MAT 552 PROBLEM SET 8

This problem uses Cartan's criterion to check that the special linear Lie algebra is semisimple. The remainder of the problem set is devoted to Levi's Theorem: every Lie algebra is a semidirect product of its solvable radical and its semisimple part. For completeness, there are some exercises on bimodules and derivations. This is used to define the semidirect product of Lie algebras, and it will be used further in the next problem set about Ado's Theorem.

Problems.

Problem 1.(The Killing form on $\mathfrak{sl}_n$.) Recall the root decomposition of $\mathfrak{g} = \mathfrak{sl}_n$ from lecture. The diagonal matrices form a maximal Abelian subalgebra $\mathfrak{h}_n$. For the induced adjoint representation of this Abelian subalgebra on all of $\mathfrak{sl}_n$, there is a direct sum decomposition,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h}_n \oplus \mathfrak{n}_+ = \left(\bigoplus_{\alpha\in\Phi_n^-} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h}'_n \oplus \left(\bigoplus_{\alpha\in\Phi_n^+} \mathfrak{g}_\alpha\right)$$

where $\Phi_n \subset \mathfrak{h}_n^\vee$ is the set of roots, and where Φ_n^+ and Φ_n^- are the subsets of positive and negative roots. Precisely, Φ_n^+ equals $\{\alpha_{i,j} | 1 \leq i < j \leq n\}$, where $\alpha_{i,j} \in$ is the linear functional that is the difference $\chi_i - \chi_j$ between the (i,i)-entry and the (j,j)-entry of a diagonal matrix, and Φ_n^- equals $-\Phi_n^+$, where $-\alpha_{i,j}$ equals $\alpha_{j,i}$. For every $\alpha = \alpha_{i,j}$, the root space $\mathfrak{g}_\alpha$ is spanned by the elementary matrix X_α that has (i,j)-entry equal to 1 and all other entries equal to 0.

(a) Since $\mathfrak{h}'_n$ is Abelian and since every $\mathfrak{g}_\alpha$ is 1-dimensional, conclude that for every $X \in \mathfrak{h}'_n$,

$$\mathrm{ad}_X \circ \mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$$

has trace equal to

$$B_K(X,X) = \sum_{\alpha\in\Phi} \langle \alpha, X\rangle^2 = 2\sum_{\alpha\in\Phi^+} \langle \alpha, X\rangle^2.$$

(b) Since the trace of X equals 0, compute that

$$B_K(X,X) = 2\sum_{1\leq i<j\leq n} \left(\chi_i(X)^2 - 2\chi_i(X)\chi_j(Y) + \chi_j(Y)^2\right) =$$

$$-4\sum_{1\leq i<j\leq n} \chi_i(X)\chi_j(X) + 2(n-1)\sum_{1\leq i\leq n} \chi_i(X)^2 =$$

$$-2\left(\sum_{1\leq i\leq n} \chi_i(X)\right)^2 + 2n\sum_{1\leq i\leq n} \chi_i(X)^2 = 2n\mathrm{Tr}_{\mathbb{C}^n}(X\circ X).$$

(c) For every diagonalizable element of $\mathrm{Mat}_{n\times n}$, there exists a basis such that the element is in $\mathfrak{h}_n$. Thus, every diagonalizable element X of $\mathfrak{sl}_n$ is conjugate to an element of $\mathfrak{h}'_n$. For such elements, conclude that $B_K(X,X) = 2n\mathrm{Tr}_{\mathbb{C}^n}(X\circ X)$.

(d) Finally, since the subset of diagonalizable elements in $\mathfrak{sl}_n$ is dense, conclude that $B_K(X,X)$ equals $2n\mathrm{Tr}_{\mathbb{C}^n}(X\circ X)$ for every X in $\mathfrak{sl}_n$. Thus, the Killing from $B_K = B_{\mathrm{ad}}$ equals a positive scalar multiple of the invariant symmetric form B_{std} of the standard representation of $\mathfrak{sl}_n$ on $\mathbb{C}^n$.

(e) For every matrix X, for the matrix $\overline{X}_{\mathrm{ss}}$ from the proof of Cartan's Solvability Criterion, the trace of $\overline{X}_{\mathrm{ss}}$ equals the complex conjugate of the trace of X. Thus, if X is in $\mathfrak{sl}_n$, so is $\overline{X}_{\mathrm{ss}}$. Since $\mathrm{Tr}_{\mathbb{C}^n}(X\circ\overline{X}_{\mathrm{ss}})$ equals 0 if and only if X is nilpotent, conclude that the kernel of the Killing form consists only of nilpotent matrices. By the Jordan canonical form, there exists a basis of $\mathbb{C}^n$ such that the nilpotent matrix X has nonzero entries only on the main superdiagonal, and these equal 0 or 1. For the transpose matrix $X^\dagger$, compute $\mathrm{Tr}_{\mathbb{C}^n}(X\circ X^\dagger)$. Conclude that the kernel of the Killing form is trivial, and thus $\mathfrak{sl}_n$ is a semisimple Lie algebra by Cartan's Semisimplicity Criterion.

Problem 2. (**Bimodules.**) Let k be a field, either $k=\mathbb{R}$ or $k=\mathbb{C}$. Let $(A, m_A : A\times A\to A)$ and $(B, m_B : B\times B\to B)$ be associative k-algebras, not necessarily unital. For a k-vector space M, an A-B-**bimodule** structure on M is a pair of k-bilinear operations,

$$L : A\times M\to M,\quad R : M\times B\to B,$$

such that the induced k-linear maps,

$$\widetilde{L} : A\to \mathrm{Hom}_k(M,M),\quad \widetilde{L}_a(y) = L(a,y),$$

$$\widetilde{R} : B\to \mathrm{Hom}_k(M,M),\quad \widetilde{R}_b(x) = R(x,b),$$

are homomorphisms of associative k-algebras and such that for every $a\in A$ and for every $b\in B$,

$$\widetilde{L}_a\circ\widetilde{R}_b = \widetilde{R}_b\circ\widetilde{L}_a.$$

If A and B are unital, an A-B-bimodule structure is **unital** if both $\widetilde{L}$ and $\widetilde{R}$ are homomorphisms of associative, unital k-algebras, i.e., they also send the multiplicative identity to the multiplicative identity.

(a) Check that an A-B-bimodule structure on M is equivalent to a module structure for the k-algebra $A\otimes_k B$ (usual product structure, not the $\mathbb{Z}$-graded variant). Thus the usual operations for modules over an algebra all make sense: direct sum, direct product, kernel, cokernel, etc.

(b) As usual, for a right B-module $(M, R_M : M\times B\to M)$ and for a left B-module $(N, L_N : B\times N\to N)$, denote by

$$\beta_{M,B,N} : M\times N\to M\otimes_B N,$$

the universal biadditive map that is B-*balanced*, i.e., for every $x\in M$, for every $b\in B$, and for every $y\in N$,

$$\beta_{M,B,N}(R_M(x,b),y) = \beta_{M,B,N}(x, L_N(b,y)).$$

For associative k-algebras A, B, and C, for an A-B-bimodule (M, L_M, R_M), and for a B-C-bimodule (N, L_N, R_N), check that the following maps are B-balanced biadditive maps,

$$\beta_{M,B,N}\circ(L_M(a,-)\times\mathrm{Id}_N) : M\times N\to M\times N\to M\otimes_B N,\ \ (x,y)\mapsto \beta_{M,B,N}(L_M(a,x),y),$$

$$\beta_{M,B,N}\circ(\mathrm{Id}_M\times R_N(-,c)) : M\times N\to M\times N\to M\otimes_B N,\ \ (x,y)\mapsto \beta_{M,B,N}(x, R_N(y,c)).$$

By universality, deduce that there exist unique additive maps,

$$L_{M\otimes_B N}(a,-) : M\otimes_B N \to M\otimes_B N,$$
$$R_{M\otimes_B N}(-,c) : M\otimes_B N \to M\otimes_B N,$$

such that $L_{M\otimes_B N}(a,-)\circ\beta_{M,B,N}$ equals $\beta_{M,B,N}\circ(L_M(a,-)\times \mathrm{Id}_N)$ and such that $R_{M\otimes_B N}(-,c)\circ\beta_{M,B,N}$ equals $\beta_{M,B,N}\circ(\mathrm{Id}_M\times R_N(-,c))$. Deduce that there are binary maps,

$$L_{M\otimes_B N} : A\times(M\otimes_B N)\to M\otimes_B N,$$
$$R_{M\otimes_B N} : (M\otimes_B N)\times C\to M\otimes_B N.$$

Check that these binary maps form an A-C-bimodule structure on $M\otimes_C N$.

(c) As usual, denote by $-\otimes_B-$ the bifunctor,

$$-\otimes_B - : \mathrm{Mod}-B\times B-\mathrm{Mod}\to\mathbb{Z}-\mathrm{Mod},\quad (M,N)\mapsto M\otimes_B N,$$
$$(\mu : M\to M',\nu : N\to N')\mapsto \mu\otimes\nu : M\otimes_B N\to M'\otimes_B N'.$$

For the A-C-bimodule structure defined above, prove that this bifunctor "restricts" to a bifunctor,

$$-\otimes_B - : A-\mathrm{Mod}-B\times B-\mathrm{Mod}-C\to A-\mathrm{Mod}-C.$$

Also, for every associative k-algebra D, for the trifunctors,

$$(-\otimes_B-)\otimes_C - : A-\mathrm{Mod}-B\times B-\mathrm{Mod}-C\times C-\mathrm{Mod}-D\to A-\mathrm{Mod}-D,$$
$$-\otimes_B(-\otimes_C -_: A-\mathrm{Mod}-B\times B-\mathrm{Mod}-C\times C-\mathrm{Mod}-D\to A-\mathrm{Mod}-D,$$

check that there is a natural equivalence,

$$\theta_{A,B,C,D} : (-\otimes_B-)\otimes_C-\Rightarrow -\otimes_B(-\otimes_C-).$$

Formulate and prove the evident higher associativity compatibility for $\theta_{A,B,C,D}$.

(d) For every associative algebra $(B, m_B : B\times B\to B)$, the **left regular module structure** and the **right regular module structure** are defined by

$$L_a(c) = m_B(a,c),\quad R_b(c) - m_B(c,b).$$

Check that these define a structure of B-B-bimodule on B, called the **regular bimodule structure**. For every B-C-bimodule N, check that the induced B-C-bimodule structure on $B\otimes_B N$ is naturally isomorphic to N as a B-C-bimodule. Similarly, for every A-B-bimodule M, check that the induced A-B-bimodule structure on $M\otimes_B B$ is naturally isomorphic to M as an A-B-bimodule.

(e) For every homomorphism of associative algebras,

$$f : B\to A,\quad m_A(f(b),f(b')) = f(m_B(b,b'))\ \forall b,b'\in A,$$

for every A-A-bimodule $(M, L : A\times M\to M, R : M\times A\to A)$, define,

$$L^f : B\times M\to M,\quad L^f(b,m) := L(f(b),m),$$
$$R^f : M\times R\to M,\quad R^f(m,b) := R(m,f(b)).$$

Check that this is a structure of B-B-bimodule on M, sometimes denoted M^f. For A-A-bimodules M and N, check that every morphism of A-A-bimodules from M to N is also a morphism of B-B-modules, $M^f\to N^f$. Finally, giving A and B their regular bimodule structures, check that the B-B-bimodule morphisms from B to A^f are isomorphic to the B-B-submodule $Z_A(f(B))$ of A of all elements that centralize $f(B)$.

(f) For every associative algebra $(B, m_B : B \times B \to B)$, for the associated Lie algebra structure,

$$[a,b]_B := m_B(a,b) - m_B(b,a),$$

check that the left regular representation is a Lie algebra representation,

$$L_{[a,b]} = L_a \circ L_b - L_b \circ L_a,$$

and similarly for the right regular representation. Finally, the usual adjoint representation is defined by,

$$\mathrm{ad}_B : B \to \mathrm{Hom}_k(B,B), \quad \mathrm{ad}_{B,a} : b \mapsto m_B(a,b) - m_B(b,a).$$

Check that this is also a Lie algebra representation (this is the Jacobi identity for this Lie algebra).

Problem 3. (**Derivations**.) For every associative k-algebra $(A, m_A : A \times A \to A)$ and for every A-A-bimodule $(M, L : A \times M \to M, R : M \times A \to M)$, denote the direct sum of the regular A-A-bimodule and M by

$$A \oplus (M \cdot \epsilon) = A \oplus M.$$

The symbol ϵ is just a placeholder. When confusion is unlikely, this is just denoted $A \oplus M \cdot \epsilon$.

(a) Check that the following k-bilinear operation is a structure of associative k-algebra on $A \oplus M \cdot \epsilon$,

$$(A \oplus M \cdot \epsilon) \times (A \oplus M \cdot \epsilon) \to A \oplus M \cdot \epsilon, \ \ (a + x\epsilon, b + y\epsilon) \mapsto m_A(a,b) + (L(a,y) + R(x,b))\,\epsilon.$$

Check that the inclusion of the summand A is a homomorphism of associative k-algebras,

$$i : A \to (A \oplus M \cdot \epsilon)\,, \quad i(a) = a + 0\epsilon.$$

Finally, check that projection to the summand A is a homomorphism of associative k-algebras,

$$r : (A \oplus M \cdot \epsilon) \to A, \quad r(a + x\epsilon) = a.$$

(b) For every left A-module $(\widetilde{M}, \widetilde{L} : A \times \widetilde{M} \to \widetilde{M})$, with its left and right operations a **compatible structure** of left module for $A \oplus M \cdot \epsilon$ is a structure of left module,

$$\widetilde{L}_M : (A \oplus M \cdot \epsilon) \times \widetilde{M} \to \widetilde{M},$$

such that $\widetilde{L}_M^i$ equals $\widetilde{L}$. Check that every compatible structure of left module for $A \oplus M \cdot \epsilon$ is of the form

$$\widetilde{L}_M(a + x\epsilon, y) = \widetilde{L}(a,y) + \widetilde{L}_\epsilon(x,y),$$

for a unique A-balanced map

$$\widetilde{L}_\epsilon : M \times N \to N,$$

that is also left A-linear, i.e., for every $x \in M$, for every $a \in A$, and for every $y \in N$,

$$\widetilde{L}_\epsilon(R(x,a), y) = \widetilde{L}_\epsilon(x, L(a,y)), \quad \widetilde{L}(a, \widetilde{L}_\epsilon(x,y)) = \widetilde{L}_\epsilon(L(a,x), y),$$

and such that for every $x, x' \in M$ and for every $y \in Y$,

$$\widetilde{L}_\epsilon(x, \widetilde{L}_\epsilon(x', y)) = 0.$$

Conversely, given $\widetilde{L}_\epsilon$ as above, prove that this defines a compatible structure of left module for $A \oplus M \cdot \epsilon$. Formulate and prove the analogous results for right modules and for bimodules.

(c) For every associative k-algebra $(B, m_B : B \times B \to B)$, for every morphism of associative k-algebras, $f : B \to A$, a **compatible morphism** of associative k-algebras to $A \otimes M \cdot \epsilon$ is a morphism of associative k-algebras,

$$f_M : B \to A \oplus M \cdot \epsilon,$$

such that $r \circ f_M$ equals f. Check that every compatible morphism is of the form

$$f_{M,\phi}(b) = f(b) + \phi(b)\epsilon,$$

for a unique k-linear map,

$$\phi : B \to M,$$

such that for every $b, b' \in B$,

$$\phi(m_B(b, b')) = R(\phi(b), f(b')) + L(f(b), \phi(b')).$$

Such a k-linear map ϕ is called an k-**derivation** from B to M^f. (When f equals the identity map from A to itself, this is a k-derivation from A to M.) Conversely, check that for every k-derivation from B to M^f, the map $f_{M,\phi}$ is a compatible morphism. Denote by $\mathrm{Der}_k(B, M^f)$ the set of all k-derivations from B to M^f. Check that the A-A-bimodule structure on M^f induces a natural structure of $Z_A(f(B))$-$Z_A(f(B))$-bimodule structure on $\mathrm{Der}_k(B, M^f)$.

(d) For an associative subalgebra R of B, a k-derivation from B to M^f is an R-**derivation** if the kernel of the derivation contains R. Check that a k-derivation is an R-derivation if and only if the compatible morphism of associative k-algebras restricts on R as $i \circ f$. Conclude that the subset $\mathrm{Der}_R(B, M^f)$ of R-derivations is a $Z_A(f(B))$-$Z_A(f(B))$-bimodule in $\mathrm{Der}_k(B, M^f)$ that is also naturally a R-R-bimodule.

(e) For every pair $\phi, \psi \in \mathrm{Der}_R(B, B)$, check that the following R-module homomorphism from B to itself is an R-derivation,

$$[\phi, \psi]_{\mathrm{Der}(B,B)}(b) = \phi(\psi(b)) - \psi(\phi(b)).$$

Check that this operation makes $\mathrm{Der}_R(B, B)$ into a Lie algebra.

Problem 4. (**Derivations and Lie algebras.**) This proble continues the previous problem. Now assume that A is unital.

(a) For every k-vector space V, and for every pair of k-linear transformations,

$$f_1 : V \to A, \quad \phi_1 : V \to M,$$

prove that there exists a unique pair of morphisms of associative, unital k-algebras,

$$f : T_k^\bullet(V) \to A, \quad f_M : T^\bullet(V) \to A \oplus M \cdot \epsilon,$$

whose restrictions to $T_k^1(V) = V$ equal f_1 and $f_1 \oplus \phi_1 \cdot \epsilon$. Conclude that there exists a unique derivation ϕ from $T_k^\bullet(V)$ to M^f whose restriction to $T_k^1(V)$ equals V.

(b) Now let $(\mathfrak{g}, [\bullet, \bullet]_\mathfrak{g})$ be a k-Lie algebra. Let M be a bimodule for the universal enveloping algebra $U_k(\mathfrak{g})$. This is equivalent to giving a left k-linear $\mathfrak{g}$-representation on M and a right k-linear $\mathfrak{g}$-representation on M,

$$L : \mathfrak{g} \times M \to M, \quad R : M \times \mathfrak{g} \to M,$$

such that for every $X, Z \in \mathfrak{g}$ and for every $y \in M$,

$$L(X, R(y, Z)) = R(L(X, y), Z).$$

Check that the following binary operation is also a left k-linear $\mathfrak{g}$-representation on M,

$$\mathrm{ad}_{L,R} : \mathfrak{g} \times M \to M, \quad (X, y) \mapsto L(X, y) - R(y, X).$$

This is the **adjoint representation** of (L, R). Conclude that there is an associated left module structure for $U_k(\mathfrak{g})$. (This is the module structure associated to the bimodule structure via the comultiplication of the Hopf algebra structure).

(c) Via the quotient homomorphism, $q : T_k^\bullet(\mathfrak{g}) \to U_k(\mathfrak{g})$, also M^q is a bimodule for $T_k^\bullet(\mathfrak{g})$. Let $f_1 : \mathfrak{g} \to T_k^\bullet(\mathfrak{g})$ be the inclusion to $T_k^1(\mathfrak{g})$. For every k-linear transformation,

$$\phi_1 : \mathfrak{g} \to M,$$

prove that the induced compatible morphism of associative k-algebras,

$$f_M : T_k^\bullet(\mathfrak{g}) \to U_k(\mathfrak{g}) \oplus M \cdot \epsilon,$$

factors through $q : T_k^\bullet(\mathfrak{g}) \to U_k(\mathfrak{g})$ if and only if ϕ_1 is a **Lie algebra derivation** with respect to the bimodule structure (L, R), i.e., for every $X, Y \in \mathfrak{g}$,

$$\phi_1([X, Y]_\mathfrak{g}) = \mathrm{ad}_{L,R}(X, \phi_1(Y)) - \mathrm{ad}_{L,R}(Y, \phi_1(X)).$$

Conclude that the k-derivations of $U_k(\mathfrak{g})$ to the $U_k(\mathfrak{g})$-bimodule M are the same as the k-linear Lie algebra derivations from $\mathfrak{g}$ to the $\mathfrak{g}$-bimodule M.

(d) Now let M equal $U_k(\mathfrak{g})$ with its regular $U_k(\mathfrak{g})$-bimodule structure. Assume that ϕ_1 factors through the k-subspace $\mathfrak{g} \xrightarrow{i} U_k(\mathfrak{g})$. Check that ϕ_1 is a Lie algebra derivation of this bimodule if and only if, for every $X, Y \in \mathfrak{g}$,

$$\phi_1([X, Y]_\mathfrak{g}) = [\phi_1(X), Y]_\mathfrak{g} + [X, \phi_1(Y)]_\mathfrak{g}.$$

This is the usual definition of a Lie algebra derivation from $\mathfrak{g}$ to $\mathfrak{g}$. Thus, the Lie algebra k-derivations from $U_k(\mathfrak{g})$ to itself that map $i(\mathfrak{g})$ to $i(\mathfrak{g})$ are the usual Lie algebra k-derivations from $\mathfrak{g}$ to $\mathfrak{g}$.

(e) In the k-Lie algebra of all k-derivations from $U_k(\mathfrak{g})$ to itself, check that the k-subspace of Lie algebra k-derivations from $\mathfrak{g}$ to $\mathfrak{g}$ is a k-Lie subalgebra, $\mathrm{Der}_k(\mathfrak{g}, \mathfrak{g})$.

(f) For every k-Lie algebra $\mathfrak{h}$ and for every morphism of k-Lie algebras,

$$\theta : \mathfrak{h} \to \mathrm{Der}_k(\mathfrak{g}, \mathfrak{g}),$$

$$\theta_z([X, Y]_\mathfrak{g}) = [\theta_z(X), Y]_\mathfrak{g} + [X, \theta_z(Y)]_\mathfrak{g}, \quad \theta_{[z,w]_\mathfrak{h}} = \theta_z \circ \theta_w - \theta_w \circ \theta_z,$$

define the **semidirect product Lie bracket** on $\mathfrak{g} \times \mathfrak{h}$ by

$$[(X, z), (Y, w)]_{\mathfrak{g},\mathfrak{h},\theta} := ([X, Y]_\mathfrak{g} + \theta_z(Y) - \theta_w(X), [z, w]_\mathfrak{h}).$$

Check that this defines a k-Lie algebra structure on $\mathfrak{g} \times \mathfrak{h}$. Check that the k-subspace $\mathfrak{g} \times \{0\} \cong \mathfrak{g}$ is a Lie ideal whose induced Lie algebra structure is the given Lie algebra structure on $\mathfrak{g}$. Check that the k-subspace $\{0\} \times \mathfrak{h} \cong \mathfrak{h}$ is a Lie subalgebra that is isomorphic to the given Lie algebra structure. Finally, check that the restriction to $\{0\} \times \mathfrak{h}$ of the adjoint action of the Lie algebra on the Lie ideal $\mathfrak{g} \times \{0\}$ is θ. This is the **semidirect product** of $\mathfrak{g}$ and $\mathfrak{h}$ via θ.

Problem 5. (**Levi's Theorem, I.**) Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra with trivial center whose solvable radical $\mathfrak{r}$ is a nonzero Abelian Lie algebra. Also assume that the adjoint action of $\mathfrak{g}$ on $\mathfrak{r}$ is an irreducible representation of $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/\mathfrak{r}$.

(a) Prove that the adjoint representation is faithful, so that $\mathfrak{g}$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$ associated to the associative (unital) algebra $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$.

(b) Define $\mathfrak{a}$ to be the subspace of $\mathfrak{gl}(\mathfrak{g})$ of linear endomorphisms of $\mathfrak{g}$ with image contained in $\mathfrak{r}$ and whose restriction to $\mathfrak{r}$ is a $\mathbb{C}$-multiple of $\mathrm{Id}_{\mathfrak{r}}$. Define $\mathfrak{b} \subset \mathfrak{a}$ to be the subspace of such linear endomorphisms whose restriction to $\mathfrak{r}$ is the zero map. Check that $\mathfrak{a}$ and $\mathfrak{b}$ are associative subalgebras of the associative algebra $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$ (neither of these subalgebras is unital). Thus, the commutator bracket on each of these subalgebras realizes each as a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Also check that $\mathfrak{b}$ is an Abelian Lie ideal in $\mathfrak{a}$, and the quotient Lie algebra is 1-dimensional (hence Abelian).

(c) Define $L_{\mathfrak{g}}$ to be the restriction to $\mathfrak{g}$ of the left regular representation of $\mathfrak{gl}(\mathfrak{g})$ on itself, i.e., for every $X \in \mathfrak{g}$ and for every $\phi \in \mathrm{Hom}(\mathfrak{g},\mathfrak{g})$, the element $L_{\mathfrak{g}}(X)\cdot\phi$ in $\mathrm{Hom}(\mathfrak{g},\mathfrak{g})$ equals

$$L_{\mathfrak{g}}(X)\cdot\phi : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto \mathrm{ad}_X \circ \phi(Y) = [X,\phi(Y)]_{\mathfrak{g}}.$$

Check that the commutator of ad_X and ϕ in $\mathfrak{gl}(\mathfrak{g})$ equals

$$[\mathrm{ad}_X,\phi]_{\mathfrak{gl}(\mathfrak{g})} : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X,\phi(Y)]_{\mathfrak{g}} - \phi([X,Y]_{\mathfrak{g}}).$$

(d) Now consider the restriction to $\mathfrak{g}$ of the adjoint representation of $\mathfrak{gl}(\mathfrak{g})$ on itself,

$$\mathrm{ad}' : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{gl}(\mathfrak{g})), \quad \mathrm{ad}'_X(\phi) := [\mathrm{ad}_X,\phi]_{\mathfrak{gl}(\mathfrak{g})}.$$

Check that every ad'_X is a derivation of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$.

(e) Check that $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{g}$-subrepresentations of the $\mathfrak{g}$-represenation on $\mathfrak{gl}(\mathfrak{g})$ determined by ad'.

Problem 6. (**Levi's Theorem, II.**) This problem continues the previous problem.

(a) Check that for every $X \in \mathfrak{g}$ and for every $\phi \in \mathfrak{a}$, the element $\mathrm{ad}'_X(\phi)$ is contained in $\mathfrak{b}$. Conclude that the induced $\mathfrak{g}$-representation on the quotient $\mathfrak{a}/\mathfrak{b}$ is the trivial 1-dimensional $\mathfrak{g}$-representation.

(b) Check that the image under $\mathrm{ad}^{\mathfrak{g}}$ of $\mathfrak{r}$ is a $\mathfrak{g}$-subrepresentation of $\mathfrak{b}$, i.e., $\mathfrak{b}$ contains every element $\phi = \mathrm{ad}_X$ for $X \in \mathfrak{r}$.

(c) Check that on the associated quotient spaces $\mathfrak{a}/\mathrm{ad}^{\mathfrak{g}}(\mathfrak{r})$ and $\mathfrak{b}/\mathrm{ad}^{\mathfrak{g}}(\mathfrak{r})$, the $\mathfrak{g}$-representation restricts as the zero representation on the Lie subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Thus, the natural short exact sequence of $\mathfrak{g}$-representations,

$$0 \to \mathfrak{a}/\mathrm{ad}^{\mathfrak{g}}(\mathfrak{r}) \to \mathfrak{b}/\mathrm{ad}^{\mathfrak{g}}(\mathfrak{r}) \to \mathfrak{a}/\mathfrak{b} \to 0,$$

is actually a short exact sequence of $\mathfrak{g}_{\mathrm{ss}}$-representations.

(d) Finally, use complete reducibility and triviality of the representation $\mathfrak{a}/\mathfrak{b}$ to conclude that there exists $\phi \in \mathfrak{a} \subset \mathrm{Hom}(\mathfrak{g},\mathfrak{r})$ restricting as the identity on $\mathfrak{r}$ such that for every $X \in \mathfrak{g}$,

$$[\mathrm{ad}_X,\phi]_{\mathfrak{gl}(\mathfrak{g})} = \mathrm{ad}_{-\psi(X)},$$

for a unique linear map $\psi \in \mathrm{Hom}(\mathfrak{g},\mathfrak{r})$. Thus, for every $X, Y \in \mathfrak{g}$,

$$\phi([X,Y]_{\mathfrak{g}}) = [X,\phi(Y)]_{\mathfrak{g}} + [\psi(X),Y]_{\mathfrak{g}}.$$

Define $\mathfrak{g}'$ to be the kernel of ψ.

(e) Check that $\psi(X)$ equals 0 if and only if, for every $Y \in \mathfrak{g}$,

$$\phi([X,Y]_{\mathfrak{g}}) = [X,\phi(Y)]_{\mathfrak{g}}.$$

For $X_1, X_2 \in \mathfrak{g}$, since
$$[[X_1, X_2]_\mathfrak{g}, Y]_\mathfrak{g} = [X_1, [X_2, Y]_\mathfrak{g}]_\mathfrak{g} - [X_2, [X_1, Y]_\mathfrak{g}]_\mathfrak{g},$$
deduce that for every $X_1, X_2 \in \mathfrak{g}$,
$$\phi([[X_1, X_2]_\mathfrak{g}, Y]_\mathfrak{g}) = \phi([X_1, [X_2, Y]_\mathfrak{g}]_\mathfrak{g}) - \phi([X_2, [X_1, Y]_\mathfrak{g}]_\mathfrak{g}) =$$
$$[X_1, \phi([X_2, Y]_\mathfrak{g})]_\mathfrak{g} - [X_2, \phi([X_1, Y]_\mathfrak{g})]_\mathfrak{g} = [X_1, [X_2, \phi(Y)]_\mathfrak{g}]_\mathfrak{g} - [X_2, [X_1, \phi(Y)]_\mathfrak{g}]_\mathfrak{g} =$$
$$[[X_1, X_2]_\mathfrak{g}, \phi(Y)]_\mathfrak{g}.$$
Thus, also $[X_1, X_2]_\mathfrak{g}$ is in $\mathfrak{g}'$. Conclude that $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$.

(f) Finally, since $\mathfrak{r}$ is an Abelian Lie algebra, check that for every $X \in \mathfrak{r}$ and for every $Y \in \mathfrak{g}$,
$$[X, Y]_\mathfrak{g} = [\psi(X), Y]_\mathfrak{g}.$$
Since the adjoint representation is faithful, conclude that also ψ is an element of $\mathfrak{a}$ that restricts as the identity on $\mathfrak{r}$. Therefore the kernel $\mathfrak{g}'$ is a complementary subspace to $\mathfrak{r}$ in $\mathfrak{g}$. Altogether, for every complex Lie algebra $\mathfrak{g}$ of finite dimension whose solvable radical is Abelian and gives an irreducible representation of $\mathfrak{g}_{\text{ss}}$ via the adjoint action, the Lie algebra is the semidirect product of the kernel $\mathfrak{g}' \cong \mathfrak{g}_{\text{ss}}$ and of the solvable radical.

Problem 7. (**Levi's Theorem, III.**) Now for every finite dimensional Lie algebra $\mathfrak{g}$ prove that $\mathfrak{g}$ is a semidirect product of its solvable radical $\mathfrak{r}$ and its semisimple part $\mathfrak{g}_{\text{ss}} := \mathfrak{g}/\mathfrak{r}$ by induction on the dimension of $\mathfrak{g}$.

If either $\mathfrak{r}$ or $\mathfrak{g}_{\text{ss}}$ is trivial, the result holds tautologically. Thus, assume that both of these are nontrivial.

If $\mathfrak{r}$ is solvable but not Abelian, then for the quotient of $\mathfrak{g}$ by the nonzero commutator Lie ideal $[\mathfrak{r}, \mathfrak{r}]_\mathfrak{g}$, conclude that there is a Lie subalgebra of the quotient that is isomorphic to $\mathfrak{g}_{\text{ss}}$. The inverse image of this Lie subalgebra in $\mathfrak{g}$ is proper in $\mathfrak{g}$ (thus has smaller dimension), and it has the same semisimple part. Use the induction hypothesis to conclude that there is a Lie subalgebra $\mathfrak{g}'$ complementary to $\mathfrak{r}$ inside this proper Lie subalgebra that maps isomorphically to $\mathfrak{g}_{\text{ss}}$. Thus, Levi's Theorem holds in this setting.

Finally, if $\mathfrak{r}$ is Abelian, yet the adjoint action of $\mathfrak{g}_{\text{ss}}$ on $\mathfrak{r}$ is reducible, then the quotient of $\mathfrak{g}$ by a proper, nonzero subrepresentation of $\mathfrak{r}$ has smaller dimension, thus has a Levi subalgebra. The inverse image in $\mathfrak{g}$ of this Levi subalgebra is a proper Lie subalgebra of $\mathfrak{g}$ that has the same semisimple part. Once again use the induction hypothesis to conclude that there exists a Levi subalgebra in $\mathfrak{g}$.