MAT 543 FALL 2025 PROBLEM SET 4

Problem 1. For every homomorphism f of associative, unital $\mathbb{C}$-algebras from A to B, for every **quasi-idempotent element** a in A, i.e., $a \cdot a = na$ for a scalar n in $\mathbb{C}$, prove that also $f(a)$ is quasi-idempotent. Moreover, for every sequence of quasi-idempotent elements $(a_1, \dots, a_r)$ of A that is **orthogonal**, i.e., $a_i a_j = 0$ for $i \neq j$, prove that also $(f(a_1), \dots, f(a_r))$ is orthogonal. In particular, for every finite group Γ and for every subgroup Π, for the orthogonal sequence of central idempotents $a_i = a_{\Pi,(V_i,\rho_i)}$ of $\mathbb{C}[\Pi]$ associated to the irreducible Π-representations, the images in $\mathbb{C}[\Gamma]$ form an orthogonal sequence of idempotents in $\mathbb{C}[\Gamma]$. Thus, for the sequence of group homomorphisms χ from Π to $\mathbb{C}^\times$, the elements $a_{\Pi,\chi} := \sum_{p\in\Pi} \chi(p)\mathbf{b}_{p^{-1}}$ in $\mathbb{C}[\Gamma]$ form an orthogonal sequence of quasi-idempotents.

Problem 2. Prove that for every finite group Γ, for every normal subgroup Π, and for every Γ-orbit of irreducible representations of Π, the sum over this orbit of the images in $\mathbb{C}[\Gamma]$ of the central quasi-idempotents in $\mathbb{C}[\Pi]$ is a central quasi-idempotent in $\mathbb{C}[\Gamma]$ that corresponds to the induced representation from Π to Γ.

Definition 0.1. For every finite group Γ, pairs (Π, χ) and (Π', χ') of a subgroup of Γ and a group homomorphism to $\mathbb{C}^\times$ from that subgroup are **semiorthogonal** if and only if both the product map from $\Pi \times \Pi'$ to its image $\Pi \cdot \Pi'$ in Γ is a bijection, and, for every element g in $\Gamma \setminus (\Pi \cdot \Pi')$, there exists an element $p' = gpg^{-1}$ of $(g\Pi g^{-1}) \cap \Pi'$ such that $\chi(p)$ is not equal to $\chi'(p')$. Stated differently, for each element g of Γ, there exists *no* element $p' = gpg^{-1}$ of $(g\Pi g^{-1}) \cap \Pi'$ with $\chi(p) \neq \chi'(p')$ if and only if there exists an element (p, p') of $\Pi \times \Pi'$ with $g = pp'$, and this is unique.

Problem 3. For semiorthogonal pairs (Π, χ) and (Π', χ'), for the image quasi-idempotents $a_{\Pi,\chi}$ and $a_{\Pi',\chi'}$, prove that the product $c_{(\Pi,\chi),(\Pi',\chi')} = a_{\Pi,\chi} a_{\Pi',\chi'}$ is the unique element c of $\mathbb{C}[\Gamma]$, normalized so that the coefficient of $\mathbf{b}_e$ equals 1, satisfying the following absorption identities,

$$\forall p \in \Pi,\ \forall p' \in \Pi',\ \mathbf{b}_p \cdot c \cdot \mathbf{b}_{p'} = \chi(p)\chi'(p')c.$$

More precisely, for every element c satisfying the absorption identities whose coefficient of $\mathbf{b}_e$ equals 1, prove that for every element $g = pp'$ in $\Pi \cdot \Pi'$, the coefficient of $\mathbf{b}_g$ in c equals the coefficient $\chi(p)^{-1}\chi'(p')^{-1}$ of $\mathbf{b}_g$ in $c_{(\Pi,\chi),(\Pi',\chi')}$. Also prove that for every element g in $\Gamma \setminus (\Pi \cdot \Pi')$, for every element $p' = gpg^{-1}$ in $(g\Pi g^{-1}) \cap \Pi'$ such that $\chi(p)$ is not equal to $\chi'(p')$, the coefficient of $\mathbf{b}_{gp}$ in $\chi(p)^{-1}\mathbf{b}_p \cdot c$ equals the coefficient of $\mathbf{b}_{p'g}$ in $\chi'(p')^{-1} c \cdot \mathbf{b}_{p'}$ if and only if the coefficient of $\mathbf{b}_g$ in c equals 0.

Problem 4. Continuing the previous problem, since every element of $c\mathbb{C}[\Gamma]c$ satisfies the absorption identities, prove that these elements are scalar multiples of $c = c_{(\Pi,\chi),(\Pi',\chi')}$. In particular, c is a quasi-idempotent with scalar factor $n = n_{(\Pi,\chi),(\Pi',\chi')}$ equal to an algebraic integer (even a sum of roots of unity of order dividing $\#\Gamma$).

Problem 5. Continuing the previous problem, for the basis $(\mathbf{b}_g)_{g\in\Gamma}$ of $\mathbb{C}[\Gamma]$, the coefficient of $\mathbf{b}_g$ in the product $c \cdot \mathbf{b}_g$ equals the coefficient 1 of $\mathbf{b}_e$ in c. For the

$\mathbb{C}$-linear endomorphism of right multiplication by c, the matrix representative with respect to this basis has every diagonal entry equal to 1. Thus, the trace equals $\#\Gamma$. On the other hand, since $c \cdot c$ equals nc, the minimal polynomial $m(t)$ of this endomorphism divides $t^2 - nt$. So, if n equals 0, then the minimal polynomial divides t^2. Since the irreducible factors of the characteristic polynomial equal the irreducible factors of the minimal polynomial, the characteristic polynomial also equals $t^{\#\Gamma}$, implying that the trace equals 0. Conclude that the algebraic integer n is nonzero, and is even a positive integer divisor of $\#\Gamma$.

Problem 6. For every $\mathbb{C}$-vector space V with associated $\mathbb{C}$-algebra $\text{End}_{\mathbb{C}}(V)$ of $\mathbb{C}$-linear endomorphisms, prove that every cyclic left ideal in $\text{End}_{\mathbb{C}}(V)$ is the left annihilator $\text{Ann}(U) = \{T | T(U) = \{0\}\}$ for a unique subspace U. If V has infinite dimension, prove that the finite rank endomorphisms form a left ideal that is not cyclic. On the other hand, if V has finite dimension, prove that every left ideal is a cyclic left ideal, and the cyclic generators are precisely those elements c in the left ideal that have minimal kernel (with respect to inclusion). Prove that an element c of $\text{End}_{\mathbb{C}}(V)$ is a square-zero element if and only if $\text{Image}(c)$ is a subspace of $\text{Ker}(c)$. Finally, prove that an element c is quasi-idempotent and non-nilpotent (i.e., the scalar factor is nonzero) if and only if both V is the direct sum of $\text{Image}(c)$ and $\text{Ker}(c)$ and the restriction of c to $\text{Image}(c)$ is a nonzero scalar multiple of the the identity.

Problem 7. For every finite-dimensional $\mathbb{C}$-vector space V, for every pair of left ideals $\text{Ann}(U)$ and $\text{Ann}(U')$, prove that every morphism of left $\text{End}_{\mathbb{C}}(V)$-modules from $\text{Ann}(U)$ to $\text{Ann}(U')$ equals right multiplication by an element x of $\text{End}_{\mathbb{C}}(V)$ such that $x(U')$ is a subspace of U. Prove that there exists such a morphism that is injective, respectively surjective, bijective, if and only if we have $\dim(U) \geq \dim(U')$, resp. $\dim(U) \leq \dim(U')$, $\dim(U) = \dim(U')$. Conclude that the minimal nonzero left ideals are precisely the ideals $\text{Ann}(U)$ for U a codimension-one subspace U of V, and these are precisely the left ideals that are simple left $\text{End}_{\mathbb{C}}(V)$-modules.

Problem 8. For every finite-dimensional $\mathbb{C}$-vector space V, for every nonzero element c of $\text{End}_{\mathbb{C}}(V)$, prove that the cyclic left ideal $\text{End}_{\mathbb{C}}(V) \cdot c$ is a minimal nonzero left ideal if and only if the subspace $c \cdot \text{End}_{\mathbb{C}}(V) \cdot c$ equals the span of c. In this case, for the unique scalar n (possibly zero) such that $c \cdot c$ equals nc, deduce that the trace of right multiplication by c on $\text{End}_{\mathbb{C}}(V)$ equals $n\dim_{\mathbb{C}}(\text{End}_{\mathbb{C}}(V) \cdot c)$. In particular, n is nonzero if and only if this trace is nonzero.

Problem 9. Finally, generalize the previous problem to a semisimple $\mathbb{C}$-algebra A that is a product of finitely many finite-dimensional matrix $\mathbb{C}$-algebras: for every nonzero element c in A, the left ideal $A \cdot c$ is a minimal nonzero left ideal (i.e., a left ideal that is a simple left module) if and only if the subspace $c \cdot A \cdot c$ equals the span of c. In this case, $c \cdot c$ equals nc where the trace of right multiplication by c on A equals $n\dim_{\mathbb{C}}(A \cdot c)$.

Problem 10. In particular, using the Wedderburn-Artin theorem, Schur's lemma and Maschke's theorem, for every finite group Γ and for pairs (Π, χ) and (Π', χ') that are semiorthogonal, prove that the quasi-idempotent element $c_{(\Pi,\chi),(\Pi',\chi')}$ gives an irreducible $\mathbb{C}$-linear Γ-representation $V_{(\Pi,\chi),(\Pi',\chi')} := \mathbb{C}[\Gamma] \cdot c_{(\Pi,\chi),(\Pi',\chi')}$ and the nonzero algebraic integer $n_{(\Pi,\chi),(\Pi',\chi')}$ is actually a positive integer that satisfies

$$n_{(\Pi,\chi),(\Pi',\chi')}\dim_{\mathbb{C}} V_{(\Pi,\chi),(\Pi',\chi')} = \#\Gamma.$$

For every integer $n \geq 1$, for the group $\mathfrak{C}$-algebra $\mathbb{C}[\mathfrak{S}_n] := \bigoplus_{\sigma \in \mathfrak{S}_n} \mathbb{C}\mathbf{b}_\sigma$, for each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 1)$ of length ℓ and of size $\lambda_1 + \cdots + \lambda_\ell = n$, the **dual partition** is the unique partition $\lambda^* = (\lambda_1^* \geq \cdots \geq \lambda_{\ell^*}^* \geq 1)$ of length $\ell^* = \lambda_1$ such that for every $j = 1, \ldots, \lambda_1$ we have $\lambda_j^* = \min\{i \geq 0 | \forall i' \geq i\ \lambda_{i'} \leq j\}$, i.e., for each (i,j) with $1 \leq i \leq \ell$ and with $1 \leq j \leq \lambda_1$, we have $j \leq \lambda_i$ if and only if $i \leq \lambda_j^*$. Since the number of such pairs (i,j) equals the size n of λ, this also equals the size of the dual partition λ^*. For each Young tableau $T = (t_{i,j})_{1\leq i\leq \ell, 1\leq j\leq \lambda_i}$ of shape λ, the dual Young tableau of shape λ^* is $T^* = (t_{i,j}^* = t_{j,i})$. The ordered partition of $\{1, \ldots, n\}$ determined by (λ, T) (unique up to reordering by a symmetry of λ) is

$$\Sigma_{\lambda,T} = (\Sigma_{(\lambda,T),1}, \ldots, \Sigma_{(\lambda,T),\ell}), \ \ \Sigma_{(\lambda,T),i} = \{t_{i,j} | 1 \leq j \leq \lambda_i\}.$$

The ordered partition of the dual Young tableau (λ^*, T^*) is

$$\Sigma'_{\lambda,T} = \Sigma_{\lambda^*,T^*} = (\Sigma'_{(\lambda,T),1}, \ldots, \Sigma'_{(\lambda,T),\ell^*}), \ \ \Sigma'_{(\lambda,T),j} = \{t_{i,j} | 1 \leq i \leq \lambda *_j\}.$$

For the partition $\Sigma_{\lambda,T}$ of $\{1, \ldots, n\}$, the corresponding **Young subgroup** of $\mathfrak{S}_n$ is

$$\Pi_{\lambda,T} := \left\{ p \in \mathfrak{S}_n | \forall i = 1, \ldots, \ell\ p(\Sigma_{(\lambda,T),i}) = \Sigma_{(\lambda,T),i} \right\}.$$

The dual Young subgroup is

$$\Pi'_{\lambda,T} := \Pi_{\lambda^*,T^*} = \left\{ p' \in \mathfrak{S}_n | \forall j = 1, \ldots, \ell^*\ p'(\Sigma'_{(\lambda,T),j}) = \Sigma'_{(\lambda,T),j} \right\}.$$

prop-semi

Proposition 0.2. *For every integer $n \geq 1$, for every partition λ of n, for every Young tableau T of shape λ, fhe pairs $(\Pi_{\lambda,T}, triv)$ and $(\Pi'_{\lambda,T}, sgn)$ are semiorthogonal.*

Proof. The partitions $\Sigma_{\lambda,T}$ and $\Sigma'_{\lambda,T}$ are semiorthogonal in the sense that the intersection of each partition set of $\Sigma_{(\lambda,T),i}$ with a partition set of $\Sigma'_{(\lambda,T),j}$ is either empty, if $j > \lambda_i$, i.e., $i > \lambda_j$, or is a singleton set $\{t_{i,j}\}$. Thus, the unique element of $\mathfrak{S}_n$ that preserves both ordered partitions is the identity permutation. So the multiplication map from $\Pi_{\lambda,T} \times \Pi'_{\lambda,T}$ to $\Pi_{\lambda,T} \cdot \Pi'_{\lambda,T}$ is injective.

To complete the proof, we claim that for every element g of $\mathfrak{S}_n$ such that for every unordered pair $\{t_1, t_2\}$ of elements of $\{1, \ldots, n\}$, either t_1 and t_2 belong to two different partition sets of $\Sigma_{\lambda,T}$ or the elements $g(t_1)$ and $g(t_2)$ belong to two different partition sets of $\Sigma'_{\lambda,T}$, then there exist (unique) elements p of $\Pi_{\lambda,T}$ and p' of $\Pi'_{\Lambda,T}$ such that g equals pp'. Assuming the claim, then for every g that is not in $\Pi_{\lambda,T} \cdot \Pi'_{\lambda,T}$, there exists a transposition $p = (t_1, t_2)$ in $\Pi_{\lambda,T}$ such that the transposition $p' = gpg^{-1} = (g(t_1), g(t_2))$ is also in $\Pi'_{\lambda,T}$. Since $\text{triv}(p)$ equals 1 and $\text{sgn}(p')$ equals -1, this proves that the pairs are semiorthogonal.

The claim is proved by induction on the number of rows and columns of λ. Of course if λ has a unique row or a unique column, then one of $\Pi_{\lambda,T}$, $\Pi'_{\lambda,T}$ is the trivial group and the other element is all of $\mathfrak{S}_n$, so that every element g of $\mathfrak{S}_n$ is uniquely decomposable as pp'. By way of induction, assume that the numbers of rows and columns of λ are both greater than 1, and assume the result is proved for smaller n and also for n with partitions that have smaller numbers of rows and columns.

Let g be an element as above. Then, in particular, for the elements $t_{1,j}$ of $\Sigma_{(\lambda,T),1}$, the elements $g(t_{1,1}), \ldots, g(t_{1,\lambda_1})$ are contained in different columns of λ. Thus, for each $j = 1, \ldots, \lambda_1$, either the identity permutation moves $g(t_{1,j})$ into the first row, or there is a transposition in $\Pi'_{\lambda,T}$ that moves $g(t_{1,j})$ into the first row. The product of these elements of $\Pi'_{\lambda,T}$ (which commute with each other) is an element p' of $\Pi'_{\lambda,T}$ that permutes all of the elements $g(t_{1,j})$ into the first row. Then there exists a permutation p of the first row, that permutes these elements into the correct order, and p is an element of $\Pi_{\lambda,T}$. After modifying g by multiplying by these elements, we may assume that g is as above and preserves the first row of the tableau. Of course the remaining elements of the tableau form a set of strictly smaller size than n. By the induction hypothesis, this modified elements is in $\Pi_{\lambda,T} \cdot \Pi'_{\lambda,T}$. Thus, multiplying by p and p' appropriately, we deduce that also g is an element of $\Pi_{\lambda,T} \cdot \Pi'_{\lambda,T}$. $\square$

Thus the $\mathbb{C}$-linear $\mathfrak{S}_n$-subrepresentations $V_{\lambda,T} = \mathbb{C}[\mathfrak{S}_n] \cdot c_{\lambda,T}$ are irreducible of dimension $n!/n_{\lambda,T}$. By a simpler version of the proof above, we proved in lecture that the quasi-idempotents are orthogonal for distinct partitions λ. Thus, the number of such pairwise non-isomorphic representations $V_{\lambda,T}$ produced by varying the partitions λ equals the number of partitions of n, and this equals the number of conjugacy classes of $\mathfrak{S}_n$. Therefore, for every integer $n \geq 1$, the irreducible $\mathbb{C}$-linear $\mathfrak{S}_n$-representations $(V_\lambda)_{\lambda \vdash n}$ give a complete set of pairwise nonisomorphic representatives of the isomorphism classes of all irreducible $\mathbb{C}$-linear $\mathfrak{S}_n$-representations.