MAT 552 PROBLEM SET 6

Problems. This problem set completes the analytic proof of the Peter-Weyl Theorem. It is intended for those students with some background in Hilbert spaces and functional analysis.

Here is a quick reminder of the basics of complex Hilbert spaces including the statement of the spectral theorem. A **complex Hilbert space** is a Hermitian inner product space (V, β) whose associated metric space is complete (all Cauchy sequences converge). For Hermitian inner product spaces (V, β) and (W, γ), a **bounded linear transformation** (resp. a **compact linear transformation**) is a $\mathbb{C}$-linear transformation,

$$T : V \to W,$$

sending closed balls in (V, β) to bounded (resp. compact) subsets of W. The **operator norm**, $\|T\|_{\text{op}}$, of T is the supremum of the γ-lengths of all elements in the T-image of the closed unit ball of (V, β).

If the domain and target are complex Hilbert spaces, then the *Closed Graph Theorem* implies that T is bounded if and only if the graph of T is closed. In this case, there exists a unique bounded linear transformation,

$$T^* : (W, \gamma) \to (V, \beta),$$

such that for every $v \in V$ and for every $w \in W$,

$$\gamma(w, T(v)) = \beta(T^*(w), v).$$

This is the **adjoint** of T. Note that $\|T^*\|_{\text{op}}$ equals $\|T\|_{\text{op}}$.

The operation of adjoint makes $B((V, \beta), (V, \beta))$ and $B((W, \gamma), (W, \gamma))$ into (unital) C^*-algebras. Together with the operations sending T to $T^* \circ T \in B((V, \beta), (V, \beta))$, resp. to $T \circ T^* \in B((W, \gamma), (W, \gamma))$, also $B((V, \beta), (W, \gamma))$ is a right Hilbert C^*-module, resp. left Hilbert C^*-module, for these respective C^*-algebras. An operator $T \in B((V, \beta), (V, \beta))$ is **normal**, resp. **self-adjoint**, if T commutes with T^*, resp. if T equals T^*.

By the *Open Mapping Theorem*, if V and W are complete, then every surjective bounded linear transformation is an open mapping. If T is also injective, then T is a homeomorphism whose inverse is also a bounded operator. Denote by $\text{Inv}((V, \beta), (W, \gamma))$ the set of all bounded linear operators from V to W having a two-sided inverse that is also a bounded linear operator. Denote $\text{Inv}((V, \beta), (V, \beta))$ by $\mathbf{GL}_{\mathbb{C}}(V, \beta)$; this is the group (and open subset) of invertible elements in the C^*-algebra $B((V, \beta), (V, \beta))$.

For every nonzero Hilbert space (V, β) and for every bounded operator T from (V, β) to itself, the **spectrum of** T is

$$\text{spec}(T) := \{\lambda \in \mathbb{C} | \lambda \text{Id}_V - T \notin \mathbf{GL}_{\mathbb{C}}(V, \beta)\}.$$

This is a compact subset of $\mathbb{C}$. The **resolvent function**,

$$R(z; T) : \mathbb{C} \setminus \text{spec}(T) \to \mathbf{GL}_{\mathbb{C}}(V, \beta), \quad R(z; T) = (T - z\text{Id}_V)^{-1},$$

is a holomorphic map to $B((V,\beta),(V,\beta))$. By Liouville's theorem, the spectrum is *nonempty*.

For every polynomial function in one variable z,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_d z^d,$$

the associated bounded operator $f(T)$ is defined by,

$$f(T) = a_0 \mathrm{Id}_V + a_1 T + a_2 T \circ T + \cdots + a_d (T \circ \cdots \circ T).$$

Every bounded continuous function f on $\mathrm{spec}(T)$ is a uniform limit of a sequence of polynomial functions f_n. The operators $f_n(T)$ converge to a bounded operator $f(T)$ independent of the choice of convergent sequence of polynomials (f_n). Denote $C^o(\mathrm{spec}(T),\mathbb{C})$ the $\mathbb{C}$-vector space of bounded continuous functions on $\mathrm{spec}(T)$. There is a well-defined $\mathbb{C}$-linear map,

$$\mathrm{subs}_T : C^0(\mathrm{spec}(T),\mathbb{C}) \to B((V,\beta),(V,\beta)).$$

For every $f(z) \in C^0(\mathrm{spec}(T),\mathbb{C})$, denote by $E_{T,f}$ the kernel of $f(T)$ as a closed $\mathbb{C}$-linear subspace of V. For every closed subset $\Sigma \subset \mathrm{spec}(T)$, denote by $E_{T,\Sigma}$ the intersection of $E_{T,f}$ over all $f(z)$ that vanish on Σ.

Hypothesis 0.1. The operator $T \in B((V,\beta),(V,\beta))$ is self-adjoint.

Then the $\mathbb{C}$-linear map subs_T is a homomorphism of commutative, unital C^*-algebras, i.e., it sends function multiplication to composition, and it sends complex conjugation of functions to the adjoint operator.

Lemma 0.2 (Real spectrum, orthogonal eigenspaces)**.** *The spectrum of every self-adjoint operator T is real. If $\Sigma,\Theta \subset \mathrm{spec}(T)$ are disjoint closed subsets, then $E_{T,\Sigma}$ and $E_{T,\Theta}$ are pairwise orthogonal closed subspaces.*

Proof. For the polynomial $p_\lambda(z) = z - \lambda$ and associated norm-squared polynomial $\|p\|^2(z) := p_\lambda(z)\cdot\overline{p_\lambda(\overline{z})}$, observe

$$p_\lambda(z)\cdot\overline{p_\lambda(\overline{z})} = \mathrm{Im}(\lambda)^2 + (z - \mathrm{Re}(\lambda))^2.$$

Thus, for a self-adjoint operator T,

$$(T - \lambda\mathrm{Id}_V)\circ(T-\lambda\mathrm{Id}_V)^* = \mathrm{Im}(\lambda)^2\mathrm{Id}_V + (T - \mathrm{Re}(\lambda)\mathrm{Id}_V)^2 \geq \mathrm{Im}(\lambda)^2\mathrm{Id}_V.$$

Combined with the open mapping theorem, this implies that $T - \lambda\mathrm{Id}_V$ is invertible whenever $\mathrm{Im}(\lambda)$ is nonzero.

Next, by Urysohn's Lemma, there exist bounded, continuous, nonnegative real-valued functions $f(z)$ and $g(z)$ such that f vanishes on Σ, such that g vanishes on Θ, and such that $f+g$ equals 1. Thus, for every $v \in E_{T,\Sigma}$ and for every $w \in E_{T,\Theta}$,

$$\langle v,w\rangle = \langle (f(T)+g(T))v, w\rangle = \langle f(T)v,w\rangle + \langle g(T)v,w\rangle =$$

$$\langle f(T)v,w\rangle + \langle v, g(T)w\rangle = \langle 0,w\rangle + \langle v,0\rangle = 0.$$

□

For every $v \in V$, denote by $\mathrm{subs}_{T,v}$ the following $\mathbb{C}$-linear map,

$$\mathrm{subs}_{T,v} : C^0(\mathrm{spec}(T),\mathbb{C}) \to V, \quad f(z) \mapsto f(T)v.$$

The linear functional,

$$\int_{\mathrm{spec}(T)} (-)d\pi_{T,v} : C^0(\mathrm{spec}(T),\mathbb{C}) \to \mathbb{C}, \quad f(z) \mapsto \langle \mathrm{subs}_{T,v}(f), v\rangle = \langle f(T)v, v\rangle,$$

defines a positive Borel measure $d\pi_{T,v}$ on $\mathrm{spec}(T)$ that is even a Radon measure. Denote by $L^2(\mathrm{spec}(T), d\pi_{T,v})$ the corresponding Lebesgue space of square-integrable functions on $\mathrm{spec}(T)$ with respect to $d\pi_{T,v}$.

Theorem 0.3 (Spectral Theorem for Self-Adjoint Operators)**.** *For every nonzero complex Hilbert space (V, β), for every bounded, self-adjoint operator T on (V, β), for every $v \in V$, the $\mathbb{C}$-linear map $subs_{T,v}$ extends to an isometric embedding of Hilbert spaces,*

$$subs_{T,v} : L^2(spec(T), d\pi_{T,v}) \to V,$$

whose image is the smallest closed, T-stable subspace of V containing v.

Theorem 0.4 (Spectral Theorem for Self-Adjoint Compact Operators)**.** *Further, T is compact if and only if $spec(T) \setminus \{0\}$ contains no accumulation points, if the eigenspace of each $\lambda \in spec(T)\setminus\{0\}$ has finite dimension, and, together with $Ker(T)$, these eigenspaces span a dense subspace of V.*

Corollary 0.5. *A bounded, self-adjoint operator on a nonzero complex Hilbert space is a scalar multiple of the identity if and only if the spectrum is a singleton set.*

Proof. If T equals $\lambda \mathrm{Id}_V$ for a real number λ, then $\mathrm{spec}(T)$ equals $\{\lambda\}$. Conversely, assume that $\mathrm{spec}(T)$ equals $\{\lambda\}$. For every nonzero vector $v \in V$, since $\lambda - z$ restricts to zero on $\mathrm{spec}(T) = \{\lambda\}$, the restriction of this polynomial in $L^2(\mathrm{spec}(T), d\pi_{T,v})$ is zero. Thus, $\lambda \mathrm{Id}_V - T$ acts as the zero operator on v, i.e., $T(v) = \lambda v$. Since this holds for every $v \in V$, the operator T equals $\lambda \mathrm{Id}_V$. □

Problem 1. (Schur's Lemma, Part 1.) For a Lie group G, a **unitary representation** in a complex Hilbert space (V, β) is a continuous group homomorphism to the group of unitary (i.e., norm-preserving) $\mathbb{C}$-linear automorphisms of (V, β) with its norm topology,

$$\rho : G \to U(V, \beta).$$

This representation is **irreducible** if the only closed, $\rho(G)$-invariant subspaces of V are V and $\{0\}$.

(a) For unitary G-representations (V, β, ρ) and (W, γ, σ), for every bounded morphism of G-representations,

$$S : V \to W, \quad S \circ \rho_g = \sigma_g \circ S, \ \forall g \in G,$$

prove that also the adjoint S^* is a bounded morphism of G-representations.

(b) Also prove that the kernel of S and the kernel of S^* are closed subrepresentations. Similarly, the orthogonal complements of $\mathrm{Ker}(S^*)$ and $\mathrm{Ker}(S)$ are closed subrepresentations. These orthogonal complements equal the closures of the images of S and S^*.

(c) Check that $T := S^* \circ S$ is a bounded, self-adjoint operator on (V, β) that is a morphism of G-representations.

(d) Now assume that (V, β, ρ) and (W, γ, σ) are both irreducible unitary representations. If T is surjective, conclude that S^* is an isomorphism, and thus also the adjoint $S = (S^*)^*$ is an isomorphism. Thus, to prove Schur's Lemma for unitary representations, it suffices to prove that every bounded, self-adjoint morphism from an irreducible unitary representation (V, ρ) to itself equals a multiple of the identity operator.

Problem 2. (Schur's Lemma, Part 2.) Let (V, β, ρ) be an irreducible unitary G-representation. Let T be a bounded, self-adjoint operator of (V, β) that is a morphism of G-representations.

(a) Prove that every element of $\text{subs}_T(C^0(\text{spec}(T), \mathbb{C}))$ is a bounded operator on (V, β) that is a self-morphism of unitary G-representations.

(b) For a nonzero vector $v \in V$, assume by way of contradiction that the measure space $(\text{spec}(T), d\pi_{T,v})$ is not a singular measure supported at a single point. Use Urysohn's Lemma to find continuous functions $f(z), g(z) \in C^0(\text{spec}(T), \mathbb{C})$ with $f(z) \cdot g(z) = 0$ and with images in $L^2(\text{spec}(T), d\pi_{T,v})$ that are each nonzero. Since $f(T) \circ g(T)$ and $g(T) \circ f(T)$ equal 0, conclude that at least one of $f(T)$ or $g(T)$ has nonzero kernel, say $f(T)$. On the other hand, since $f(T)v$ is nonzero by the spectral theorem, conclude a contradiction. Altogether, conclude that for every nonzero vector $v \in V$, the measure space $(\text{spec}(T), d\pi_{T,v})$ is a singular metric supported at a single point λ_v. Repeat the proof of the corollary to conclude that $T(v)$ equals $\lambda_v \cdot v$.

(c) For a $\mathbb{C}$-linear operator on a $\mathbb{C}$-vector space V, if every vector is an eigenvector for some eigenvalue, conclude that the operator is a scalar multiple of the identity. Thus, for T as above, conclude that there exists $\lambda \in \mathbb{R}$ with $T = \lambda \text{Id}_V$.

Problem 3. (Eigenspaces of convolution operators.) Assume now that G is a compact (real) Lie group with normalized Haar measure $d\text{vol}_G$. For every $g \in G$, define

$$\lambda_g : L^2(G, d\text{vol}_G) \to L^2(G, d\text{vol}_G), \quad (\lambda_g u)(h) := u(g^{-1}h),$$
$$\rho_g : L^2(G, d\text{vol}_G) \to L^2(G, d\text{vol}_G), \quad (\lambda_g u)(h) := u(hg^{-1}),$$

For all continuous functions $u, v \in C^0(G, \mathbb{C})$, define the **convolution function** $u * v$ on G by

$$u * v(h) = \int_{g \in G} u(g)(\lambda_g v)(h) d\text{vol}_G(g) = \int_{g \in G} (\rho_g u)(h) v(g) d\text{vol}_G(g).$$

(a) Prove that λ_g and ρ_g are isometries. Prove that these define left, resp. right, unitary representations $\lambda : G \to U(L^2(G, d\text{vol}_G))$ and $\rho : G^{\text{opp}} \to U(L^2(G, d\text{vol}_G))$. Prove that these commute with one another, $\lambda_g(\rho_h u) = \rho_h(\lambda_g u)$.

(b) Prove that the L^∞ norm of $u * v$ is bounded above by $\|u\|_2 \cdot \|v\|_2$. (**Hint.** Use that the group inversion preserves the Haar measure. Thus the L^2-norm of $g \mapsto \lambda_g v(h)$ equals the L^2-norm of v.)

(c) Since G is a finite measure space, L^∞ is a subspace of L^2. Conclude that convolution extends to a continuous $\mathbb{C}$-bilinear operation,

$$* : L^2(G, d\text{vol}_G) \times L^2(G, d\text{vol}_G) \to L^2(G, d\text{vol}_G), \quad \|u * v\|_2 \leq \|u * v\|_\infty \leq \|u\|_2 \cdot \|v\|_2 .$$

In particular, for every $w \in L^2(G, d\text{vol}_G)$, deduce that the following operators are bounded operators,

$$\lambda_w : L^2(G, d\text{vol}_G) \to L^2(G, d\text{vol}_G), \quad v \mapsto w * v,$$

$$\rho_w : L^2(G, d\text{vol}_G) \to L^2(G, d\text{vol}_G), \quad u \mapsto u * w.$$

For the "heuristic" Dirac delta function δ_g of $g \in G$, this gives identities,

$$\lambda_g(v) = \lambda_{\delta_g}(v), \quad \rho_g(u) = \rho_{\delta_g}(u).$$

(d) For every $u, v, w \in L^2(G, d\text{vol}_G)$ and every $g \in G$, check the following identities,

$$(u * v) * w = u * (v * w),$$
$$\lambda_g(v * w) = (\lambda_g(v)) * w, \quad \rho_g(u * v) = u * (\rho_g(v)),$$
$$\lambda_u(v * w) = (\lambda_u(v)) * w, \quad \rho_w(u * v) = u * (\rho_w(v)).$$

(e) For every $w \in L^2(G, d\text{vol}_G)$, define $\widetilde{w} \in L^2(G, d\text{vol}_G)$ by

$$\widetilde{w}(g) = \overline{w(g^{-1})}.$$

Prove that the adjoint of λ_w equals $\lambda_{\widetilde{w}}$, and prove that the adjoint of ρ_w equals $\rho_{\widetilde{w}}$. In particular, conclude that λ_w, resp. ρ_w, is self-adjoint if and only if $\widetilde{w}$ equals w.

(f) Read about *Hilbert-Schmidt operators*. Conclude that λ_w and ρ_w are Hilbert-Schmidt operator, thus they are compact. When $\widetilde{w}$ equals w, conclude that these are compact self-adjoint operators. Since $\lambda_u(\rho_w(v))$ equals $\rho_w(\lambda_u(v))$, conclude that the eigenspaces of ρ_w, resp. of λ_w, are left G-subrepresentations of $L^2(G, d\text{vol}_G)$, resp. right G-subrepresentations of $L^2(G, d\text{vol}_G)$. Since the eigenspaces of a compact operator for nonzero eigenvalues have finite dimension, conclude that these eigenspaces for ρ_w, resp. for λ_w, are direct sums of finitely many irreducible left, resp. right, G-subrepresentations that have finite dimension.

(g) A sequence $(w_n)_{n \geq 0}$ of continuous, nonnegative real-valued functions on G is a **balanced Dirac sequence** if each $\widetilde{w_n}$ equals w_n, if each $\int_g w_n(g) d\text{vol}_G(g)$ equals 1, and if for every $\epsilon > 0$ and every open neighborhood of $e \in G$, for all $n \gg 0$, we have $|w_n(g)| < \epsilon$ for all g outside the open neighborhood. Prove that there exists a balanced Dirac sequence.

(h) For every $v \in C^0(G, \mathbb{C})$, prove that $\rho_{w_n}(v)$ converges uniformly to v on G, and thus converges to v in $L^2(G, d\text{vol}_G)$. For every $u \in L^2(G, d\text{vol}_G)$, use self-adjointness of ρ_{w_n} to prove that

$$\lim_{n \to \infty} \langle \rho_{w_n}(u), v \rangle_{L^2} = \langle u, v \rangle.$$

Since the continuous functions are dense in $L^2(G, d\text{vol}_G)$, prove that this holds for every $v \in L^2(G, d\text{vol}_G)$, i.e., $\rho_{w_n}(u)$ converges weakly to u. In particular, if $\rho_{w_n}(u)$ equals 0 for all $n \gg 0$, conclude that also u equals 0.

(i) Conclude that for every nonzero $u \in L^2(G, d\text{vol}_G)$, for all $n \gg 0$, the element u is not in $\text{Ker}(\rho_{w_n})$. Thus, u has nonzero orthogonal projection to at least one of the eigenspaces of ρ_{w_n} with nonzero eigenvalue. Since this is a direct sum of finitely many irreducible (left) G-subrepresentations, conclude that u has nonzero projection to at least one irreducible (left) G-subrepresentation of finite dimension. Thus, the sum in $L^2(G, d\text{vol}_G)$ of all irreducible (left) G-subrepresentations of finite dimension is dense in $L^2(G, d\text{vol}_G)$. This completes the proof of surjectivity in the Peter-Weyl Theorem.

Problem 4. (Irreducible unitary representations of compact groups have finite dimension.) Let G be a compact (real) Lie group. Let (V, β) be a nonzero complex

Hilbert space, and let $\rho : G \to U(V, \beta)$ be a unitary representation that is irreducible. For any nonzero vector $v \in V$, and for the orthogonal projection to the span of v,

$$\text{proj}_v : V \to \text{span}(v) \subseteq V,$$

consider the $\mathbb{C}$-linear operator on V,

$$T = \int_{g \in G} \rho_g \circ \text{proj}_v \circ \rho_g^{-1} d\text{vol}_G(g).$$

(a) Prove that T is a bounded linear operator that is a morphism of G-representations. By Schur's Lemma, conclude that T equals λId_V for some real number λ.

(b) Compute that

$$\langle T(v), v \rangle = \int_{g \in G} \langle \text{proj}_v \circ \rho_g^{-1}(v), \rho_g^{-1}(v) \rangle d\text{vol}_G(g) =$$

$$\int_{g \in G} \langle \text{proj}_v \circ \rho_g^{-1}(v), \text{proj}_v \circ \rho_g^{-1}(v) \rangle d\text{vol}_G(t).$$

Prove that the function $g \mapsto \langle \text{proj}_v \circ \rho_g^{-1}(v), \text{proj}_v \circ \rho_g^{-1}(f) \rangle$ is continuous and nonzero at $g = e$. Conclude that the integral is a positive real number, and thus also λ is positive.

(c) Since T is defined as a limit of Riemann sums, prove that T is in the closure of the finite-rank operators, i.e., T is a compact operator. Thus the identity operator on V is a compact operator. Conclude that V has finite dimension. Thus, every irreducible (left) unitary G-representation has finite dimension, and hence occurs in the Peter-Weyl Theorem.

Problem 5. (Compact Lie groups have faithful representations of finite dimension.) Let G be a compact (real) Lie group. Let $W \subset L^2(G, d\text{vol}_G)$ be a finite dimensional subspace containing a system of coordinate functions of G relative to an embedding of G as a submanifold of the real manifold $\mathbb{C}^n$. Use the previous problems to prove that there exists a unitary representation (V, β, ρ) that is a finite direct sum of irreducible unitary representations such that W is contained in the image of $V^\vee \otimes_{\mathbb{C}} V$. Since the span of the matrix entries of ρ contain coordinate functions, conclude that ρ is injective. Thus, every compact (real) Lie group has a faithful (unitary) representation of finite dimension.