MAT 543 FALL 2025 PROBLEM SET 2

Problem 1. For every finite group Γ, prove that every 1-dimensional irreducible $\mathbb{C}$-linear representation of Γ is isomorphic to $(\mathbb{C}, \rho)$ for a unique group homomorphism $\rho : \Gamma \to \mathbb{C}^\times$, which factors uniquely through the quotient homomorphism to the Abelianization, $\Gamma \twoheadrightarrow \Gamma^{\text{ab}} := \Gamma/[\Gamma, \Gamma]$. Thus, they are all "pulled back" from Γ^{ab}.

Problem 2. For every finite Abelian group Γ, prove that every irreducible $\mathbb{C}$-linear Γ-representation is 1-dimensional. Conclude that the set of isomorphism classes of irreducible, $\mathbb{C}$-linear Γ-representations naturally has the structure of the Abelian group $\widehat{\Gamma} := \text{Hom}_{\textbf{Group}}(\Gamma, \mathbb{C}^\times)$ with valuewise multiplication. This is the **Pontrjagin dual group** of Γ.

Problem 3. For every finite Abelian group of the form $\prod_{\lambda\in\Lambda} \mathbb{Z}/n_\lambda\mathbb{Z}$, check that the following pairing into $\mathbb{C}^\times$ gives an isomorphism with the Pontrjagin dual group,

$$\prod_{\lambda\in\Lambda} \mathbb{Z}/n_\lambda\mathbb{Z} \times \prod_{\lambda\in\Lambda} \mathbb{Z}/n_\lambda\mathbb{Z} \to \mathbb{C}^\times, \quad ((\overline{a_n})_{\lambda\in\Lambda}, (\overline{b_\lambda})_{\lambda\in\Lambda}) \mapsto \exp\left(2\pi\sqrt{-1}\sum_{\lambda\in\Lambda}\frac{a_\lambda b_\lambda}{n_\lambda}\right).$$

Problem 4. For every group Γ with finite order ℓ, for every $\mathbb{C}$-linear representation (V, ρ) of finite dimension $n \geq 1$, for every integer $d \geq 0$, prove that induced Γ-action commutes with the natural action of the symmetric group $\mathfrak{S}_d$ on the tensor product $V^{\otimes d}$. Thus, every $\mathfrak{S}_d$-isotypic component of $V^{\otimes d}$ is also a Γ-subrepresentation. In particular, this gives Γ-subrepresentations $(\bigwedge_{\mathbb{C}}^d V, \bigwedge_{\mathbb{C}}^d \rho)$ and $(\text{Sym}_{\mathbb{C}}^d V, \text{Sym}_{\mathbb{C}}^d \rho)$.

Problem 5. Continuing the previous problem, for each element g of Γ, for the characteristic polynomial of $\rho(g)$,

$$\det_V(t\text{Id}_V - \rho(g)) = (t-\zeta_1)\cdots(t-\zeta_n) = t^n + \cdots + (-1)^d\sigma_d(\zeta_1, \ldots, \zeta_n)t^{n-d} + \cdots + (-1)^n\sigma_n(\zeta_1, \ldots, \zeta_n),$$

check that the trace of $\bigwedge_{\mathbb{C}}^d \rho(g)$ equals $\sigma_d(\zeta_1, \ldots, \zeta_n)$. Deduce that, for every integer $d \geq 1$, there exists a degree-d, $\mathbb{Z}$-coefficient, homogeneous polynomial $p_d(s_1, \ldots, s_d)$, such that for every Γ, for every (V, ρ), and for every element g of Γ, the trace of the endomorphism $\rho(g^d)$ equals

$$\zeta_1^d + \cdots + \zeta_n^d = p_d(\sigma_1(\zeta_1, \ldots, \zeta_n), \ldots, \sigma_n(\zeta_1, \ldots, \zeta_n)).$$

Prove also that the polynomial homomorphism $(p_1, \ldots, p_n)$ of $\mathbb{Q}[s_1, \ldots, s_n]$ has an inverse polynomial homomorphism, say $(q_1, \ldots, q_n)$. Deduce that each q_d is a homogenous, $\mathbb{Q}$-coefficient, degree-d polynomial in $(s_1, \ldots, s_d)$ such that

$$\text{trace}_{\bigwedge^d V}(\bigwedge_{\mathbb{C}}^d \rho(g)) = q_d(\text{trace}_V(\rho(g)), \ldots, \text{trace}_V(\rho(g^d))).$$

Check the following cases,

$$p_1(s_1) = s_1,\ p_2(s_1, s_2) = s_1^2 - 2s_2,\ p_3(s_1, s_2, s_3) = s_1^3 - 3s_1s_2 + 3s_3,$$

$$q_1(s_1) = s_1,\ q_2(s_1, s_2) = (s_1^2 + s_2)/2,\ q_3(s_1, s_2, s_3) = (s_1^3 + 3s_1s_2 + 2s_3)/6.$$

Problem 6. Continuing the previous problem, for every $\mathfrak{S}_d$-isotypic component of $V^{\otimes d}$, for every g in Γ, the trace of g on that subrepresentation is a symmetric, degree-d, homogeneous, $\mathbb{Z}$-coefficient polynomial evaluated on the eigenvalues $\zeta_1, \dots, \zeta_n$, thus a homogeneous, $\mathbb{Q}$-coefficient, degree-d polynomial in the traces of $\rho(g), \dots, \rho(g^d)$. In particular, deduce that for every integer $d \geq 1$, there exists a degree-d, $\mathbb{Z}$-coefficient, homogeneous polynomial $r_d(s_1, \dots, s_d)$ such that, for every finite group Γ, for every finite-dimensional, $\mathbb{C}$-linear Γ-representation (V, ρ), for every g in Γ, we have,

$$\text{trace}_{\text{Sym}^d V}(\text{Sym}^d_{\mathbb{C}} \rho(g)) = r_d(\text{trace}_V(\rho(g)), \dots, \text{trace}_V(\rho(g^d))).$$

Check the following cases,

$$r_1(s_1) = s_1, \ r_2(s_1, s_1) = (s_1^2 + s_2)/2, \ r_3(s_1, s_2, s_3) = (s_1^3 + 3s_1 s_2 + 2s_3)/6.$$

Problem 7. Let Γ be a finite group. Recall from Exercise 10(b) on Problem Set 1, for every *irreducible* $\mathbb{C}$-linear Γ-representation V of finite dimension $n \geq 1$, and for every conjugacy class C in Γ, the element $\mathbf{b}_C := \sum_{g \in G} \mathbf{b}_g$ acts on V as a scalar ρ_C. The finite free $\mathbb{Z}$-submodule

$$\text{Center}(\mathbb{Z}[\Gamma]) := \bigoplus_C \mathbb{Z} \cdot \mathbf{b}_C$$

of the $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$ is a $\mathbb{Z}$-subalgebra, and thus the $\mathbb{Z}$-linear transformation of multiplication by $\mathbf{b}_C$ on this finite free $\mathbb{Z}$-submodule has a characteristic polynomial with integer coefficients. Since ρ_C also satisfies this characteristic polynomial, the complex number ρ_C is actually an algebraic integer. Taking traces,

$$\text{trace}_V(\widetilde{\rho}(\mathbf{b}_C)) = \dim_{\mathbb{C}} V \cdot \rho_C = n\rho_C,$$

so that the trace is n times an algebraic integer, and likewise for the inverse conjugacy class. Of course, for each g in C, since $\rho(g)$ can be diagonalized with root of unity eigenvalues, also $\text{trace}_V(\rho(g))$ is an algebraic integer, as is $\text{trace}_V(\rho(g^{-1}))$. Since trace is additive and conjugacy-invariant, choosing one representative g_C in each conjugacy class C, we have

$$\sum_{g \in C} \text{trace}_V(\rho(g)) \text{trace}_V(\rho(g^{-1})) = \text{trace}_V(\widetilde{\rho}(\mathbf{b}_C)) \text{trace}_V(\rho(g_C^{-1})) = n \cdot \rho_C \cdot \text{trace}_V(\rho(g_C^{-1})).$$

By the Schur orthogonality relations, we also have,

$$\#\Gamma = \sum_{g \in \Gamma} \text{trace}_V(\rho(g)) \text{trace}_V(\rho(g^{-1}) = \sum_C \text{trace}_V(\widetilde{\rho}(\mathbf{b}_C)) \text{trace}_V(\rho(g_C^{-1})) = n \cdot \sum_C \rho_C \cdot \text{trace}_V(\rho(g_C^{-1})).$$

Thus, $\#\Gamma/n$ is an algebraic integer that is also an element of $\mathbb{Q}$, i.e., it is an element of $\mathbb{Z}$. Finally, deduce Burnside's theorem: for every finite group Γ, for every irreducible, $\mathbb{C}$-linear Γ-representation (V, ρ), the dimension of V divides $\#\Gamma$.

Problem 8. For every finite cyclic group, for every subgroup, for every character of the subgroup, determine the irreducible decomposition of the induced representation of the entire cyclic group.

Problem 9. For every finite dihedral group, for its normal, index 2 cyclic subgroup, explicitly compute induction and restriction of $\mathbb{C}$-linear representations.

Problem 10. For the symmetric group on 4 letters and its subgroup of the symmetric group on 3 letters, explicitly compute induction and restriction of $\mathbb{C}$-linear representations.