# MAT 543 FALL 2025 PROBLEM SET 3

**Problem 1.** For a subgroup $\Gamma'$ of a finite group $\Gamma$, define $\mathrm{proj}^{\Gamma}_{\Gamma'}$ to be the $\mathbb{C}$-linear transformation from $\mathbb{C}[\Gamma]$ to $\mathbb{C}[\Gamma']$ that sends $\mathbf{b}_g$ to $\mathbf{b}_g$ if $g$ is an element of $\Gamma'$ and to 0 otherwise. Prove that this is compatible with both the left and right $\mathbb{C}[\Gamma']$-module structures on $\mathbb{C}[\Gamma]$. Define a left $\mathbb{C}[\Gamma]$-module structure on the $\mathbb{C}$-vector space of morphisms of left $\mathbb{C}[\Gamma']$-modules,

$$\mathrm{Hom}_{\mathbb{C}[\Gamma']-\mathrm{mod}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma']) = \mathrm{Hom}_{\mathbb{C}-\mathrm{mod}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma'])^{\Gamma'},$$

by precomposition with *right* multiplication on $\mathbb{C}[\Gamma]$, i.e., for every element $a$ in $\mathbb{C}[\Gamma]$, for every $\mathbb{C}[\Gamma']$-module homomorphism $T$ from $\mathbb{C}[\Gamma]$ to $\mathbb{C}[\Gamma']$, for every element $b$ of $\mathbb{C}[\Gamma]$, define $(a \cdot T)(b) := T(b \cdot a)$. Prove that left multiplication on $\mathrm{proj}^{\Gamma}_{\Gamma'}$ defines a *injective* homomorphism of left $\mathbb{C}[\Gamma]$-modules,

$$i^{\Gamma'}_{\Gamma} : \mathbb{C}[\Gamma] \to \mathrm{Hom}_{\mathbb{C}[\Gamma']-\mathrm{mod}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma']), \quad a \mapsto a \cdot \mathrm{proj}^{\Gamma'}_{\Gamma}.$$

Since both the domain and target have the same finite dimension as $\mathbb{C}$-vector spaces, deduce that this is an isomorphism. Finally, check that this isomorphism also respects the natural right $\mathbb{C}[\Gamma']$-module structures. Deduce that for every left $\mathbb{C}[\Gamma']$-module $M$, the following is an isomorphism of left $\mathbb{C}[\Gamma]$-modules,

$$\mathrm{Ind}^{\Gamma}_{\Gamma'}(M) = \mathbb{C}[\Gamma] \otimes_{\mathbb{C}[\Gamma']} M \to \mathrm{Hom}_{\mathbb{C}[\Gamma']-\mathrm{mod}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma']) \otimes_{\mathbb{C}[\Gamma']} M \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}[\Gamma']-\mathrm{mod}}(\mathbb{C}[\Gamma], M).$$

This gives an alternative interpretation of the induced $\Gamma$-representation of $M$ that is occasionally useful.

**Problem 2.** For every $\mathbb{C}$-algebra $(A, \cdot)$ that has finite $\mathbb{C}$-dimension, the **norm** of $A$ is the polynomial map,

$$A \to \mathrm{Hom}_{\mathbb{C}-\mathrm{mod}}(A, A) \to \mathbb{C}, \quad a \mapsto \det_A(L_a)$$

where the first map sends $a$ to left multiplication $L_a$ by $a$ on $A$, and where the second map is the determinant for $\mathbb{C}$-linear endomorphisms of the vector space $A$. For $A$ isomorphic to a $n \times n$ matrix algebra, check that this is the usual determinant raised to the power $n$. More generally, for $A$ isomorphic to a product of matrix algebras, check that this is the product of powers of the usual determinants on each factor. Since the determinant polynomial is irreducible, conclude that the norm, considered as a polynomial, has irreducible components in one-to-one correspondence with the matrix algebra factors, and the degrees of the irreducible components determine the sizes of these matrix algebra factors. Use this to give a (wildly inefficient) algorithm to determine the number and dimensions of the irreducible $\mathbb{C}$-linear representations of a finite group $\Gamma$ in terms of the irreducible factors of the $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$.

**Problem 3.** Prove that the number $p(n)$ of conjugacy classes in the symmetric group $\mathfrak{S}_n$ satisfies the following product expansion as formal power series,

$$1 + p(1)t + p(2)t^2 + \cdots + p(n)t^n + \cdots = \prod_{d=1}^{\infty} \frac{1}{1 - t^d}.$$

Compute $p(n)$ for $n = 1$ to 7. For each of these cases, also draw diagrams of the corresponding Young diagrams.

**Problem 4.** For every integer $n \geq 1$, for every partition $\lambda$ of $n$, for every Young tableau $T$, check that both $a_\lambda$ and $b_\lambda$ are quasi-idempotent, i.e., find positive integers $r_\lambda$ and $s_\lambda$ such that $a_\lambda \cdot a_\lambda$ equals $r_\lambda a_\lambda$ and $b_\lambda \cdot b_\lambda$ equals $s_\lambda b_\lambda$. Deduce that, for all positive integers $d$ and $e$, the left $\mathbb{C}[\Gamma]$-ideal generated by $c_\lambda = a_\lambda \cdot b_\lambda$ equals the ideal generated by $a_\lambda^d b_\lambda^e$ for any positive integers $d$ and $e$.

**Problem 5.** Continuing the previous problem, prove that the left ideal generated by $c_\lambda$ is isomorphic to the image under *right* multiplication by $b_\lambda$ from the left ideal generated by $a_\lambda$ to the left ideal generated by $b_\lambda$. Use this to give one (common) interpretation of the Schur functor of a $\mathbb{C}$-vector space $W$ as the image of a natural $\mathbb{C}$-linear transformation from a tensor product of symmetric powers of $W$ to a tensor product of exterior powers of $W$. Also use this to prove that $V_{(\lambda,T)^*}$ is isomorphic as a $\mathbb{C}$-linear $\mathfrak{S}_n$-representation to the tensor product of $V_{(\lambda,T)}$ and the 1-dimensional sign representation.

**Problem 6.** Continuing the previous problem, assuming the theorem from lecture that both $a_\lambda b_\lambda$ and $b_\lambda a_\lambda$ are quasi-idempotent elements whose left ideals are irreducible $\mathbb{C}$-linear $\mathfrak{S}_n$-representations, deduce that the restriction to the left ideal $\mathbb{C}[\mathfrak{S}_n]b_\lambda a_\lambda$ of the morphism of right multiplication by $b_\lambda$ from $\mathbb{C}[\mathfrak{S}_n]a_\lambda$ to $\mathbb{C}[\mathfrak{S}_n]\cdot b_\lambda$ is nonzero (since, in particular, $a_\lambda b_\lambda a_\lambda$ is mapped to $a_\lambda b_\lambda a_\lambda b_\lambda = n_\lambda a_\lambda b_\lambda$). Deduce that this gives an isomorphism of $\mathbb{C}$-linear $\Gamma$-representations from $\mathbb{C}[\mathfrak{S}_n]b_\lambda a_\lambda$ to $\mathbb{C}[\mathfrak{S}_n]a_\lambda b_\lambda$.

**Problem 7.** For $n = 3$, explicitly compute $c_\lambda$ for $\lambda = (2, 1)$. For $n = 4$, explicitly compute $c_\lambda$ for $\lambda = (2, 1)$ and for $\lambda = (2, 2)$. Also explicitly compute the corresponding Schur functors. Use this to give a decomposition of the $\mathbb{C}$-linear $\mathfrak{S}_3$-representation $W \otimes_\mathbb{C} W \otimes_\mathbb{C} W$ and of the $\mathbb{C}$-linear $\mathfrak{S}_4$-representation $W \otimes_\mathbb{C} W \otimes_\mathbb{C} W \otimes_\mathbb{C} W$.

**Problem 8.** For every integer $n > 1$, for the self-dual partition $\lambda = (n-1, 1)$, prove that $V_\lambda$ is isomorphic to the quotient of standard permutation representation (of dimension $n$) by the diagonal, i.e., $V_\lambda$ is a "standard" representation.

**Problem 9.** For every integer $n \geq 1$, for every partition $\mu = (\mu_1, \dots, \mu_m)$ of $n$, there is a set $\Pi_{n,\mu}$ of partitions of the set $\{1, \dots, n\}$ into subsets of sizes $(\mu_1, \dots, \mu_m)$. For every partition $\lambda$ of $n$, define $K_{\lambda,\mu}$ to be the $\mathbb{C}$-dimension of $\mathrm{Hom}_{CC[\mathfrak{S}_n]-\mathrm{mod}}(V_\lambda, \mathbb{C}[\Pi_{n,\mu}])$. These are the **Kostka numbers**. For $n = 2, 3$, and 4, compute the Kostka numbers.

**Problem 10.** The Kostka number $K_{\lambda,\lambda}$ always equals one, and $K_{\lambda,\mu}$ equals zero if $\mu$ is lexicographically larger than $\lambda$. This gives another inductive construction of the irreducible representations $V_\lambda$ as the quotient of $\mathbb{C}[\Pi_{n,\lambda}]$ by all of the irreducible subrepresentations of type $V_\mu$ for $\mu$ lexicographically larger than $\lambda$ (and the base case $V_{(n)}$ is the trivial 1-dimensional representation). Check that this works for $n = 2, 3$, and 4. For every finite dihedral group, for its normal, index 2 cyclic subgroup, explicitly compute induction and restriction of $\mathbb{C}$-linear representations.