MAT 552 PROBLEM SET 4

Problem 1. Recall that for the field $\mathbf{F}$ equal to $\mathbb{R}$ or $\mathbb{C}$, an **associative F-algebra** is a pair $(A, \cdot)$ of an $\mathbf{F}$-vector space A and a $\mathbf{F}$-bilinear map,

$$\cdot : A \times A \to A, \quad (a, b) \mapsto a \cdot b,$$

that is associative: for every $a, b, c \in A$, the following equality holds,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

The operation $(a, b) \mapsto a \cdot b$ is called the **multiplication operation**. We do not assume that there exists a multiplicative identity; when a multiplicative identity exists, the algebra is called **unital**. Also, we do not assume that multiplication is commutative; when mutiplication is commutative, the algebra is called a **commutative algebra** (some authors use this term only when multiplication is commutative and there exists a multiplicative identity).

Recall that the Lie bracket operation on A associated to $\cdot$ is defined to be the commutator,

$$[\bullet, \bullet]_A : A \times A \to A, \quad (a, b) \mapsto a \cdot b - b \cdot a.$$

(a) Please quickly check that the Lie bracket operation is $\mathbf{F}$-bilinear, that it is skew-symmetric, and that the Jacobi identity holds. Thus, the Lie bracket operation defines a Lie algebra structure. This is called the **associated Lie algebra** of $(A, \cdot)$.

(b) Recall that for every $\mathbf{F}$-Lie algebra $(\mathfrak{g}, [\bullet, \bullet]_{\mathfrak{g}})$, the **center** of the Lie algebra is defined to be

$$\mathfrak{z}(\mathfrak{g}) := \{Y \in \mathfrak{g} | \forall X \in \mathfrak{g},\ [X, Y]_{\mathfrak{g}} = 0\}.$$

Recall that the **center** of an associative algebra $(A, \cdot)$ is defined to be

$$Z(A) := \{b \in A | \forall a \in A,\ ab = ba\}.$$

For every associative $\mathbf{F}$-algebra $(A, \cdot)$, check that the center of the associative algebra equals the center of the associated Lie algebra.

(c) Check that the center of the (associative) matrix algebra $\text{Mat}_{n \times n}(\mathbf{F})$ equals the $\mathbf{F}$-span of the identity matrix. In particular, it is 1-dimensional as an $\mathfrak{F}$-vector space.

(d) For all $\mathbf{F}$-associative algebras $(A, \cdot)$ and $(B, \cdot)$, for every $\mathbf{F}$-algebra morphism,

$$\phi : B \to A, \quad \forall b, b' \in B, \quad \phi(b \cdot b') = \phi(b) \cdot \phi(b'),$$

check that also ϕ is also a morphism of $\mathbf{F}$-Lie algebra. Also, the $\mathbf{F}$-Lie algebra morphism associated to an identity $\mathbf{F}$-associative algebra morphism equals the identity morphism of the associated $\mathbf{F}$-Lie algebra morphism. Finally, the $\mathbf{F}$-Lie algebra morphism of a composition of $\mathbf{F}$-associative algebra morphisms equals the composition of the associated $\mathbf{F}$-Lie algebra morphisms.

Altogether, this defines a covariant functor from the category of $\mathbf{F}$-associative algebras to the category of $\mathbf{F}$-Lie algebras. This functor sends products of $\mathbf{F}$-associative algebras to products of the associated **F**-Lie algebras (more generally, the functor preserves all categorical limits).

(e) In particular, conclude that for every **F**-associative subalgebra B of $(A, \cdot)$, also B is an **F**-Lie subalgebra of the associated **F**-Lie subalgebra $(A, [\bullet, \bullet]_A)$. Since every 1-dimensional **F**-subspace of every **F**-Lie algebra is an **F**-Lie subalgebra, prove that there exists an **F**-associative algebra $(A, \cdot)$ and a **F**-Lie subalgebra of $(A, [\bullet, \bullet]_A)$ that is not an **F**-associative subalgebra of $(A, \cdot)$.

(f) For an associative **F**-algebra $(A, \cdot)$ an **F**-subspace I is a **left ideal**, resp. **right ideal**, **two-sided ideal**, if for every $b \in I$ and for every $a \in A$, also $a \cdot b$ is in I, resp. also $b \cdot a$ is in I, also $a \cdot b$ and $b \cdot a$ are in I. Check that every two-sided ideal I is also a **F**-Lie ideal in the associated **F**-Lie algebra $(A, [\bullet, \bullet]_A)$. In particular, the kernel of every **F**-algebra homomorphism between **F**-associative algebras is a **F**-Lie ideal. On the other hand, since the center of $\text{Mat}_{n\times n}(\mathbf{F})$ is not a two-sided ideal for $n \geq 2$, conclude that there exists an **F**-associative algebra $(A, \cdot)$ such that the **F**-Lie ideal $\mathfrak{z}(A)$ in the associated Lie algebra $(A, [\bullet, \bullet]_A)$ is not a two-sided ideal in $(A, \cdot)$.

Problem 2. Part of this problem is covered in Dummit and Foote. Please only do those parts of this problem that are new to you.

For a group Γ, the **F-group algebra** is defined to be the free **F**-vector space $\mathbf{F}[\Gamma]$ with free basis $(\mathbf{b}_\gamma)_{\gamma\in\Gamma}$. For every element a of $\mathbf{F}[\Gamma]$, the **support** of a, $\text{supp}(a)$, is defined to be the finite subset of Γ of all elements γ such that the coefficient of $\mathbf{b}_\gamma$ in a is nonzero.

The multiplication operation on $\mathbf{F}[\Gamma]$ is defined to be the unique **F**-bilinear map that acts as follows on basis elements,

$$* : \mathbf{F}[\Gamma] \times \mathbf{F}[\Gamma] \to \mathbf{F}[\Gamma], \quad (b_\gamma, b_\delta) \mapsto b_{\gamma\cdot\delta}.$$

(a) Check that the multiplication operation is associative, and thus $(\mathbf{F}[\Gamma], *)$ is an **F**-associative algebra. Moreover, for the identity element e of the group Γ, check that $\mathbf{b}_e$ is a multiplicative identity in $\mathbf{F}[\Gamma]$.

(b) Check that the center of $\mathbf{F}[\Gamma]$ is the **F**-vector subspace $\text{Class}(\Gamma, \mathbf{F})$ of all elements a whose support is a union of conjugacy classes in Γ and such that for every $\delta \in \text{supp}(a)$, for every $\gamma \in \Gamma$, the coefficients of $\mathbf{b}_\delta$ and $\mathbf{b}_{\gamma\cdot\delta\cdot\gamma^{-1}}$ are equal. Said differently, the coefficients of a define a function from Γ to **F** whose support is finite and that is constant on every conjugacy class. In particular, the **F**-dimension of the center equals the number of finite conjugacy classes in Γ. (If Γ is a finite group, this equals the number of all conjugacy classes in Γ, e.g., the partition number of n if Γ equals the symmetric group on n letters.)

(c) Prove that for every $\gamma \in \Gamma$, the element $\mathbf{b}_\gamma$ is a (left-right) multiplicatively invertible element of $\mathbf{F}[\Gamma]$, i.e., an element of the multiplicative group $\mathbf{F}[\Gamma]^\times$ of (left-right) multiplicatively invertible elements. Check that the induced set map,

$$\mathbf{b}^\Gamma : \Gamma \to \mathbf{F}[\Gamma]^\times, \quad \gamma \mapsto \mathbf{b}_\gamma,$$

is a morphism of groups.

(d) Conversely, for every $\mathbf{F}$-associative algebra $(A,\cdot)$, for every morphism of groups to the multiplicative group $A^\times$ of $(A,\cdot)$,

$$\rho : \Gamma \to A^\times,$$

prove that there is a unique morphism of $\mathbf{F}$-associative unital algebras,

$$\widetilde{\rho} : (\mathbf{F}[\Gamma], *) \to (A,\cdot),$$

such that $\widetilde{\rho}\circ \mathbf{b}^\Gamma$ equals ρ.

(e) Now give Γ the discrete topology, and consider this discrete topological space as a Lie group in which every connected component is a singleton set, i.e., a connected, 0-dimensional manifold. For every finite dimensional $\mathbf{F}$-vector space V and every representation,

$$\rho : \Gamma \to \mathbf{GL}(V,\mathbf{F}),$$

conclude that there exists a unique morphism of $\mathbf{F}$-associative unital algebras,

$$\widetilde{\rho} : (\mathbf{F}[\Gamma], *) \to (\mathrm{Mat}(V,\mathbf{F}),\cdot),$$

such that $\widetilde{\rho}\circ\mathbf{b}^\Gamma$ equals ρ. Conclude that finite dimensional $\mathbf{F}$-linear Γ-representations are equivalent to left $\mathbf{F}[\Gamma]$-modules having finite dimension as an $\mathbf{F}$-vector space.

(f) For every morphism of groups,

$$\psi : \Gamma \to \Delta,$$

prove that there exists a unique morphism of $\mathbf{F}$-associative unital algebras,

$$\mathbf{F}[\psi] : \mathbf{F}[\Gamma] \to \mathbf{F}[\Delta],$$

such that $\mathbf{F}[\psi]\circ \mathbf{b}^\Gamma$ equals $\mathbf{b}^\Delta \circ \psi$. Thus, the rule $\psi \mapsto \mathbf{F}[\psi]$ sends compositions to compositions and identity morphisms to identity morphisms. Also, the composition of $\mathbf{F}[\psi]$ with each $\mathbf{F}$-linear representation,

$$\sigma : \Delta \to \mathbf{GL}(V,\mathbf{F}),$$

is a $\mathbf{F}$-linear representation of Γ,

$$\sigma\circ\psi : \Gamma \to \mathbf{GL}(V,\mathbf{F}),$$

sometimes called the **restriction representation** (typically only when ψ is injective).

Altogether, this defines a covariant functor from the category of groups to the category of $\mathbf{F}$-associative unital algebras sending every group Γ to the $\mathbf{F}$-associative unital algebra $\mathbf{F}[\Gamma]$ and sending every morphism of groups ψ to the morphism of $\mathbf{F}$-associative unital algebras $\mathbf{F}[\psi]$.

Later in the course, as a consequence of Schur's Lemma, Maschke's Theorem, and Wedderburn's Theorem, we will prove that for every finite group Γ, the $\mathbb{C}$-associative unital algebra $\mathbb{C}[\Gamma]$ is isomorphic to a product of matrix algebras,

$$\mathbb{C}[\Gamma] \cong \mathrm{Mat}_{n_1\times n_1}(\mathbb{C}) \times \cdots \times \mathrm{Mat}_{n_r\times n_r}(\mathbb{C}).$$

From the above, the integer r equals the number of conjugacy classes in Γ. Also, for every $i=1,\dots,r$, the unique nonzero, simple, left $\mathrm{Mat}_{n_1\times n_1}(\mathbb{C})$-module of $\mathbb{C}$-vector space dimension n_i is an irreducible $\mathbb{C}$-linear Γ-representation V_i of $\mathbb{C}$-vector space dimension n_i, the irreducible $\mathbb{C}$-linear Γ-representations $V_1,\dots,V_r$ are pairwise non-isomorphic, and every irreducible $\mathbb{C}$-linear Γ-representation is isomorphic to one of these. In particular, the $\mathbb{C}$-vector space dimension $\#\Gamma$ of $\mathbb{C}[\Gamma]$ equals the sum $n_1^2 +$

$\cdots + n_r^2$ of the squares of the dimensions of the irreducible representations. Together with the Frobenius orthogonality relations, this greatly simplifies the problem of classifying the finitely many irreducible $\mathbb{C}$-linear Γ-representations.

Problem 3. For every pair of $\mathbb{R}$-Lie groups, resp. $\mathbb{C}$-Lie groups,

$$(G, e, m : G \times G \to G), \quad (H, \epsilon, \mu : H \times H \to H),$$

the **product Lie group** is defined to be the product manifold $G \times H$ with the product binary operation,

$$m \times \mu : (G \times H) \times (G \times H) \to G \times H, \quad ((g, h), (g', h')) \mapsto (m(g, g'), \mu(h, h')).$$

(a) Check that this binary operation is a morphism of Lie groups.

(b) check that this is the unique structure of Lie group on the product manifold $G \times H$ such that both of the following projections are morphisms of Lie groups,

$$\mathrm{pr}_1 : G \times H \to G, \quad (g, h) \mapsto g,$$

$$\mathrm{pr}_2 : G \times H \to H, \quad (g, h) \mapsto h.$$

Also check that this is the unique structure of Lie group on the product manifold $G \times H$ such that both of the following maps are morphisms of Lie groups whose images commute through each other,

$$q_1 : G \to G \times H, \quad g \mapsto (g, \epsilon),$$

$$q_2 : H \to G \times H, \quad h \mapsto (e, h),$$

$$\forall g \in G, \forall h \in H, \quad q_1(g)q_2(h) = q_2(h)q_1(g).$$

(c) Check the pair of morphisms of Lie groups $(\mathrm{pr}_1 : G \times H \to G, \mathrm{pr}_2 : G \times H \to H)$ is final among all pairs of morphisms of Lie groups to G and H. Precisely, for every Lie group K and for every pair of morphisms of Lie groups $(p_1 : K \to G, p_2 : K \to H)$, prove that there exists a unique morphism of Lie groups,

$$p : K \to G \times H,$$

such that p_i equals $\mathrm{pr}_i \circ p$ for $i = 1$ and $i = 2$. Thus, this structure of Lie group on $G \times H$ forms a **categorical product** in the category of Lie groups.

(d) Similarly, check that the pair of morphisms of Lie groups $(q_1 : G \to G \times H, q_2 : H \to G \times H)$ is initial among all pairs of morphisms from G and H to a Lie group whose images commute through each other. Precisely, for every Lie group L and for every pair of morphisms of Lie groups $(r_1 : G \to L, r_2 : H \to L)$ such that

$$\forall g \in G, \forall h \in H, \quad r_1(g)r_2(h) = r_2(h)r_1(g),$$

prove that there exists a unique morphism of Lie groups,

$$r : G \times H \to L,$$

such that r_i equals $r \circ q_i$ for $i = 1$ and $i = 2$.

(e) In particular, for $\mathbf{F}$ equal to $\mathbb{R}$, resp. to $\mathbb{C}$, when L is $\mathbf{GL}(V, \mathbf{F})$ for a finite dimensional $\mathbf{F}$-vector space, conclude that a $\mathbf{F}$-linear representation of the product Lie group $G \times H$ is equivalent to a pair (σ, ρ) of $\mathbf{F}$-linear representations,

$$\sigma : G \to \mathbf{GL}(V, \mathbf{F}),$$

$$\rho : H \to \mathbf{GL}(V, \mathbf{F}),$$

such that
$$\forall g \in G, \forall h \in H, \quad \sigma(g)\cdot\rho(h) = \rho(h)\cdot\sigma(g).$$
In particular, the morphism σ factors through the closed Lie subgroup,
$$textIsom_{\text{Rep}_H^{\mathbf{F}}}((V,\rho),(V,\rho)) \subset \mathbf{GL}(V,\mathbf{F}),$$
and similarly ρ factors through the closed Lie subgroup,
$$textIsom_{\text{Rep}_G^{\mathbf{F}}}((V,\sigma),(V,\sigma)) \subset \mathbf{GL}(V,\mathbf{F}).$$

(f) Use Schur's Lemma to prove that the irreducible $\mathbf{F}$-linear representations of $G\times H$ are precisely the representations of the form $(U\otimes_{\mathbf{F}} W, (\sigma'\otimes \text{Id}_W), (\text{Id}_U\otimes\rho')$ for irreducible $\mathbf{F}$-linear representations,
$$\sigma' : G \to \mathbf{GL}(U,\mathbf{F}),$$
$$\rho' : H \to \mathbf{GL}(W,\mathbf{F}).$$

(g) In particular, let (V,σ) and (V,ρ) be representations that are completely reducible. Denote the isotypic components by
$$(V,\sigma) = \bigoplus_{i\in I}(V_i,\sigma_i), \quad (V,\rho) = \bigoplus_{j\in J}(V_j,\rho_j),$$
where I, resp. J, denotes the set of isomorphism classes of irreducible $\mathbf{F}$-linear H-representations U_i, resp. irreducible $\mathbf{F}$-linear G-representations W_j, that appear as subrepresentations of (V,σ), resp. of (V,ρ). Prove that each V_i is preserved by ρ, and prove that each V_j is preserved by σ. Since every subrepresentation of a completely reducible representation is also completely reducible, conclude that there is a simultaneous decomposition,
$$V = \bigoplus_{(i,j)\in I\times J} V_{i,j},$$
where $V_{i,j}$ is simultaneously a direct sum of irreducible G-representations of type i and a direct sum of irreducible H-representations of type j. Finally, use Schur's Lemma to conclude that $V_{i,j}$ is a direct sum of copies of the irreducible $\mathbf{F}$-linear $G\times$-representation $U_i\otimes_{\mathbf{F}} W_j$.

Problem 4 Repeat the previous exercise for Lie algebras in place of Lie groups.

Problem 5 Recall from lecture that the adjoint representation,
$$\text{Ad} : \mathbf{SL}_2 \to \mathfrak{gl}(\mathfrak{sl}_2) \cong \mathfrak{gl}_3$$
factors through the orthogonal subgroup of $\mathfrak{gl}(\mathfrak{sl}_2)$ associated to the quadratic form $q = -\det_2|_{\mathfrak{sl}_2}$, and this factorization contains the center of $\mathbf{SL}_2$ in its kernel. Thus, there is an induced morphism of split Lie groups,
$$\phi : \mathbf{PGL}_2 \to \mathbf{SO}(\mathfrak{sl}_2, q),$$
and this is an isomorphism of Lie groups. Thus, there is an induced isomorphism of the simply connected forms. In this sense, "B_1 equals A_1".

(a) Use this to prove that the $\mathbf{F}$-linear representations of $\mathbf{SO}(\mathfrak{sl}_2, q)$ are precisely the representations of $\mathbf{SL}_2$ on which the center acts trivially. Check that for the standard 2-dimensional $\mathbf{F}$-representation V of $\mathbf{SL}_2$, the symmetric product representation $\text{Sym}^d_{\mathbf{F}}(V)$ is trivial on the center of $\mathbf{SL}_2$ if and only if the nonnegative integer d is even.

(b) Find a "compact form" of this isomorphism, i.e., prove that there exists a positive definite inner product B on the Lie algebra $\mathfrak{su}(2, \mathbb{R})$ of the compact Lie group $\mathbf{SU}(2, \mathbb{R})$ (this is just the Killing form) such that the adjoint representation preserves B and such that the induced morphism of Lie groups,

$$\mathbf{SU}(2, \mathbb{R}) \to \mathbf{SO}(\mathfrak{su}(2, \mathbb{R}), B),$$

is surjective with kernel equal to the center Z.

(c) With respect to the isomorphism of $\mathbf{SU}(2, \mathbb{R})/Z$ and the compact Lie group $\mathbf{SO}(\mathfrak{su}(2, \mathbb{R}), B)$ from (b), repeat part (a) characterizing those representations of $\mathbf{SU}(2, \mathbb{R})$ that factor through representations of $\mathbf{SO}(\mathfrak{su}(2, \mathbb{R}), q)$.

Problem 6. On the 4-dimensional vector space $\mathrm{Mat}_{2\times 2}$, the quadratic $-\det_2$ comes from a nondegenerate, symmetric, bilinear pairing that is indefinite. The associated orthogonal group $\mathbf{SO}(\mathrm{Mat}_{2\times 2}, -\det_2)$ is a split special orthogonal group.

(a) For each $(g, h) \in \mathbf{SL}_2 \times \mathbf{SL}_2$, prove that the following $\mathbf{F}$-linear map of $\mathrm{Mat}_{2\times 2}$ is an isometry with respect to $-\det_2$,

$$\rho(g, h) : \mathrm{Mat}_{2\times 2} \to \mathrm{Mat}_{2\times 2}, \quad X \mapsto gXh^{-1}.$$

Also check that $\rho(gg', hh')$ equals $\rho(g, h) \circ \rho(g', h')$. Conclude that ρ is a morphism of Lie groups,

$$\rho : \mathbf{SL}_2 \times \mathbf{SL}_2 \to \mathbf{SO}(\mathrm{Mat}_{2\times 2}, -\det_2).$$

(b) If $\rho(g, h)$ is the identity on $\mathrm{Mat}_{2\times 2}$, use the special choice $X = h$ or $X = g^{-1}$ to conclude that g equals h. Conversely, for g equal to h, conclude that $\rho(g, g)$ is the identity if and only if g is in the center Z of $\mathbf{SL}_2$. Thus, the kernel of ρ equals the diagonally embedded copy of the center, $\Delta(Z)$. Conclude that ρ factors through an injective morphism of Lie groups,

$$\phi : (\mathbf{SL}_2 \times \mathbf{SL}_2)/\Delta(Z) \to \mathbf{SO}(\mathrm{Mat}_{2\times 2}, -\det_2).$$

(c) The induced morphism of Lie algebras,

$$\mathrm{Lie}(\phi) : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \to \mathfrak{so}(\mathrm{Mat}_{2\times 2}, -\det_2),$$

is an injective $\mathbf{F}$-linear map. Compute that both the domain vector space and the target vector space have dimension 6. Use the Rank-Nullity Theorem to conclude that $\mathrm{Lie}(\phi)$ is an isomorphism of $\mathbf{F}$-Lie algebras. Since $\mathbf{SO}(\mathrm{Mat}_{2\times 2}, -\det_2)$ is connected, also conclude that ϕ is surjective, hence an isomorphism of Lie groups. Thus, there is an induced isomorphism of simply connected forms. In this sense, "D_2 equals $A_1 \times A_1$".

(d) Repeat part (b) of the previous exercise to determine which pairs (U, W) of irreducible representations of $\mathbf{SL}_2$ give an irreducible representation $U \otimes_{\mathbf{F}} V$ of $\mathbf{SL}_2 \times \mathbf{SL}_2$ that factors through a representation of $\mathbf{SO}(\mathrm{Mat}_{2\times 2}, -\det_2)$.

(e) What happens when you try to find a "compact form" of the isomorphism ϕ?