MAT 552 PROBLEM SET 1

Problem 1. (**Complex Lie group representations of the complex multiplicative group.**) Recall that the complex multiplicative group $\mathbb{G}_m(\mathbb{C})$ is $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ as a multiplicative group.

For every finite Abelian group A, the **Pontrjagin dual** of A is

$$\widehat{A} := \mathrm{Hom}_{\mathbf{Group}}(A, \mathbb{G}_m(\mathbb{C})).$$

This is the same as the set of 1-dimensional $\mathbb{C}$-linear representations of A with a specified basis via the rule that associates to every $\chi \in \widehat{A}$ the 1-dimensional $\mathbb{C}$-vector space and action,

$$\mathbb{C}_\chi := \mathbb{C}, \quad \forall a \in A, \quad a \bullet z := \chi(a)z.$$

(a) Define the identity element of the Pontrjagin dual to be the constant group homomorphism with image $1 \in \mathbb{G}_m(\mathbb{C})$. Prove that this corresponds to the trivial 1-dimensional $\mathbb{C}$-linear representation of A. Also, for every pair of elements, $\chi, \chi' \in \widehat{A}$, define the product by

$$(\chi \cdot \chi')(a) = \chi(a)\chi'(a), \quad \forall a \in A.$$

Prove that this product is an element of $\widehat{A}$ and corresponds to the 1-dimensional $\mathbb{C}$-linear representation,

$$\mathbb{C}_{\chi\cdot\chi'} = \mathbb{C}_\chi \otimes_{\mathbb{C}} \mathbb{C}_{\chi'}.$$

With these operations, prove that $\widehat{A}$ is a finite Abelian group that is (non-canonically) isomorphic to A. Also, show that for elements $\chi, \chi' \in \widehat{A}$, the set of $\mathbb{C}[A]$-module homomorphisms (i.e., A-equivariant, $\mathbb{C}$-linear maps) from $\mathbb{C}_\chi$ to $\mathbb{C}_{\chi'}$ is the 1-dimensional $\mathbb{C}$-vector space $\mathbb{C} \cdot \mathrm{Id}$ if χ equals χ', and otherwise it is the zero vector space.

(b) For every finite dimensional, $\mathbb{C}$-linear representation of A,

$$\rho : A \to \mathbf{GL}(V),$$

for every $\chi \in \widehat{A}$, define $V_{\rho,\chi}$ to be the following subset of V,

$$V_{\rho,\chi} := \{v \in V | \forall a \in A, \ \rho(a) \bullet v = \chi(a)v\} \cong \mathrm{Hom}_{\mathbb{C}[A]-\mathrm{mod}}(\mathbb{C}_\chi, (V, \rho)).$$

Prove that $V_{\rho,\chi}$ is a $\mathbb{C}[A]$-submodule of V. Prove that the following natural map is an isomorphism of $\mathbb{C}[A]$-modules,

$$\bigoplus_{\chi \in \widehat{A}} V_{\rho,\chi} \to V.$$

For every pair (V, ρ) and (W, σ) of finite dimensional $\mathbb{C}[A]$-modules, prove that these direct sum decompositions define a direct sum decomposition of $\mathbb{C}$-vector spaces,

$$\mathrm{Hom}_{\mathbb{C}[A]-\mathrm{mod}}((V, \rho), (W, \sigma)) = \bigoplus_{\chi \in \widehat{A}} \mathrm{Hom}_{\mathbb{C}-\mathrm{mod}}(V_{\rho,\chi}, W_{\sigma,\chi}),$$

$$(V \otimes_{\mathbb{C}} W)_{\rho\otimes\sigma,\chi} = \bigoplus_{(\zeta,\eta)\in\widehat{A}\times\widehat{A},\zeta\cdot\eta=\chi} V_{\rho,\zeta} \otimes_{\mathbb{C}} W_{\sigma,\eta}.$$

(c) For every positive integer n, define $\mu_n \subset \mathbb{G}_m(\mathbb{C})$ to be the finite subgroup of n^{th} roots of unity. Prove that the inclusion of μ_n in $\mathbb{G}_m(\mathbb{C})$ is a cyclic generator for $\widehat{\mu}_n$. Via this canonical generator, show that $\widehat{\mu}_n$ is canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

(d) Show that the inclusion partial order on subgroups of $\mathbb{G}_m(\mathbb{C})$ restricts on the set of subgroups $\{\mu_n | n \in \mathbb{Z}_{\geq 1}\}$ as the divisibility partial order on $\mathbb{Z}_{\geq 1}$. Prove that for every inclusion $\mu_\ell \subseteq \mu_n$, the restriction map $\widehat{\mu}_n \to \widehat{\mu}_\ell$ is just reduction modulo ℓ,

$$\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z}, \quad \overline{a} \mapsto \overline{a}.$$

(e) Define $\mu_\infty \subset \mathbb{G}_m(\mathbb{C})$ to be the union over all $n \in \mathbb{Z}_{\geq 1}$ of μ_n as a subgroup. Give μ_∞ the subspace topology induced as a subset of $\mathbb{G}_m(\mathbb{C})$ (with its usual Euclidean metric topological structure). Show that the group operations on μ_∞ are continuous with respect to this topological structure. Show that every subgroup μ_n is a closed subgroup of μ_∞ that is even compact.

(f) By restricting to closed subgroups μ_n, conclude that the **continuous** group homomorphisms from μ_∞ to $\mathbb{G}_m(\mathbb{C})$ are precisely of the form,

$$\chi_d : \mu_\infty \to \mathbb{G}_m(\mathbb{C}), \quad \chi_d(z) = z^d,$$

for integers $d \in \mathbb{Z}$. Thus, the **continuous Pontrjagin dual** of μ_∞ is canonically isomorphic to $\mathbb{Z}$. Finally, show that for every **continuous** group homomorphism to the group of $\mathbb{C}$-automorphisms of a finite dimensional $\mathbb{C}$-vector space,

$$\rho : \mu_\infty \to \mathbf{GL}(V),$$

the following subspaces define a direct sum decomposition of V as a $\mathbb{C}$-vector space with a continuous $\mathbb{C}$-linear action of μ_∞,

$$V_{\rho,d} := \{v \in V | \forall z \in \mu_\infty,\ \rho(z) \bullet v = z^d v\}.$$

(g) For every holomorphic group homomorphism,

$$\rho : \mathbb{G}_m(\mathbb{C}) \to \mathbf{GL}(V),$$

by restricting to the topological subgroup μ_∞, prove that the following subspaces for all $d \in \mathbb{Z}$ define a direct sum decomposition of V as a finite dimensional $\mathbb{C}$-vector space with a holomorphic $\mathbb{C}$-linear action of $\mathbb{G}_m(\mathbb{C})$,

$$V_{\rho,d} := \{v \in V | \forall z \in \mathbb{G}_m(\mathbb{C}),\ \rho(z) \bullet v = z^d v\}.$$

Problem 2. (**Linear complex tori in a general linear group.**) Recall that a complex Lie group T is a **linear complex torus** (as opposed to a compact complex torus) if it is isomorphic as a complex Lie group to the r-fold product $\mathbb{G}_m(\mathbb{C})^r$ for some nonnegative integer r.

(a) Use the previous exercise to prove that the following two sets are dual finitely generated free Abelian groups (under value-wise multiplication),

$$X^*(T) := \text{Hom}_{\mathbb{C}-\text{Lie Group}}(T, \mathbb{G}_m(\mathbb{C})) \cong \mathbb{Z}^r,$$

$$X_*(T) := \text{Hom}_{\mathbb{C}-\text{Lie Group}}(\mathbb{G}_m(\mathbb{C}), T) \cong \mathbb{Z}^r.$$

The duality pairing is the natural composition pairing

$$X^*(T) \times X_*(T) \to \text{Hom}_{\mathbb{C}-\text{Lie Group}}(\mathbb{G}_m(\mathbb{C}), \mathbb{G}_m(\mathbb{C})) = \mathbb{Z}, \quad (\chi, \rho) \mapsto \chi \circ \rho.$$

The first free Abelian group is the **character lattice** of T, and the second is the **cocharacter lattice** of T. By convention, the group operations on each are written additively (even through the group operation is value-wise multiplication).

(b) Prove that for every morphism of complex Lie groups,

$$\rho : T \to \mathbf{GL}(V),$$

the following subspaces for all $\chi \in X^*(T)$ define a direct sum decomposition of V as a finite dimensional $\mathbb{C}$-vector space with a holomorphic $\mathbb{C}$-linear action of T,

$$V_{\rho,\chi} := \{v \in V | \forall z \in T, \quad \rho(z) \bullet v = \chi(z)v\}.$$

(c) For every (V, ρ), for the finite subset of $X^*(T)$,

$$\mathrm{Supp}(V, \rho) := \{\chi \in X^*(T) | \dim_{\mathbb{C}}(V_{\rho,\chi}) > 0\},$$

prove that the Abelian subgroup $\langle \mathrm{Supp}(V, \rho)\rangle$ of $X^*(T)$ generated by $\mathrm{Supp}(V, \rho)$ is a finitely generated free Abelian group. For every choice of $\mathbb{Z}$-module basis $(\chi_1, \dots, \chi_s)$ of $\langle \mathrm{Supp}(V, \rho)\rangle$, prove that ρ factors as the composition of a submersive morphism of complex Lie groups,

$$(\chi_1, \dots, \chi_s) : T \to \mathbb{G}_m(\mathbb{C})^s,$$

and an injective morphism of complex Lie groups,

$$\rho' : \mathbb{G}_m(\mathbb{C})^s \to \mathbf{GL}(V).$$

In particular, conclude that ρ is injective if and only if $\langle \mathrm{Supp}(V, \rho)\rangle$ equals $X^*(T)$.

(d) For fixed V of dimension n, show that n equals the maximum possible dimension of the image $\rho(T)$ of an injective complex Lie group morphisms ρ from a linear complex torus to $\mathbf{GL}(V)$. For a linear complex torus T of dimension n, show that a complex Lie group morphism ρ from T to $\mathbf{GL}(V)$ is injective if and only if the finite set $\mathrm{Supp}(V, \rho)$ is a basis for $X^*(T)$ as a free $\mathbb{Z}$-module. The image of any such ρ is called a **maximal torus** in $\mathbf{GL}(V)$.

(e) Conclude that the set of maximal tori $\rho(T)$ in V is in natural bijection with the set of (unordered) direct sum decompositions of V into 1-dimensional $\mathbb{C}$-linear subspaces $(V_{\rho,\chi})_{\chi \in \mathrm{Supp}(V,\rho)}$. In particular, conclude that any two maximal tori are conjugate by an element of $\mathbf{GL}(V)$.

(f) Finally, show that the normalizer $N(T)$ in $\mathbf{GL}(V)$ contains T as a normal subgroup (by definition) and equals a semidirect product of T by a finite group $W(T)$ that is canonically isomorphic to the group of permutations of the n-element set $\mathrm{Supp}(V, \rho)$. For each choice of lifting of the (unordered) direct sum decomposition to an (unordered) $\mathbb{C}$-basis for V, there is an associated lift of $W(T)$ to a subgroup of $N(T)$: the subgroup of permutation matrices that permute the vectors of the basis. This lift of $W(T)$ to a subgroup of $N(T)$ depends on the unordered basis only up to simultaneous nonzero scaling of all basis vectors (an unordered basis up to such scaling is equivalent to a **pinning**).

(g) Returning to the factorization of a general morphism of complex Lie groups,

$$\rho : T \to \mathbf{GL}(V),$$

as a composition of $(\chi_1, \dots, \chi_s)$ and ρ', conclude that every torus $\rho(T) = \rho'(\mathbb{G}_m(\mathbb{C})^s)$ is contained in a maximal torus.

Problem 3. (**General Linear Groups, Special Linear Groups, Maximal Tori, and Lie Algebras.**) Let $n \geq 1$ be an integer. Let V be the n-dimensional $\mathbb{C}$-vector space $\mathbb{C}^n$ with its standard ordered basis $(e_1, \dots, e_n)$. Denote by $\text{Mat}_{n\times n}(\mathbb{C})$ the $\mathbb{C}$-algebra of $\mathbb{C}$-linear endomorphisms of $\mathbb{C}^n$. Denote the determinant holomorphic map by

$$\det_n : \text{Mat}_{n\times n}(\mathbb{C}) \to \mathbb{C}, \quad \det_n([a_{i,j}]) = \sum_{\sigma\in\mathfrak{S}_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}.$$

Let $\mathbf{GL}_n(\mathbb{C}) \subset \text{Mat}_{n\times n}(\mathbb{C})$ denote the dense open subset where $\det_n$ is nonzero.

(a) Use the properties of the determinant to prove that $\mathbf{GL}_n(\mathbb{C})$ is a complex Lie group with group operation given by matrix multiplication and with identity element $\text{Id}_{n\times n}$. Also prove that the restriction of the determinant map is a submersive morphism of complex Lie groups,

$$\det_n : \mathbf{GL}_n(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C}).$$

Let T_n denote the linear complex torus

$$T_n = \mathbb{G}_m(\mathbb{C})^n = \{(z_1, \dots, z_n) | z_1, \dots, z_n \in \mathbb{G}_m(\mathbb{C})\}.$$

Let ρ_n denote the following morphism of complex Lie groups,

$$\rho_n : T_n \to \mathbf{GL}_n(\mathbb{C}), \quad \rho_n(z_1, \dots, z_n) \cdot e_i = z_i e_i, \forall i = 1, \dots, n.$$

For every $i = 1, \dots, n$, denote by $\rho_{n,i}$ the restriction of ρ_n to the i^{th} factor of T_n,

$$\rho_{n,i} : \mathbb{G}_m(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C}), \quad \rho_{n,i}(z_i) \cdot e_j = z_i^{\delta_{i,j}} e_j,$$

where $\delta_{i,j}$ is the usual Kronecker delta function: equal to 1 if $i = j$ and equal to 0 otherwise. For every $i = 1, \dots, n$, denote by $\chi_{n,i}$ the morphism of complex Lie groups,

$$\chi_{n,i} : T_n \to \mathbb{G}_m(\mathbb{C}), \chi_{n,i}(z_1, \dots, z_n) = z_i.$$

(b) Check that $(\chi_{n,1}, \dots, \chi_{n,n})$ and $(\rho_{n,1}, \dots, \rho_{n,n})$ are dual ordered bases of $X^*(T_n)$ and $X_*(T_n)$. Following standard convention, we write elements of these lattices additively, i.e.,

$$d_1\rho_{n,1} + \cdots + d_n\rho_{n,n} : \mathbb{G}_m(\mathbb{C}) \to T_n, \quad z \mapsto \rho(z^{d_1}, \dots, z^{d_n}),$$

$$e_1\chi_{n,1} + \cdots + e_n\chi_{n,n} : T_n \to \mathbb{G}_m(\mathbb{C}), \quad (z_1, \dots, z_n) \mapsto z_1^{e_1} \cdots z_n^{e_n}.$$

Denote by $\mathbf{SL}_n(\mathbb{C})$ the kernel of $\det_n$ on $\mathbf{GL}_n(\mathbb{C})$. By convention, $\mathbf{SL}_1(\mathbb{C})$ is the group with just one element.

(c) Check that the intersection $T'_n := T_n \cap \mathbf{SL}_n(\mathbb{C})$ is the subtorus of T_n whose cocharacter sublattice $X_*(T'_n)$ in $X_*(T_n)$ equals the span of all cocharacters $\rho_{n,i} - \rho_{n,j}$ for $1 \leq i < j \leq n$. Also check that the restriction map of character lattices,

$$X^*(T_n) \to X^*(T'_n),$$

is surjective with kernel equal to the span of the character $\chi_{n,1} + \cdots + \chi_{n,n}$ (this character is the restriction of $\det_n$ to T_n).

(d) For every $\mathbb{C}$-vector space W, for every $w \in W$, the following flow gives a tangent vector field τ_w on W,

$$\phi_w : \mathbb{C} \times W \to W, \quad \phi_w(t, v) = v + tw.$$

Prove that the tangent vector fields τ_w for $w \in W$ give a trivialization of the tangent bundle of W identifying the $\mathbb{C}$-vector space W with the $\mathbb{C}$-tangent space of W at each point. In particular, the $\mathbb{C}$-tangent space of $\text{Mat}_{n\times n}(\mathbb{C})$ at every point is identified with the $\mathbb{C}$-vector space $\text{Mat}_{n\times n}(\mathbb{C})$. Thus, also the $\mathbb{C}$-tangent space at $\text{Id}_{n\times n}$ of the open subset $\mathbf{GL}_n(\mathbb{C})$ equals

$$\mathfrak{gl}_n(\mathbb{C}) := \text{Mat}_{n\times n}(\mathbb{C}).$$

(e) Check that the derivative of $\det_n(\text{Id}_{n\times n} + tM)$ at $t = 0$ equals the trace $\text{tr}(M)$. Conclude that the $\mathbb{C}$-tangent space at $\text{Id}_{n\times n}$ of $\mathbf{SL}_n(\mathbb{C})$, as a $\mathbb{C}$-linear subspace of $\mathfrak{gl}_n(\mathbb{C})$, equals

$$\mathfrak{sl}_n(\mathbb{C}) := \{M \in \text{Mat}_{n\times n}(\mathbb{C}) : \text{tr}(M) = 0\}.$$

Similarly, the $\mathbb{C}$-tangent space at $\text{Id}_{n\times n}$ of T_n equals the $\mathbb{C}$-subspace $\mathfrak{h}_n$ of all diagonal matrices in $\mathfrak{gl}_n(\mathbb{C})$. Finally, the $\mathbb{C}$-tangent space $\text{Id}_{n\times n}$ of T'_n equals the $\mathbb{C}$-subspace $\mathfrak{h}'_n = \mathfrak{h}_n \cap \mathfrak{sl}_n(\mathbb{C})$.

Problem 4 (**Centralizers and Root Data**) For every $1 \le i, j \le n$, denote by $E_{i,j} \in \text{Mat}_{n\times n}(\mathbb{C})$ the matrix

$$E_{i,j} \cdot e_\ell = \delta_{j,\ell} e_i.$$

Thus $(E_{i,j})_{1\le i,j\le n}$ is a $\mathbb{C}$-basis for $\text{Mat}_{n\times n}(\mathbb{C})$. The **conjugation action** on a complex Lie group G by a complex Lie subgroup H is defined by

$$c_h : G \to G, \quad c_h(t) = hgh^{-1}$$

for every h in H. In particular, the conjugation action of $\mathbf{GL}_n(\mathbb{C})$ on $\mathbf{GL}_n(\mathbb{C})$ is the restriction to the open subset $\mathbf{GL}_n(\mathbb{C})$ of a $\mathbb{C}$-linear action of $\mathbf{GL}_n(\mathbb{C})$ on the $\mathbb{C}$-vector space $\text{Mat}_{n\times n}(\mathbb{C})$. Since this is $\mathbb{C}$-linear, the induced action on the $\mathbb{C}$-tangent space $\mathfrak{gl}_n(\mathbb{C})$ at $\text{Id}_{n\times n}$ is the same $\mathbb{C}$-linear action. This induced action is the **adjoint action**.

(a) Compute that the span of $E_{i,j}$ is a $\mathbb{C}$-eigenspace for the adjoint action of $\rho(z_1, \dots, z_n)$ with corresponding eigenvalue $z_j^{-1} z_i$, i.e., with character $\chi_i - \chi_j$ (written additively). Conclude that there is a direct sum decomposition of $\mathfrak{gl}_n(\mathbb{C})$ as an adjoint representation of T_n,

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{h}_n \oplus \bigoplus_{1\le i<j\le n} \mathbb{C}\cdot E_{i,j} \oplus \bigoplus_{1\le j<i\le n} \mathbb{C}\cdot E_{i,j}.$$

There is a corresponding direct sum decomposition of $\mathfrak{sl}_n(\mathbb{C})$ as an adjoint representation of T'_n,

$$\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h}'_n \oplus \bigoplus_{1\le i<j\le n} \mathbb{C}\cdot E_{i,j} \oplus \bigoplus_{1\le j<i\le n} \mathbb{C}\cdot E_{i,j}.$$

Thus, the nonzero characters of T_n that occur in the adjoint action on $\mathfrak{gl}_n(\mathbb{C})$ are $\chi_{n,i} - \chi_{n,j}$ and $\chi_{n,j} - \chi_{n,i}$ for $1 \le i < j \le n$, and the associated root spaces are $\mathbb{C}\cdot E_{i,j}$ and $\mathbb{C}\cdot E_{j,i}$. Similarly, the nonzero characters of T'_n that occur in the adjoint action of T'_n on $\mathfrak{sl}_n(\mathbb{C})$ are $\overline{\chi}_{n,i} - \overline{\chi}_{n,j}$ and $\overline{\chi}_{n,j} - \overline{\chi}_{n,i}$ for $1 \le i < j \le n$.

(b) Check that the $\mathbb{C}$-subspace of $\text{Mat}_{n\times n}(\mathbb{C})$ of elements centralized by $\rho(z_1, \dots, z_n)$ is a direct sum of $\mathbb{C}\cdot E_{i,j}$ for every $1 \le i, j \le n$ such that z_i equals z_j. In particular, the center of $\mathbf{GL}_n(\mathbb{C})$ equals $\mathbb{G}_m(\mathbb{C}) \cdot \text{Id}_{n\times n}$, i.e., the image of the cocharacter $\rho_{n,1} + \dots + \rho_{n,n}$. Also, the centralizer of $\rho(z_1, \dots, z_n)$ always contains the subset of diagonal matrices.

(c) For a nonzero character α of T_n, for the kernel $T_{n,\alpha} := \mathrm{Ker}(\alpha) \subset T_n$, check that the simultaneous centralizer of $T_{n,\alpha}$ in $\mathrm{Mat}_{n\times n}(\mathbb{C})$ is strictly larger than the subset of diagonal matrices if and only if α equals $\chi_{n,i} - \chi_{n,j}$ or $\chi_{n,j} - \chi_{n,i}$ for some $1 \leq i < j \leq n$. In this case, the intersection of the centralizer with $\mathbf{GL}_n(\mathbb{C})$ is denoted $\mathbf{GL}_n(\mathbb{C})_\alpha$. For the commutator subgroup $\mathcal{D}_\alpha$ of $\mathbf{GL}_n(\mathbb{C})_\alpha$, check that $\mathbf{GL}_n(\mathbb{C})_\alpha$ equals $T_{n,\alpha} \cdot \mathcal{D}_\alpha$. Also check that $\mathcal{D}_\alpha$ equals the image of a submersive morphism of complex Lie groups,

$$f_{n,\alpha} : \mathbf{SL}_2(\mathbb{C}) \to \mathcal{D}_\alpha,$$

that is uniquely determined by the requirement that the composition $f_{n,\alpha} \circ (\rho_{2,1} - \rho_{2,2})$ is a cocharacter $\alpha^\vee$ of T_n with $\langle \alpha, \alpha^\vee \rangle$ positive. Check that the pairing $\langle \alpha, \alpha^\vee \rangle$ equals 2.

A character $\alpha \in X^*(T_n)$ as above is a **root** of $(\mathbf{GL}_n(\mathbb{C}), T_n)$, the cocharacter $\alpha^\vee \in X_*(T_n)$ is a **coroot** of $(\mathbf{GL}_n(\mathbb{C}), T_n)$, and the **root group** of α is the image

$$U_\alpha := f_{n,\alpha}(U_+)$$

where $U_+ \subset \mathbf{SL}_2(\mathbb{C})$ is the unipotent complex Lie subgroup of upper triangular unipotent matrices whose Lie algebra in $\mathfrak{sl}_2(\mathbb{C})$ is the root space for the unique root with positive pairing against $\rho_{2,1} - \rho_{2,2}$. The set of all roots of $(\mathbf{GL}_n(\mathbb{C}), T_n)$ is denoted $\Phi(\mathbf{GL}_n(\mathbb{C}), T_n) \subset X^*(T_n)$. The set of all coroots of $(\mathbf{GL}_n(\mathbb{C}), T_n)$ is denoted $\Phi^\vee(\mathbf{GL}_n(\mathbb{C}), T_n) \subset X_*(T_n)$. There is a natural bijection from $\Phi(\mathbf{GL}_n(\mathbb{C}), T_n)$ to $\Phi^\vee(\mathbf{GL}_n(\mathbb{C}), T_n)$ sending each root α to the coroot $\alpha^\vee$.

A **root datum** is a 4-tuple $(X, R, X^\vee, R^\vee)$ of finitely generated free Abelian groups X and $X^\vee$ and a perfect $\mathbb{Z}$-bilinear pairing,

$$\langle \bullet, \bullet \rangle : X \times X^\vee \to \mathbb{Z},$$

together with finite subsets $R \subset X \setminus \{0\}$ and $R^\vee \subset X^\vee \setminus \{0\}$ for which there exists a bijection,

$$R \leftrightarrow R^\vee, \ \alpha \leftrightarrow \alpha^\vee,$$

satisfying the axioms

(i) $\forall \alpha \in R, \quad \langle \alpha, \alpha^\vee \rangle = 2,$
(ii) $\forall \alpha \in R,\ s_{\alpha,\alpha^\vee}(R) = R$ and $s_{\alpha^\vee,\alpha}(R^\vee) = R^\vee$,

where the $\mathbb{Z}$-linear involutions $s_{\alpha,\alpha^\vee}$ and $s_{\alpha^\vee,\alpha}$ are defined as follows,

$$s_{\alpha,\alpha^\vee} : X \to X, \quad s_{\alpha,\alpha^\vee}(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha,$$
$$s_{\alpha^\vee,\alpha} : X^\vee \to X^\vee, \quad s_{\alpha^\vee,\alpha}(\beta^\vee) = \beta^\vee - \langle \alpha, \beta^\vee \rangle \alpha^\vee.$$

The root datum is **reduced** if for every root $\alpha \in R$, the only $\mathbb{Q}$-multiples that are in R are $\pm\alpha$.

(d) Check that for $X = X^*(T_n)$, for $X^\vee = X_*(T_n)$, for $R = \Phi(\mathbf{GL}_n(\mathbb{C}), T_n)$, and for $R^\vee = \Phi^\vee(\mathbf{GL}_n(\mathbb{C}), T_n)$, the 4-tuple $(X, R, X^\vee, R^\vee)$ is a reduced root datum. Check that the subgroup of $\mathrm{Hom}_\mathbb{Z}(X, X)$ generated by the involutions $s_{\alpha,\alpha^\vee}$ is precisely the isomorphic image of $W(T_n) = N(T_n)/T_n$ for its $\mathbb{Z}$-linear action on $X^*(T_n)$ induced by the conjugation action of $N(T_n)$ on T_n. Check that the simultaneous kernel $X_0 \subset X$ of all coroots is the span of the weight $\chi_{n,1} + \cdots + \chi_{n,n}$ of $\det_n$ restricted to T_n. Check that the $\mathbb{Z}$-span Q of R together with X_0 give a direct sum decomposition of a sublattice of X whose quotient is a finite cyclic group of order n. Check that the Pontrjagin dual of this finite cyclic group inside T_n is precisely the center $\mu_n \cdot \mathrm{Id}_{n\times n}$ of the commutator subgroup $\mathcal{D}(\mathbf{GL}_n(\mathbb{C})) = \mathbf{SL}_n(\mathbb{C})$. The finite index sublattice $X_0 \oplus Q$ corresponds to the character lattice of the image maximal torus $(\det_n, q_n)(T_n)$ in the quotient group

$$(\det_n, q_n) : \mathbf{GL}_n(\mathbb{C}) \to \mathbb{G}_m(\mathbb{C}) \times \mathbf{PGL}_n(\mathbb{C}),$$

having kernel $\mu_n \cdot \mathrm{Id}_{n\times n}$. The span Q of R is the **root lattice**. Inside the $\mathbb{Q}$-span of Q in $X \otimes \mathbb{Q}$, the **weight lattice** is the finitely generated free Abelian group of elements that have integer pairing with $R^\vee$.

(e) Repeat the previous parts for the pair $(\mathbf{SL}_n(\mathbb{C}), T'_n)$ to explicitly find the root datum of this pair.

(f) Define $\mathbf{PGL}_n(\mathbb{C})$ to be the quotient complex Lie group

$$\mathbf{PGL}_n(\mathbb{C}) := \mathbf{GL}_n(\mathbb{C})/\mathbb{G}_m(\mathbb{C}) \cdot \mathrm{Id}_{n\times n} = \mathbf{SL}_n(\mathbb{C})/\mu_n \cdot \mathrm{Id}_{n\times n}.$$

Define $\overline{T}_n$ to be the image of T_n in $\mathbf{PGL}_n(\mathbb{C})$. Repeat the previous parts for the pair $(\mathbf{PGL}_n(\mathbb{C}), \overline{T}_n)$ to find the root datum of this pair.

Problem 5 (**Borel Subgroups, Positive Roots and Root Systems**) Denote by $B_n \subset \mathbf{GL}_n(\mathbb{C})$ the complex Lie subgroup of all upper triangular matrices. Thus B_n contains T_n as a complex Lie subgroup. Similarly, denote by B'_n the intersection $B_n \cap \mathbf{SL}_n(\mathbb{C})$. Finally, denote by $\overline{B}_n$ the image of B_n in the quotient group $\mathbf{PGL}_n(\mathbb{C}) = \mathbf{GL}_n(\mathbb{C})/\mathbb{G}_m(\mathbb{C}) \cdot \mathrm{Id}_{n\times n}$.

(a) Use the Jordan canonical form to prove that B_n is a maximal connected solvable subgroup of $\mathbf{GL}_n(\mathbb{C})$ containing T_n. Prove that every other maximal connected solvable subgroup of $\mathbf{GL}_n(\mathbb{C})$ containing T_n is of the form

$$B_{n,[w]} := wB_nw^{-1}$$

for a unique element $[w] \in W(T_n)$. These are the **Borel subgroups** of $\mathbf{GL}_n(\mathbb{C})$ that contain T_n.

(b) For a specified triple $(\mathbf{GL}_n(\mathbb{C}), T_n, B_n)$, a root α is **positive** if the root group U_α is contained in B_n. Check that the positive roots are the roots $\chi_{n,i} - \chi_{n,j}$ with $1 \le i < j \le n$. In particular, for every root α, precisely one of α and $-\alpha$ is a positive root. Denote the set of positive roots by $\Phi(\mathbf{GL}_n(\mathbb{C}), T_n, B_n)^+ \subset \Phi(\mathbf{GL}_n(\mathbb{C}), T_n)$. Check that the root groups U_α of the positive roots cumulatively generate the subgroup $U_n \subset B_n$ of all upper triangular unipotent matrices. This is a maximal connected normal complex Lie subgroup of B_n that is unipotent, the **unipotent radical**. The maximal torus T_n maps isomorphically to the quotient complex Lie group B_n/U_n, i.e., T_n is a **Levi factor** of B_n.

(c) A positive root is a **positive simple root** if it is not a sum of two or more positive roots. Check that the positive simple roots of $(\mathbf{GL}_n(\mathbb{C}), T_n, B_n)$ are precisely the positive roots $\chi_{n,i} - \chi_{n,i+1}$ for $1 \le i < n$. The set of positive simple roots is denoted $\Delta(\mathbf{GL}_n(\mathbb{C}), T_n, B_n) \subset \Phi(\mathbf{GL}_n(\mathbb{C}), T_n, B_n)^+$.

(d) Check that the following symmetric $\mathbb{R}$-bilinear form on the $\mathbb{R}$-vector space $V := X \otimes \mathbb{R}/X_0 \otimes \mathbb{R}$ is positive definite and invariant under the action of the Weyl group $W(T_n)$,

$$B_R(\bullet, \bullet) : V \times V \to \mathbb{R}, \quad B_R(\beta, \beta') = \sum_{\alpha \in R} \langle \beta, \alpha^\vee \rangle \langle \beta', \alpha^\vee \rangle.$$

Up to $\mathbb{R}^\times_{>0}$-scaling, such a bilinear form is unique. Conclude that also B_R is a scaling of the inner product on V induced by the standard Euclidean inner product on $X^*(T_n) = \mathbb{Z}^n$ with its usual ordered basis as orthogonal basis.

For the root system arising from $(\mathbf{GL}_n(\mathbb{C}), T_n, B_n)$, the **standard normalization** of B_R is the scaling so that every root in V has inner product 2. The pair $((V, B_R), R)$ of a finite dimensional, positive definite, real inner product space (V, B_R) and a finite subset $R \subset V \setminus \{0\}$ is a **root system**. This satisfies the following axioms.

(i) The finite subset R spans V as an $\mathbb{R}$-vector space.

(iii) For every $\alpha \in R$, the finite set R is preserved by reflection σ_α through the orthogonal complement of $\mathbb{R} \cdot \alpha$,

$$\sigma_\alpha(\beta) := \beta - \frac{2B_R(\beta, \alpha)}{B_R(\alpha, \alpha)}\alpha.$$

(iv) For every pair of roots $\alpha, \beta \in R$, the real number $2B_R(\alpha, \beta)/B_R(\alpha, \alpha)$ is an integer.

The root system is **reduced** if for every root $\alpha \in R$, the only $\mathbb{R}$-multiples of α in R are α and $-\alpha$. The root system is **reducible** if it is isomorphic to an orthogonal direct sum of nonzero root systems; otherwise it is **irreducible**. The **Weyl group** of the root system is the finite subgroup of $\mathbb{R}$-linear isometries of (V, B_R) generated by the reflections σ_α. A partition $R = R^+ \sqcup R^-$ by real half-spaces is a set of **positive roots** if for every root α, precisely one of α or $-\alpha$ is in R^+. For a set of positive roots, the **positive simple roots** are those positive roots that cannot be written as a sum of two positive roots. The set of positive simple roots is denoted Δ.

For a root system with a set of positive roots $((V, B_R), R, R^+)$ the associated **Dynkin diagram** is the graph with vertex set equal to Δ where a pair of positive simple roots (α, β) with $B_R(\alpha, \alpha) \leq B_R(\beta, \beta)$ has no edge if α and β are orthogonal, they have a single undirected edge if the angle between them is $2\pi/3$, they have a double edge, directed from β to α (directed toward the **short root**) if the angle equals $3\pi/4$, and they have a triple edge, directed toward the short root, if the angle equals $5\pi/6$ (these are the only possible angles). The Dynkin diagram is connected if and only if the root system is irreducible.

(e) The common root system arising from $(\mathbf{GL}_n(\mathbb{C}), T_n, B_n)$, from $(\mathbf{SL}_n(\mathbb{C}), T'_n, B'_n)$ and from $(\mathbf{PGL}_n(\mathbb{C}), \overline{T}_n, \overline{B}_n)$ is called the A_{n-1} root system. Check that the Weyl group of this root system is $W(T_n)$, a symmetric group on n elements. Also check that the Dynkin diagram is the one drawn in lecture.