18.725 PROBLEM SET 2

Due date: Friday, September 24 in lecture. Late work will be accepted only with a medical note or for another Institute-approved reason. You are strongly encouraged to work with others, but the final write-up should be entirely your own and based on your own understanding.

Read through all the problems. Write solutions to the "Required Problems", 1, 3, 4, and 5, together with 2 others of your choice to a total of 6 problems. Difficult problems are labeled "Difficult Problems", and there is one "Very Difficult Problem".

Products: For every pair of objects X, Y in a category $\mathcal{C}$, a *product of* (X, Y) is a triple (U, π_1, π_2) of an object U, a morphism $\pi_1 : U \to X$ and a morphism $\pi_2 : U \to Y$ such that for every object T the following is a bijection,

$$(\pi_1, \pi_2) : \mathrm{Hom}_{\mathcal{C}}(T, U) \to \mathrm{Hom}_{\mathcal{C}}(T, X) \times \mathrm{Hom}_{\mathcal{C}}(T, Y), \quad f \mapsto (\pi_1 \circ f, \pi_2 \circ f).$$

Required Problem 1 Problem 3 from PS# 1 proves every pair in the category of affine algebraic sets has a product. Prove this affine algebraic set is also a product of the pair in the category of quasi-affine algebraic sets, i.e., the universal property holds for every quasi-affine algebraic set T. (**Hint:** Every quasi-affine algebraic set is a union of open affine sets. Combine with the gluing lemma and Problem 3(c).)

Problem 2 Prove every pair in the category of quasi-affine algebraic sets has a product.

Fiber products: For every pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in a category $\mathcal{C}$, a *fiber product of* (f, g) is a triple (U, g', f') of an object U and morphisms $g' : U \to X$, $f' : U \to Y$ such that,

(i) $f \circ g' = g \circ f'$, and
(ii) for every triple (V, g'', f'') satisfying $f \circ g'' = g \circ f''$ there exists a unique morphism $u : V \to U$ such that $g'' = g' \circ u$ and $f'' = f' \circ u$.

Let $\mathcal{C}$ be a category in which every pair (X, Y) has a product, denoted $(X \times Y, \pi_1, \pi_2)$ (this hypothesis holds in Problem 3), and for every pair of morphisms $f : U \to X$, $g : U \to Y$, denote by $f \times g : U \to X \times Y$ the unique morphism such that $\pi_1 \circ (f \times g) = f$, $\pi_2 \circ (f \times g) = g$; this is not standard notation, but will be less confusing for the following problem. For every object Z, the *diagonal morphism of* Z is $\mathrm{Id}_Z \times \mathrm{Id}_Z : Z \to Z \times Z$.

Required Problem 3 (a) Let (U, g', f') be a fiber product of (f, g). Denote by $h : U \to Z$ the morphism $f \circ g' = g \circ f'$. Prove $(U, g' \times f', h)$ is a fiber product of the pair of morphisms $(f \circ \pi_1) \times (g \circ \pi_2) : X \times Y \to Z \times Z$ and $\Delta_Z : Z \to Z \times Z$.

(b) Conversely, i.e., without assuming existence of a fiber product of (f, g), let (U, e, h) be a fiber product of the pair of morphisms $(f \circ \pi_1) \times (g \circ \pi_2) : X \times Y \to Z \times Z$ and $\Delta_Z : Z \to Z \times Z$. Define $g' = \pi_1 \circ e$ and $f' = \pi_2 \circ e$. Prove (U, g', f') is a fiber product of (f, g).

Coproducts: For every pair of objects X, Y of a category $\mathcal{C}$, a *coproduct of* (X,Y) is a triple (U,q_1,q_2) of an object U and a pair of morphisms $q_1 : X \to U$, $q_2 : Y \to U$ such that for every object T the following is a bijection,

$$(q_1,q_2) : \mathrm{Hom}_{\mathcal{C}}(U,T) \to \mathrm{Hom}_{\mathcal{C}}(X,T) \times \mathrm{Hom}_{\mathcal{C}}(Y,T), \quad f \mapsto (f \circ q_1, f \circ q_2).$$

Required Problem 4(a) Let $n \geq 0$ be an integer, let $U = \mathbb{V}(x_{n+1}(x_{n+1}-1)) \subset \mathbb{A}^{n+1}_k$, let $q_1 : \mathbb{A}^n_k \to U$ be $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_n,0)$ and let $q_2 : \mathbb{A}^n_k \to U$ be $(a_1,\dots,a_n) \mapsto (a_1,\dots,a_n,1)$. Prove (U,q_1,q_2) is a coproduct of $(\mathbb{A}^n_k,\mathbb{A}^n_k)$ in the category of quasi-affine algebraic sets. (**Hint:** For every pair of regular functions f_1 and f_2 on $\mathbb{A}^n_k$, $f(x_1,\dots,x_n,x_{n+1}) = x_{n+1}f_2(x_1,\dots,x_n) + (1-x_{n+1})f_1(x_1,\dots,x_n)$ is a regular function on $\mathbb{A}^{n+1}_k$ such that $q_1^*f = f_1$ and $q_2^*f = f_2$.)

(b) Assuming part (a), deduce every pair (X,Y) of quasi-affine algebraic sets has a coproduct (U,q_1,q_2). (**Hint:** Embed in a large affine variety and use (a).)

Some problems on irreducibility:

Required Problem 5(a) Prove every nonempty open subset of an irreducible topological space is dense.

(b) Let $Y \subset X$ be a subset of a topological space, irreducible with the relative topology. Prove the closure of Y is also irreducible with the relative topology.

(c) Prove the image of an irreducible topological space under a continuous map is irreducible with the relative topology from the target.

Problem 6 Assuming Problem 5, prove the irreducible components of $\mathbb{V}(\langle x_1 - x_2x_3, x_1x_3 - x_2^2\rangle) \subset \mathbb{A}^3_k$ are $V_1 = \{(0,0,a) | a \in \mathbb{A}^1_k\}$ and $V_2 = \{(b^3,b^2,b) | b \in \mathbb{A}^1_k\}$. This is the "affine hyperplane section $x_4 = 1$" of the example from lecture on 9/13.

Problem 7 Find the irreducible components of $\mathbb{V}(\langle x_1x_2, x_1x_3, x_2x_3\rangle) \subset \mathbb{A}^3_k$.

Difficult Problem 8 For every integer $n \geq 1$ and every collection S of nonempty subsets of $\{1,\dots,n\}$, define $m(S) \subset S$ to be the collection of subsets of $\{1,\dots,n\}$ minimal among those in S, and define $S^\vee$ to be the collection of all nonempty subsets $A \subset \{1,\dots,n\}$ such that for every $B \in S$, $A \cap B \neq \emptyset$.

(a) Prove $m(S)^\vee = S^\vee$, $S \subset (S^\vee)^\vee$ and $m((S^\vee)^\vee) = m(S)$.

(b) Define $I_S \subset k[x_1,\dots,x_n]$ to be the ideal $\langle m_A | A \in S\rangle$, where $m_A = \prod_{i \in A} x_i$. Prove the set of irreducible components of $\mathbb{V}(I_S)$ is in bijection with $m(S^\vee)$.

Images of some morphisms:

Problem 9 For every pair of integers $m,n \geq 0$, the *affine Segre mapping* $F : \mathbb{A}^m_k \times \mathbb{A}^n_k \to \mathbb{A}^{mn}_k$ is as follows. Let $x_1,\dots,x_m$ be coordinates on $\mathbb{A}^m_k$, let $y_1,\dots,y_n$ be coordinates on $\mathbb{A}^n_k$ and let $z_{i,j}, 1 \leq i \leq m, 1 \leq j \leq n$ be coordinates on $\mathbb{A}^{mn}_k$. Then $F^*z_{i,j} = x_iy_j$. Find an ideal $I \subset k[z_{i,j}]$ such that $\mathbb{V}(I) = \mathrm{Image}(F)$. (**Hint:** The generators of I are homogeneous degree 2 binomials.)

Problem 10 For every pair of integers $n,d \geq 0$, define $N = \binom{n+d}{d}$, and define the *affine Veronese mapping* $F : \mathbb{A}^n_k \to \mathbb{A}^N_k$ as follows. Let $x_1,\dots,x_n$ be coordinates on $\mathbb{A}^n_k$ and let $z_{i_1,\dots,i_n}$ be coordinates on $\mathbb{A}^N_k$ where $(i_1,\dots,i_n)$ runs through all n-tuples of nonnegative integers with $i_1 + \cdots + i_n = d$. Then $F^*z_{i_1,\dots,i_n} = x_1^{i_1}x_2^{i_2}\dots x_n^{i_n}$. Find an ideal $I \subset k[z_{i_1,\dots,i_n}]$ such that $\mathbb{V}(I) = \mathrm{Image}(F)$. (**Hint:** The generators of I are homogeneous degree 2 binomials.)

Difficult Problem 11 For every integer $n \geq 2$, define $N = \binom{n}{2}$ and define $F : \mathbb{A}_k^{2n} \to \mathbb{A}_k^N$ as follows. Let $x_{1,1}, \ldots, x_{1,n}, x_{2,1}, \ldots, x_{2,n}$ be coordinates on $\mathbb{A}_k^{2n}$ and let $z_{i,j}, 1 \leq i < j \leq n$ be coordinates on $\mathbb{A}_k^N$. Then $F^* z_{i,j} = x_{1,i}x_{2,j} - x_{1,j}x_{2,i}$. The image of this morphism is the *affine cone over the Grassmannian* $\boldsymbol{Grass}(2,n)$. Find an ideal $I \subset k[z_{i,j}]$ such that $\mathbb{V}(I) = \text{Image}(F)$. (**Hint:** Interpret elements of $\mathbb{A}_k^{2n}$ as $2 \times n$ matrices; interpret elements of $\mathbb{A}_k^N$ as elements of the exterior square of the n-space, which also give anti-symmetric $n \times n$ matrices, and take Pfaffians of appropriate 4×4-submatrices of this $n \times n$-matrix. The generators are homogeneous degree 2 trinomials.)

Problem 12 Give an example of a regular morphism of affine varieties $F : V \to W$ whose image is not a quasi-affine algebraic set.

Problem 13: Proposition 4.8 can fail if W is not affine Let $V = \mathbb{A}_k^2$, let $W = \mathbb{A}_k^2 - \mathbb{V}(x_1, x_2)$ and let $i : W \to V$ be the inclusion. Prove that $i^* : \mathcal{O}_V(V) \to \mathcal{O}_W(W)$ is an isomorphism, but there is no inverse of i, i.e., Proposition 4.8 fails for V and W.

Very Difficult Problem 14 Prove there exists a quasi-affine algebraic set V such that $\mathcal{O}_V(V)$ is not a finitely-generated k-algebra. The examples I am aware of all have dimension ≥ 4. (**Warning:** This problem would be more appropriate at the end of 18.726. I mention it now because you can understand it, and it is a problem to keep in mind as the semester goes on.)

Problem 15 Prove the k-algebra $\mathcal{O}_V(V)$ of every quasi-affine algebraic set V is a *subalgebra* of a finitely-generated k-algebra.

Problem 16, An open affine that is not a basic open affine, I Together with the next problem, this problem gives an open subset of an affine algebraic set, itself isomorphic to an affine algebraic set, but not a basic open affine $D(s)$. In both problems, assume $\text{char}(k) \neq 2$ and let i denote a solution of $x^2 + 1$ in k. Let $C \subset \mathbb{A}_k^2$ be the *affine nodal plane cubic*, $C = \mathbb{V}(y^2 - x^2(x-1))$ Let $(a_0, b_0) \in C$ and define $F : D(x - a_0) \to \mathbb{A}_k^3$ by $F(a,b) = (a, b, (b + b_0)/(a - a_0))$.

(a) Prove there exists a regular morphism $G : C - \{(a_0, b_0)\} \to \mathbb{A}_k^3$ whose restriction to $D(x - a_0$ equals F. (**Hint:** Expand the defining equation of C in the coordinates $x - a_0$ and $y - b_0$.)

(b) Prove the image of G is an affine algebraic subset of $\mathbb{A}_k^3$.

(c) Prove the projection $\pi : \mathbb{A}_k^3 \to \mathbb{A}_k^2$, $\pi(a,b,c) = (a,b)$ restricts on the image of G to an inverse morphism to G. Therefore $C - \{a_0, b_0\}$ is an open subset of C, itself isomorphic to an affine algebraic set.

Difficult Problem 17, An open affine that is not a basic open affine, II This problem continues Problem 16; again $\text{char}(k) \neq 2$. Consider the morphism $H : \mathbb{A}_k^1 \to C$ by $H(u) = (u^2 + 1, u(u^2 + 1))$. Let t be a coordinate on $\mathbb{A}_k^1$.

(a) Prove $H^* : k[C] \to k[t]$ maps $k[C]$ isomorphically to the subalgebra of functions $f(t) \in k[t]$ such that $f(i) = f(-i)$.

(b) For (b), (c) and (d), assume $(a_0, b_0) \in C - \{(0,0)\}$. Prove the ideal of $k[t]$ generated by $H^*(\langle x - a_0, y - b_0 \rangle)$ is the principal ideal $\langle a_0 t - b_0 \rangle$.

(c) If there is an element $s \in k[V]$ such that $\mathbb{V}(s) = \{(a_0, b_0)\}$, $H^*(s) = c(at - b)^n$ for some nonzero constant $c \in k$ and integer $n \geq 1$. (**Hint:** Consider the image of

s in $k[V][1/xy] \cong k[t][1/(t^2+1)]$. Use this to express H^*s as $c(t^2+1)^r(at-b)^n$ for some $r \geq 0$, and then use that $s(0,0) \neq 0$.)

(d) Deduce that $(a_0 i - b_0)^n = (-a_0 i - b_0)^n$, because $c(at-b)^n$ is in the image of H^*. Therefore for every $(a_0, b_0) \in C - \{(0,0)\}$, if $(b_0 - ia_0)/(b_0 + ia_0)$ is not a root of unity, then $C - \{(a_0, b_0)\}$ is of the form $D(s)$ for *no* element $s \in k[V]$ (in fact these are equivalent conditions).