \in \Sigma_{n,r+1}$ is the unique element such that the set $|\underline{i}'| = |\underline{i}| \cup \{j\}$. The elements $\underline{i}'$ are linearly independent. Since $\mathbf{v}_1, \dots, \mathbf{v}_r$ are in the kernel, for every $\underline{i} \in \Sigma_{n,r}$ such that $c_{\underline{i}} \neq 0$, $1, \dots, r$ are in $|\underline{i}|$. Since $|\underline{i}|$ has size r, this means that $c_{\underline{i}} = 0$ if $\underline{i} \neq (1, \dots, r)$. Therefore $\mathbf{w} = c\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$ for some $c \in k$.

(iii) Let $n = 4$ and $r = 2$. Let $(\mathbf{v}_1, \dots, \mathbf{v}_4)$ be an ordered basis for V and let $(\mathbf{v}_{1,2}, \mathbf{v}_{1,3}, \mathbf{v}_{1,4}, \mathbf{v}_{2,3}, \mathbf{v}_{2,4}, \mathbf{v}_{3,4})$ be an ordered basis for $\bigwedge^2(V)$. Denote by $(x_{1,2}, \dots, x_{2,4}$ the dual ordered basis for $(\bigwedge^2(V))^\vee$. Let $(\mathbf{v}_{1,2,3}, \mathbf{v}_{1,2,4}, \mathbf{v}_{1,3,4}, \mathbf{v}_{2,3,4})$ be an ordered basis for $\bigwedge^3(V)$. With respect to these ordered bases, write down the linear transformation $\widetilde{L}$ as a 4×4 matrix whose entries are linear polynomials in $x_{1,2}, \dots, x_{3,4}$.

Solution: With respect to the ordered basis $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ for V and the ordered basis for $\bigwedge^3(V)$, $\mathcal{C} = (\mathbf{v}_{1,2,3}, \mathbf{v}_{1,2,4}, \mathbf{v}_{1,3,4}, \mathbf{v}_{2,3,4})$, the matrix of $\widetilde{L}$ is,

$$[\widetilde{L}]_{\mathcal{C},\mathcal{B}} = \begin{pmatrix} x_{2,3} & -x_{1,3} & x_{1,2} & 0 \\ x_{2,4} & -x_{1,4} & 0 & x_{1,2} \\ x_{3,4} & 0 & -x_{1,4} & x_{1,3} \\ 0 & x_{3,4} & -x_{2,4} & x_{2,3} \end{pmatrix}.$$

(iv) After performing elementary row and column operations, reduce this matrix to a skew-symmetric matrix. The rank of a skew-symmetric matrix is always even, therefore the 3×3 minors vanish iff the determinant vanishes. Prove there exists a quadratic polynomial in $x_{1,2}, \dots, x_{3,4}$ such that the determinant of the skew-symmetric matrix is the square of this polynomial. The polynomial is called the *Pfaffian*, and generates the homogeneous ideal of $F(\text{Grass}(2,4)) \subset \mathbb{P}^5_k$.

Solution: The columns of the new matrix are related to the columns of the original matrix by $C_1' = -C_4, C_2' = C_3, C_3' = -C_2, C_4' = C_1$. This gives the row equivalent, skew-symmetric matrix,

$$\begin{pmatrix} 0 & x_{1,2} & x_{1,3} & x_{2,3} \\ -x_{1,2} & 0 & x_{1,4} & x_{2,4} \\ -x_{1,3} & -x_{1,4} & 0 & x_{3,4} \\ -x_{2,3} & -x_{2,4} & -x_{3,4} & 0 \end{pmatrix}.$$

The determinant of this matrix is the square of the Pfaffian, $x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}$. This is well-defined only up to ± 1, but this is the standard normalization. Observe this is essentially the same as the polynomial in Problem 11 from Problem Set 2 (= Problem 12 from Problem Set 3).

Problem 9: Serre's criterion says that an irreducible variety X is normal if,

(i) the singular locus of X has codimension at least 2, and
(ii) for every pair of open subset $U \subset V \subset X$, if $V - U \subset V$ has codimension at least 2, the restriction map is an isomorphism, $\rho^V_U : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Here is an example of a non-normal variety that satisfies the first condition, but not the second. Let $A \subset k[x,y]$ be the set of polynomials $f(x,y)$ such that $f(1,0) = f(0,1)$.

(i) Prove that A is a finitely generated k-subalgebra of $k[x,y]$.

Solution: It is clear that A is a k-subalgebra. Moreover, A contains the subalgebra $k[x,y]^{S_2}$ of symmetric polynomials. By a standard algebra theorem, $k[x,y]^{S_2} = k[x+y, xy]$, and $k[x,y]$ is a free module over $k[x+y,xy]$ generated by 1 and x. Therefore $A \subset k[x,y]$ is a finitely generated module over $k[x+y,xy]$. So $k[x+y,xy] \to A$ is a finitely generated ring extension, proving A is a finitely generated k-algebra. However, this does not identify generators.

Clearly $x((x+y)-1), x^2y \in A$. Every element in $k[x,y]$ equals $f + xg$ for unique $f, g \in k[x+y,xy]$. At $(1,0)$, $f + xg$ has value $f(1,0) + g(1,0)$, and at $(0,1)$, it has value $f(0,1) = f(1,0)$. Therefore $f + xg$ is in A iff $g(1,0) = 0$, i.e., iff $g \in \langle (x+y)-1, xy \rangle k[x+y,xy]$. So, as a module over $k[x+y,xy]$, $\{xg | xg \in A\}$ is generated by $x((x+y)-1)$ and x^2y. Therefore A is generated by,

$$A = k[x+y, xy, x((x+y)-1), x^2y] \cong k[z_1, z_2, z_3, z_4]/I,$$
$$I = \langle z_3^2 - z_1(z_1-1)z_3 + (z_1-1)^2 z_2, z_3 z_4 - z_1 z_2 z_3 + (z_1-1)z_2^2,$$
$$z_4^2 - z_1 z_2 z_4 + z_2^3, z_2 z_3 - z_4(z_1-1)\rangle.$$

(ii) Let X be an affine variety with $k[X] \cong A$, and let $F : \mathbb{A}_k^2 \to X$ be the unique morphism such that $F^{\#}$ induces the inclusion $A \subset k[x,y]$. Prove that F is a birational, finite morphism that is not an isomorphism. Therefore X is not normal.

Solution: First of all $x = (x^2y)/xy$ is in the function field of A, and thus also $y = (x+y) - x$ is in the function field of A. So $K(A) = k(x,y)$. Moreover x satisfies the monic polynomial $x^2 - x(x+y) + (xy)$ over A. So x is in the integral closure of A. Thus also y is in the integral closure of A. So $F : \mathbb{A}_k^2 \to X$ is finite and birational. But $A \neq k[x,y]$, so F is not an isomorphism.

(iii) Let $U = \mathbb{A}_k^2 - \{(1,0),(0,1)\}$. Prove that $F(U) \subset X$ is an open set and $F : U \to F(U)$ is an isomorphism. In particular $F(U)$ is smooth, and $X - F(U)$ is finite because the inverse image $\mathbb{A}_k^2 - U$ is finite. So the singular locus of X has codimension 2.

Let $V_1 = D(x+y-1) \subset \mathbb{A}_k^2$ and let $V_2 = D(xy) \subset \mathbb{A}_k^2$. Then $V_1 \cup V_2 = U$. Of course $F^{-1}(D((x+y)-1)) = V_1$ and $F^{-1}(D(xy)) = V_2$. For $F : V_1 \to D((x+y)-1)$, the induced map on algebras is,

$$k[x+y, xy, x(x+y-1), x^2y][1/(x+y-1)] \to k[x,y][1/(x+y-1)].$$

In particular, $x(x+y-1)/(x+y-1)$ maps to x, $x+y-x(x+y-1)/(x+y-1)$ maps to y, and $1/(x+y-1)$ maps to $1/(x+y-1)$. So the map of algebras is an isomorphism, i.e., $F : V_1 \to D((x+y)-1)$ is an isomorphism. Similarly, for $F : V_2 \to D(xy)$, the induced map on algebras is,

$$k[x+y, xy, x(x+y-1), x^2y][1/xy] \to k[x,y][1/xy].$$

In particular, x^2y/xy maps to x, $(x+y) - x^2y/xy$ maps to y, and $1/xy$ maps to $1/xy$. So the map of algebras is an isomorphism, i.e., $F : V_2 \to D(xy)$ is an isomorphism. Therefore $F(U) = D((x+y)-1) \cup D(xy)$ is an open subset of X and $F : U \to F(U)$ is an isomorphism.

(iv) Prove that the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(F(U))$ is not an isomorphism.

Solution: Of course $\mathcal{O}_X(X) = A = k[x+y, xy, x((x+y)-1), x^2y]$. By the isomorphism, $\mathcal{O}_X(F(U)) = \mathcal{O}_{\mathbb{A}_k^2}(U)$. By the same argument as in Problem 13 from Problem Set 2 (or by Serre's criterion), $\mathcal{O}_{\mathbb{A}_k^2}(U) = \mathcal{O}_{\mathbb{A}_k^2}(\mathbb{A}_k^2) = k[x,y]$. So the restriction map is $A \to k[x,y]$, which is not an isomorphism.

Problem 10: In a commutative algebra textbook, read the proof that an integrally closed, Noetherian local ring of dimension 1 is a DVR, and thus is regular. Sketch a proof that every normal 1-dimensional variety is smooth.