

MAT 552 PROBLEM SET 5

Problem 1. (Duals of irreducibles are irreducible.) For every representation of a Lie group, prove that the representation is indecomposable, resp. irreducible, resp. completely reducible, if and only if the dual representation also has this property. Show by example that a tensor product of two indecomposable representations, resp. irreducible representations, need not have this property. (Hint. Consider the identity and the trace for the tensor product of a representation with its dual.)

Problem 2. (Natural decomposition into irreducibles.) Let G be a Lie group. Let (U, σ) and (V, ρ) be \mathbb{C} -linear (left) G -representation. The following \mathbb{C} -bilinear map,

$$\text{Hom}_{\mathbb{C}}(U, V) \times U \rightarrow V, \quad ((T : U \rightarrow V), u) \mapsto T(u),$$

induces a \mathbb{C} -linear map,

$$c_{U,V} : \text{Hom}_{\mathbb{C}}(U, V) \otimes_{\mathbb{C}} U \rightarrow V.$$

This is natural in both U and V .

(a) For the induced \mathbb{C} -linear (left) G -representations on Hom and tensor product of representations, prove that $c_{U,V}$ is a morphism of \mathbb{C} -linear (left) G -representations. In particular, conclude that for the G -invariant subrepresentation,

$$\text{Hom}_{\text{Rep}_{\mathbb{C}}^G}((U, \sigma), (V, \rho)) \subseteq \text{Hom}_{\mathbb{C}}(U, V),$$

the following restriction of $c_{U,V}$ is a morphism of \mathbb{C} -linear (left) G -representations,

$$H \otimes_{\mathbb{C}} U \rightarrow V, \quad H := \text{Hom}_{\text{Rep}_{\mathbb{C}}^G}((U, \sigma), (V, \rho)).$$

(b) Denote by $I = I(V, \rho)$ the finite set of isomorphism classes of irreducible \mathbb{C} -linear G -representations (V_i, ρ_i) that are isomorphic to a \mathbb{C} -linear G -subrepresentation of (V, ρ) . Conclude the existence of a natural morphism of \mathbb{C} -linear G -representations,

$$a_{(V,\rho)} : \bigoplus_{i \in I} H_i \otimes_{\mathbb{C}} (V_i, \rho_i) \rightarrow (V, \rho), \quad H_i := \text{Hom}_{\text{Rep}_{\mathbb{C}}^G}((V_i, \rho_i), (V, \rho)).$$

(c) Complete the proof from lecture of the corollary of Schur's Lemma: (V, ρ) is completely reducible if and only if $a_{(V,\rho)}$ is an isomorphism.

(d) When (V, ρ) is completely reducible, use Schur's Lemma to prove that the isomorphism $a_{(V,\rho)}$ induces a natural decomposition of unital, associative, \mathbb{C} -algebras,

$$\tilde{a}_{(V,\rho)} : \text{Hom}_{\text{Rep}_{\mathbb{C}}^G}((V, \rho), (V, \rho)) \xrightarrow{\cong} \prod_{i \in I} \text{Hom}_{\mathbb{C}}(H_i, H_i).$$

In particular, conclude an isomorphism of the centers of these algebras,

$$Z(\tilde{a}_{(V,\rho)}) : Z(\text{Hom}_{\text{Rep}_{\mathbb{C}}^G}((V, \rho), (V, \rho))) \xrightarrow{\cong} \prod_{i \in I} \mathbb{C} \cdot \text{Id}_{H_i}.$$

(e) In particular, conclude that the cardinality of I equals the \mathbb{C} -vector space dimension of the center of this algebra. For each $i \in I$, the **multiplicity** of (V_i, ρ_i) in (V, ρ) is defined to be $m_i := \dim_{\mathbb{C}}(H_i)$. Conclude that the algebra has \mathbb{C} -vector space dimension equal to

$$\sum_{i \in I} m_i^2.$$

Problem 3. (Schur's Lemma and Weyl's Trick for Lie algebras.) Formulate the analogue of Schur's Lemma, the Weyl Unitarian Trick, etc., for real Lie algebras instead of real Lie groups. In particular, for a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ whose associated simply connected, real Lie group $G_{\mathbb{R}}$ is compact, conclude that all finite dimensional \mathbb{C} -linear, (left) \mathfrak{g} -modules are completely reducible. Also conclude that for the associated complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$, the same holds for finite dimensional, \mathbb{C} -linear, (left) $\mathfrak{g}_{\mathbb{C}}$ -modules. Finally, conclude that the same holds for every simply connected, complex Lie group $G_{\mathbb{C}}$ whose Lie algebra is a complexification of $\mathfrak{g}_{\mathbb{R}}$.

Problem 4. (Hopf algebra structure on the group algebra.) Let Γ be a finite group considered as a (compact, totally disconnected) Lie group of dimension 0. As usual, denote the identity element by e . Denote the unital, associative group \mathbb{C} -algebra by $(\mathbb{C}[\Gamma], \mathbf{b}_e, *)$.

The **trace** (sometimes called the **counit**) is defined to be

$$\mathrm{Tr}_{\Gamma} : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}, \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g.$$

The **comultiplication** is defined to be

$$\Delta_{\Gamma} : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma], \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g (\mathbf{b}_g \otimes \mathbf{b}_g).$$

The **antipode** is defined to be

$$S_{\Gamma} : \mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma], \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g \mathbf{b}_{g^{-1}}.$$

Check that these operations (together with the usual unital, associated \mathbb{C} -algebra operations) make $\mathbb{C}[\Gamma]$ into a *Hopf \mathbb{C} -algebra*. Precisely, check all of the following.

(a) The comultiplication is coassociative, i.e., the following two compositions are equal,

$$\begin{aligned} \mathbb{C}[\Gamma] &\xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma} \otimes \mathrm{Id}} (\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma], \\ \mathbb{C}[\Gamma] &\xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Id} \otimes \Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} (\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]). \end{aligned}$$

(b) The counit is a left-right coidentity, i.e., the following two compositions both equal the identity map,

$$\begin{aligned} \mathbb{C}[\Gamma] &\xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Tr}_{\Gamma} \otimes \mathrm{Id}} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] = \mathbb{C}[\Gamma], \\ \mathbb{C}[\Gamma] &\xrightarrow{\Delta_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Id} \otimes \mathrm{Tr}_{\Gamma}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[\Gamma]. \end{aligned}$$

(c) The unital, associative \mathbb{C} -algebra structure and the counital, coassociative \mathbb{C} -coalgebra structure satisfy the axioms of a **bialgebra**, i.e., each of the following diagram commute.

$$\begin{array}{ccc}
 \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{\Delta_{\Gamma \circ (-*-)}} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \\
 \Delta \otimes \Delta_{\Gamma} \otimes \Delta_{\Gamma} \downarrow & & \uparrow (-*-) \otimes (-*-) \\
 \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{\text{pr}_1 \otimes \text{pr}_3 \otimes \text{pr}_2 \otimes \text{pr}_4} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \\
 & \xrightarrow{(-*-)} & \mathbb{C}[\Gamma] \\
 \text{Tr}_{\Gamma} \otimes \text{Tr}_{\Gamma} \downarrow & & \downarrow \text{Tr}_{\Gamma} \\
 \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} & \xrightarrow{\cong} & \mathbb{C} \\
 \mathbb{C} \xrightarrow{\cong} & \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} & \\
 \mathbf{b}_e \downarrow & & \downarrow \mathbf{b}_e \otimes \mathbf{b}_e \\
 \mathbb{C}[\Gamma] \xrightarrow{\Delta_{\Gamma}} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \\
 \mathbb{C} \xrightarrow{\text{Id}} & \mathbb{C} & \\
 \mathbf{b}_e \downarrow & & \uparrow \text{Tr}_{\Gamma} \\
 \mathbb{C}[\Gamma] \xrightarrow{\text{Id}} & \mathbb{C}[\Gamma] &
 \end{array}$$

(d) The antipode S satisfies the axioms of a **Hopf algebra**, i.e., the following diagram commutes.

$$\begin{array}{ccc}
 \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{S_{\Gamma} \otimes \text{Id}} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \\
 \Delta_{\Gamma} \uparrow & & \downarrow -*- \\
 \mathbb{C}[\Gamma] & \xrightarrow{\text{Tr}_{\Gamma}(-)\mathbf{b}_e} & \mathbb{C}[\Gamma] \\
 \Delta_{\Gamma} \downarrow & & \uparrow -*- \\
 \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{\text{Id} \otimes S_{\Gamma}} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]
 \end{array}$$

(e) For every pair (U, σ) and (V, ρ) of left modules over a Hopf \mathbb{C} -algebra R , for every element $r \in R$ with

$$\Delta_R(t) = \sum_{\alpha} s_{\alpha} \otimes r_{\alpha},$$

there is an associated left R -module structure on $U \otimes_{\mathbb{C}} V$ defined by

$$(\sigma \otimes \rho)(t) \cdot (u \otimes v) := \sum_{\alpha} (\sigma(s_{\alpha}) \cdot u) \otimes (\rho(r_{\alpha}) \cdot v).$$

Check that for the comultiplication Δ_{Γ} defined above, this equals the structure of Γ -representation on $U \otimes_{\mathbb{C}} V$ as defined in lecture. Also, check that the trivial representation (i.e., the left-right identity for the tensor product operation on \mathbb{C} -linear left Γ -representations) is the unique representation such that the associated trace on $\mathbb{C}[\Gamma]$ equals Tr_{Γ} .

(f) Similarly, for every left module (V, ρ) over R , define a left R -module on the dual \mathbb{C} -vector space V^\vee of \mathbb{C} -linear functional χ on V by

$$(t \cdot \chi)(v) := \chi(S(t) \cdot v).$$

Check that for the antipode S_Γ defined above, this equals the structure of Γ -representation on V^\vee as defined in lecture. Thus, the “extra structures” on \mathbb{C} -linear Γ -representations are explained by the Hopf algebra structure on $\mathbb{C}[\Gamma]$. Conversely, these extra structures uniquely determine the Hopf algebra structures Δ_Γ , Tr_Γ , and S_Γ on the group \mathbb{C} -algebra $\mathbb{C}[\Gamma]$.

(g) Finally, check that the comultiplication is cocommutative, i.e., Δ equals its postcomposition with the involution

$$\text{pr}_2 \otimes \text{pr}_1 : \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma], \quad a_1 \otimes a_2 \mapsto a_2 \otimes a_1.$$

For every Hopf \mathbb{C} -algebra that is cocommutative and that is finite dimensional as a \mathbb{C} -vector space, there is an associated group Γ consisting of all elements b such that $\Delta(b)$ equals $b \otimes b$. Moreover, the Hopf \mathbb{C} -algebra is canonically isomorphic, as a Hopf \mathbb{C} -algebra, to the group \mathbb{C} -algebra of this group.

In particular, we can recover the finite group Γ from the structure of the group \mathbb{C} -algebra $\mathbb{C}[\Gamma]$ as a Hopf algebra, and we can recover this from the group \mathbb{C} -algebra as a unital, associative \mathbb{C} -algebra together with the extra structures on tensor product and duals of \mathbb{C} -linear, (left) Γ -representations.

Problem 5. (A universal property of the group algebra as a representation.) As in the previous problem, let $\mathbb{C}[\Gamma]$ be the group \mathbb{C} -algebra of a finite group Γ . Give $\mathbb{C}[\Gamma]$ its natural structure of \mathbb{C} -linear (left) G -representation, i.e., $(g, \mathbf{b}_h) \mapsto \mathbf{b}_{gh}$.

(a) Prove the following claim from lecture. For every \mathbb{C} -linear Γ -representation (V, ρ) , the following \mathbb{C} -linear map is an isomorphism,

$$\text{Hom}_{\text{Rep}_{\Gamma}^{\mathbb{C}}}(\mathbb{C}[\Gamma], (V, \rho)) \rightarrow V, \quad (T : \mathbb{C}[\Gamma] \rightarrow V) \mapsto T(\mathbf{b}_e).$$

Also, show that this isomorphism is natural in (V, ρ) . Stated in terms of category theory, there is a **fiber functor**,

$$F : \text{Rep}_{\Gamma}^{\mathbb{C}} \rightarrow \mathbb{C} - \text{Vect}, \quad (V, \rho) \mapsto V, \quad \text{Hom}_{\text{Rep}_{\Gamma}^{\mathbb{C}}}((U, \sigma), (V, \rho)) \hookrightarrow \text{Hom}_{\mathbb{C}}(U, V).$$

This is a covariant functor, and it is represented by $\mathbb{C}[\Gamma]$.

(b) Invert the isomorphism above to get a \mathbb{C} -linear map,

$$V \xrightarrow{\cong} \text{Hom}_{\text{Rep}_{\Gamma}^{\mathbb{C}}}(\mathbb{C}[\Gamma], (V, \rho)) \subseteq \text{Hom}_{\mathbb{C}}(\mathbb{C}[\Gamma], V).$$

Use adjointness of Hom and tensor product to obtain an associated \mathbb{C} -linear map,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} V \rightarrow V.$$

Prove that this \mathbb{C} -linear map is a morphism of \mathbb{C} -linear G -representations for the following structures of \mathbb{C} -linear G -representation,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} (V, \text{triv}) \rightarrow (V, \rho).$$

(c) Apply this in the special case that (V, ρ) equals $\mathbb{C}[\Gamma]$ itself, and deduce that the \mathbb{C} -linear map,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \rightarrow \mathbb{C}[\Gamma],$$

is the usual \mathbb{C} -algebra multiplication on the group \mathbb{C} -algebra.

Thus, the natural isomorphism above is enough to deduce the algebra structure. Altogether, this means that we can recover the group \mathbb{C} -algebra as a unital, associative \mathbb{C} -algebra from the data of the category of finite dimensional, \mathbb{C} -linear (left) Γ -representations as a category whose Hom sets are binaturally given structures of finite dimensional \mathbb{C} -vector spaces with \mathbb{C} -bilinear composition operations and the fiber functor.

The previous problem sketched how the extra operations of tensor product of representations, the trivial (1-dimensional) representation, and Hom objects define the additional Hopf algebra structures. Altogether, we can (explicitly) recover the finite group Γ from the category of \mathbb{C} -linear (left) Γ -representations with these extra operations (a structure of “rigid, symmetric, monoidal category”) and the fiber functor (a structure of “Tannakian category”).

(d) By Maschke’s Theorem, every finite dimensional \mathbb{C} -linear Γ -representation is completely reducible. Combine this with Schur’s Lemma and (a) above to conclude that every finite dimensional, irreducible, \mathbb{C} -linear Γ -representation (V_i, ρ_i) is isomorphic to a \mathbb{C} -linear subrepresentation of $\mathbb{C}[\Gamma]$.

(e) Use the morphism of \mathbb{C} -linear Γ -representations from (c) to conclude that right multiplication of $\mathbb{C}[\Gamma]$ on itself (the “right regular representation”) gives an isomorphism of unital, associative \mathbb{C} -algebras,

$$\mathbb{C}[\Gamma]^{\text{opp}} \xrightarrow{\cong} \text{Hom}_{\text{Rep}_{\mathbb{C}}^{\Gamma}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma]).$$

Here, for every unital, associative \mathbb{C} -algebra A , the **opposite algebra** A^{opp} is the same \mathbb{C} -vector space but with multiplication defined by $a \bullet b := ba$.

(f) Combine the isomorphism in (e) with the previous exercise and Problem 2 from the previous problem set to conclude that the number $\#I$ of isomorphism classes $[(V_i, \rho_i)]$ of irreducible \mathbb{C} -linear Γ -representations equals the number of conjugacy classes in Γ , that every \mathbb{C} -bilinear pairing below is a perfect pairing of \mathbb{C} -vector spaces,

$$H_i \times V_i \rightarrow \mathbb{C}[\Gamma] \xrightarrow{\text{Tr}_{\Gamma}} \mathbb{C}, \quad H_i := \text{Hom}_{\text{Rep}_{\mathbb{C}}^{\Gamma}}((V_i, \rho_i), \mathbb{C}[\Gamma]),$$

that each multiplicity m_i equals the \mathbb{C} -vector space dimensions $n_i = \dim_{\mathbb{C}}(V_i)$, and that there is an equality of positive integers,

$$\#\Gamma = \sum_{i \in I} n_i^2.$$

Problem 6. (Central idempotents of the group algebra give the irreducible representations.) Inside the center $Z(\mathbb{C}[\Gamma])$ considered as a \mathbb{C} -algebra, an element e is an **idempotent** if e^2 equals e . For each idempotent e , the **annihilator** of e is

$$\text{Ann}(e) := \{b \in \mathbb{C}[\Gamma] \mid eb = 0\}.$$

(a) With respect to the \mathbb{C} -algebra isomorphism

$$Z(\tilde{a}) : Z(\mathbb{C}[\Gamma]) = Z(\mathbb{C}[\Gamma]^{\text{opp}}) \xrightarrow{\cong} \prod_{i \in I} \mathbb{C} \cdot \text{Id}_{H_i},$$

check that the idempotent elements correspond to those elements $(a_i \text{Id}_{H_i})_{i \in I}$ such that every a_i^2 equals a_i , i.e., such that a_i equals 1 or 0.

(b) For an idempotent e , define the **support**, $\text{supp}(e)$, to be the subset of I such that a_i equals 1. Check that the annihilator of e maps isomorphically to the left-right ideal,

$$\{(M_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_{\mathbb{C}}(H_i, H_i) \mid \forall i \in \text{supp}(e), M_i = 0\}.$$

(c) In particular, check that the unique nonzero idempotents with codimension-1 annihilator in $Z(\mathbb{C}[\Gamma])$ are the **primitive idempotents** \mathbf{e}_i for each $i \in I$,

$$\mathbf{e}_i \mapsto (a_j \text{Id}_{H_j})_{j \in I}, \quad a_i = 1, \quad a_j = 0, \forall j \neq i.$$

Moreover, for each $i \in I$, check that the common annihilator in $\mathbb{C}[\Gamma]$ of \mathbf{e}_j for all $j \neq i$ maps isomorphically to the left-right ideal that is the factor $\text{Hom}_{\mathbb{C}}(H_i, H_i)$. As a \mathbb{C} -linear left Γ -representation, this \mathbb{C} -algebra is a direct sum of n_i copies of the irreducible representation (V_i, ρ_i) . Thus, we can construct the irreducible, \mathbb{C} -linear, Γ -representations from the full list of idempotents in $Z(\mathbb{C}[\Gamma])$ having codimension-1 annihilator in $Z(\mathbb{C}[\Gamma])$.

Problem 7 (Schur's Orthogonality Relations and idempotents in the group algebra.) This exercise reconstructs the primitive idempotents (and thus the irreducible representations) in the group algebra from the information of the irreducible characters. The key is a natural Hermitian inner product on the center of the group algebra, together with Schur's Orthogonality Relations.

(a) For every \mathbb{C} -linear (left) Γ -representation (V, ρ) and for every conjugacy class C in Γ , check that the following \mathbb{C} -linear operator on V is a morphism of \mathbb{C} -linear (left) Γ -representations,

$$\sum_{g \in C} \rho(g) \in \text{Hom}_{\mathbb{C}}(V, V).$$

(b) If (V, ρ) is a finite dimensional, irreducible representation, use Schur's Lemma to conclude that this morphism is a multiple of the identity, say $\rho_C \text{Id}_V$. Taking traces, deduce the identity,

$$\rho_C \cdot \dim_{\mathbb{C}}(V) = \sum_{g \in C} \text{Tr}_V(\rho(g)).$$

(c) Similarly, conclude that the image of the following \mathbb{C} -linear operator is contained in the \mathbb{C} -linear subrepresentation of invariant elements,

$$\sum_{g \in \Gamma} \rho(g) \in \text{Hom}_{\mathbb{C}}(V, V).$$

If (V, ρ) is a finite dimensional representation whose invariant subspace is zero, conclude that

$$\sum_C \rho_C = 0, \quad \text{i.e.,} \quad \sum_{g \in \Gamma} \text{Tr}_V(\rho(g)) = 0.$$

In particular, this holds if (V, ρ) is a nontrivial irreducible representation. Conversely, if (V, ρ) is a trivial representation, show that the sum equals $\dim_{\mathbb{C}}(V) \# \Gamma$. Thus, since trace is additive for direct sum decomposition, for a general (V, ρ) , conclude the identity

$$\frac{1}{\#\Gamma} \sum_{g \in \Gamma} \text{Tr}_V(\rho(g)) = \dim_{\mathbb{C}}(V^{\Gamma}).$$

(d) For every finite dimensional, \mathbb{C} -linear, (left) Γ -representation (V, ρ) , the **character** of this representation is the class function,

$$\chi_{(V, \rho)} : \Gamma \rightarrow \mathbb{C}, \quad g \mapsto \text{Tr}_V(\rho(g)).$$

Every $\rho(g)$ is diagonalizable with eigenvalues ζ that are roots of unity with $\zeta^{-1} = \bar{\zeta}$. Conclude the identity

$$\chi_{(V^\vee, \rho^\vee)}(g) = \chi_{(V, \rho)}(g^{-1}) = \overline{\chi_{(V, \rho)}(g)}.$$

Similarly, for representations (U, σ) and (V, ρ) , for the tensor product of the eigen-decompositions to prove the identity,

$$\chi_{(U \otimes V, \sigma \otimes \rho)}(g) = \chi_{(U, \sigma)}(g)\chi_{(V, \rho)}(g).$$

Consequently, conclude the identity,

$$\chi_{\text{Hom}_{\mathbb{C}}(U, V)}(g) = \overline{\chi_{(U, \sigma)}(g)}\chi_{(V, \rho)}(g).$$

Sum over g and use (c) to deduce the identity,

$$\dim_{\mathbb{C}} \text{Hom}_{\text{Rep}_{\Gamma}^{\mathbb{C}}}((U, \sigma), (V, \rho)) = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \overline{\chi_{(U, \sigma)}(g)} \cdot \chi_{(V, \rho)}(g).$$

(e) Now assume that (V, ρ) is irreducible. For every class function,

$$\alpha : \Gamma \rightarrow \mathbb{C},$$

with **associated central element**,

$$\frac{1}{\#\Gamma} \sum_{g \in \Gamma} \alpha(g) \mathbf{b}_{g^{-1}},$$

conclude that the associated \mathbb{C} -linear operator on V ,

$$\sum_{g \in \Gamma} \alpha(g) \rho(g^{-1}),$$

equals λId_V where λ satisfies the identity

$$\lambda \dim_{\mathbb{C}}(V) = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \alpha(g) \chi_{(V, \rho)}(g^{-1}) = \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \alpha(g) \overline{\chi_{(V, \rho)}(g)}.$$

In particular, this central element annihilates the primitive idempotent corresponding to (V, ρ) if and only if the class function α is orthogonal to the class function $\chi_{(V, \rho)}$ with respect to the Hermitian inner product on the \mathbb{C} -vector space of class functions defined by

$$\langle \alpha, \beta \rangle := \frac{1}{\#\Gamma} \sum_{g \in \Gamma} \alpha(g) \overline{\beta(g)}.$$

(f) Let (U, σ) and (V, ρ) be finite dimensional, \mathbb{C} -linear, (left) Γ -representations. Reinterpret (d) as an identity,

$$\dim_{\mathbb{C}} \text{Hom}_{\text{Rep}_{\Gamma}^{\mathbb{C}}}((U, \sigma), (V, \rho)) = \langle \chi_{(V, \rho)}, \chi_{(U, \sigma)} \rangle.$$

In particular, if (U, σ) and (V, ρ) are irreducible representations, conclude that $\langle \chi_{(V, \rho)}, \chi_{(U, \sigma)} \rangle$ equals 0 unless the representations are isomorphic, in which case $\langle \chi_{(V, \rho)}, \chi_{(U, \sigma)} \rangle$ equals 1. Thus, the characters of irreducible representations form an orthonormal subset of the \mathbb{C} -vector space of \mathbb{C} -valued class functions with respect to the Hermitian inner product defined above.

(g) Finally, since the dimension of the space of class functions equals the dimension of $Z(\mathbb{C}[\Gamma])$, and since this equals the number $\#I$ of isomorphism classes of irreducible representations, conclude that the characters of irreducible representations form an orthonormal basis for the \mathbb{C} -vector space of \mathbb{C} -valued class functions with respect to the Hermitian inner product defined above. Altogether this is the **Schur orthogonality relations**.

(h) Deduce that for the irreducible representations (V_i, ρ_i) for $i \in I$ with character $\chi_i = \chi_{(V_i, \rho_i)}$, the corresponding central elements,

$$\mathbf{e}_i := \frac{\chi_i(e)}{\#\Gamma} \sum_{g \in \Gamma} \chi_i(g) \mathbf{b}_{g^{-1}},$$

are the primitive idempotents. (**Hint.** First use (e) and (f) to show that the primitive idempotent is a scalar multiple of this sum. Then square the element and examine the coefficient of \mathbf{b}_e to conclude that the idempotent equals the sum.)

Combined with the previous problem, the data of the characters of the irreducible representations as a subset of the \mathbb{C} -vector space of \mathbb{C} -valued class functions on Γ is sufficient to reconstruct all irreducible, \mathbb{C} -linear, (left) Γ -representations via the corresponding isotypic factor of $\mathbb{C}[\Gamma]$.

Problem 8. For a finite Abelian group Γ , interpret the previous problems as an eigendecomposition of the group \mathbb{C} -algebra as a direct sum of 1-dimensional subrepresentations indexed by the elements of the Pontrjagin dual group.

Problem 9. Also explicitly work out the decomposition of the group \mathbb{C} -algebra $\mathbb{C}[\Gamma]$ for the symmetric groups $\Gamma = \mathfrak{S}_3$ and $\Gamma = \mathfrak{S}_4$. In fact, there is a different method for reconstructing the irreducible, \mathbb{C} -linear, (left) \mathfrak{S}_n -representations of \mathfrak{S}_n as left $\mathbb{C}[\mathfrak{S}_n]$ -submodules of $\mathbb{C}[\mathfrak{S}_n]$. These submodules are cyclic submodules $V_\lambda := \mathbb{C}[\mathfrak{S}_n]c_\lambda$ generated by the **Young symmetrizer** c_λ associated to each partition λ of n . There are combinatorial formulas for the character of V_λ , but often the Young symmetrizer provides a more direct approach.