MAT 543 FALL 2025 PROBLEM SET 1

Parts of these problem are also covered in Dummit and Foote. Please focus on those parts of the problems that are new to you.

For every associative, unital ring R, for every left, respectively right, R-module M, the module M is **simple** if (and only if) the unique nonzero submodule is the entire module. A module is **semisimple** if (and only if) it is a direct sum of finitely many simple submodules (the zero module is an empty direct sum). Since the kernel, respectively image, of a morphism of R-modules is an R-submodule, we have the following.

Lemma 0.1 (Schur)**.** *For simple (right) R-modules M and N, the Abelian group $Hom_{mod-R}(M,N)$ is zero unless M is isomorphic to N. Moreover, the associative, unital ring $Hom_{mod-R}(M,M)$ (under composition) is a division algebra, i.e., every nonzero element is invertible.*

For every nonzero, associative, unital ring R, the ring R is **left semisimple**, respectively **right semisimple**, if (and only if) R is semisimple as a left R-module, resp. as a right R-module. This is equivalent to the condition that every finitely generated left R-module, resp. right R-module, is semisimple. Similarly, R is **simple** if it has no nonzero two-sided ideals. A simple ring is left semisimple if and only if it s right semisimple if and only if it is a matrix algebra over a division algebra D, i.e., $\text{Hom}_{D-\text{mod}}(M,M)$ for M a nonzero, finite free, left D-module.

Theorem 0.2 (Wedderburn – Artin)**.** *For every associative, unital ring R, for every simple right R-module M, for the natural left module structure on M by the division algebra $D_M := Hom_{mod-R}(M,M)$, the induced homomorphism $R \to R_M := Hom_{D_M-mod}(M,M)$ is a surjection whose kernel is the annihilator of M in R. For every nonzero, right semisimple ring R, for every finite collection $(M_i)_{i\in I}$ of representatives of the distinct isomorphism classes of simple right R-submodules of R, the induced homomorphism of associative, unital rings $R \to \prod_{i\in I} R_{M_i}$ is an isomorphism.*

Of course the analogous theorem holds for left modules and left semisimple rings. One corollary of the theorem is that every left semisimple ring is also right semisimple (since both are nonzero, finite products of matrix algebras over division algebras). Thus, these rings are called **semisimple**.

For every field $\mathbf{F}$, for every semisimple, associative, unital $\mathbf{F}$-algebra R, the dimension of R as an $\mathbf{F}$-vector space is finite if and only if the dimension of each simple right R-submodule M is finite. In that case, the division $\mathbf{F}$-algebra D_M is an $\mathbf{F}$-subalgebra of the simple $\mathbf{F}$-algebra $\text{Hom}_{\mathbf{F}}(M,M)$. In particular, again D_M has finite $\mathbf{F}$-dimension.

Since every finite field extension of $\mathbf{F}$ is a division $\mathbf{F}$-algebra with finite $\mathbf{F}$-dimension, the field $\mathbf{F}$ is algebraically closed if every finite-dimensional, division $\mathbf{F}$-algebra is isomorphic to $\mathbf{F}$ as an $\mathbf{F}$-algebra. Conversely, over an algebraically closed field

$\mathbf{F}$, the *reduced norm* of every $\mathbf{F}$-division algebra of finite $\mathbf{F}$-dimension > 1 has a nonzero root. This contradicts the hypothesis that this is a division algebra. Therefore, every division $\mathbf{F}$-algebra with finite $\mathbf{F}$-dimension over an algebraically closed field $\mathbf{F}$ is isomorphic to $\mathbf{F}$ itself as an $\mathbf{F}$-algebra.

Thus, for every algebraically closed field, for every semisimple $\mathbf{F}$-algebra R with finite $\mathbf{F}$-dimension, for each simple right R-submodule M of R, the division $\mathbf{F}$-algebra D_M is $\mathbf{F}$ itself. Therefore the factor ring R_M is the matrix $\mathbf{F}$-algebra $\text{Hom}_{\mathbf{F}}(M, M)$.

Corollary 0.3 (Wedderburn – Artin)**.** *For every algebraically closed field* $\mathbf{F}$*, for every semisimple* $\mathbf{F}$*-algebra with finite* $\mathbf{F}$*-dimension, for every collection* $\{M_i\}_{i\in I}$ *of representatives of the distinct isomorphism classes of simple, right* R*-submodules of* R*, the induced homomorphism of associative, unital* $\mathbf{F}$*-algebras is an isomorphism,*

$$R \xrightarrow{\cong} \prod_{i\in I} Hom_{\mathbf{F}-mod}(M_i, M_i).$$

Finally, Maschke's Theorem gives the following.

Theorem 0.4 (Maschke)**.** *For every field* $\mathbf{F}$*, for every finite group* Γ *such that the integer* $\#\Gamma$ *is invertible in* $\mathbf{F}$*, the associated* $\mathbf{F}$*-algebra is semisimple; namely,* $\mathbf{F}[\Gamma] := \bigoplus_{g\in\Gamma} \mathbf{F}\mathbf{b}_g$ *with* $\mathbf{F}$*-bilinear multiplication extending the rule* $(\mathbf{b}_g, \mathbf{b}_h) \mapsto \mathbf{b}_{g\cdot h}$*.*

Putting the pieces together gives the following.

Corollary 0.5. *For every algebraically closed field* $\mathbf{F}$*, for every finite group* Γ *such that* $\#\Gamma$ *is invertible in* $\mathbf{F}$*, for every finite set* $\{(V_i, \rho_i)\}_{i\in I}$ *of irreducible* $\mathbf{F}$*-linear* Γ*-subrepresentations of* $\mathbf{F}[\Gamma]$ *representing (uniquely) every isomorphism class, the induced homomorphism* $\mathbf{F}[\Gamma] \to \prod_{i\in I} Hom_{\mathbf{F}}(V_i, V_i)$ *is an isomorphism of associative, unital* $\mathbf{F}$*-algebras and of* $\mathbf{F}$*-linear* Γ*-representations. In particular,* $\#I$ *equals the* $\mathbf{F}$*-dimension of the center of* $\mathbf{F}[\Gamma]$*, i.e., the number* $\#\Gamma/inner(\Gamma)$ *of conjugacy classes of* Γ*. Also, the sum* $\sum_{i\in I} (dim_{\mathbf{F}}(V_i))^2$ *equals the* $\mathbf{F}$*-dimension of* $\mathbf{F}[\Gamma]$*, i.e.,* $\#\Gamma$*.*

Problem 1. Check that the $\mathbf{F}$-bilinear operation on $\mathbf{F}[\Gamma]$ defined above is associative, and thus $(\mathbf{F}[\Gamma], *)$ is an $\mathbf{F}$-associative algebra. Moreover, for the identity element e of the group Γ, check that $\mathbf{b}_e$ is a multiplicative identity in $\mathbf{F}[\Gamma]$.

Problem 2. Check that the center of $\mathbf{F}[\Gamma]$ is the $\mathbf{F}$-vector subspace $\text{Class}(\Gamma, \mathbf{F})$ of all elements $\sum_{g\in\Gamma} \alpha_g \mathbf{b}_g$ such that the coefficients α_g are constant on conjugacy classes.

Problem 3. Prove that for every $\gamma \in \Gamma$, the element $\mathbf{b}_\gamma$ is a (left-right) multiplicatively invertible element of $\mathbf{F}[\Gamma]$, i.e., an element of the multiplicative group $\mathbf{F}[\Gamma]^\times$ of (left-right) multiplicatively invertible elements. Check that the induced set map,

$$\mathbf{b}^\Gamma : \Gamma \to \mathbf{F}[\Gamma]^\times, \quad \gamma \mapsto \mathbf{b}_\gamma,$$

is a morphism of groups.

Problem 4. Conversely, for every $\mathbf{F}$-associative algebra $(A, \cdot)$, for every morphism of groups to the multiplicative group $A^\times$ of $(A, \cdot)$,

$$\rho : \Gamma \to A^\times,$$

prove that there is a unique morphism of **F**-associative unital algebras,

$$\widetilde{\rho} : (\mathbf{F}[\Gamma], *) \to (A, \cdot),$$

such that $\widetilde{\rho} \circ \mathbf{b}^\Gamma$ equals ρ.

Problem 5. For every finite dimensional **F**-vector space V and for every **F**-linear Γ-representation,

$$\rho : \Gamma \to \mathbf{GL}(V, \mathbf{F}),$$

conclude that there exists a unique morphism of **F**-associative unital algebras,

$$\widetilde{\rho} : (\mathbf{F}[\Gamma], *) \to \mathrm{Hom}_{\mathbf{F}}(V, V),$$

such that $\widetilde{\rho} \circ \mathbf{b}^\Gamma$ equals ρ. Conclude that finite dimensional **F**-linear Γ-representations are equivalent to left $\mathbf{F}[\Gamma]$-modules having finite dimension as an **F**-vector space.

Problem 6. For every morphism of groups,

$$\psi : \Gamma \to \Delta,$$

prove that there exists a unique morphism of **F**-associative unital algebras,

$$\mathbf{F}[\psi] : \mathbf{F}[\Gamma] \to \mathbf{F}[\Delta],$$

such that $\mathbf{F}[\psi] \circ \mathbf{b}^\Gamma$ equals $\mathbf{b}^\Delta \circ \psi$. Thus, the rule $\psi \mapsto \mathbf{F}[\psi]$ sends compositions to compositions and identity morphisms to identity morphisms. Also, the composition of $\mathbf{F}[\psi]$ with each **F**-linear representation,

$$\sigma : \Delta \to \mathbf{GL}(V, \mathbf{F}),$$

is a **F**-linear representation of Γ,

$$\sigma \circ \psi : \Gamma \to \mathbf{GL}(V, \mathbf{F}),$$

sometimes called the **restriction representation** (typically only when ψ is injective).

Altogether, this defines a covariant functor from the category of groups to the category of **F**-associative unital algebras sending every group Γ to the **F**-associative unital algebra $\mathbf{F}[\Gamma]$ and sending every morphism of groups ψ to the morphism of **F**-associative unital algebras $\mathbf{F}[\psi]$.

By the theorems above for the algebraically closed field $\mathbb{C}$ of characteristic 0, for every finite group Γ, for every finite collection $\{(V_i, \rho_i)\}_{i \in I}$ of representatives of the distinct isomorphism class of irreducible $\mathbb{C}$-linear Γ-representations, the associated homomorphism is an isomorphism,

$$\mathbb{C}[\Gamma] \xrightarrow{\cong} \prod_{i \in I} \mathrm{Hom}_{\mathbb{C}}(V_i, V_i).$$

Of course $\mathrm{Hom}_{\mathbb{C}-\mathrm{mod}}(M_i, M_i)$ is isomorphic as a $\mathbb{C}$-algebra to the algebra of $n_i \times n_i$-matrices with entries in $\mathbb{C}$, where n_i is the $\mathbb{C}$-dimension of V_i.

Problem 7. (**Hopf algebra structure on the group algebra**.) The **trace** (sometimes called the **counit**) of $\mathbb{C}[\Gamma]$ is defined to be

$$\mathrm{Tr}_\Gamma : \mathbb{C}[\Gamma] \to \mathbb{C}, \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g \in \Gamma} z_g.$$

The **comultiplication** is defined to be

$$\Delta_\Gamma : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma], \quad \sum_{g \in \Gamma} z_g \mathbf{b}_g \mapsto \sum_{g\in\Gamma} z_g(\mathbf{b}_g \otimes \mathbf{b}_g).$$

The **antipode** is defined to be

$$S_\Gamma : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma], \quad \sum_{g\in\Gamma} z_g \mathbf{b}_g \mapsto \sum_{g\in\Gamma} z_g \mathbf{b}_{g^{-1}}.$$

Check that these operations (together with the usual unital, associative $\mathbb{C}$-algebra operations above) make $\mathbb{C}[\Gamma]$ into a *Hopf* $\mathbb{C}$*-algebra*. Precisely, check all of the following.

(a) The comultiplication is coassociative, i.e., the following two compositions are equal,

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_\Gamma} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\Delta_\Gamma \otimes \mathrm{Id}} (\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]) \otimes_{\mathbb{C}} \mathbb{C}[\Gamma],$$

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_\Gamma} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Id} \otimes \Delta_\Gamma} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} (\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]).$$

(b) The counit is a left-right coidentity, i.e., the following two compositions both equal the identity map,

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_\Gamma} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Tr}_\Gamma \otimes \mathrm{Id}} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] = \mathbb{C}[\Gamma],$$

$$\mathbb{C}[\Gamma] \xrightarrow{\Delta_\Gamma} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \xrightarrow{\mathrm{Id} \otimes \mathrm{Tr}_\Gamma} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[\Gamma].$$

(c) The unital, associative $\mathbb{C}$-algebra structure and the counital, coassociative $\mathbb{C}$-coalgebra structure satisfy the axioms of a **bialgebra**, i.e., each of the following diagram commute.

$$\begin{array}{ccc}
\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{\Delta_\Gamma \circ (-*-)} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \\
{\scriptstyle \Delta \otimes \Delta_\Gamma \otimes \Delta_\Gamma} \big\downarrow & & \big\uparrow {\scriptstyle (-*-) \otimes (-*-)} \\
\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow[\mathrm{pr}_1 \otimes \mathrm{pr}_3 \otimes \mathrm{pr}_2 \otimes \mathrm{pr}_4]{} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]
\end{array} .$$

$$\begin{array}{ccc}
\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] & \xrightarrow{(-*-)} & \mathbb{C}[\Gamma] \\
{\scriptstyle \mathrm{Tr}_\Gamma \otimes \mathrm{Tr}_\Gamma} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{Tr}_\Gamma} \\
\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} & \xrightarrow[\cong]{} & \mathbb{C}
\end{array}$$

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\cong} & \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \\
{\scriptstyle \mathbf{b}_e} \big\downarrow & & \big\downarrow {\scriptstyle \mathbf{b}_e \otimes \mathbf{b}_e} \\
\mathbb{C}[\Gamma] & \xrightarrow[\Delta_\Gamma]{} & \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma]
\end{array} .$$

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\mathrm{Id}} & \mathbb{C} \\
{\scriptstyle \mathbf{b}_e} \big\downarrow & & \big\uparrow {\scriptstyle \mathrm{Tr}_\Gamma} \\
\mathbb{C}[\Gamma] & \xrightarrow[\mathrm{Id}]{} & \mathbb{C}[\Gamma]
\end{array}$$

(d) The antipode S satisfies the axioms of a **Hopf algebra**, i.e., the following diagram commutes.

$$\begin{array}{ccc}
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] & \xrightarrow{S_\Gamma\otimes \mathrm{Id}} & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \\
{\scriptstyle\Delta_\Gamma}\uparrow & & \downarrow{\scriptstyle -*-} \\
\mathbb{C}[\Gamma] & \xrightarrow{\mathrm{Tr}_\Gamma(-)\mathbf{b}_e} & \mathbb{C}[\Gamma] \\
{\scriptstyle\Delta_\Gamma}\downarrow & & \uparrow{\scriptstyle -*-} \\
\mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] & \xrightarrow{\mathrm{Id}\otimes S_\Gamma} & \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma]
\end{array} .$$

(e) For every pair (U,σ) and (V,ρ) of left modules over a Hopf $\mathbb{C}$-algebra R, for every element $r\in R$ with

$$\Delta_R(t) = \sum_\alpha s_\alpha \otimes r_\alpha,$$

there is an associated left R-module structure on $U\otimes_{\mathbb{C}} V$ defined by

$$(\sigma\otimes\rho)(t)\cdot(u\otimes v) := \sum_\alpha (\sigma(s_\alpha)\cdot u)\otimes(\rho(r_\alpha)\cdot v).$$

Check that for the comultiplication Δ_Γ defined above, this equals the structure $\sigma\otimes\rho$ of Γ-representation on $U\otimes_{\mathbb{C}} V$ as defined in lecture. Also, check that the trivial representation (i.e., the left-right identity for the tensor product operation on $\mathbb{C}$-linear left Γ-representations) is the unique representation such that the associated trace on $\mathbb{C}[\Gamma]$ equals Tr_Γ.

(f) Similarly, for every left module (V,ρ) over R, define a left R-module on the dual $\mathbb{C}$-vector space $V^\vee$ of $\mathbb{C}$-linear functional χ on V by

$$(t\cdot\chi)(v) := \chi(S(t)\cdot v).$$

Check that for the antipode S_Γ defined above, this equals the structure of Γ-representation on $V^\vee$ as defined in lecture. Thus, the "extra structures" on $\mathbb{C}$-linear Γ-representations are explained by the Hopf algebra structure on $\mathbb{C}[\Gamma]$. Conversely, these extra structures uniquely determine the Hopf algebra structures Δ_Γ, Tr_Γ, and S_Γ on the group $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$.

(g) Finally, check that the comultiplication is cocommutative, i.e., Δ equals its postcomposition with the involution

$$\mathrm{pr}_2\otimes\mathrm{pr}_1 : \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]\otimes_{\mathbb{C}}\mathbb{C}[\Gamma], \quad a_1\otimes a_2 \mapsto a_2\otimes a_1.$$

For every Hopf $\mathbb{C}$-algebra that is cocommutative and that is finite dimensional as a $\mathbb{C}$-vector space, there is an associated group Γ consisting of all **grouplike** elements b such that $\Delta(b)$ equals $b\otimes b$. Moreover, the Hopf $\mathbb{C}$-algebra is canonically isomorphic, as a Hopf $\mathbb{C}$-algebra, to the group $\mathbb{C}$-algebra of the group Γ of grouplike elements.

In particular, we can recover the finite group Γ from the structure of the group $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$ as a Hopf algebra. Using the next exercise, we can recover this from the $\mathbb{C}$-linear category of finite-dimensional $\mathbb{C}$-linear representations of Γ together with the extra structures of tensor products and duals of such representations. This is a first instance of *Tannaka(-Krein) duality*.

Problem 8. (**A universal property of the group algebra as a representation.**) As in the previous problem, let $\mathbb{C}[\Gamma]$ be the group $\mathbb{C}$-algebra of a finite group Γ. Give $\mathbb{C}[\Gamma]$ its natural structure of $\mathbb{C}$-linear (left) G-representation, i.e., $(g, \mathbf{b}_h) \mapsto \mathbf{b}_{g \cdot h}$.

(a) Prove the following claim from lecture. For every $\mathbb{C}$-linear Γ-representation (V, ρ), the following $\mathbb{C}$-linear map is an isomorphism,

$$\mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}(\mathbb{C}[\Gamma], (V, \rho)) \to V, \quad (T : \mathbb{C}[\Gamma] \to V) \mapsto T(\mathbf{b}_e).$$

Also, show that this isomorphism is natural in (V, ρ). Stated in terms of category theory, there is a **fiber functor**,

$$F : \mathrm{Rep}_\Gamma^{\mathbb{C}} \to \mathbb{C} - \mathrm{Vect}, \quad (V, \rho) \mapsto V, \quad \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}((U, \sigma), (V, \rho)) \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(U, V).$$

This is a covariant functor, and it is represented by $\mathbb{C}[\Gamma]$.

(b) Invert the isomorphism above to get a $\mathbb{C}$-linear map,

$$V \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}(\mathbb{C}[\Gamma], (V, \rho)) \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[\Gamma], V).$$

Use adjointness of Hom and tensor product to obtain an associated $\mathbb{C}$-linear map,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} V \to V.$$

Prove that this $\mathbb{C}$-linear map is a morphism of $\mathbb{C}$-linear G-representations for the following structures of $\mathbb{C}$-linear G-representation,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} (V, \mathrm{triv}) \to (V, \rho).$$

(c) Apply this in the special case that (V, ρ) equals $\mathbb{C}[\Gamma]$ itself, and deduce that the $\mathbb{C}$-linear map,

$$\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma],$$

is the usual $\mathbb{C}$-algebra multiplication on the group $\mathbb{C}$-algebra. Thus, as a $\mathbb{C}$-linear Γ-representation, the group $\mathbb{C}$-algebra is determined up to unique isomorphism by its universal property above, and the tensor and Hom operations even recover the structure of associative, unital $\mathbb{C}$-algebra on $\mathbb{C}[\Gamma]$.

(d) By Maschke's Theorem, every finite dimensional $\mathbb{C}$-linear Γ-representation is completely reducible. Combine this with Schur's Lemma and (a) above to conclude that every finite dimensional, irreducible, $\mathbb{C}$-linear Γ-representation (V_i, ρ_i) is isomorphic to a $\mathbb{C}$-linear subrepresentation of $\mathbb{C}[\Gamma]$ (explaining why we defined I as we did).

(e) Use the morphism of $\mathbb{C}$-linear Γ-representations from (c) to conclude that right multiplication of $\mathbb{C}[\Gamma]$ on itself (the "right regular representation") gives an isomorphism of unital, associative $\mathbb{C}$-algebras,

$$\mathbb{C}[\Gamma]^{\mathrm{opp}} \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}(\mathbb{C}[\Gamma], \mathbb{C}[\Gamma]).$$

Here, for every unital, associative $\mathbb{C}$-algebra A, the **opposite algebra** A^{opp} is the same $\mathbb{C}$-vector space but with multiplication defined by $a \bullet b := ba$. The antipode S_Γ is a $\mathbb{C}$-algebra antihomomorphism that is an involution, so that the Hopf $\mathbb{C}$-algebra $\mathbb{C}[\Gamma]$ is even involutive.

(f) Combine the isomorphism in (e) with the previous exercise and the Wedderburn – Artin Theorem to conclude that the number $\#I$ of isomorphism classes $[(V_i, \rho_i)]$ of irreducible $\mathbb{C}$-linear Γ-representations equals the number of conjugacy classes in

Γ, that every (composite) $\mathbb{C}$-bilinear pairing below is a perfect pairing of $\mathbb{C}$-vector spaces,

$$H_i \times V_i \to \mathbb{C}[\Gamma] \xrightarrow{\text{Tr}_\Gamma} \mathbb{C}, \quad H_i := \text{Hom}_{\mathbb{C}[\Gamma]}((V_i, \rho_i), \mathbb{C}[\Gamma]),$$

that each multiplicity m_i of (V_i, ρ_i) in the $\mathbb{C}$-linear Γ-representation $\mathbb{C}[\Gamma]$ equals the $\mathbb{C}$-vector space dimension $n_i = \dim_{\mathbb{C}}(V_i)$, and that there is an equality of positive integers,

$$\#\Gamma = \sum_{i \in I} n_i^2.$$

Problem 9. (**Central idempotents of the group algebra give the irreducible representations**.) Inside the center $Z(\mathbb{C}[\Gamma])$ considered as a $\mathbb{C}$-algebra, an element z is a **central idempotent** if z^2 equals z. For each central idempotent z, the **annihilator** of z is

$$\text{Ann}(z) := \{b \in \mathbb{C}[\Gamma] | zb = 0\}.$$

(a) With respect to the $\mathbb{C}$-algebra isomorphism

$$Z(\widetilde{a}) : Z(\mathbb{C}[\Gamma]) = Z(\mathbb{C}[\Gamma]^{\text{opp}}) \xrightarrow{\cong} \prod_{i \in I} \mathbb{C} \cdot \text{Id}_{V_i},$$

check that the idempotent elements correspond to those elements $(a_i \text{Id}_{V_i})_{i \in I}$ such that every a_i^2 equals a_i, i.e., such that a_i equals 1 or 0.

(b) For a central idempotent z, define the **support**, $\text{supp}(z)$, to be the subset of I such that a_i equals 1. Check that the annihilator of z maps isomorphically to the left-right ideal,

$$\{(T_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_{\mathbb{C}}(V_i, V_i) | \forall i \in \text{supp}(z),\ T_i = 0\}.$$

(c) In particular, check that the unique nonzero central idempotents with codimension-1 annihilator in $Z(\mathbb{C}[\Gamma])$ are the **primitive idempotents** $\mathbf{e}_i$ for each $i \in I$,

$$\mathbf{e}_i \mapsto (a_j \text{Id}_{H_j})_{j \in I}, \quad a_i = 1, \quad a_j = 0, \forall j \neq i.$$

Moreover, for each $i \in I$, check that the common annihilator in $\mathbb{C}[\Gamma]$ of $\mathbf{e}_j$ for all $j \neq i$ maps isomorphically to the left-right ideal that is the factor $\text{Hom}_{\mathbb{C}}(H_i, H_i)$. As a $\mathbb{C}$-linear left Γ-representation, this $\mathbb{C}$-algebra is a direct sum of n_i copies of the irreducible representation (V_i, ρ_i). Thus, we can construct the irreducible, $\mathbb{C}$-linear, Γ-representations from the full list of idempotents in $Z(\mathbb{C}[\Gamma])$ having codimension-1 annihilator in $Z(\mathbb{C}[\Gamma])$ (slightly indirectly, since we have to extract one of the irreducible factors in this direct sum of n_i copies of that factor).

Problem 10. (**Schur's Orthogonality Relations and idempotents in the group algebra**.) This exercise reconstructs the primitive idempotents (and thus the irreducible representations) in the group algebra from the information of the irreducible characters. The key is a natural Hermitian inner product on the center of the group algebra, together with Schur's Orthogonality Relations.

(a) For every $\mathbb{C}$-linear Γ-representation (V, ρ) and for every conjugacy class C in Γ, check that the following $\mathbb{C}$-linear operator on V is a morphism of $\mathbb{C}$-linear Γ-representations,

$$\sum_{g \in C} \rho(g) \in \text{Hom}_{\mathbb{C}}(V, V).$$

(b) If (V,ρ) is a finite dimensional, irreducible $\mathbb{C}$-linear Γ-representation, use Schur's Lemma to conclude that this morphism is a multiple of the identity, say $\rho_C \text{Id}_V$. Taking traces, deduce the identity,

$$\rho_C \cdot \dim_{\mathbb{C}}(V) = \sum_{g\in C} \text{Tr}_V(\rho(g)).$$

(c) Similarly, conclude that the image of the following $\mathbb{C}$-linear operator is contained in the $\mathbb{C}$-linear Γ-subrepresentation of invariant elements,

$$\sum_{g\in\Gamma} \rho(g) \in \text{Hom}_{\mathbb{C}}(V,V).$$

If (V,ρ) is a finite dimensional representation whose invariant subspace is zero, conclude that

$$\sum_{C} \rho_C = 0, \quad \text{i.e.,} \ \sum_{g\in\Gamma} \text{Tr}_V(\rho(g)) = 0.$$

In particular, this holds if (V,ρ) is a nontrivial irreducible representation. Conversely, if (V,ρ) is a trivial representation (of arbitrary finite dimension), show that the sum equals $\dim_{\mathbb{C}}(V)\#\Gamma$. Thus, since trace is additive for direct sum decomposition, for a general (V,ρ), conclude the identity

$$\frac{1}{\#\Gamma}\sum_{g\in\Gamma} \text{Tr}_V(\rho(g)) = \dim_{\mathbb{C}}(V^{\Gamma}).$$

(d) For every finite dimensional, $\mathbb{C}$-linear, (left) Γ-representation (V,ρ), the **character** of this representation is the class function,

$$\chi_{(V,\rho)} : \Gamma \to \mathbb{C}, \quad g \mapsto \text{Tr}_V(\rho(g)).$$

Every $\rho(g)$ is diagonalizable with eigenvalues ζ that are roots of unity with $\zeta^{-1} = \overline{\zeta}$. Conclude the identity

$$\chi_{(V^\vee,\rho^\vee)}(g) = \chi_{(V,\rho)}(g^{-1}) = \overline{\chi_{(V,\rho)}(g)}.$$

Similarly, for representations (U,σ) and (V,ρ), use the tensor product of the eigendecompositions for $\sigma(g)$, respectively $\rho(g)$, to prove the identity,

$$\chi_{(U\otimes V,\sigma\otimes\rho)}(g) = \chi_{(U,\sigma)}(g)\chi_{(V,\rho)}(g).$$

Consequently, conclude the identity,

$$\chi_{\text{Hom}_{\mathbb{C}}(U,V)}(g) = \overline{\chi_{(U,\sigma)}(g)}\chi_{(V,\rho)}(g).$$

Sum over g and use (c) to deduce the identity,

$$\dim_{\mathbb{C}}\text{Hom}_{\text{Rep}_\Gamma^{\mathbb{C}}}((U,\sigma),(V,\rho)) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma} \overline{\chi_{(U,\sigma)}(g)} \cdot \chi_{(V,\rho)}(g).$$

(e) Now assume that (V,ρ) is irreducible. For every class function,

$$\alpha : \Gamma \to \mathbb{C},$$

with **associated central element**,

$$\frac{1}{\#\Gamma}\sum_{g\in\Gamma} \alpha(g)\mathbf{b}_{g^{-1}},$$

conclude that the associated $\mathbb{C}$-linear operator on V,

$$\sum_{g\in\Gamma} \alpha(g)\rho(g^{-1}),$$

equals $\lambda \mathrm{Id}_V$ where λ satisfies the identity

$$\lambda \dim_{\mathbb{C}}(V) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\chi_{(V,\rho)}(g^{-1}) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\overline{\chi_{(V,\rho)}(g)}.$$

In particular, this central element annihilates the primitive idempotent corresponding to (V,ρ) if and only if the class function α is orthogonal to the class function $\chi_{(V,\rho)}$ with respect to the Hermitian inner product on the $\mathbb{C}$-vector space of class functions defined by

$$\langle \alpha, \beta\rangle := \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\alpha(g)\overline{\beta(g)}.$$

(f) Let (U,σ) and (V,ρ) be finite dimensional, $\mathbb{C}$-linear, (left) Γ-representations. Reinterpret (d) as an identity,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Rep}_\Gamma^{\mathbb{C}}}((U,\sigma),(V,\rho)) = \langle \chi_{(V,\rho)}, \chi_{(U,\sigma)}\rangle.$$

In particular, if (U,σ) and (V,ρ) are irreducible representations, conclude that $\langle \chi_{(V,\rho)}, \chi_{(U,\sigma)}\rangle$ equals 0 unless the representations are isomorphic, in which case $\langle \chi_{(V,\rho)}, \chi_{(U,\sigma)}\rangle$ equals 1. Thus, the characters of irreducible representations form an orthonormal subset of the $\mathbb{C}$-vector space of $\mathbb{C}$-valued class functions with respect to the Hermitian inner product defined above.

(g) Finally, since the dimension of the space of class functions equals the dimension of $Z(\mathbb{C}[\Gamma])$, and since this equals the number $\#I$ of isomorphism classes of irreducible representations, conclude that the characters of irreducible representations form an orthonormal basis for the $\mathbb{C}$-vector space of $\mathbb{C}$-valued class functions with respect to the Hermitian inner product defined above. Altogether this is the **Schur orthogonality relations** (sometimes also attributed to Frobenius).

(h) Deduce that for the irreducible representations (V_i,ρ_i) for $i\in I$ with character $\chi_i = \chi_{(V_i,\rho_i)}$, the corresponding central elements are the primitive idempotents,

$$\mathbf{e}_i := \frac{\chi_i(e)}{\#\Gamma}\sum_{g\in\Gamma}\chi_i(g)\mathbf{b}_{g^{-1}}.$$