MAT 552 PROBLEM SET 3

Problem 1. For a Lie group (G, m, e) denote by $\mathfrak{g}$ the (abstract) Lie algebra of T_eG together with the bracket defined in class, which equals

$$[X,Y]_{\mathfrak{g}} := \frac{1}{2}\left(\frac{d}{ds}\frac{d}{dt}\mathrm{Exp}_G(sX)\cdot\mathrm{Exp}_G(tY)\cdot\mathrm{Exp}_G(-sX)\cdot\mathrm{Exp}_G(-tY)\right|_{s=t=0}.$$

For a vector field A on a manifold M, recall that the associated flow is defined on all sufficiently small neighborhoods of the zero section in the trivial rank 1 bundle over M,

$$\Phi_A : \mathbb{A}^1 \times M \supseteq U \to M, \quad (t,p) \mapsto \Phi_A^t(p),$$

satisfying the axioms that $\Phi_A^t(\Phi_A^s(p)) = \Phi_A^{s+t}(p)$ and $(d/dt)\Phi_A^t(p)|_{t=0}$ equals the tangent vector A_p of A at p for all p in M and for all s and t such that $(-|s|-|t|, |s|+|t|)\times\{p\}$ is contained in U. For vector fields A and B on M the "vector field Lie bracket" is defined by,

$$[A,B]_M := \frac{1}{2}\left(\frac{d}{ds}\frac{d}{dt}\Phi_{-s}^B \circ \Phi_{-t}^A \circ \Phi_s^B \circ \Phi_t^A\right|_{s=t=0}.$$

For $X \in \mathfrak{g}$ with its G-left invariant exponential flow,

$$\Phi_{G,t}^X : G \to G, \quad g \mapsto g\cdot\mathrm{Exp}_G(tX),$$

check that the two sign conventions agree after multiplying by -1 (so they do not agree "on the nose", but do agree after correcting the sign).

Problem 2. For a finite dimensional Lie algebra $(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$, define $\mathrm{Aut}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$ to be the subgroup of the Lie group $\mathbf{GL}(\mathfrak{g})$ of all linear automorphisms $\Lambda : \mathfrak{g} \to \mathfrak{g}$ such that for every $X, Y \in \mathfrak{g}$,

$$[\Lambda(X), \Lambda(Y)]_{\mathfrak{g}} = [X,Y]_{\mathfrak{g}}.$$

Similarly, define $\mathrm{Der}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$ to be the linear subspace of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$ of all linear endomorphisms $\lambda : \mathfrak{g} \to \mathfrak{g}$ such that for every $X, Y \in \mathfrak{g}$,

$$\lambda([X,Y]_{\mathfrak{g}}) = [\lambda(X), Y]_{\mathfrak{g}} + [X, \lambda(Y)]_{\mathfrak{g}}.$$

(a) Check that $\mathrm{Aut}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$ is a closed Lie subgroup of the Lie group $\mathbf{GL}(\mathfrak{g})$. Find an example where this is not a normal subgroup.

(b) Check that $\mathrm{Der}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$. Find an example where this is not a Lie ideal.

(c) Inside the Lie algebra $\mathfrak{gl}(\mathfrak{g})$ of the Lie group $\mathbf{GL}(\mathfrak{g})$, check that the Lie subalgebra associated to the closed Lie subgroup $\mathrm{Aut}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$ equals $\mathrm{Der}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$.

(d) For every $X \in \mathfrak{g}$ and for every $\lambda \in \mathrm{Der}(\mathfrak{g}, [\bullet,\bullet]_{\mathfrak{g}})$, check that

$$[\lambda, \mathrm{ad}_X^{\mathfrak{g}}]_{\mathfrak{gl}(\mathfrak{g})} = \mathrm{ad}_{\lambda(X)}^{\mathfrak{g}}.$$

(e) Conclude that the adjoint morphism of Lie algebras,

$$\mathrm{ad}^{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),$$

factors through the Lie subalgebra of derivations. Find examples when the image of the adjoint representation equals the Lie subalgebra of derivations, and find examples where the image is a proper Lie subalgebra of the Lie subalgebra of derivations.

Problem 3. This exercise is for those readers that know about affine algebraic groups G over $\mathbb{C}$ with the corresponding finitely generated, commutative, unital $\mathbb{C}$-algebra $\mathbb{C}[G]$ of polynomial maps from G to $\mathbb{C}$ and its comultiplication,

$$m^* : \mathbb{C}[G] \to \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[G].$$

(Please note: even though the notation appears similar, typically this is not the $\mathbb{C}$-group algebra of G, which is typically a noncommutative $\mathbb{C}$-algebra.)

For a $\mathbb{C}$-vector space V, a $\mathbb{C}$-**linear coaction** of G on V is a $\mathbb{C}$-linear map,

$$\phi : V \to V \otimes_{\mathbb{C}} \mathbb{C}[G],$$

such that the composition with evaluation at e,

$$\mathrm{Id}_V \otimes \mathrm{ev}_e : V \otimes_{\mathbb{C}} \mathbb{C}[G] \to V \otimes_{\mathbb{C}} \mathbb{C} = V$$

is the identity map on V, and such that the composition of ϕ with the following two $\mathbb{C}$-linear maps are equal,

$$\phi \otimes \mathrm{Id}_{\mathbb{C}[G]} : V \otimes_{\mathbb{C}} \mathbb{C}[G] \to (V \otimes_{\mathbb{C}} \mathbb{C}[G]) \otimes_{\mathbb{C}} \mathbb{C}[G],$$
$$\mathrm{Id}_V \otimes m^* : V \otimes_{\mathbb{C}} \mathbb{C}[G] \to V \otimes_{\mathbb{C}} (\mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[G]).$$

(a) For every $\mathbb{C}$-vector space V, for every $\mathbb{C}$-linear coaction ϕ, and for every $\mathbb{C}$-subspace W of V, prove that there exists a unique minimal $\mathbb{C}$-vector subspace $W' \subset V$ such that the image $\phi(W)$ is contained in the subspace $W' \otimes_{\mathbb{C}} \mathbb{C}[G]$.

(b) Use the axioms of a coaction to prove that $(W')'$ equals W'. Thus, the restriction of ϕ to W' defines a $\mathbb{C}$-linear coaction of G on W'.

(c) If W is a finite dimensional $\mathbb{C}$-vector space, prove that also W' is finite dimensional. Conclude that V is an increasing union of finite dimensional $\mathbb{C}$-subspaces on which ϕ restricts to a coaction.

(d) In particular, setting V equal to the $\mathbb{C}$-vector space $\mathbb{C}[G]$ with the coaction m^*, for every finite subset $S \subset \mathbb{C}[G]$ of $\mathbb{C}$-algebra generators, conclude that there is a finite dimensional $\mathbb{C}$-vector subspace $W' \subset \mathbb{C}[G]$ containing S and such that ϕ restricts to a coaction on W'.

(e) For every finite dimensional $\mathbb{C}$-vector space V with a $\mathbb{C}$-linear coaction ϕ, define the following map,

$$\rho : G \to \mathbf{GL}(V), \quad g \mapsto (v \mapsto (\mathrm{Id}_V \otimes \mathrm{ev}_g)(\phi(v))).$$

Prove that this is a $\mathbb{C}$-linear action of G on V. These are precisely the "algebraic representations" of the algebraic group G.

(f) For every finite dimensional $\mathbb{C}$-vector subspace W' of $\mathbb{C}[G]$ that contains a set of $\mathbb{C}$-algebra generators of $\mathbb{C}[G]$, prove that the corresponding $\mathbb{C}$-linear action of G on W' is faithful. Thus, every affine algebraic group is a closed algebraic subgroup of $\mathbf{GL}(W')$ for some finite dimensional $\mathbb{C}$-vector space W'. This is an explicit form of Lie's Third Theorem for affine algebraic groups.

Problem 4. Let (G, m, e) be a Lie group. For every integer n, denote by $\mathcal{O}_{G,e}/\mathfrak{m}^{n+1}$ the finite dimensional vector space of germs of analytic functions on G at e up to order n. For every $g \in G$, denote by $\mathrm{Ad}_{G,n,g}$ the induced linear map

$$\mathcal{O}_{G,e}/\mathfrak{m}^{n+1} \to \mathcal{O}_{G,e}/\mathfrak{m}^{n+1},$$

induced by the conjugation map near the fixed point e,

$$\mathrm{Inner}_g : G \to G, \quad h \mapsto ghg^{-1}, \quad e \mapsto e.$$

(a) Prove that this gives a Lie group morphism

$$\mathrm{Ad}_{G,n} : G \to \mathbf{GL}_{\mathbb{C}}(\mathcal{O}_{G,e}/\mathfrak{m}^{n+1}),$$

such that $\mathrm{Ad}_{G,1}$ is the dual linear representation of the usual adjoint (linear) representation Ad_G of G on T_eG.

(b) When G is a complex Lie group, prove that $\mathrm{Ad}_{G,n}$ is a morphism of complex Lie groups (you can do this in coordinates, or you can use a similar diagram-chasing argument to that in lecture for Ad_G, where now we restrict the bundle isomorphism to the product in $G \times G$ of G times the "n^{th} infinitesimal neighborhood of e in G").

(c) When G is a compact, complex Lie group, what does the maximum modulus principle imply about holomorphic maps from G to the affine $\mathbb{C}$-space $\mathbf{Mat}_{\mathbb{C}}(\mathcal{O}_{G,e}/\mathfrak{m}^{n+1})$?

(d) Conclude that every connected, compact, complex Lie group G is commutative. These are usually called "compact complex tori". Use the same argument to prove that every $\mathbb{C}$-linear representation on a finite dimensional $\mathbb{C}$-vector space by a compact complex torus is a direct sum of subrepresentations, each of which is isomorphic to the trivial one-dimensional representation (so the finite dimensional linear representations are semisimple, but for trivial reasons).

Problem 5. For $\mathbf{SL}_n(\mathbb{C})$ with its standard maximal torus T, standard Borel, standard pinning, etc., use the derivatives of the standard basis T of cocharacters to write an explicit basis of the "Cartan subalgebra" $\mathfrak{h} = \mathrm{Lie}(T)$ inside $\mathfrak{sl}_n(\mathbb{C})$. Also use the pinning to write out a $\mathbb{C}$-basis for each root space of $\mathfrak{sl}_n(\mathbb{C})$. Combine these to form a vector space basis for the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. For each pair of basis vectors, explicitly compute the Lie bracket of those two elements of $\mathfrak{sl}_n(\mathbb{C})$ as a linear combination of the basis vectors. Write this out explicitly when $n = 2$ and $n = 3$. Are the coefficients contained in the subfield $\mathbb{R}$ of $\mathbb{C}$? Are they contained in $\mathbb{Q}$? Are they contained in $\mathbb{Z}$? What does this suggest to you about the possibility of extending Lie theory to affine algebraic groups over a more general field than $\mathbb{R}$ and $\mathbb{C}$?

Problem 6. Use the previous problem to prove that the real Lie algebra of $\mathbf{SL}_n(\mathbb{R})$ is a real form of the complex Lie algebra of $\mathbf{SL}_n(\mathbb{C})$. Also repeat the problem for the subgroup $\mathbf{SU}(n, \mathbb{R})$ of $\mathbf{SL}_n(\mathbb{C})$. Use this to check that $\mathfrak{su}(n, \mathbb{R})$ is also a real form of $\mathfrak{sl}_n(\mathbb{C})$. Finally, explicitly check that there is an isomorphism from $\mathfrak{su}(2, \mathbb{R})$ to $\mathfrak{so}_3(\mathbb{R})$ that complexifies to the isomorphism of complex Lie algebras associated to the isomorphism of complex Lie groups $\mathbf{PGL}_2(\mathbb{C}) \to \mathbf{SO}_3(\mathbb{C})$ discussed in lecture.