MAT 552 PROBLEM SET 9

Problems.

Problem 0.(**Lie's Theorem.**) Let $\mathfrak{g}$ be a finite-dimensional Lie k-algebra, let $\mathfrak{h}$ be a Lie ideal in $\mathfrak{g}$, and let (V, ρ) be a finite-dimensional representation of $\mathfrak{g}$. Let λ denote a morphism of Lie algebras from $\mathfrak{h}$ to the unique 1-dimensional Lie algebra,

$$\lambda : \mathfrak{h} \to k, \quad X \mapsto \langle \lambda, X \rangle \in k.$$

For every integer $r \geq 0$, denote by $V^r_{\mathfrak{h},\lambda}$ the simultaneous kernel in V over all $X \in \mathfrak{h}$ of the k-linear endomorphisms $(\rho_X - \langle \lambda, X\rangle \mathrm{Id}_V)^{1+r}$. The subspace $V^0_{\mathfrak{h},\lambda}$ is the $\mathfrak{h}$-**eigenspace** of V with **weight** λ. The nondecreasing sequence of k-subspaces $(V^r_{\mathfrak{h},\lambda})_{r=0,1,\dots}$ stabilizes to the $\mathfrak{h}$-**generalized eigenspace** $V^{\mathrm{gen}}_{\mathfrak{h},\lambda}$ of V.

(a) Prove that each subspace $V^r_{\mathfrak{h},\lambda}$ is an $\mathfrak{h}$-subrepresentation of V.

(b) For every $Y \in \mathfrak{g}$, since $\mathrm{ad}_Y(X)$ is in $\mathfrak{h}$ for every $X \in \mathfrak{h}$, use the identity,

$$\rho_Y \circ \rho_X - \rho_X \circ \rho_Y = \rho_{\mathrm{ad}_Y(X)},$$

to conclude that ρ_Y maps $V^r_{\mathfrak{h},\lambda}$ to $V^{1+r}_{\mathfrak{h},\lambda}$. Conclude that $V^{\mathrm{gen}}_{\mathfrak{h},\lambda}$ is a $\mathfrak{g}$-subrepresentation.

(c) Prove that Lie's Theorem is equivalent to Lie's Lemma: each eigenspace $V^r_{\mathfrak{h},\lambda}$ is a $\mathfrak{g}$-subrepresentation of V. Also show that this is equivalent to the claim that for every λ with $V^{\mathrm{gen}}_{\mathfrak{h},\lambda}$ nonzero (i.e., for each $\mathfrak{h}$-weight of the representation), for every $X \in \mathfrak{h}$ and for every $Y \in \mathfrak{g}$, the pairing $\langle \lambda, \mathrm{ad}_Y(X)\rangle$ is zero.

(d) For a nonzero element v in $V^0_{\mathfrak{h},\lambda}$, prove that the smallest ρ_Y-stabilized $\mathfrak{h}$-subrepresentation W that contains v has a basis of the form $(\rho^0_Y(v), \dots, \rho^{m-1}_Y(v))$ for some positive integer m.

(e) Check that W is a generalized eigenspace of $\rho_{\mathrm{ad}_Y(X)}$ with eigenvalue $\langle \lambda, \mathrm{ad}_Y(X)\rangle$, so that the trace of $\rho_{\mathrm{ad}_Y(X)}$ on W equals $m\langle \lambda, \mathrm{ad}_Y(X)\rangle$. However, since $\rho_{\mathrm{ad}_Y(X)}$ equals a commutator of k-linear endomorphisms of W, namely $\rho_Y \circ \rho_X - \rho_X \circ \rho_Y$, conclude that the trace equals 0. Since the characteristic of k equals 0, conclude that $\langle \lambda, \mathrm{ad}_Y(X)\rangle$ is zero, proving Lie's Lemma (and thus Lie's Theorem).

(f) Finally, if $\mathfrak{h}$ is solvable, use induction along the lower central series to prove that for every Jordan-Hölder filtration of (V, ρ) by $\mathfrak{g}$-subrepresentations, every simple factor is an $\mathfrak{h}$-eigenspace for some weight λ, and thus every k-subspace of the representation is a $\mathfrak{h}$-subrepresentation. This is equivalent to Lie's Theorem.

Problem 1.(**Engel's Theorem.**) Consider the following assertion (the weak form of Engel's Theorem). An action of a Lie algebra $\mathfrak{g}$ on a finite-dimensional vector space V is a **nilpotent action** if the image of $\mathfrak{g}$ in $\mathfrak{gl}(V)$ is contained in the nilpotent cone of $\mathfrak{gl}(V)$, i.e., every image element is a nilpotent linear transformation of V.

Theorem 0.1 (Weak Engel's Theorem)**.** *Every nilpotent action of a Lie algebra on a vector space of finite, positive dimension annihilates a nonzero vector in the vector space.*

Obviously this is a property only of the image of $\mathfrak{g}$ in $\mathfrak{gl}(V)$, which is a Lie algebra of finite dimension. Thus, it suffices to prove the result for Lie algebras that have finite dimension and faithful representation that have finite dimension.

(a) For a Lie algebra $\mathfrak{g}$ as above, for every $\mathfrak{g}$-subrepresentation W of V, prove that the images of $\mathfrak{g}$ in both $\mathfrak{gl}(W)$ and $\mathfrak{gl}(V/W)$ are contained in the nilpotent cones. Up to replacing $\mathfrak{g}$ by its image in $U = \mathfrak{gl}(V)$, assume that the action on V is faithful. For the adjoint action of $\mathfrak{g}$ on $U = \mathfrak{gl}(V)$, check that the image of $\mathfrak{g}$ is contained in the nilpotent cone of $\mathfrak{gl}(U)$. In particular, the adjoint image of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$ is contained in the nilpotent cone, so that $\mathfrak{g}$ is a nilpotent Lie algebra. In the not necessarily faithful case, the quotient of $\mathfrak{g}$ by the kernel of the representation is a nilpotent Lie algebra.

(b) If $\mathfrak{g}$ has dimension 0 or 1, prove the weak form of Engel's Theorem.

Now, by way of induction, assume that $\mathfrak{g}$ has dimension > 1, and assume the weak Engel's Theorem is true for all Lie subalgebras that have strictly smaller dimension than the dimension of $\mathfrak{g}$.

(c) For every proper Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing the kernel of ρ that is maximal among proper Lie subalgebras of $\mathfrak{g}$ containing the kernel of ρ, conclude that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is nilpotent and preserves $\mathfrak{h}$. Thus the induced representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ is nilpotent. By the induction hypothesis, conclude that there exists an element X of $\mathfrak{g} \setminus \mathfrak{h}$ such that the adjoint action of $\mathfrak{h}$ on X has image contained in $\mathfrak{h}$, i.e., $[X, \mathfrak{h}] \subset \mathfrak{h}$. Deduce that $\mathfrak{h} + \text{span}(X)$ is a Lie subalgebra of $\mathfrak{g}$ containing the kernel of ρ and that strictly contains $\mathfrak{h}$. Since $\mathfrak{h}$ was maximal among proper Lie subalgebras, deduce that $\mathfrak{h} + \text{span}(X)$ equals $\mathfrak{g}$. Thus, $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ of codimension 1, and it is a Lie ideal.

(d) Continuing the previous part, use the induction hypothesis to conclude that there exists a nonzero vector w of V that is annihilated by $\mathfrak{h}$. If also w is annihilated by the action of X, deduce that $v = w$ satisfies the weak form of Engel's Theorem. If w is not annihilated by the action of X, deduce that $v = X \cdot w$ satisfies the weak form of Engel's Theorem. Thus, the weak form of Engel's Theorem holds by induction on the dimension of $\mathfrak{g}$.

(e) Use the weak form of Engel's Theorem and induction on the dimension of V to conclude the strong form of Engel's Theorem:

Theorem 0.2 (Engel's Theorem)**.** *Every nilpotent action of a Lie algebra $\mathfrak{g}$ on a vector space of finite dimension admits a maximal flag of subspaces that are $\mathfrak{g}$-subrepresentations whose associated graded one-dimensional $\mathfrak{g}$-representations are each trivial.*

There is a slightly sharper version. Now let $\mathfrak{g}$ be a finite-dimensional Lie algebra, let $\mathfrak{n}$ be a Lie ideal in $\mathfrak{g}$, and let (V, ρ) be a nonzero, finite-dimensional $\mathfrak{g}$-representation whose restriction to $\mathfrak{n}$ acts nilpotently on V. By the weak form of Engel's Theorem, the annihilator $V^{\mathfrak{n}}$ in V of $\mathfrak{n}$ is nonzero. Of course $V^{\mathfrak{n}}$ is a $\mathfrak{n}$-subrepresentation of the $\mathfrak{n}$-representation V (the "invariant subrepresentation").

(f) Since $\mathfrak{n}$ is a Lie ideal in $\mathfrak{g}$, prove that $V^{\mathfrak{n}}$ is, in fact, a $\mathfrak{g}$-subrepresentation of V. By considering the induced action of $\mathfrak{g}$ on the quotient $V/V^{\mathfrak{n}}$ and using induction on the dimension of V, conclude the following variant of Engel's Theorem.

Corollary 0.3. *For every finite-dimensional representation (V,ρ) of a Lie algebra $\mathfrak{g}$ and for every Lie ideal $\mathfrak{n}$ of $\mathfrak{g}$ that acts nilpotently on V, for every Jordan-Hölder filtration of (V,ρ) by $\mathfrak{g}$-subrepresentations, every simple factor is a trivial $\mathfrak{n}$-subrepresentation.*

(g) For a finite-dimensional representation (V,ρ) of a Lie algebra $\mathfrak{g}$, and for Lie ideals $\mathfrak{m}$ and $\mathfrak{n}$ that both act nilpotently on V, for the flag of $\mathfrak{g}$-subrepresentations as above such that $\mathfrak{n}$ acts trivially on the associated graded $\mathfrak{g}$-representations, conclude that the $\mathfrak{m}$-action on each associated graded $\mathfrak{g}$-representation is nilpotent. Thus, there exists a flag of $\mathfrak{g}$-subrepresentations of each associated graded $\mathfrak{g}$-subrepresentations, such that $\mathfrak{m}$ also acts trivially on the new associated graded $\mathfrak{g}$-subrepresentations. Conclude that there exists a refinement of the original flag to a flag of $\mathfrak{g}$-subrepresentations of V such that the action of $\mathfrak{m}+\mathfrak{n}$ on each associated graded $\mathfrak{g}$-representation is trivial. Altogether, this proves the following.

Corollary 0.4. *For every finite dimensional representation of a Lie algebra $\mathfrak{g}$, for every pair of Lie ideals, $\mathfrak{m}$ and $\mathfrak{n}$, that both act nilpotently on the representation, also the Lie ideal $\mathfrak{m}+\mathfrak{n}$ acts nilpotently on the representation. Thus, there exists a maximal Lie ideal of $\mathfrak{g}$ that acts nilpotently on the representation.*

The maximal Lie ideal of $\mathfrak{g}$ that acts nilpotently on a given finite-dimensional representation (V,ρ) is the **nilradical of the representation**, $\mathrm{nil}_\rho(\mathfrak{g})$.

(h) In particular, apply this to the adjoint representation $(\mathfrak{g},\mathrm{ad}_\mathfrak{g})$ to conclude that there exists a flag of Lie ideals in $\mathfrak{g}$ whose associated graded Lie algebras are each trivial representations when restricted to the nilradical of the Lie algebra, $\mathrm{nil}(\mathfrak{g}) = \mathrm{nil}_{\mathrm{ad}}(\mathfrak{g})$.

(i) Let (V,ρ) be a finite-dimensional representation of a finite-dimensional Lie algebra $\mathfrak{g}$ such that the associated representation $V/V^\mathfrak{g}$ of the quotient Lie algebra $\mathfrak{g}/\mathrm{nil}_\rho(\mathfrak{g})$ is nilpotent. Use induction on the dimension of V to prove that $\mathrm{nil}_\rho(\mathfrak{g})$ equals all of $\mathfrak{g}$. Conclude the following corollary.

Corollary 0.5. *A Lie algebra acts nilpotently on a finite-dimensional representation if the Lie algebra is the sum of a Lie ideal and a Lie subalgebra, each of which act nilpotently on the representation.*

Problem 2. (**Universal Enveloping Algebras are Noetherian**.) Read about the (left, resp. right) **Noetherian property** for an associative, unital ring: every ascending chain of (left, resp. right) ideals stabilizes. For an associative, unital ring that is filtered, the ring is (left, resp. right) Noetherian if the associated graded ring of the filtration is (left, resp. right) Noetherian. Read about the Hilbert Basis Theorem: a commutative, unital ring is (both left and right) Noetherian if it is a quotient of a finitely generated polynomial ring over a commutative, unital ring that is (both left and right) Noetherian. Since the associated graded ring of the filtered universal enveloping algebra of a Lie algebra $\mathfrak{g}$ is a quotient of the polynomial algebra with first graded piece equal to $\mathfrak{g}$, conclude that the universal enveloping algebra of every finite dimensional Lie algebra is both left and right Noetherian. (Conversely, if the universal enveloping algebra of a Lie algebra is Noetherian, then the Lie algebra is finite dimensional.) Thus, a $\mathfrak{g}$-module has a (finite) Jordan-Hölder filtration by $\mathfrak{g}$-submodules whose associated graded $\mathfrak{g}$-modules are simple if and only if the $\mathfrak{g}$-module is Artinian, e.g., this holds if the $\mathfrak{g}$-module has finite dimension as a vector space. The length of the Jordan-Hölder filtration, and the sequence of simple modules (up to permutation) are independent of the choice of Jordan-Hölder filtration.

Prove that in this case, the nilradical $\text{nil}_\rho(\mathfrak{g})$ is the simultaneous annihilator in $\mathfrak{g}$ of all of these simple $\mathfrak{g}$-modules, and the "nilpotency degree" for the induced nilpotent action of the nilradical is bounded above by the length ℓ of the Jordan-Hölder filtration. Define the **nilradical** in $U_k(\mathfrak{g})$, $\text{Nil}_\rho(U_k(\mathfrak{g}))$, to be the simultaneous annihilator in $U_k(\mathfrak{g})$ of all of these simple $\mathfrak{g}$-modules as a two-sided ideal, so that the $\iota_{\mathfrak{g}}$-preimage of $\text{Nil}_\rho(U_k(\mathfrak{g}))$ equal $\text{nil}_\rho(\mathfrak{g})$, and the power of the ideal, $\text{Nil}_\rho(U_k(\mathfrak{g}))^\ell$ annihilates the module V.

Problem 3.(**Derivations and Ideals in the Universal Enveloping Algebra**.) For a Lie k-algebra $\mathfrak{g}$ and its associated universal enveloping k-algebra $\iota_{\mathfrak{g}} : \mathfrak{g} \to U_k(\mathfrak{g})$, recall from Problem 4 of Problem Set 8 that the Lie algebra k-derivations of $\mathfrak{g}$ are precisely the k-derivations of the associative, unital algebra $U_k(\mathfrak{g})$ that map the subspace $\iota_{\mathfrak{g}}(\mathfrak{g})$ back to itself. For such a derivation θ, check that the image of the derivation applied to $U_k(\mathfrak{g})$ is contained in the two-sided ideal of $U_k(\mathfrak{g})$ generated by $\iota_{\mathfrak{g}}(\theta(\mathfrak{g}))$. Thus, also for every Lie subalgebra $\mathfrak{s}$ of the Lie algebra of such k-derivations, the subspace of $U_k(\mathfrak{g})$ spanned by the images of all derivations of $\mathfrak{s}$ is contained in the two-sided ideal of $U_k(\mathfrak{g})$ generated by $\iota_{\mathfrak{g}}(\theta(\mathfrak{g}))$. Similarly, for every two-sided ideal N of $U_k(\mathfrak{g})$ that is mapped back to itself by the derivations in $\mathfrak{s}$, check that for each integer $m \geq 0$, the two-sided ideal N^m is also mapped back to itself by the derivations in $\mathfrak{s}$.

Problem 4.(**The Zassenhaus Extension Lemma**.) This problem is taken from notes by Theo Johnson-Freyd on Lie groups and Lie algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra that equals a semidirect product of a Lie ideal $\mathfrak{r}$ and a Lie subalgebra $\mathfrak{s}$ via a derivation,

$$\theta : \mathfrak{s} \to \text{Der}_k(\mathfrak{r}, \mathfrak{r}).$$

Let (V, ρ) be a representation of $\mathfrak{r}$ whose nilradical $\text{nil}_\rho(\mathfrak{r})$ contains the image of the derivation θ_X on $\mathfrak{r}$ for every X in $\mathfrak{s}$, and thus also the nilradical $\text{Nil}_\rho(U_k(\mathfrak{r}))$ contains the image of the extended derivation θ_X on $U_k(\mathfrak{r})$ for every X in $\mathfrak{s}$.

Lemma 0.6 (Zassenhaus Extension Lemma)**.** *For all integers m that are sufficiently positive, the left $U_k(\mathfrak{r})$-module map,*

$$\widetilde{\rho} : U_k(\mathfrak{r}) \otimes_k V \twoheadrightarrow V, \quad (a, v) \mapsto \widetilde{\rho}_a(v),$$

factors through the quotient,

$$\overline{\rho} : (U_k(\mathfrak{r})/N^m) \otimes_k V \twoheadrightarrow V,$$

the left $\mathfrak{r}$-module structure on the domain has a natural extension to a left $\mathfrak{g}$-module structure, and the nilradical in $\mathfrak{g}$ of this left $\mathfrak{g}$-module contains the nilradical in $\mathfrak{r}$ of the left $\mathfrak{r}$-module (V, ρ).

(a) Since $\text{nil}_\rho(\mathfrak{g})$ contains the image of each derivation θ_X, show that also the two-sided ideal $N = \text{Nil}_\rho(U_k(\mathfrak{r})$ is mapped back to itself by the extended derivation θ_X on $U_k(\mathfrak{g})$. Thus, the ideal N is mapped back to itself by all of $\mathfrak{s}$. Use the previous exercise to conclude that $\mathfrak{s}$ also maps each power of the ideal, N^m, back to itself.

Conclude that there is an induced action of $\mathfrak{s}$ on the quotient ring $U_k(\mathfrak{g})/N^m$ by derivations that maps each quotient ideal N^n/N^m back to itself for $n = 0, \dots, m$.

(b) For each integer $m \geq 0$, use the natural left-module action of $U_k(\mathfrak{r})$ on $U_k(\mathfrak{r})/N^m$ and the action of $\mathfrak{s}$ by derivations from the previous part to conclude that there is an action on $U_k(\mathfrak{r})/N^m$ of the semidirect product Lie algebra $\mathfrak{g}$ whose restriction to the submodule $\mathfrak{r}$ is the usual left-module action.

(c) Since $U_k(\mathfrak{r})$ is Noetherian, the two-sided ideal $N = \text{Nil}_\rho(U_k(\mathfrak{r}))$ as well as each of its powers, N^m, is finitely generated. Conclude that each associated graded N^m/N^{1+m} is also finitely generated, both as a $U_k(\mathfrak{r})$-module and as a $U_k(\mathfrak{r})/N$-module. By definition of $\text{Nil}_\rho(U_k(\mathfrak{r}))$, the quotient ring $U_k(\mathfrak{r})/N$ is the image of $U_k(\mathfrak{r})$ in the product of the $U_k(\mathfrak{r})$-endomorphism ring of the finitely many simple modules V_i in a Jordan-Hölder filtration of the $U_k(\mathfrak{r})$-module V. Of course each of these endomorphism rings is a subring of the finite dimensional k-vector space $\text{Hom}_k(V_i, V_i)$. Conclude that also that the quotient ring $U_k(\mathfrak{r})/N$ is a finite-dimensional k-vector space. Therefore, the associated graded N^{m-1}/N^m is a finite dimensional k-vector space for each integer $m \geq 0$ (since it is a finitely generated module over a k-algebra that is a finite-dimensional k-vector space). Also, for the length ℓ of the Jordan-Hölder filtration, the power N^ℓ is in the annihilator of V. Thus the action of $U_k(\mathfrak{r})$ on V factors through the quotient ring $U_k(\mathfrak{r})/N^\ell$, and this is a finite-dimensional k-vector space, since the filtration by powers of N is a finite filtration whose associated graded pieces are each finite-dimensional k-vector spaces. Finally, since $\mathfrak{s}$ acts on $U_k(\mathfrak{r})/N^\ell$ by derivations, conclude that this induces an action of the semidirect product Lie algebra $\mathfrak{g}$ on $U_k(\mathfrak{r})/N^\ell$ whose restriction to $\mathfrak{r}$ is the usual action.

(d) Since the action of $U_k(\mathfrak{r})$ on V factors $U_k(\mathfrak{r})/N^\ell$, conclude that the natural k-bilinear module action,

$$\widetilde{\rho} : U_k(\mathfrak{r}) \times V \to V, \quad (a, v) \mapsto a \cdot v,$$

factors through a k-linear transformation,

$$\overline{\rho} : \left(U_k(\mathfrak{r})/N^\ell\right) \otimes_k V \to V.$$

Define W to be the domain of this k-linear transformation. Since both V and $U_k(\mathfrak{r})/N^\ell$ are finite-dimensional, conclude that also W is a finite-dimensional k-vector space. Make the tensor product k-vector space W into a left $\mathfrak{g}$-module via the $\mathfrak{g}$-module action on $U_k(\mathfrak{r})/N^\ell$ constructed above and the trivial action on V; denote by σ the induced representation of $\mathfrak{g}$ on W. For the $\mathfrak{g}$-representation σ on W with its restriction to the Lie ideal $\mathfrak{r}$ of $\mathfrak{g}$, and for the $\mathfrak{r}$-module representation ρ on V, check that $\overline{\rho}$ is a morphism of left $\mathfrak{r}$-modules.

(e) Check that the intersection with the Lie ideal $\mathfrak{r}$ of the nilradical $\text{nil}_\sigma(\mathfrak{g})$ contains the nilradical $\text{nil}_\rho(\mathfrak{r})$. This completes the proof of the Zassenhaus Extension Lemma.

(f) As a bonus, check that the nilradical in $\mathfrak{g}$ of the action on $(U_k(\mathfrak{r})/N^\ell) \otimes_k V$ also contains $\mathfrak{s}$ if the derivation θ_X of $\mathfrak{r}$ is a nilpotent endomorphism of $\mathfrak{r}$ for each X in $\mathfrak{s}$. (Hint. Note that the associated graded $\mathfrak{g}$-representations of a Jordan-Hölder filtration on $(U_k(\mathfrak{r})/N^\ell) \otimes_k V$ factors through the action of the quotient Lie algebra $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{s}$.)

Problem 5.(**Derivations of a solvable Lie algebra**.) For a finite-dimensional, solvable Lie algebra $\mathfrak{r}$ with nilradical $\text{nil}(\mathfrak{r})$, the adjoint representation of $\mathfrak{r}$ preserves a maximal flag of subspaces that are actually $\mathfrak{r}$-subrepresentations. Moreover, the Lie ideal $\text{nil}(\mathfrak{r})$ equals the kernel of the morphism of Lie algebras from $\mathfrak{r}$ to the direct sum of the trivial Lie algebra $\text{End}_k(V_i, V_i) = k$ taken over all 1-dimension associated graded $\mathfrak{r}$-representations V_i in this flag. Thus, the quotient Lie algebra $\mathfrak{r}/\text{nil}(\mathfrak{r})$ is identified with a Lie subalgebra of this finite-dimensional, Abelian Lie algebra.

(a) Conclude that $\mathfrak{r}/\text{nil}(\mathfrak{r})$ is an Abelian Lie algebra. Equivalently, conclude that $\text{nil}(\mathfrak{r})$ contains the commutator ideal.

(b) Assume that $\mathfrak{r}/\text{nil}(\mathfrak{r})$ is nonzero. Denote by $\Phi \subset \text{Hom}_k(\mathfrak{r}/\text{nil}(\mathfrak{r}), k)$ the set of all **nontrivial** k-linear transformations that occur among the 1-dimensional subquotients above. Deduce that Φ is a spanning set of the k-vector space $\text{Hom}_k(\mathfrak{r}/\text{nil}(\mathfrak{r}), k)$ that is a finite set whose size is bounded above the dimension of $\text{nil}(\mathfrak{r})$.

(c) Deduce that the filtration of $\mathfrak{r}$ by powers of the nilradical is a "characteristic" filtration by $\mathfrak{r}$-Lie subalgebras, i.e., it is preserved by all automorphisms of the Lie algebra $\mathfrak{r}$. For each associated graded $\mathfrak{r}$-representation for this filtration, deduce that the $\mathfrak{r}$-action factors through the Abelian Lie algebra $\mathfrak{r}/\text{nil}(\mathfrak{r})$. For the action of this finite-dimensional Lie algebra on this finite-dimensional vector space, deduce that the generalized eigenspace decomposition of this representation has nonzero terms only for characters in the finite set Φ, and each character in Φ has a nonzero term for some associated graded $\mathfrak{r}$-representation of this filtration. Deduce that the direct sum decomposition by generalized eigenspaces for Φ is also characteristic in the sense that every automorphism of $\mathfrak{r}$ permutes the set Φ and permutes the corresponding generalized eigenspaces.

(d) Since the group of automorphisms is an algebraic subgroup of $\mathbf{GL}(\mathfrak{g})$, conclude that it has finitely many connected components. Show that every automorphism in the connected component of the identity acts as the identity permutation on Φ, and thus preserves (setwise) each generalized eigenspace in the direct sum decomposition above. Thus, the filtration by powers of the nilradical, refined by the generalized eigenspaces, gives a decomposition of the adjoint representation as $\mathfrak{r}$-subrepresentations that is "characteristic" for automorphisms of $\mathfrak{r}$ in the same connected component as the identity automorphism.

(e) In particular, since the automorphism acts as the identity permutation on Φ, and since the elements of Φ span the dual k-vector space of $\mathfrak{r}/\text{nil}(\mathfrak{r})$, deduce that each automorphism of $\mathfrak{r}$ in the connected component of the identity acts as the identity automorphism on the quotient Lie algebra $\mathfrak{r}/\text{nil}(\mathfrak{r})$.

(f) Since derivations of $\mathfrak{r}$ exponentiate to automorphisms of $\mathfrak{r}$ in the connected component of the identity, conclude that every derivation of $\mathfrak{r}$ induces the zero derivation of the quotient Lie algebra $\mathfrak{r}/\text{nil}(\mathfrak{r})$. Deduce that every derivation of a finite-dimensional, solvable Lie algebra (in characteristic zero), maps the solvable Lie algebra to its nilradical Lie ideal.

(g) In the Zassenhaus Extension Lemma, additionally assume that the Lie ideal $\mathfrak{r}$ is a solvable Lie algebra. Use the result above to show that a finite-dimensional representation (V, ρ) of $\mathfrak{r}$ satisfies the hypothesis of the lemma if the nilradical of ρ contains $\text{nil}(\mathfrak{r})$. Thus, such (V, ρ) is a quotient $\mathfrak{r}$-representation of the restriction to $\mathfrak{r}$ of a finite-dimensional $\mathfrak{g}$-representation whose nilradical in $\mathfrak{g}$ contains both $\mathfrak{nil}_\rho(\mathfrak{r})$ and $\mathfrak{s}$.

Problem 6.(**Ado's Theorem**.) This proof of Ado's theorem also follows the notes by Theo Johnson-Freyd, and the strategy traces back (at least) to the work of Harish-Chandra in his article *Faithful representations of Lie algebras*.

Theorem 0.7 (Ado's Theorem). *Over a field k of chereacteristic 0, every Lie k-algebra $(\mathfrak{g}, [\bullet, \bullet]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g})$ that has finite dimension as a k-vector space admits a faithful representation whose nilradical contains nil$(\mathfrak{g})$.*

(a) First prove this when the Lie algebra $\mathfrak{g}$ is itself nilpotent by induction on the dimension of the Lie algebra. The base case is when $\mathfrak{g}$ is the zero vector space, in which case the result is tautological. If the dimension is strictly positive, use Engel's Theorem to prove that there exists a Lie ideal $\mathfrak{r}$ of codimension 1 in $\mathfrak{g}$ that contains the commutator ideal. Let $\mathfrak{s}$ be the span of any element of $\mathfrak{g}$ not in $\mathfrak{r}$. By the induction hypothesis, there exists a faithful representation of $\mathfrak{r}$. By the Zassenhaus Extension Lemma, there is an extension of this representation to a $\mathfrak{g}$-representation. The kernel of this extension has trivial intersection with $\mathfrak{r}$. Therefore we get a faithful representation of all of $\mathfrak{g}$ by taking the direct sum of this faithful representation with any faithful, nilpotent representation of the quotient Lie algebra $\mathfrak{g}/\mathfrak{r} \cong \mathfrak{s}$, e.g., the unique 2-dimensional faithful, nilpotent representation of the 1-dimensional Lie algebra $\mathfrak{s}$.

(b) Next, when $\mathfrak{g}$ is a solvable Lie algebra, adapt the proof above to the case where the Lie ideal $\mathfrak{r}$ is a codimension 1 Lie ideal that contains the nilradical of $\mathfrak{g}$ (which, in turn, contains the commutator ideal).

(c) Finally, in the general case, let $\mathfrak{r}$ denote the solvable radical of $\mathfrak{g}$, and let $\mathfrak{s}$ be a Levi factor (which exists by Levi's Theorem). Use the Zassenhaus Extension Lemma to prove that there exists a finite-dimensional $\mathfrak{g}$-representation whose kernel has trivial intersection with $\mathfrak{r}$ and whose nilradical contains $\mathfrak{nil}(\mathfrak{r})+\mathfrak{s}$. Take the direct sum of this representation with the adjoint representation of $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$. By Cartan's Semisimplicity Criterion, conclude that the kernel of this direct sum is trivial, so that this gives a faithful representation.