MAT 543 FALL 2025 PROBLEM SET 5

Problem 1. For every integer $n \geq 1$, for every partition $\mu = (\mu_1, \dots, \mu_m)$ of the integer n, define the μ**-power sum polynomial** to be the homogeneou, degree-n, symmetric polynomial in m variables (each of degree 1),

$$p_\mu(t_1, \dots, t_n) = \prod_{k=1}^{n} p_k(t_1, \dots, t_n)^{m_k(\mu)} = \prod_{k=1}^{n} (t_1^k + \cdots + t_n^k)^{m_k(\mu)},$$

where $m_k(\mu)$ is the **multiplicity** of k in μ,

$$m_k(\mu) = \#\{1 \leq i \leq m | \mu_i = k\}.$$

For every finite-dimensional, $\mathbb{C}$-linear representation (V, ρ) of $\mathfrak{S}_n$, define the **associated symmetric polynomial** in $(t_1, \dots, t_n)$ to be,

$$p_{V,\rho}(t_1, \dots, t_n) = \frac{1}{\#\mathfrak{S}_n} \sum_{\sigma \in \mathfrak{S}_n} \text{trace}_V(\rho(g)) p_{[g]}(t_1, \dots, t_n),$$

where $[g]$ denotes the cycle type of g (or, equivalently, the conjugacy class of g). Use the Frobenius formula to prove that the associated symmetric polynomial of each Specht module V_λ is the **Schur polynomial**,

$$S_\lambda(t_1, \dots, t_n) = \det[t_i^{\mu_j + n - j}]_{1 \leq i,j \leq n} / \det[t_i^{n-j}]_{1 \leq i,j \leq n}.$$

Since the Schur polynomials S_λ of partitions λ of n give a free $\mathbb{Z}$-basis of the $\mathbb{Z}$-module Λ_n of homogeneous, degree-n, symmetric polynomials in $(t_1, \dots, t_n)$ with $\mathbb{Z}$-coefficients, deduce that the function p sending every representation to its associated symmetric polynomial extends to an isomorphism of $\mathbb{Z}$-modules from the module $R(\mathfrak{S}_n)$ of virtual representations of $\mathfrak{S}_n$ to the module Λ_n.

Problem 2. Recall that each **Kostka number** $K_{\lambda,\mu}$ is the (nonnegative integer) multiplicity of the Specht modules V_λ in the representation U_μ of $\mathfrak{S}_n$ induced from the trivial representations of any Young subgroup of shape μ. Use the known identities of the Kostka numbers to prove that the symmetric function of U_μ is the product $H_{\mu_1} \cdots H_{\mu_m}$, where H_d is the **complete symmetric polynomial** of degree-d, i.e., the sum (with coefficient 1) of every monomial in $(t_1, \dots, t_n)$ of degree d. Deduce that, under Schur-Weyl duality, the representation U_μ of $\mathfrak{S}_n$ corresponds to the following representation of $\mathbf{SL}_{\mathbb{C}}(W)$,

$$\otimes_{k=1}^{m} \text{Sym}^{\mu_k}(W).$$

Problem 3. For every pair of integers $m, n \geq 1$, for every $\mathbb{C}$-linear $\mathfrak{S}_m$-representation (U, σ), and for every $\mathbb{C}$-linear $\mathfrak{S}_n$-representation (V, ρ), the $\mathbb{C}$-linear $\mathfrak{S}_{m+n}$-representation induced from the representation of a Young subgroup $\mathfrak{S}_m \times \mathfrak{S}_n$ of $(U \otimes_{\mathbb{C}} V, \sigma \boxtimes \rho)$ is denoted $(U, \sigma) \circ (V, \rho)$, the **outer product** of (U, σ) and (V, ρ), as a representation of $\mathfrak{S}_{m+n}$. Prove that this defines a $\mathbb{Z}$-bilinear map from $R(\mathfrak{S}_m) \times R(\mathfrak{S}_n)$ to $R(\mathfrak{S}_{m+n})$. Moreover, prove that the symmetric function $p_{(U,\sigma)\circ(V,\rho)}$ equals the product of symmetric functions $p_{U,\sigma} \cdot p_{V,\rho}$. Deduce that the maps p assemble into an algebra isomorphism from the algebra $R := \oplus_{n=0}^{\infty} R(\mathfrak{S}_n)$ (with outer product as the multiplication law) to the ring $\Lambda = \oplus_{n=0}^{\infty}\Lambda_n$ of symmetric functions. Finally, prove that restriction of representations from $\mathfrak{S}_{m+n}$ to each Young subgroup $\mathfrak{S}_m \times \mathfrak{S}_n$ altogether gives a coproduct on R that makes R into a $\mathbb{Z}$-graded Hopf algebra.

Problem 4. For every partition λ of m, for every partition μ of n, for the Specht module V_λ of $\mathfrak{S}_m$, for the Specht module V_μ of $\mathfrak{S}_n$, define each **Littlewood-Richardson coefficient** $N^{\nu}_{\lambda,\mu}$, for every partition ν of $m+n$, to be the multiplicity of the Specht module V_ν of $\mathfrak{S}_{m+n}$ in $V_\lambda \circ V_\mu$. Use the previous results to prove that also $N^{\nu}_{\lambda,\mu}$ is the coefficient of the Schur polynomial S_ν in the expansion of the product $S_\lambda \cdot S_\mu$ of Schur polynomials as a $\mathbb{Z}$-linear combination of Schur polynomials of degree $m+n$.

Problem 5. Prove **Pieri's Rule**: for every partition λ of m, for every partition μ of $n \geq m$, the multiplicity of the Specht module V_μ in the $\mathfrak{S}_n$-representation induced from a representation of the Specht module V_λ of the subgroup $\mathfrak{S}_m$ equals 0 unless the Young diagram of μ contains the Young diagram of λ, and then the multiplicity equals the number of ways to arrange the integers from $m+1$ to n in the complement of Young diagrams (the **skew diagram**) so that the integers increase both in every row and every column (that happens to contain two or more boxes of the skew diagram).

Problem 6. Use the Schur character formula to deduce a formula for the dimension of each Schur functor $\mathbb{S}^\lambda(W)$ in terms of the dimension of W and the partition λ.

Problem 7. As a left representation of $\mathbf{GL}_{\mathbb{C}}(W)$, the $\mathbb{C}$-subalgebra of $\text{Hom}_{\mathbb{C}}(W^{\otimes n}, W^{\otimes n})$ of morphisms of $\mathfrak{S}_n$-representations from $W^{\otimes n}$ (with trivial action) to $W^{\otimes n}$ with the natural action of $\mathbf{GL}_{\mathbb{C}}(W)$, equals $\text{Sym}^n(\text{Hom}_{\mathbb{C}}(W, W)) = \text{Sym}^n(\text{Hom}_{\mathbb{C}}(U, W))$ as a $\mathbb{C}$-vector space where U is a copy of W with trivial action of $\mathbf{GL}_{\mathbb{C}}(W)$. By Schur-Weyl duality, this also equals the following finite direct product of simple algebras,

$$\prod_{\lambda \vdash n} \text{Hom}_{\mathbb{C}}(\mathfrak{S}^\lambda(W), \mathfrak{S}^\lambda(W)).$$

By taking characters for a maximal torus in $\mathbf{GL}_{\mathbb{C}}(W)$, deduce an identity of symmetric functions. For $n = 1$, 2 and 3, check directly that this identity holds.

Problem 8. Use Schur-Weyl duality to reinterpret each Littlewood-Richardson coefficient $N^{\nu}_{\lambda,\mu}$ as the multiplicity of the Schur function $\mathbb{S}^\nu(W)$ in the tensor product of $\mathbf{GL}_{\mathbb{C}}(W)$-representations, $\mathbb{S}^\lambda(W) \otimes_C \mathbb{S}^\mu(W)$.

Problem 9. Combine this with Pieri's Rule to prove that $\mathbb{S}^\lambda(W) \otimes_{\mathbb{C}} \text{Sym}^n(W)$ is isomorphic to the direct sum of $\mathbb{S}^\nu(W)$ over all partitions ν whose Young diagram is obtained from the Young diagram of λ by adding n boxes, no two in the same column.

Problem 10. Read more about these ideas, and their direct reinterpretation via counting problems for Young diagrams and Young tableau, either in the textbook (especially Appendix A), or in some other source.