MAT 543 FALL 2025 PROBLEM SET 7

Problem 1. (**The $\mathbb{C}$-Lie algebra of derivations**.) For every associative $\mathbb{C}$-algebra $(A, \cdot : A \times A \to A)$ and for every A-A-bimodule $(M, A \times M \xrightarrow{L} M, M \times A \xrightarrow{R} M)$ with $a \cdot (m \cdot b) = (a \cdot m) \cdot b)$, i.e., $L(a, R(m, b)) = R(L(a, m), b)$, for every $(a, b) \in A \times A$ and for every $m \in M$, a $\mathbb{C}$-**derivation of associative algebras** from A to M is a $\mathbb{C}$-linear map θ from A to M such that every element (a, b) of $A \times A$ satisfies the *Leibniz rule*,

$$\theta(a \cdot b) = \theta(a) \cdot b + a \cdot \theta(b).$$

Prove that the subset $\mathrm{Der}_{\mathbb{C}}((A, \cdot, M)$ of $\mathrm{Hom}_{\mathbb{C}}(A, M)$ of all $\mathbb{C}$-derivations from A to M is a $\mathbb{C}$-vector subspace. In particular, a $\mathbb{C}$-**derivation of associative algebras from** A **to itself** is a derivation from A to A with its regular representation A-A-bimodule structure. **Prove** that the corresponding subspace $\mathrm{Der}_{\mathbb{C}}(A, \cdot)$ of the associative $\mathbb{C}$-algebra $\mathrm{End}_{\mathbb{C}}(A)$ is a $\mathbb{C}$-Lie subalgebra (for the usuall commutator bracket on the associative algebra).

Problem 2.(**Derivations of $\mathbb{C}$-Lie algebras**.) For every $\mathbb{C}$-Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and for every $\mathfrak{g}$-$\mathfrak{g}$-birepresentation $(M, \mathfrak{g} \times M \xrightarrow{L} M, M \times \mathfrak{g} \xrightarrow{R} M)$ (or, equivalently, for every bimodule for the universal enveloping algebra $U_{\mathbb{C}}(\mathfrak{g})$ of $\mathfrak{g}$), a $\mathbb{C}$-**derivation of Lie algebras** from $\mathfrak{g}$ to the birepresentation M is a $\mathbb{C}$-linear map θ from $\mathfrak{g}$ to M such that every element (X, Y) of $\mathfrak{g} \times \mathfrak{g}$ satisfies the *Lie algebra birepresentation Leibniz rule*,

$$\theta([X, Y]) = \theta(X) \cdot Y - X \cdot \theta(Y).$$

Prove that every $\mathbb{C}$-Lie algebra derivation from $\mathfrak{g}$ to M extends uniquely to an associative algebra derivation from $U_{\mathbb{C}}(\mathfrak{g})$ to M. In particular, by the previous problem, the $\mathbb{C}$-vector subspace $\mathrm{Der}((\mathfrak{g}, [\cdot, \cdot]), U_{\mathbb{C}}(\mathfrak{g}))$ is a $\mathbb{C}$-Lie algebra. A **derivation of a $\mathbb{C}$-Lie algebra to itself** is a $\mathbb{C}$-Lie algebra derivation θ from $\mathfrak{g}$ to $U_{\mathbb{C}}(\mathfrak{g})$ such that $\theta(\mathfrak{g})$ is contained in the image subspace $\mathfrak{g} \xhookrightarrow{\iota_{\mathfrak{g}}} U_{\mathbb{C}}(\mathfrak{g})$. **Prove** that the $\mathbb{C}$-vector subspace of these is a $\mathbb{C}$-Lie subalgebra $\mathrm{Der}(\mathfrak{g}, [\cdot, \cdot])$ of the $\mathbb{C}$-Lie algebra $\mathrm{Der}((\mathfrak{g}, [\cdot, \cdot]), U_{\mathbb{C}}(\mathfrak{g}))$. **Prove** also that a derivation of $\mathfrak{g}$ to itself is equivalent to a $\mathbb{C}$-linear endomorphism θ of $\mathfrak{g}$ such that every element (X, Y) of $\mathfrak{g} \times \mathfrak{g}$ satisfies the *Lie algebra Leibniz rule*,

$$\theta([X, Y]) = [\theta(X), Y] + [X, \theta(Y)].$$

Finally, **prove** that the adjoint representation defines a morphism of $\mathbb{C}$-Lie algebras $\mathrm{ad}^{\mathfrak{g}}$ from $\mathfrak{g}$ to $\mathrm{Der}_{\mathbb{C}}(\mathfrak{g}, [\cdot, \cdot])$.

Problem 3.(**Semidirect products.**) For every $\mathbb{C}$-Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, for every $\mathbb{C}$-Lie algebra $(\mathfrak{h}, [\cdot, \cdot])$, and for every morphism ϕ of $\mathbb{C}$-Lie algebras from $\mathfrak{h}$ to $\mathrm{Der}_{\mathbb{C}}(\mathfrak{g})$, **prove** that the following operation on the product $\mathbb{C}$-vector space $\mathfrak{g} \times \mathfrak{h}$ defines a $\mathbb{C}$-Lie algebra, the **semidirect product Lie algebra**,

$$[(X, x), (Y, y)]_{\mathfrak{g}, \mathfrak{h}, \phi} := ([X, Y]_{\mathfrak{g}} + \phi_x(Y) - \phi_y(X), [x, y]_{\mathfrak{h}}).$$

The $\mathbb{C}$-Lie ideal $\mathfrak{g} \times \{0\}$ is identified with $\mathfrak{g}$. The quotient $\mathbb{C}$-Lie algebra by this $\mathbb{C}$-Lie ideal is identified with $\mathfrak{h}$, which is also identified with the $\mathbb{C}$-Lie subalgebra

$\{0\} \times \mathfrak{h}$. The adjoint action of the semidirect product Lie algebra on the Lie ideal $\mathfrak{g}$ restricts on the $\mathbb{C}$-Lie subalgebra $\mathfrak{h}$ to the Lie algebra homomorphism ϕ. Finally, **prove** that for every $\mathbb{C}$-vector space V, the restriction map gives a bijection between representations on V of the semidirect product $\mathbb{C}$-Lie algebra and ordered pair of a representation of $\mathfrak{g}$ on V and a representation of $\mathfrak{h}$ on V that normalizes the $\mathfrak{g}$-representation via ϕ, i.e., for every $X \in \mathfrak{g}$, for every $x \in \mathfrak{H}$, and for every $v \in V$,

$$x \cdot (X \cdot v) - X \cdot (x \cdot v) = \phi_x(X) \cdot v.$$

Problem 4. (**Levi's Theorem, I.**) Let $\mathfrak{g}$ be a finite dimensional $\mathbb{C}$-Lie algebra with trivial center whose solvable radical $\mathfrak{r}$ is a nonzero Abelian Lie algebra. Also assume that the adjoint action of $\mathfrak{g}$ on $\mathfrak{r}$ is an irreducible representation of $\mathfrak{g}_{\text{ss}} = \mathfrak{g}/\mathfrak{r}$.

(a) Prove that the adjoint representation is faithful, so that $\mathfrak{g}$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}(\mathfrak{g})$ associated to the associative (unital) algebra $\text{End}_{\mathbb{C}}(\mathfrak{g})$.

(b) Define $\mathfrak{a}$ to be the subspace of $\text{End}_{\mathbb{C}}(\mathfrak{g})$ of linear endomorphisms of $\mathfrak{g}$ with image contained in $\mathfrak{r}$ and whose restriction to $\mathfrak{r}$ is a $\mathbb{C}$-multiple of $\text{Id}_{\mathfrak{r}}$. Define $\mathfrak{b} \subset \mathfrak{a}$ to be the subspace of such linear endomorphisms whose restriction to $\mathfrak{r}$ is the zero map. Prove that $\mathfrak{a}$ and $\mathfrak{b}$ are associative $\mathbb{C}$-subalgebras of the associative $\mathbb{C}$-algebra $\text{End}_{\mathbb{C}}(\mathfrak{g})$ (neither of these subalgebras is unital). Thus, the commutator bracket on each of these subalgebras realizes each as a Lie subalgebra of $\text{End}_{\mathbb{C}}(\mathfrak{g})$. Also check that $\mathfrak{b}$ is an Abelian Lie ideal in $\mathfrak{a}$, and the quotient Lie algebra is 1-dimensional (hence Abelian).

(c) Define $L_{\mathfrak{g}}$ to be the restriction to $\mathfrak{g}$ of the left regular representation of $\text{End}_{\mathbb{C}}(\mathfrak{g})$ on itself, i.e., for every $X \in \mathfrak{g}$ and for every $\phi \in \text{End}_{\mathbb{C}}(\mathfrak{g})$, the element $L_{\mathfrak{g}}(X) \cdot \phi$ in $\text{End}_{\mathbb{C}}(\mathfrak{g})$ equals

$$L_{\mathfrak{g}}(X) \cdot \phi : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto \text{ad}_X \circ \phi(Y) = [X, \phi(Y)]_{\mathfrak{g}}.$$

Prove that the commutator of ad_X and ϕ in $\text{End}_{\mathbb{C}}(\mathfrak{g})$ equals

$$[\text{ad}_X, \phi]_{\mathfrak{gl}(\mathfrak{g})} : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X, \phi(Y)]_{\mathfrak{g}} - \phi([X, Y]_{\mathfrak{g}}).$$

(d) Now consider the restriction to $\mathfrak{g}$ of the adjoint representation of $\text{End}_{\mathbb{C}}(\mathfrak{g})$ on itself,

$$\text{ad}' : \mathfrak{g} \to \text{End}_{\mathbb{C}}(\text{End}_{\mathbb{C}}(\mathfrak{g})), \quad \text{ad}'_X(\phi) := [\text{ad}_X, \phi]_{\text{End}_{\mathbb{C}}(\mathfrak{g})}.$$

For every element X of $\mathfrak{g}$, prove that ad'_X is a $\mathbb{C}$-derivation from the $\mathbb{C}$-Lie algebra $\text{End}_{\mathbb{C}}(\mathfrak{g})$ to itself.

(e) Check that $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{g}$-subrepresentations of the $\mathfrak{g}$-represenation on $\text{End}_{\mathbb{C}}(\mathfrak{g})$ determined by ad'.

Problem 5. (**Levi's Theorem, II.**) This problem continues the previous problem.

(a) Check that for every $X \in \mathfrak{g}$ and for every $\phi \in \mathfrak{a}$, the element $\text{ad}'_X(\phi)$ is contained in $\mathfrak{b}$. Conclude that the induced $\mathfrak{g}$-representation on the quotient $\mathfrak{a}/\mathfrak{b}$ is the trivial 1-dimensional $\mathfrak{g}$-representation.

(b) Check that the image under $\text{ad}^{\mathfrak{g}}$ of $\mathfrak{r}$ is a $\mathfrak{g}$-subrepresentation of $\mathfrak{b}$, i.e., $\mathfrak{b}$ contains the element $\phi = \text{ad}_X$ for every $X \in \mathfrak{r}$.

(c) Check that on the associated quotient spaces $\mathfrak{a}/\text{ad}^{\mathfrak{g}}(\mathfrak{r})$ and $\mathfrak{b}/\text{ad}^{\mathfrak{g}}(\mathfrak{r})$, the $\mathfrak{g}$-representation restricts as the zero representation on the Lie subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Thus, the natural short exact sequence of $\mathfrak{g}$-representations,

$$0 \to \mathfrak{a}/\text{ad}^{\mathfrak{g}}(\mathfrak{r}) \to \mathfrak{b}/\text{ad}^{\mathfrak{g}}(\mathfrak{r}) \to \mathfrak{a}/\mathfrak{b} \to 0,$$

is actually a short exact sequence of $\mathfrak{g}_{\text{ss}}$-representations.

(d) Finally, use complete reducibility and triviality of the representation $\mathfrak{a}/\mathfrak{b}$ to conclude that there exists $\phi \in \mathfrak{a} \subset \text{Hom}(\mathfrak{g}, \mathfrak{r})$ restricting as the identity on $\mathfrak{r}$ such that for every $X \in \mathfrak{g}$,

$$[\text{ad}_X, \phi]_{\mathfrak{gl}(\mathfrak{g})} = \text{ad}_{-\psi(X)},$$

for a unique linear map $\psi \in \text{Hom}(\mathfrak{g}, \mathfrak{r})$. Thus, for every $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$,

$$\phi([X, Y]_{\mathfrak{g}}) = [X, \phi(Y)]_{\mathfrak{g}} + [\psi(X), Y]_{\mathfrak{g}}.$$

Define $\mathfrak{g}'$ to be the kernel of ψ.

(e) Check that $\psi(X)$ equals 0 if and only if, for every $Y \in \mathfrak{g}$,

$$\phi([X, Y]_{\mathfrak{g}}) = [X, \phi(Y)]_{\mathfrak{g}}.$$

For $X_1, X_2 \in \mathfrak{g}$, since

$$[[X_1, X_2]_{\mathfrak{g}}, Y]_{\mathfrak{g}} = [X_1, [X_2, Y]_{\mathfrak{g}}]_{\mathfrak{g}} - [X_2, [X_1, Y]_{\mathfrak{g}}]_{\mathfrak{g}},$$

deduce that for every $(X_1, X_2) \in \mathfrak{g} \times \mathfrak{g}$,

$$\phi([[X_1, X_2]_{\mathfrak{g}}, Y]_{\mathfrak{g}}) = \phi([X_1, [X_2, Y]_{\mathfrak{g}}]_{\mathfrak{g}}) - \phi([X_2, [X_1, Y]_{\mathfrak{g}}]_{\mathfrak{g}}) =$$
$$[X_1, \phi([X_2, Y]_{\mathfrak{g}})]_{\mathfrak{g}} - [X_2, \phi([X_1, Y]_{\mathfrak{g}})]_{\mathfrak{g}} = [X_1, [X_2, \phi(Y)]_{\mathfrak{g}}]_{\mathfrak{g}} - [X_2, [X_1, \phi(Y)]_{\mathfrak{g}}]_{\mathfrak{g}} =$$
$$[[X_1, X_2]_{\mathfrak{g}}, \phi(Y)]_{\mathfrak{g}}.$$

Thus, also $[X_1, X_2]_{\mathfrak{g}}$ is in $\mathfrak{g}'$. Conclude that $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$.

(f) Finally, since $\mathfrak{r}$ is an Abelian Lie algebra, check that for every $X \in \mathfrak{r}$ and for every $Y \in \mathfrak{g}$,

$$[X, Y]_{\mathfrak{g}} = [\psi(X), Y]_{\mathfrak{g}}.$$

Since the adjoint representation is faithful, conclude that also ψ is an element of $\mathfrak{a}$ that restricts as the identity on $\mathfrak{r}$. Therefore the kernel $\mathfrak{g}'$ is a complementary subspace to $\mathfrak{r}$ in $\mathfrak{g}$. Altogether, for every complex Lie algebra $\mathfrak{g}$ of finite dimension whose solvable radical is Abelian and gives an irreducible representation of $\mathfrak{g}_{\text{ss}}$ via the adjoint action, the Lie algebra is the semidirect product of the kernel $\mathfrak{g}' \cong \mathfrak{g}_{\text{ss}}$ and of the solvable radical.

Problem 6. (**Levi's Theorem, III.**) Now for every finite dimensional $\mathbb{C}$-Lie algebra $\mathfrak{g}$ prove that $\mathfrak{g}$ is a semidirect product of its solvable radical $\mathfrak{r}$ and its semisimple part $\mathfrak{g}_{\text{ss}} := \mathfrak{g}/\mathfrak{r}$ by induction on the dimension of $\mathfrak{g}$.

If either $\mathfrak{r}$ or $\mathfrak{g}_{\text{ss}}$ is trivial, the result holds tautologically. Thus, assume that both of these are nontrivial.

If $\mathfrak{r}$ is solvable but not Abelian, then for the quotient of $\mathfrak{g}$ by the nonzero commutator Lie ideal $[\mathfrak{r}, \mathfrak{r}]_{\mathfrak{g}}$, conclude that there is a Lie subalgebra of the quotient that is isomorphic to $\mathfrak{g}_{\text{ss}}$. The inverse image of this Lie subalgebra in $\mathfrak{g}$ is proper in $\mathfrak{g}$ (thus has smaller dimension), and it has the same semisimple part. Use the induction hypothesis to conclude that there is a Lie subalgebra $\mathfrak{g}'$ complementary to $\mathfrak{r}$ inside this proper Lie subalgebra that maps isomorphically to $\mathfrak{g}_{\text{ss}}$. Thus, Levi's Theorem holds in this setting.

Finally, if $\mathfrak{r}$ is Abelian, yet the adjoint action of $\mathfrak{g}_{\text{ss}}$ on $\mathfrak{r}$ is reducible, then the quotient of $\mathfrak{g}$ by a proper, nonzero subrepresentation of $\mathfrak{r}$ has smaller dimension, thus has a Levi subalgebra. The inverse image in $\mathfrak{g}$ of this Levi subalgebra is a proper Lie subalgebra of $\mathfrak{g}$ that has the same semisimple part. Once again use the induction hypothesis to conclude that there exists a Levi subalgebra in $\mathfrak{g}$.

Problem 7.(Lie's Theorem.) Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{C}$-Lie algebra, let $\mathfrak{h}$ be a $\mathbb{C}$-Lie ideal in $\mathfrak{g}$, and let (V, ρ) be a $\mathbb{C}$-linear representation of $\mathfrak{g}$ of finite dimension. Let λ denote a morphism of $\mathbb{C}$-Lie algebras from $\mathfrak{h}$ to the unique 1-dimensional $\mathbb{C}$-Lie algebra,

$$\lambda : \mathfrak{h} \to \mathbb{C}, \quad X \mapsto \langle \lambda, X \rangle \in \mathbb{C}.$$

For every integer $r \geq 0$, denote by $V^r_{\mathfrak{h},\lambda}$ the simultaneous kernel in V over all $X \in \mathfrak{h}$ of the $\mathbb{C}$-linear endomorphisms $(\rho_X - \langle \lambda, X \rangle \text{Id}_V)^{1+r}$. The subspace $V^0_{\mathfrak{h},\lambda}$ is the **$\mathfrak{h}$-eigenspace** of V with **weight** λ. The nondecreasing sequence of $\mathbb{C}$-subspaces $(V^r_{\mathfrak{h},\lambda})_{r=0,1,\ldots}$ stabilizes to the **$\mathfrak{h}$-generalized eigenspace** $V^{\text{gen}}_{\mathfrak{h},\lambda}$ of V.

(a) Prove that each $\mathbb{C}$-subspace $V^r_{\mathfrak{h},\lambda}$ is an $\mathfrak{h}$-subrepresentation of V.

(b) For every $Y \in \mathfrak{g}$, since $\text{ad}_Y(X)$ is in $\mathfrak{h}$ for every $X \in \mathfrak{h}$, use the identity,

$$\rho_Y \circ \rho_X - \rho_X \circ \rho_Y = \rho_{\text{ad}_Y(X)},$$

to conclude that ρ_Y maps $V^r_{\mathfrak{h},\lambda}$ to $V^{1+r}_{\mathfrak{h},\lambda}$. Conclude that $V^{\text{gen}}_{\mathfrak{h},\lambda}$ is a $\mathfrak{g}$-subrepresentation.

(c) Prove that Lie's Theorem is equivalent to Lie's Lemma: each eigenspace $V^r_{\mathfrak{h},\lambda}$ is a $\mathfrak{g}$-subrepresentation of V. Also show that this is equivalent to the claim that for every λ with $V^{\text{gen}}_{\mathfrak{h},\lambda}$ nonzero (i.e., for each $\mathfrak{h}$-weight of the representation), for every $X \in \mathfrak{h}$ and for every $Y \in \mathfrak{g}$, the pairing $\langle \lambda, \text{ad}_Y(X) \rangle$ is zero.

(d) For a nonzero element v in $V^0_{\mathfrak{h},\lambda}$, prove that the smallest ρ_Y-stabilized $\mathfrak{h}$-subrepresentation W that contains v has a basis of the form $(\rho^0_Y(v), \ldots, \rho^{m-1}_Y(v))$ for some positive integer m.

(e) Check that W is a generalized eigenspace of $\rho_{\text{ad}_Y(X)}$ with eigenvalue $\langle \lambda, \text{ad}_Y(X) \rangle$, so that the trace of $\rho_{\text{ad}_Y(X)}$ on W equals $m\langle \lambda, \text{ad}_Y(X) \rangle$. However, since $\rho_{\text{ad}_Y(X)}$ equals a commutator of $\mathbb{C}$-linear endomorphisms of W, namely $\rho_Y \circ \rho_X - \rho_X \circ \rho_Y$, conclude that the trace equals 0. Since the characteristic of $\mathbb{C}$ equals 0, conclude that $\langle \lambda, \text{ad}_Y(X) \rangle$ is zero, proving Lie's Lemma (and thus Lie's Theorem).

(f) Finally, if $\mathfrak{h}$ is solvable, use induction along the lower central series to prove that for every Jordan-Hölder filtration of (V, ρ) by $\mathfrak{g}$-subrepresentations, every simple factor is an $\mathfrak{h}$-eigenspace for some weight λ, and thus every $\mathbb{C}$-subspace of the representation is a $\mathfrak{h}$-subrepresentation. This is equivalent to Lie's Theorem.

Problem 8.(Engel's Theorem.) Consider the following assertion (the weak form of Engel's Theorem). A finite-dimensional $\mathbb{C}$-linear representation V of a $\mathbb{C}$-Lie algebra $\mathfrak{g}$ is a **nilpotent representation** if the image of $\mathfrak{g}$ in $\text{End}_{\mathbb{C}}(V)$ is contained in the nilpotent cone of $\text{End}_{\mathbb{C}}(V)$, i.e., every image element is a nilpotent linear transformation of V.

Theorem 0.1 (Weak Engel's Theorem)**.** *For every finite-dimensional, nilpotent representation of a $\mathbb{C}$-Lie algebra, the $\mathbb{C}$-Lie algebra annihilates a nonzero vector.*

Obviously this is a property only of the image of $\mathfrak{g}$ in $\mathrm{End}_{\mathbb{C}}(V)$, which is a $\mathbb{C}$-Lie algebra of finite dimension. Thus, it suffices to prove the result for finite-dimensional $\mathbb{C}$-Lie algebras that have a faithful, finite-dimensional $\mathbb{C}$-linear representation.

(a) For a Lie algebra $\mathfrak{g}$ as above, for every $\mathfrak{g}$-subrepresentation W of V, prove that the images of $\mathfrak{g}$ in both $\mathfrak{gl}(W)$ and $\mathfrak{gl}(V/W)$ are contained in the nilpotent cones. Up to replacing $\mathfrak{g}$ by its image in $U = \mathfrak{gl}(V)$, assume that the action on V is faithful. For the adjoint action of $\mathfrak{g}$ on $U = \mathfrak{gl}(V)$, check that the image of $\mathfrak{g}$ is contained in the nilpotent cone of $\mathfrak{gl}(U)$. In particular, the adjoint image of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$ is contained in the nilpotent cone, so that $\mathfrak{g}$ is a nilpotent Lie algebra. In the not necessarily faithful case, the quotient of $\mathfrak{g}$ by the kernel of the representation is a nilpotent Lie algebra.

(b) If $\mathfrak{g}$ has dimension 0 or 1, prove the weak form of Engel's Theorem.

Now, by way of induction, assume that $\mathfrak{g}$ has dimension > 1, and assume the weak Engel's Theorem is true for all Lie subalgebras that have strictly smaller dimension than the dimension of $\mathfrak{g}$.

(c) For every proper Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing the kernel of ρ that is maximal among proper Lie subalgebras of $\mathfrak{g}$ containing the kernel of ρ, conclude that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is nilpotent and preserves $\mathfrak{h}$. Thus the induced representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$ is nilpotent. By the induction hypothesis, conclude that there exists an element X of $\mathfrak{g} \setminus \mathfrak{h}$ such that the adjoint action of $\mathfrak{h}$ on X has image contained in $\mathfrak{h}$, i.e., $[X, \mathfrak{h}] \subset \mathfrak{h}$. Deduce that $\mathfrak{h} + \mathrm{span}(X)$ is a Lie subalgebra of $\mathfrak{g}$ containing the kernel of ρ and that strictly contains $\mathfrak{h}$. Since $\mathfrak{h}$ was maximal among proper Lie subalgebras, deduce that $\mathfrak{h} + \mathrm{span}(X)$ equals $\mathfrak{g}$. Thus, $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ of codimension 1, and it is a Lie ideal.

(d) Continuing the previous part, use the induction hypothesis to conclude that there exists a nonzero vector w of V that is annihilated by $\mathfrak{h}$. If also w is annihilated by the action of X, deduce that $v = w$ satisfies the weak form of Engel's Theorem. If w is not annihilated by the action of X, deduce that $v = X \cdot w$ satisfies the weak form of Engel's Theorem. Thus, the weak form of Engel's Theorem holds by induction on the dimension of $\mathfrak{g}$.

(e) Use the weak form of Engel's Theorem and induction on the dimension of V to conclude the strong form of Engel's Theorem:

Theorem 0.2 (Engel's Theorem)**.** *Every finite-dimensional, nilpotent representation of a $\mathbb{C}$-Lie algebra $\mathfrak{g}$ admits a maximal flag of $\mathbb{C}$-linear subspaces that are $\mathfrak{g}$-subrepresentations whose associated graded 1-dimensional $\mathfrak{g}$-representations are each trivial.*

There is a slightly sharper version, as follows. Now let $\mathfrak{g}$ be a finite-dimensional Lie algebra, let $\mathfrak{n}$ be a Lie ideal in $\mathfrak{g}$, and let (V, ρ) be a nonzero, finite-dimensional $\mathfrak{g}$-representation whose restriction to $\mathfrak{n}$ acts nilpotently on V. By the weak form of Engel's Theorem, the annihilator $V^{\mathfrak{n}}$ in V of $\mathfrak{n}$ is nonzero. Of course $V^{\mathfrak{n}}$ is a $\mathfrak{n}$-subrepresentation of the $\mathfrak{n}$-representation V (the "invariant subrepresentation").

(f) Since $\mathfrak{n}$ is a Lie ideal in $\mathfrak{g}$, prove that $V^{\mathfrak{n}}$ is, in fact, a $\mathfrak{g}$-subrepresentation of V. By considering the induced action of $\mathfrak{g}$ on the quotient $V/V^{\mathfrak{n}}$ and using induction on the dimension of V, conclude the following variant of Engel's Theorem.

Corollary 0.3. *For every finite-dimensional representation (V,ρ) of a Lie algebra $\mathfrak{g}$ and for every Lie ideal $\mathfrak{n}$ of $\mathfrak{g}$ that acts nilpotently on V, for every Jordan-Hölder filtration of (V,ρ) by $\mathfrak{g}$-subrepresentations, every simple factor is a trivial $\mathfrak{n}$-subrepresentation.*

(g) For a finite-dimensional $\mathbb{C}$-linear representation (V,ρ) of a Lie algebra $\mathfrak{g}$, and for Lie ideals $\mathfrak{m}$ and $\mathfrak{n}$ that both act nilpotently on V, for the flag of $\mathfrak{g}$-subrepresentations as above such that $\mathfrak{n}$ acts trivially on the associated graded $\mathfrak{g}$-representations, conclude that the $\mathfrak{m}$-action on each associated graded $\mathfrak{g}$-representation is nilpotent. Thus, there exists a flag of $\mathfrak{g}$-subrepresentations of each associated graded $\mathfrak{g}$-subrepresentations, such that $\mathfrak{m}$ also acts trivially on the new associated graded $\mathfrak{g}$-subrepresentations. Conclude that there exists a refinement of the original flag to a flag of $\mathfrak{g}$-subrepresentations of V such that the action of $\mathfrak{m}+\mathfrak{n}$ on each associated graded $\mathfrak{g}$-representation is trivial. Altogether, this proves the following.

Corollary 0.4. *For every finite dimensional representation of a Lie algebra $\mathfrak{g}$, for every pair of Lie ideals, $\mathfrak{m}$ and $\mathfrak{n}$, that both act nilpotently on the representation, also the Lie ideal $\mathfrak{m}+\mathfrak{n}$ acts nilpotently on the representation. Thus, there exists a maximal Lie ideal of $\mathfrak{g}$ that acts nilpotently on the representation.*

The maximal Lie ideal of $\mathfrak{g}$ that acts nilpotently on a given finite-dimensional representation (V,ρ) is the **nilradical of the representation**, $\text{nil}_\rho(\mathfrak{g})$.

(h) In particular, apply this to the adjoint representation $(\mathfrak{g},\text{ad}_\mathfrak{g})$ to conclude that there exists a flag of Lie ideals in $\mathfrak{g}$ whose associated graded Lie algebras are each trivial representations when restricted to the nilradical of the Lie algebra, $\text{nil}(\mathfrak{g}) = \text{nil}_{\text{ad}}(\mathfrak{g})$.

(i) Let (V,ρ) be a finite-dimensional representation of a finite-dimensional Lie algebra $\mathfrak{g}$ such that the associated representation $V/V^\mathfrak{g}$ of the quotient Lie algebra $\mathfrak{g}/\text{nil}_\rho(\mathfrak{g})$ is nilpotent. Use induction on the dimension of V to prove that $\text{nil}_\rho(\mathfrak{g})$ equals all of $\mathfrak{g}$. Conclude the following corollary.

Corollary 0.5. *A $\mathbb{C}$-Lie algebra acts nilpotently on a finite-dimensional $\mathbb{C}$-linear representation if the $\mathbb{C}$-Lie algebra is the sum of a $\mathbb{C}$-Lie ideal and a $\mathbb{C}$-Lie subalgebra, each of which act nilpotently on the representation.*

Problem 9.(**Jordan canonical form via polynomials**). For every $\mathbb{C}$-vector space V of finite dimension $d \geq 1$, for every $\mathbb{C}$-linear endomorphism X of V, by the Fundamental Theorem of Algebra the characteristic polynomial of X factors into a product of powers of distinct monic linear factors,

$$\det_V(t\text{Id}_V - X) = (t-\lambda_1)^{e_1}\cdots(t-\lambda_\ell)^{e_\ell}, \quad e_i \in \mathbb{Z}_{\geq 1}, \quad e_1+\cdots+e_\ell = d,$$

for a finite collection $\{\lambda_1,\dots,\lambda_\ell\}$ of pairwise distinct complex numbers. The morphism of associative, unital $\mathbb{C}$-algebras,

$$\text{ev}_X^V : \mathbb{C}[t] \to \text{End}_\mathbb{C}(V), \quad c_r t^r + \cdots + c_1 t + c_0 \mapsto c_r X^r + \cdots + c_1 X + c_0 \text{Id}_V,$$

factors through the quotient $\mathbb{C}[t]/\det_V(t\text{Id}_V - X)$ by the Cayley-Hamilton theorem. Use the Chinese Remainder Theorem to **prove** that the image of ev_X^V consists of all $\mathbb{C}$-linear endomorphisms Y of V such that for every $i = 1\dots,\ell$, both Y maps the generalized eigenspace $V_i := \text{Ker}(\lambda_i\text{Id}_V - X)^{e_i}$ to itself, and the restriction of Y to V_i equals some polynomial (possibly depending on i) applied to the restriction of X to

V_i. In particular, for every $i = 1, \dots, \ell$, both the endomorphism $X_{\text{ss},i}$, respectively the endomorphism $X_{\text{nil},i}$ are in the image, where $X_{\text{ss},i}$ and $X_{\text{nil},i}$ restrict on V_j as zero for every $j \neq i$, and where $X_{\text{ss},i}$ restricts on V_i as $\lambda_i \text{Id}_{V_i}$, resp. where $X_{\text{nil},i}$ restricts on V_i as the restriction of $X - \lambda_i \text{Id}_V$. Deduce that also the sums $X_{\text{ss}} := X_{\text{ss},1} + \cdots + X_{\text{ss},\ell}$ and $X_{\text{nil}} := X_{\text{nil},1} + \cdots + X_{\text{nil},\ell}$ are in the image. Of course X_{ss} is semisimple, X_{nil} is nilpotent, X equals the sum of these two, and X_{ss} commutes with X_{nil}; this is the **Jordan canonical form** decomposition of X. Similarly, each projection pr_i from V to V_i is in the image (since its restriction to each V_j is zero for $j \neq i$ and its restriction to V_i equals Id_{V_i}). Finally, the operator $\overline{X}_{\text{ss}} := \overline{\lambda}_1 \text{pr}_1 + \cdots + \overline{\lambda}_\ell \text{pr}_\ell$ is in the image.

Problem 10.(**Cartan's solvability criterion**.) For every $\mathbb{C}$-vector space V of finite dimension d, recall the $\mathbb{C}$-Lie algebra homomorphism,

$$\text{ad}^V : \text{End}_{\mathbb{C}}(V) \to \text{End}_{\mathbb{C}}(\text{End}_{\mathbb{C}}(V)), \quad X \mapsto (\text{ad}^V_X : Y \to X \circ Y - Y \circ X).$$

For every element X of $\text{End}_{\mathbb{C}}(V)$, **prove** that the semisimple part of ad^V_X equals $\text{ad}^V_{X_{\text{ss}}}$, and the characteristic polynomial of this endomorphism depends only on the characteristic polynomial of X (and on d, the degree of that polynomial). Thus this endomorphism is the evaluation on ad^V_X of a polynomial of degree $\leq d^2$ that only depends on the characteristic polynomial of X. Also the nilpotent part of ad^V_X, namely $\text{ad}^V_{X_{\text{nil}}}$, equals the evaluation on ad^V_X of such a polynomial. Most pertinently, the endomorphism $\text{ad}^V_{\overline{X}_{\text{ss}}}$ equals the evaluation on ad^V_X of such a polynomial whose constant coefficient equals 0. Thus, for every element Y of $\text{End}_{\mathbb{C}}(V)$, in the smallest $\mathbb{C}$-Lie subalgebra of $\text{End}_{\mathbb{C}}(V)$ containing both X and Y, there exist elements Z and W such that $[X_{\text{ss}}, Y]$ equals $[X, Z]_{\mathfrak{g}}$ and $[\overline{X}_{\text{ss}}, Y]$ equals $[X, W]_{\mathfrak{g}}$. This gives the following important result.

Theorem 0.6 (Cartan's solvability criterion, I)**.** *A $\mathbb{C}$-Lie subalgebra $\mathfrak{g}$ of $\text{End}_{\mathbb{C}}(V)$ is solvable if, for every element X of the commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ and for every element Y of $\mathfrak{g}$, the trace of the endomorphism $X \circ Y$ equals 0.*

Proof. For every commutator $[Y', Y]$ in $[\mathfrak{g}, \mathfrak{g}]$, for every element X of $\mathfrak{g}$, the trace of $\overline{X}_{\text{ss}} \circ [Y', Y]$ equals the trace of $[\overline{X}_{\text{ss}}, Y] \circ Y'$, using the invariance of the trace under cyclic invariance of factors in a composition. By the previous exercise, $[\overline{X}_{\text{ss}}, Y]$ equals $[X, Z]$ for some $Z \in \mathfrak{g}$. Thus, the the trace of $\overline{X}_{\text{ss}} \circ [Y', Y]$ equals the trace of $[X, Z] \circ Y'$ for elements X, Y' and Z of $\mathfrak{g}$, and this equals 0 by the hypothesis. In particular, if X itself is in $[\mathfrak{g}, \mathfrak{g}]$, i.e., a sum of finitely many commutators $[Y_i, Y'_i]$ for elements Y_i and Y'_i of $\mathfrak{g}$, then it follows that the trace of $\overline{X}_{\text{ss}} \circ X$ equals 0. Of course this can happen if and only if every eigenvalue of X equals 0, i.e., if and only if X is nilpotent. So the hypotheses imply that the $\mathbb{C}$-Lie ideal $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, by Engel's theorem. Thus $\mathfrak{g}$ is solvable. ∎