MAT 543 Additional Notes on Categories and Functors

Contents

MAT 543 Representation Theory — **Jason Starr**

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1 Introduction

These are additional notes on categories and functors for this course. Some of the notes are cut-and-pasted from previous courses I taught about basic algebraic objects (semigroups, monoids, groups, acts and actions, associative rings, commutative rings, and modules), elementary language of category theory, and adjoint pairs of functors. Much of the notes are exercises working through the basic results about these definitions.

2 Algebraic Objects

Definition 2.1. A **semigroup** is a pair (G, m) of a set G and a binary relation,

$$m : G \times G \to G,$$

such that m is associative, i.e., the following diagram commutes,

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{Id}_G} & G \times G \\ {\scriptstyle \mathrm{Id}_G \times m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array} .$$

The binary operation is equivalent to a set function,

$$L_\bullet : G \to \mathrm{Hom}_{\mathbf{Sets}}(G, G), \quad g \mapsto L_g,$$

such that for every $g, g' \in G$, the composition $L_g \circ L_{g'}$ equals $L_{m(g,g')}$, where $m(g, g')$ is defined to equal $L_g(g')$. When no confusion is likely, the element $m(g, g')$ is often denoted $g \cdot g'$.

For semigroups (G, m) and (G', m') a **semigroup morphism** from the first to the second is a set map

$$u : G \to G',$$

such that the following diagram commutes,

$$\begin{array}{ccc} G \times G & \xrightarrow{u \times u} & G' \times G' \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle m'} \\ G & \xrightarrow[u]{} & G' \end{array} .$$

The set of semigroup morphisms is denoted $\mathrm{Hom}_{\mathbf{Semigroups}}((G, m), (G', m'))$.

Definition 2.2. For a semigroup (G, m), an element e of G of is a **left identity element**, resp. **right identity element**, if for every $g \in G$, g equals $m(e, g)$, resp. g equals $m(e, g)$. An **identity element** is an element that is both a left identity element and a right identity element.

A **monoid** is a triple (G, m, e) where (G, m) is a semigroup and e is an identity element. For monoids (G, m, e) and (G', m', e') a **monoid morphism** from the first monoid to the second is a semigroup morphism that preserves identity elements. The set of monoid morphisms is denoted $\mathrm{Hom}_{\mathbf{Monoids}}((G, m, e), (G', m', e'))$.

Example 2.3. For every semigroup (G, m), the **opposite semigroup** is (G, m^{opp}), where $m^{\mathrm{opp}}(g, g')$ is defined to equal $m(g', g)$ for every $(g, g') \in G \times G$. A left identity element of a semigroup is equivalent to a right identity element of the opposite semigroup. In particular, the opposite semigroup of a monoid is again a monoid.

Example 2.4. For every set I and for every collection $(G_\alpha, m_\alpha)_{\alpha \in I}$ of semigroups, for the Cartesian product set $G := \prod_{\alpha \in I} G_\alpha$ with its projections,

$$\mathrm{pr}_\alpha : G \to G_\alpha,$$

there exists a unique semigroup operation m on G such that every projection is a morphism of semigroups. Indeed, for every α, the composition

$$\mathrm{pr}_\alpha \circ m : G \times G \to G_\alpha$$

equals $m_\alpha \circ (\mathrm{pr}_\alpha \times \mathrm{pr}_\alpha)$. There exists an identity element e of (G, m) if and only if there exists an identity element e_α of (G_α, m_α) for every α, in which case e is the unique element such that $\mathrm{pr}_\alpha(e)$ equals e_α for every $\alpha \in I$.

Example 2.5. For every set S, the set $\mathrm{Hom}_{\mathbf{Sets}}(S, S)$ of set maps from S to itself has a structure of monoid where the semigroup operation is set composition, $(f, g) \mapsto f \circ g$, and where the identity element of the monoid is the identity function on S. For every semigroup (G, m), a **left act** of (G, m) on S is a semigroup morphism

$$\rho : (G, m) \to (\mathrm{Hom}_{\mathbf{Sets}}(S, S), \circ).$$

For every ordered pair $((S, \rho), (T, \pi))$ of sets with left G-acts, a **left G-equivariant map** from (S, ρ) to (T, π) is a set function $u : S \to T$ such that $u(\rho(g)s)$ equals $\pi(g)u(s)$ for every $g \in G$ and for every $s \in S$.

For each set S, a **right act** of G on S is a semigroup morphism ρ from (G, m) to the opposite semigroup of $\mathrm{Hom}_{\mathbf{Sets}}(S, S)$. Note, this is equivalent to a left act of the opposite semigroup $G^{\mathbf{opp}}$ on S. For every ordered pair $((S, \rho), (T, \pi))$ of sets with a right G-act, a **right G-equivariant map** is a set function $u : S \to T$ such that $u(s\rho(g))$ equals $u(s)\pi(g)$ for every $g \in G$ and for every $s \in S$. Note, this is equivalent to a left G^{opp}-equivariant map.

For an ordered pair $((G, m), (H, n))$ of semigroups, for each set S, a $G - H$-act on S is an ordered pair (ρ, π) of a left G-act on S, ρ, and a right H-act on S, π, such that $(\rho(g)s)\pi(h)$ equals $\rho(g)(s\pi(h))$ for every $g \in G$, for every $h \in H$, and for every $s \in S$. This is equivalent to a left act on S by the product semigroup of G and H^{opp}. A $G - H$-equivariant map is a map that is left equivariant for the associated left act by $G \times H^{\mathrm{opp}}$.

For every monoid (G,m,e), a **left action** of (G,m,e) on S is a monoid morphism from (G,m,e) to $\text{Hom}_{\mathbf{Sets}}(S,S)$. There is a category $G-\mathbf{Sets}$ whose objects are pairs (S,ρ) of a set S and a left action of (G,m,e) on S, whose morphisms are left G-equivariant maps, and where composition is usual set function composition. A **right action** is a monoid morphism from (G,m,e) to the opposite monoid of $\text{Hom}_{\mathbf{Sets}}(S,S)$. There is a category $\mathbf{Sets}-G$ whose objects are pairs (S,ρ) of a set S and a right action of (G,m,e) on S, whose morphisms are left G-equivariant maps, and where composition is usual set function composition. Finally, for every ordered pair $((G,m,e),(H,n,f))$ of monoids, a $G-H$-action on S is a $G-H$ act (ρ,π) such that each of ρ and π is an action. There is a category $G-H-\mathbf{Sets}$ whose objects are sets together with a $G-H$-action, whose morphisms are $G-H$-equivariant maps, and where composition is usual set function composition.

Definition 2.6. A semigroup $(G,\cdot)$ is called **left cancellative**, resp. **right cancellative**, if for every f,g,h in G, if $f\cdot g$ equals $f\cdot h$, resp. if $g\cdot f$ equals $h\cdot f$, then g equals h. A semigroup is **cancellative** if it is both left cancellative and right cancellative. A semigroup is **commutative** if for every $f,g\in G$, $f\cdot g$ equals $g\cdot f$, i.e., the identity function from G to itself is a semigroup morphism from G to the opposite semigroup. For an element f of a monoid, a **left inverse**, resp. **right inverse**, is an element g such that $g\cdot f$ equals the identity, resp. such that $f\cdot g$ equals the identity. An **inverse** of f is an element that is both a left inverse and a right inverse. An element f is **invertible** if it has an inverse.

Definition 2.7. A **group** is a monoid such that every element is invertible. The map that associates to each element the (unique) inverse element is the **group inverse map**, $i:G\to G$. If the monoid operation is commutative, the group is **Abelian**. A monoid morphism between groups is a **group homomorphism**, and the set of monoid morphisms between two groups is denoted $\mathbf{Hom}_{\mathbf{Groups}}((G,m,e),(G',m',e'))$. If both groups happen to be Abelian, this is also denoted $\mathbf{Hom}_{\mathbb{Z}-\mathbf{mod}}((G,m,e),(G',m',e'))$. In this case, this set is itself naturally an Abelian group for the operation that associates to a pair (u,v) of group homomorphisms the group homomorphism $u\cdot v$ defined by $(u\cdot v)(g)=m'(u(g),v(g))$.

Definition 2.8. An **associative ring** is an ordered pair $((A,+,0),L_\bullet)$ of an Abelian group $(A,+,0)$ and a homomorphism of Abelian groups,

$$L_\bullet : A\to \text{Hom}_{\mathbb{Z}-\mathbf{mod}}(A,A),\quad a\mapsto (L_a:A\to A)$$

such that for every $a,a'\in A$, the composition $L_a\circ L_{a'}$ equals $L_{a\cdot a'}$, where $a\cdot a'$ denotes $L_a(a')$. The set map $L_\bullet$ is equivalent to a biadditive binary operation,

$$\cdot : A\times A\to A,\quad (a,a')=a\cdot a',$$

that is also associative, i.e., for every a,a',a'' in A, the element $(a\cdot a')\cdot a''$ equals $a\cdot(a'\cdot a'')$. In particular, $(A,\cdot)$ is a semigroup. For associative rings $(A,+,0,\cdot)$ and $(A',+',0',\cdot')$, a **ring homomorphism** from the first to the second is a set function that is simultaneously a morphism of Abelian groups from $(A,+,0)$ to $(A',+',0')$ and a morphism of semigroups from $(A,\cdot)$ to $(A',\cdot')$. For every associative ring $(A,+,0,\cdot)$, the **opposite ring** is $(A,+,0,\cdot^{\text{opp}})$.

Definition 2.9. An **associative, unital ring** is an associative ring such that the multiplication semigroup has an identity element, i.e., there exists a multiplicative identity. An **unital ring homomorphism** is a ring homomorphism that preserves multiplicative identities. For associative, unital rings $(A, +, 0, \cdot, 1)$ and $(A', +', 0', \cdot', 1')$, the set of unital ring homomorphisms from the first to the second is denoted $\text{Hom}_{\textbf{UnitalRings}}((A, +, 0, \cdot, 1), (A', +', 0', \cdot', 1'))$, or just $\text{Hom}_{\textbf{UnitalRings}}(A, A')$ if the identities and operations are understood. In particular, a **commutative, associative, unital ring** is an associative unital ring such that the multiplication monoid is commutative. The set of unital ring homomorphisms between two commutative, associative, unital rings is denoted $\text{Hom}_{\textbf{CommUnitalRings}}(A, A')$.

Definition 2.10. For every Abelian group $(F, +, 0)$, the Abelian group $\text{Hom}_{\mathbb{Z}\textbf{-mod}}(F, F)$ of group homomorphisms from the group to itself has a structure of associative, unital ring where the multiplication operation is composition, and where the identity element is the identity homomorphism. For every associative ring $(R, +, 0, \cdot)$, a (not necessarily unital) **left module structure** on F for the associative ring R is a morphism of associative rings from R to $\text{Hom}_{\mathbb{Z}\textbf{-mod}}(F, F)$. A (not necessarily unital) **right module structure** is a morphism of associative rings from R to the opposite ring of $\text{Hom}_{\mathbb{Z}\textbf{-mod}}(F, F)$. For every associative, unital ring $(R, +, 0, \cdot, 1)$, a (unital) **left module structure** on F for the associative unital ring R is a morphism of associative unital rings from R to $\text{Hom}_{\mathbb{Z}\textbf{-mod}}(F, F)$. A (unital) **right module structure** on F for R is a morphism of associative unital rings from R to the the oppsite ring of $\text{Hom}_{\mathbb{Z}\textbf{-mod}}(F, F)$. For every left module structure on F of R, the **opposite module** is the equivalent right module structure on F of the opposite ring of R.

For left R-modules F and F',

$$L_\bullet : R \to \text{Hom}_{\mathbb{Z}\textbf{-mod}}(F, F), \quad L'_\bullet : R \to \text{Hom}_{\mathbb{Z}\textbf{-mod}}(F', F'),$$

a **left R-module morphism** from F to F' is a group homomorphism,

$$\phi : (F, +, 0) \to (F', +', 0'),$$

such that for every $r \in R$, the following composition functions are equal,

$$\phi \circ L_r, L'_r \circ \phi : F \to F',$$

i.e., $\phi(r \cdot x)$ equals $r \cdot' \phi(x)$ for every $r \in R$ and for every $x \in F$. For right R-modules G and G', a **right R-module morphism** from G to G' is a left R^{opp}-module morphism from the opposite module G^{opp} to the opposite module $(G')^{\text{opp}}$.

3 Categories

Definition 3.1. A **category** $\mathcal{A}$ consists of (i) a "recognition principle" or "axiom list" (possibly depending on auxiliary sets) for determining whether a specified set a is an **object** of this category,

(ii) an assignment, for every ordered pair (a,a') of objects of $\mathcal{A}$, of a specified set $\mathrm{Hom}_{\mathcal{A}}(a,a')$, and
(iii) an assignment, for every ordered triple (a,a',a'') of objects of $\mathcal{A}$, of a specified set function

$$- \circ - : \mathrm{Hom}_{\mathcal{A}}(a',a'') \times \mathrm{Hom}_{\mathcal{A}}(a,a') \to \mathrm{Hom}_{\mathcal{A}}(a,a''), \quad (g,f) \mapsto g \circ f,$$

such that, for every object a of $\mathcal{A}$, there exists an element $\mathrm{Id}_a \in \mathrm{Hom}_{\mathcal{A}}(a,a)$ that is a left-right identity for $\circ$, and such that for every ordered 4-tuple (a,a',a'',a''') of objects of $\mathcal{A}$ and for every ordered triple

$$(g,f,e) \in \mathrm{Hom}_{\mathcal{A}}(a'',a''') \times \mathrm{Hom}_{\mathcal{A}}(a',a'') \times \mathrm{Hom}_{\mathcal{A}}(a,a'),$$

the elements $g \circ (f \circ e)$ and $(g \circ f) \circ e$ in $\mathrm{Hom}_{\mathcal{A}}(a,a''')$ are equal, i.e., $\circ$ is associative. The elements of $\mathrm{Hom}_{\mathcal{A}}(a,a')$ are **morphisms** from a to a' in $\mathcal{A}$. The set function $- \circ -$ is **composition** in $\mathcal{A}$.

Definition 3.2. For a category $\mathcal{A}$, for an ordered pair (a,a') of objects of $\mathcal{A}$, for an ordered pair of elements

$$(g,f) \in \mathrm{Hom}_{\mathcal{A}}(a,a') \times \mathrm{Hom}_{\mathcal{A}}(a',a),$$

if the composition $g \circ f \in \mathrm{Hom}_{\mathcal{A}}(a,a)$ equals Id_a, then g is a **left inverse** of f in $\mathcal{A}$ and f is a **right inverse** of g in $\mathcal{A}$. If g is both a left inverse of f and a right inverse of f, then g is an **inverse** of f in $\mathcal{A}$. An **isomorphism** in $\mathcal{A}$ is a morphism in $\mathcal{A}$ that has an inverse in $\mathcal{A}$.

Definition 3.3. For a category $\mathcal{A}$, an **initial object**, respectively a **terminal object** (or final object), is an object a such that for every object a', the set $\mathrm{Hom}_{\mathcal{A}}(a,a')$, resp. the set $\mathrm{Hom}_{\mathcal{A}}(a',a)$, is a singleton set. An object that is simultaneously an initial object and a terminal object is called a **zero object**.

Example 3.4. The category **Sets** has as objects all sets. For every ordered pair of sets, the associated set of morphisms in **Sets** is defined to be the set of all set functions from the first set to the second set. The composition in **Sets** is usual composition of functions. A set function has a left inverse, respectively a right inverse, an inverse, if and only if the set function is injective, resp. surjective, bijective. The empty set is an initial object. Every singleton set is a final object.

Example 3.5. For every category $\mathcal{A}$, for every object a of A, there is a monoid $H_a^a := \mathrm{Hom}_{\mathcal{A}}(a,a)$ whose semigroup operation is the categorical composition and whose monoid identity element is the categorical identity morphism of a. This is the $\mathcal{A}$**-monoid** of the object a. For every ordered pair (a,a') of objects of $\mathcal{A}$, the set $H_{a'}^a := \mathrm{Hom}_{\mathcal{A}}(a,a')$ the categorical composition defines a set map,

$$H_{a'}^{a'} \times H_{a'}^a \times H_a^a \to H_{a'}^a, \quad (u',f,u) \mapsto u' \circ f \circ u.$$

This is a $H_{a'}^{a'} - H_a^a$-action on $H_{a'}^a$. This is the $\mathcal{A}$**-action** of $H_{a'}^{a'} - H_a^a$ on $H_{a'}^a$. Finally, for every ordered triple (a,a',a'') of objects, the composition binary operation,

$$H_{a''}^{a'} \times H_{a'}^a \to H_{a''}^a,$$

is a $H_{a''}^{a''} - H_a^a$-equivariant map that is $H_{a'}^{a'}$-balanced, i.e., for every $u'' \in H_{a''}^{a''}$, for every $g \in H_{a''}^{a'}$, for every $u' \in H_{a'}^{a'}$, for every $f \in H_{a'}^a$, and for every $u \in H_a^a$, we have,

$$u'' \circ (g \circ f) = (u'' \circ g) \circ f, \quad (g \circ u') \circ f = g \circ (u' \circ f), \quad (g \circ f) \circ u = g \circ (f \circ u).$$

This is the $\mathcal{A}$**-equivariant binary operation**.

Example 3.6. For every monoid, there is a category with a single object whose unique categorical monoid is the specified monoid. Every category with a single object is (strictly) equivalent to such a category for a monoid (unique up to non-unique isomorphism).

Example 3.7. For every monoid (G, m, e), for every set S together with a left G-action ρ, there is an associated category, sometimes denoted $[(S,\rho)/G]$, whose objects are the elements of S, and such that for every ordered pair $(s, s') \in S \times S$ the set of morphisms is

$$G^s_{s'} := \{g \in G | \rho(g)s = s'\}.$$

For every ordered triple $(s, s', s'') \in S \times S \times S$, the semigroup operation defines a binary operation,

$$G^{s'}_{s''} \times G^s_{s'} \to G^s_{s''}, \quad (g', g) \mapsto g'g.$$

The morphism of an element $g \in G^s_{s'}$ is left invertible, respectively right invertible, invertible, in this category if and only if the element g of the monoid is left invertible, resp. right invertible, invertible. This category has an initial object if and only if the left G-action is left G-equivariantly isomorphic to the left regular representation of the monoid G on itself, in which case every invertible element is an initial object. For the left regular representation, the category has a final object if and only if the monoid is a group (every element is invertible), in which case every object is both initial and final.

Example 3.8. For every ordered pair of monoids (G, G'), for every ordered pair (M, M') where M is a set with a specified $G' - G$-action and where M is a set with a specified $G - G'$-action, for every ordered pair of biequivariant and balanced binary operations,

$$\circ_{M',M} : M' \times M \to G, \quad \circ_{M,M'} : G \times M' \to G',$$

that are associative, i.e., for all $f, f_1, f_2 \in M$ and for all $f', f'_1, f'_2 \in M'$,

$$(f_1 \circ_{M,M'} f') \cdot f_2 = f_1 \cdot (f' \circ_{M',M} f_2), \quad (f'_1 \circ_{M',M} f) \cdot f'_2 = f'_1 \cdot (f \circ_{M,M'} f'_2),$$

there is a category $\mathcal{A}$ with precisely two objects a and a' such that the categorical monoid G^a_a equals G, such that the categorical monoid $G^{a'}_{a'}$ equals G', such that the categorical $G'-G$-set $G^a_{a'}$ equals M, such that the categorical $G - G'$-set $G^{a'}_a$ equals M' and such that the composition binary relations are the specified binary operations $\circ_{M',M}$ and $\circ_{M,M'}$. Every category with precisely two objects is (strictly) equivalent to such a category for some datum as above, $(G, G', M, M', \circ_{M',m}, \circ_{M,M'})$.

Example 3.9. Continuing the previous example, let (S, S') be an ordered pair of sets, let

$$\rho : G \to \mathrm{Hom}_{\mathbf{Sets}}(S, S), \quad \rho' : G' \to \mathrm{Hom}_{\mathbf{Sets}}(S', S'),$$

be an ordered pair of left actions so that $\mathrm{Hom}_{\mathbf{Sets}}(S, S')$ has an induced $G' - G$ action and $\mathrm{Hom}_{\mathbf{Sets}}(S', S)$ has an induced $G - G'$ action. Let

$$\mu : M \to \mathrm{Hom}_{\mathbf{Sets}}(S, S'), \quad \mu' : M' \to \mathrm{Hom}_{\mathbf{Sets}}(S', S),$$

be an ordered pair of a $G' - G$ equivariant map and a $G - G'$ equivariant map that are compatible with the composition maps, i.e., for every $f \in M$ and for every $f' \in M'$,

$$\mu'(f') \circ \mu(f) = \rho(f' \circ_{M',M} f), \quad \mu(f) \circ \mu'(f') = \rho'(f \circ_{M,M'} f').$$

There is a category $[(S, S', \rho, \rho', \mu, \mu')/(G, G', M, M', \circ_{M',M}, \circ_{M,M'})]$ whose objects are elements s of S and elements s' of S', such that for every pair of elements $(s_1, s_2) \in S \times S$, resp. $(s_1', s_2') \in S' \times S'$, the morphisms are $G_{s_2}^{s_1}$, resp. $(G')_{s_2'}^{s_1'}$, as in $[(S,\rho)/G]$, resp. as in $[(S',\rho')/G']$, and such that for every $s \in S$ and for every $s' \in S'$, the morphisms from s to s', resp. the morphisms from s' to s, are those elements m of M with $\mu(m)s = s'$, resp. those elements m' of M' with $\mu'(m')s' = s$. The compositions are defined in the evident way.

Example 3.10. For every monoid M, for the associated category $\mathcal{A}$ with one object a whose monoid of self-morphisms equals M, the category $\mathrm{Hom}\mathcal{A}$ has objects (a, a, f) for every $f \in M$. For an ordered pair $(f, g) \in M \times M$, the set of morphisms from (a, a, f) to (a, a, g) equals the set of ordered pairs $(q, q') \in M \times M$ such that $g \cdot q$ equals $q' \cdot f$.

Example 3.11. For every semigroup (G, m), for every set S with a left G-act ρ on S, the identity function from S to itself is a left G-equivariant map from (S, ρ) to (S, ρ). Also, for every ordered triple $((S,\rho),(T,\pi),(U,\lambda))$ of sets with a left G-act, the composition of each left G-equivariant map from (S,ρ) to (T,π) with a left G-equivariant map from (T,π) to (U,λ) is a left G-equivariant map from (S,ρ) to (U,λ). Thus, there is a category $G - \mathbf{Act}$ whose objects are sets with a left G-act, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a left G-act, $\mathrm{Hom}_{G-\mathbf{Act}}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of left G-equivariant maps, and where composition is the usual set function composition. Similarly, there is a category $\mathbf{Act} - G$ whose objects are sets with a right G-act, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a right G-act, $\mathrm{Hom}_{\mathbf{Act}-G}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of right G-equivariant maps, and where composition is the usual set function composition. Finally, for every ordered pair $((G,m),(H,n))$ of semigroups, there is a category $G - H - \mathbf{Act}$ whose objects are sets S with a $G - H$-act, whose morphisms are $G - H$-equivariant maps, and where composition is usual set function composition.

Example 3.12. For every monoid (G, m, e), for every set S with a left G-action ρ on S, the identity function from S to itself is a left G-equivariant map from (S,ρ) to (S,ρ). Also, for every ordered triple $((S,\rho),(T,\pi),(U,\lambda))$ of sets with a left G-action, the composition of each left G-equivariant map from (S,ρ) to (T,π) with a left G-equivariant map from (T,π) to (U,λ) is a left G-equivariant map from (S,ρ) to (U,λ). Thus, there is a category $G - \mathbf{Sets}$ whose objects are sets with a left G-action, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a left G-action, $\mathrm{Hom}_{G-\mathbf{Sets}}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of left G-equivariant maps, and where composition is the usual set function composition. Similarly, there is a category $\mathbf{Sets} - G$ whose objects are sets with a right G-action, (S,ρ), where for every ordered pair $((S,\rho),(T,\pi))$ of sets with a right G-action, $\mathrm{Hom}_{\mathbf{Sets}-G}((S,\rho),(T,\pi))$ is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S,T)$ of right G-equivariant maps, and where composition is the usual set function composition. Finally, for every ordered pair $((G, m, e), (H, n, f))$ of monoids, there is a category $G - H - \mathbf{Sets}$ whose objects are sets S with a $G - H$-action, whose morphisms are $G - H$-equivariant maps, and where composition is usual set function composition.

Example 3.13. The category **Semigroups**, respectively **Monoids**, **Groups**, **Rings**, **UnitalRings**, **CommUnitalRings**, has as objects all semigroups, respectively all monoids (semigroups that have an identity element), all groups, all associative, unital rings, all associative, commutative, unital rings. For every ordered pair of objects, the set of morphisms in each of these categories is the set of all set maps between the objects that preserve the algebraic operations (and identity elements, when these are part of the structure). Composition is usual composition of set maps. In each of these categories, a morphism is an isomorphism if and only if it is a bijection, in which case the set-theory inverse of the bijection is also the inverse in the category. Each of these categories has a terminal object consisting of any object whose underlying point set is a singleton set. The trivial object is also an initial object, hence a zero object, in **Monoids** and **Groups**. The commutative, unital ring $\mathbb{Z}$ is an initial object in **UnitalRings** and **CommUnitalRings**.

Example 3.14. For every associative, unital ring A, the category $A - \mathbf{mod}$, resp. $\mathbf{mod} - A$, is the category whose objects are left A-modules, resp. right A-modules, and whose morphisms are homomorphisms of left A-modules, resp. of right A-modules. Composition is usual composition of set functions. The zero module is both an initial object and a terminal object, i.e., a zero object.

Definition 3.15. For a commutative, unital ring R, an $R-$**mod enriched category** is a category $\mathcal{A}$ together with a specified structure of (left-right) R-module on each set of morphisms such that each composition set map is R-bilinear.

Definition 3.16. For every category $\mathcal{A}$, the **arrow category of of $\mathcal{A}$** is the category $\mathcal{A}^{\to}$ whose objects are ordered triples (a_0, a_1, f) of objects a_0 and a_1 of $\mathcal{A}$ and an element $f \in \mathrm{Hom}_{\mathcal{A}}(a_0, a_1)$, such that for every ordered pair $((a_0, a_1, f), (a'_0, a'_1, f'))$ of objects the set of morphisms is

$$\mathrm{Hom}_{\mathcal{A}^{\to}}((a_0, a_1, f), (a'_0, a'_1, f')) = \{(q_0, q_1) \in \mathrm{Hom}_{\mathcal{A}}(a_0, a'_0) \times \mathrm{Hom}_{\mathcal{A}}(a_1, a'_1) | f' \circ q_0 = q_1 \circ f\},$$

and for every ordered triple of objects, $((a_0, a_1, f), (a'_0, a'_1, f'), (a''_0, a''_1, f''))$, for every morphism (q_0, q_1) from (a_0, a_1, f) to (a'_0, a'_1, f'), and for every morphism (q'_0, q'_1) from (a'_0, a'_1, f') to (a''_0, a''_1, f''), the composition $(q'_0, q'_1) \circ (q_0, q_1)$ is defined to be $(q'_0 \circ q_0, q'_1 \circ q_1)$.

Definition 3.17. For every category, the **opposite category** has the same objects, but the set of morphisms from a first object to a second object in the opposite category is defined to be the set of morphisms from the second object to the first object in the original category. With this definition, composition in the opposite category is defined to be composition in the original category, but in the opposite order. For every object, the associated categorical monoid of that object in the opposite category equals the opposite monoid of the categorical monoid in the original category. For every ordered pair of objects, the categorical biaction for the opposite category is the opposite biaction of the categorical biaction of the original category.

Example 3.18. For every commutative, unital ring R, for every category enriched over $R-\mathbf{mod}$, every categorical monoid has an associated structure of an associative, unital, central R-algebra such that the algebra product is the monoid operation. Conversely, for every central R-algebra, there is a category enriched over R-$\mathbf{mod}$ with precisely one object whose central R-algebra of self-morphisms is the specified central R-algebra. Also, for the opposite category enriched over $R-\mathbf{mod}$, every central R-algebra of self-morphisms of an object is the opposite central R-algebra of that in the original category.

Example 3.19. For every commutative, unital ring R, for every category $\mathcal{A}$ enriched over $R-\mathbf{mod}$, for every ordered pair (a,a') of objects of $\mathcal{A}$ with the associated central R-algebra structures on the monoids H_a^a and $H_{a'}^{a'}$, categorical composition defines an associated structure of R-central $H_{a'}^{a'}-H_a^a$-bimodule on $H_{a'}^a$ inducing the categorical $H_{a'}^{a'}-H_a^a$-action. Also, for every ordered triple (a,a',a'') of objects of $\mathcal{A}$, the composition binary operation defines an R-central $H_{a''}^{a''}-H_a^a$-bimodule homomorphism,

$$H_{a''}^{a'}\otimes_{H_{a'}^{a'}}H_{a'}^a\to H_{a''}^a.$$

Conversely, for every ordered pair of central R-algebras (H,H'), for every ordered pair (S,T) of an R-central $H'-H$ bimodule S, i.e., a left $H'\otimes_R H^{\mathrm{opp}}$-module, and an R-central $H-H'$ bimodule T, for every ordered pair of balanced bimodule homomorphisms,

$$\circ_{T,S}:T\otimes_{H'}S\to H,\quad \circ_{S,T}:S\otimes_H T\to H',$$

that are associative, there is a category $\mathcal{A}$ enriched over $R-\mathbf{mod}$ with precisely two objects a and a' such that the categorical central R-algebra H_a^a equals H, such that the categorical central R-algebra $H_{a'}^{a'}$ equals H', such that the categorical R-central $H'-H$ bimodule $H_{a'}^a$ equals S, such that categorical R-central $H-H'$ bimodule $H_a^{a'}$ equals T, and such that the composition binary operationss are the specified binary operations $\circ_{G,F}$ and $\circ_{F,G}$.

Also, for the opposite category enriched over $R-\mathbf{mod}$, the R-central algebras of self-morphisms of an object are replaced by their opposites, and the opposite of the R-central $H'-H$ bimodule structure on $H_{a'}^a$ is the categorical R-central $H^{\mathrm{opp}}-(H')^{\mathrm{opp}}$ bimodule structure of the opposite category.

Example 3.20. For every partially ordered set $(S,\le)$, there is a category whose objects are the elements of S, and such that for every ordered pair $(s,s')\in S\times S$, the set of morphisms is empty unless $s\le s'$, in which case the set of morphisms is a singleton set. There is a unique composition law consistent with these sets of morphisms. The opposite category is the category associated to the opposite partially ordered set $(S,\ge)$.

Definition 3.21. For a category $\mathcal{A}$, a **subcategory** of $\mathcal{A}$ is a category $\mathcal{B}$ such that every object of $\mathcal{B}$ is an object of $\mathcal{A}$, such that for every ordered pair (b,b') of objects of $\mathcal{B}$, the set $\mathrm{Hom}_{\mathcal{B}}(b,b')$ is a subset of $\mathrm{Hom}_{\mathcal{A}}(b,b')$, and such that for every ordered triple (b,b',b'') of objects of $\mathcal{B}$, the composition in $\mathcal{B}$ is the restriction of composition in $\mathcal{A}$. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is **full** if for every ordered pair (b,b') of objects of $\mathcal{B}$, the subset $\mathrm{Hom}_{\mathcal{B}}(b,b')$ equals all of $\mathrm{Hom}_{\mathcal{A}}(b,b')$.

Similarly, for a commutative, unital ring R and a category $\mathcal{A}$ enriched over $R-\mathbf{mod}$, an $R-\mathbf{mod}$ **enriched subcategory** is a subcategory $\mathcal{B}$ of $\mathcal{A}$ such that every subset $\mathrm{Hom}_{\mathcal{B}}(b,b')$ of $\mathrm{Hom}_{\mathcal{A}}(b,b')$ is an R-submodule.

Example 3.22. For every monoid M, for the associated category with one object whose categorical monoid equals M, the subcategories are precisely the categories with one object associated to the submonoids of M. For every commutative, unital ring R, for every R-central algebra A, for the associated category enriched over $R-\mathbf{mod}$ that has precisely one object whose categorical central R-algebra equals A, the $R-\mathbf{mod}$ enriched subcategories are precisely those associated to R-subalgebras of A. For every partially ordered set $(S,\le)$, the subcategories of the associated category are precisely the categories of pairs $(T,\le_T)$ of a subset T of S and a partial ordering $\le_T$ on T such that the inclusion map is order-preserving, $(T,\le_T)\to(S,\le)$. The subcategory is full if and only if $\le_T$ is the restriction of $\le$ to T.

4 Functors

Definition 4.1. For every pair of categories $\mathcal{A}$ and $\mathcal{B}$, a **covariant functor** F from $\mathcal{A}$ to $\mathcal{B}$ is defined to be a rule that associates to every object a of $\mathcal{A}$ an object $F(a)$ of $\mathcal{B}$ and that associates to every ordered pair of objects (a,a') of $\mathcal{A}$ a set map

$$F_{a,a'}:\mathrm{Hom}_{\mathcal{A}}(a,a')\to\mathrm{Hom}_{\mathcal{B}}(F(a),F(a')),$$

such that for every object a of $\mathcal{A}$, $F_{a,a}(\mathrm{Id}_a)$ equals $\mathrm{Id}_{F(a)}$, and such that for every triple of objects (a,a',a'') of $\mathcal{A}$,

$$F_{a,a''}(g\circ f)=F_{a',a''}(g)\circ F_{a,a'}(f),\quad \forall(g,f)\in\mathrm{Hom}_{\mathcal{A}}(a',a'')\times\mathrm{Hom}_{\mathcal{A}}(a,a').$$

The functor is **faithful**, resp. **fully faithful**, if every set map $F_{a,a'}$ is injective, resp. bijective. The functor is **essentially surjective** if every object of $\mathcal{B}$ is isomorphic to $F(a)$ for an object of $\mathcal{A}$. The functor is an **equivalence** if it is fully faithfuly and essentially surjective.

A **contravariant functor** from $\mathcal{A}$ to $\mathcal{B}$ is a covariant functor from the opposite category $\mathcal{A}^{\mathrm{opp}}$ to $\mathcal{B}$.

Definition 4.2. For every triple of categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, for every covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ and for every covariant functor G from $\mathcal{B}$ to $\mathcal{C}$, the **composition functor** $G\circ F$ from $\mathcal{A}$ to $\mathcal{C}$ is the covariant functor associating to every object a of $\mathcal{A}$ the object $G(F(a))$ of $\mathcal{C}$, and associating to every ordered pair of objects (a,a') of $\mathcal{A}$, the composition set map,

$$G_{F(a),F(a')}\circ F_{a,a'}:\mathrm{Hom}_{\mathcal{A}}(a,a')\to\mathrm{Hom}_{\mathcal{B}}(F(a),F(a'))\mathrm{Hom}_{\mathcal{C}}(G(F(a)),G(F(a'))).$$

For every category $\mathcal{A}$, the **identity functor** from $\mathcal{A}$ to $\mathcal{A}$ is the rule associating every object to itself, and sending each set of morphisms to itself by the identity set map.

Definition 4.3. For every triple of categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, for every pair of covariant functors, $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, the **comma category**, $F \downarrow G$, has as objects ordered triples (a, b, u) of an object a of $\mathcal{A}$, an object b of $\mathcal{B}$, and a $\mathcal{C}$-morphism $u : F(a) \to G(b)$. For an ordered pair of objects, $((a, b, u), (a', b', u'))$, a morphism in the comma category is an ordered pair (q, r) of $q \in \mathrm{Hom}_{\mathcal{A}}(a, a')$ and $r \in \mathrm{Hom}_{\mathcal{B}}(b, b')$ such that $u' \circ F(q)$ equals $G(r) \circ u$ in $\mathrm{Hom}_{\mathcal{C}}(F(a), G(b'))$. Composition is defined in the evident way. In particular, the arrow category of $\mathcal{C}$ is the comma category when $\mathcal{A}$ equals $\mathcal{B}$ equals $\mathcal{C}$ and each of F and G is the identity functor on $\mathcal{C}$. In general, there is a **domain functor** or **source functor**, $F \downarrow G \to \mathcal{A}$, associating to every object (a, b, u) the $\mathcal{A}$-object a and associating to every morphism (q, r) the $\mathcal{A}$-morphism q. There is also a **codomain functor** or **target functor**, $F \downarrow G \to \mathcal{B}$, associating to every object (a, b, u) the $\mathcal{B}$-object b and associating to every morphism (q, r) the $\mathcal{B}$-morphism r. Finally, there is an **arrow functor**, $F \downarrow G \to \mathcal{C}^{\to}$ associating to every object (a, b, u) the $\mathcal{C}^{\to}$-object $(F(a), G(b), u)$ and associating to every morphism (q, r) the $\mathcal{C}^{\to}$-morphism $(F(q), G(r))$.

Definition 4.4. For a category $\mathcal{A}$, a full subcategory is **skeletal** if every object of $\mathcal{A}$ is isomorphic to an object of the subcategory. If there exists a skeletal subcategory whose objects are indexed by a set, then $\mathcal{A}$ is a **small category**. If the objects of $\mathcal{A}$ form a set, then $\mathcal{A}$ is a **strictly small category**.

Example 4.5. Let **FinSets** be the full subcategory of **Sets** whose objects are the finite subsets. Let $\mathcal{B}$ be the full subcategory whose objects are the subsets $[1, n] = \{1, \ldots, n\}$ of $\mathbb{Z}_{\geq 1}$ for every integer $n \geq 0$. Then $\mathcal{B}$ is a strictly small category that is a skeletal subcategory of **FinSets**, but **FinSets** is not a strictly small category.

Example 4.6. For every partially ordered set $(S, \leq)$ and for every partially ordered set $(T, \leq)$, a functor from the associated category of $(S, \leq)$ to the associated category of $(T, \leq)$ is equivalent to a order-preserving function from S to T. Such a functor is always faithful. It is full if and only if the function is **strict**, i.e., for every $(s, s') \in S \times S$, the image pair $(t, t') \in T \times T$ satisfies $t \leq t'$ if and only if $s \leq s'$. The functor is essentially surjective if and only if the set function is surjective.

Example 4.7. For every pair of categories $\mathcal{A}$ and $\mathcal{B}$, for every covariant functor F from $\mathcal{A}$ to $\mathcal{B}$, the **opposite functor** F^{opp} from the opposite category $\mathcal{A}^{\mathrm{opp}}$ to the opposite category $\mathcal{B}^{\mathrm{opp}}$ associates to every object a of $\mathcal{A}^{\mathrm{opp}}$ the object $F(a)$ of $\mathcal{B}^{\mathrm{opp}}$, and associates to every ordered pair (a, a') of objects of $\mathcal{A}^{\mathrm{opp}}$ the set function $F_{a',a}$ of (a', a). For a triple of categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, for covariant functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$, the functor $(G \circ F)^{\mathrm{opp}}$ is the composition $G^{\mathrm{opp}} \circ F^{\mathrm{opp}}$, and the opposite functor of the identity functor is the identity functor of the opposite category. The opposite functor is faithful, respectively full, essentially surjective if and only if the original functor is faithful, resp. full, essentially surjective. Finally, the opposite functor of F^{opp} is the original functor F.

Example 4.8. For every set a, denote by $\mathcal{P}(a)$ the power set of a, i.e., the set whose elements are all subsets of a. For every set map $f : a \to a'$, define $\mathcal{P}_{a,a'}(f)$ to be the set map from $\mathcal{P}(a)$ to $\mathcal{P}(a')$ associating to every subset b of a the image subset $f(b)$ of a'. Similarly, define $\mathcal{P}^{a',a}(f)$ to be the set map from $\mathcal{P}(a')$ to $\mathcal{P}(a)$ that associates to every subset b' of b the preimage subset $f^{\mathrm{pre}}(b')$

of a. This defines a covariant functor $\mathcal{P}_*$ from **Sets** to itself and a contravariant functor $\mathcal{P}^*$ from **Sets** to itself. These functors preserve the full subcategory **FinSets**, but they do not preserve the skeletal subcategory $\mathcal{B}$.

Example 4.9. There is a forgetful functor from **Groups** to **Sets** that forgets the group structure. Similarly, there is a forgetful functor from $R-\mathbf{mod}$ to **Groups** that remembers only the additive group structure on the R-module. Similarly, there is a forgetful functor from **Rings** to $\mathbb{Z}$-**mod** that remembers only the additive group structure. There is a forgetful functor from **UnitalRings** to **Rings**. All of these are faithful functors. The category **CommUnitalRings** is a full subcategory of **UnitalRings**.

Example 4.10. For every ordered pair of monoids, the covariant functors between the associated categories are naturally equivalent to the morphisms of monoids. For every commutative, unital ring R, for every ordered pair of central R-algebras, the covariant functors between the associated categories that are R-linear on sets of morphisms are naturally equivalent to the R-algebra homomorphisms between these central R-algebras.

Definition 4.11. For every category $\mathcal{A}$ and for every object a of $\mathcal{A}$, the **Yoneda covariant functor** of a is the covariant functor,

$$h^a : \mathcal{A} \to \mathbf{Sets}, \quad h^a(a') = \mathrm{Hom}_{\mathcal{A}}(a, a').$$

For every ordered pair of objects (a', a''), for every morphism $g \in \mathrm{Hom}_{\mathcal{A}}(a', a'')$, and for every element $f \in h^a(a')$, i.e., for every morphism $f \in \mathrm{Hom}_{\mathcal{A}}(a, a')$, composition defines an element $g \circ f$ in $h^a(a'')$. This defines a set function,

$$h^a_{a',a''} : \mathrm{Hom}_{\mathcal{A}}(a', a'') \to \mathrm{Hom}_{\mathbf{Sets}}(h^a(a'), h^a(a'')), \quad g \mapsto (f \mapsto g \circ f).$$

In particular, $h^a_{a',a'}$ sends the identity morphism of a' to the identity set function of $h^a(a')$. Also, since composition is associative, the set maps $h^a_{a',a''}$ respect composition. Altogether, this defines a covariant functor.

Similarly, the **Yoneda contravariant functor** of a'' is the contravariant functor,

$$h_{a''} : \mathcal{A} \to \mathbf{Sets}, \quad h_{a''}(a') = \mathrm{Hom}_{\mathcal{A}}(a', a'').$$

Each set map $h^{a,a'}_{a''}$ is defined by sending $g \in \mathrm{Hom}_{\mathcal{A}}(a, a')$ to the set map

$$h_{a''}(a') \to h_{a''}(a), \quad g \mapsto g \circ f.$$

Example 4.12. For every partially ordered set $(S, \leq)$, for the associated category, for every element $a \in S$, the Yoneda functor h_a associates to each element a' the empty set unless $a' \leq a$, in which case it associates a singleton set. Similarly, the Yoneda functor $h^{a'}$ associates to each element a the empty set unless $a' \leq a$.

5 Natural Transformations

Definition 5.1. For categories $\mathcal{A}$ and $\mathcal{B}$, for covariant functors F and G from $\mathcal{A}$ to $\mathcal{B}$, a **natural transformation** from F to G is a rule θ that associates to every object a of $\mathcal{A}$ an element $\theta_a \in \text{Hom}_{\mathcal{B}}(F(a), G(a))$ such that for every ordered pair of objects (a, a') of $\mathcal{A}$, for every element $f \in \text{Hom}_{\mathcal{A}}(a, a')$, the following compositions of morphisms in $\mathcal{B}$ are equal,

$$\theta_{a'} \circ F(f) = G(f) \circ \theta_a.$$

For covariant functors, F, G and H from $\mathcal{A}$ to $\mathcal{B}$, for natural transformations from F to G and from G to H, the (vertical) **composite natural transformation** from F to H is defined in the evident way. Also, for every functor F, the identity natural transformation from F to itself is defined in the evident way. An invertible natural transformation (with respect to composition of natural transformations and the identity natural transformations) is called a **natural equivalence** or **natural isomorphism**. This holds if and only if θ_a is an invertible morphism for every object a, in which case the inverse natural transformation associates to a the inverse of θ_a.

For every natural transformation θ between covariant functors $F, G : \mathcal{A} \to \mathcal{B}$, for every natural transformation θ' between covariant functors $F', G' : \mathcal{B} \to \mathcal{C}$, the **horizontal composition natural transformation**, or **Godement product**, is the natural transformation $\theta * \theta' : F' \circ F \to G' \circ G$ associating to every object a of $\mathcal{A}$ the $\mathcal{C}$-morphism,

$$\theta'_{G(a)} \circ_{\mathcal{C}} F'_{F(a),G(a)}(\theta_a) = (\theta * \theta')_a = G'_{F(a),G(a)}(\theta_a) \circ \theta'_{F(a)}.$$

This is associative in θ and θ'. For every covariant functor $I : \mathcal{B} \to \mathcal{C}$, the I-**pushforward natural transformation**, $I_*\theta = \theta * \text{Id}_I$, is the natural transformation between the composition functors $I \circ F, I \circ G : \mathcal{A} \to \mathcal{C}$ associating to every object a of $\mathcal{A}$ the morphism $I_{F(a),G(a)}(\theta_a)$ in $\text{Hom}_{\mathcal{C}}(I(F(a)), I(G(a)))$. Similarly, for every covariant functor $E : \mathcal{D} \to \mathcal{A}$, the E-**pullback natural transformation**, $E^*\theta = \text{Id}_E * \theta$, is the natural transformation between the composition functors $F \circ E, G \circ E : \mathcal{D} \to \mathcal{B}$ that associates to every object d of $\mathcal{D}$ the morphism $\theta_{E(d)}$ in $\text{Hom}_{\mathcal{B}}(F(E(d)), G(E(d)))$. Of course the Godement product can be expanded in terms of pushforward, pullback and vertical composition,

$$G^*\theta' \circ (F')_*\theta = \theta * \theta' = G'_*\theta \circ F^*\theta'.$$

In particular,

$$I_*(E^*(\theta)) = (\text{Id}_E * \theta) * \text{Id}_I = \text{Id}_E * (\theta * \text{Id}_I) = E^*(I_*(\theta)).$$

Example 5.2. For every partially ordered set $(S, \leq)$, for every partially ordered set $(T, \leq)$, for every pair of order-preserving functions,

$$F, G : (S, \leq) \to (T, \leq),$$

there exists a natural transformation from F to G if and only if $F \leq G$, i.e., $F(s) \leq G(s)$ for every $s \in S$. In this case, the natural transformation is unique. Notice that $F \leq F$, and if both $F \leq G$

and $G \le H$ for order-preserving functions F, G, and H, then also $F \le H$, reflecting composition of natural transformations. If $F \le G$, then the natural transformation is a natural equivalence if and only if the set functions are equal. For order-preserving functions $I : (T, \le) \to (U, \le')$ and $E : (R, \le') \to (S, \le)$, if $F \le G$, then also $I \circ F \le' I \circ G$ and $F \circ E \le G \circ E$, reflecting the I-pushforward and E-pullback of the natural transformation.

Example 5.3. For categories $\mathcal{A}$ and $\mathcal{B}$, for covariant functors F and G from $\mathcal{A}$ to $\mathcal{B}$, for every natural transformation θ from F to G, the **opposite natural transformation** θ^{opp} from G^{opp} to F^{opp} associates to every object a of $\mathcal{A}$ the element θ_a in $\text{Hom}_{\mathcal{B}}(F(a), G(a)) = \text{Hom}_{\mathcal{B}^{\text{opp}}}(G(a), F(a))$. The natural transformation θ is a natural equivalence if and only if θ^{opp} is a natural equivalence. The opposite natural transformation of θ^{opp} is the original natural transformation θ. The opposite natural transformation is compatible with vertical composition and Godement product.

Example 5.4. Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$ be covariant functors, and let $F \downarrow G$ be the comma category with its domain functor, $s : F \downarrow G \to \mathcal{A}$, and its codomain functor $t : F \downarrow G \to \mathcal{B}$. For the composite functors, $F \circ s, G \circ t : F \downarrow G \to \mathcal{C}$, there is a natural transformation,

$$\theta : F \circ s \Rightarrow G \circ t, \quad \theta_{(a,b,u)} = u.$$

For every category $\mathcal{D}$, for every functor $E : \mathcal{D} \to F \downarrow G$, there is a triple (S, T, η) of functors,

$$S = s \circ E : \mathcal{D} \to \mathcal{A}, \quad T = t \circ E : \mathcal{D} \to \mathcal{B},$$

and a natural transformation $\eta = E^*\theta$ from $F \circ S$ to $G \circ T$. Conversely, for every natural transformation (S, T, η) as above, there is a unique functor $E : \mathcal{D} \to F \downarrow G$ such that $s \circ E$ (strictly) equals S, such that $t \circ E$ (strictly) equals T, and such that $E^*\theta$ equals η.

Example 5.5. As a special case of the preceding, for every category $\mathcal{A}$, for every category $\mathcal{D}$, a covariant functor to the arrow category,

$$E : \mathcal{D} \to \mathcal{A}^{\to},$$

is (strictly) equivalent to an ordered pair (S, T) of covariant functors,

$$S : \mathcal{D} \to \mathcal{A}, \quad T : \mathcal{D} \to \mathcal{A},$$

and a natural transformation $\eta : S \Rightarrow T$.

Example 5.6. For every set a, denote by $\theta_a : a \to \mathcal{P}(a)$ the set function that associates to every element $x \in a$ the singleton set of x. This defines a natural transformation from the identity functor of **Sets**, resp. **FinSets**, to the covariant functor $\mathcal{P}_*$.

Example 5.7. For every category $\mathcal{A}$, for every covariant functor $F : \mathcal{A} \to \textbf{Sets}$, for every object a of $\mathcal{A}$, for every element $t \in F(a)$, for every object a' of $\mathcal{A}$, for every element $f \in \text{Hom}_{\mathcal{A}}(a, a')$, denote by $f_*(t)$ the element of $F(a')$ that is the image of t under $F_{a,a'}(f)$. This defines a set function,

$$\widetilde{t}_{a'} : h^a(a') \to F(a'), \quad f \mapsto f_*(t).$$

This is a natural transformation $\widetilde{t}$ from the covariant functor h^a to F. Every natural transformation from h^a to F is of the form $\widetilde{t}$ for a unique element $t \in F(a)$.

Similarly, for every contravariant functor $G : \mathcal{A}^{\text{opp}} \to \mathbf{Sets}$, for every element $t \in G(a)$, for every object a' of $\mathcal{A}$, and for every element $f \in \text{Hom}_{\mathcal{A}}(a', a)$, denote by $f^*(t)$ the element of $G(a')$ that is the image of t under $G^{a',a}(f)$. This defines a set function,

$$\widetilde{t}^{a'} : h_a(a') \to G(a'), \quad f \mapsto f^*(t).$$

This is a natural transformation $\widetilde{t}$ from the contravariant functor h_a to G. Every natural transformation from h_a to G is of the form $\widetilde{t}$ for a unique element $t \in F(a)$.

Definition 5.8. For a category $\mathcal{A}$ and for a covariant functor $F : \mathcal{A} \to \mathbf{Sets}$, a **representation** of F is a pair (a, t) of an object a of $\mathcal{A}$ and an element $t \in F(a)$ such that the associated natural transformation $\widetilde{t} : h_a \Rightarrow F$ is a natural equivalence. If there exists a representation, then F is a **representable functor**. Similarly, a **representation** of a contravariant functor is a representation of the associated covariant functor from $\mathcal{A}^{\text{opp}}$ to **Sets**, and the contravariant functor is a **representable functor** if there exists a representation.

Example 5.9. For a covariant functor $F : \mathcal{A} \to \mathcal{B}$, for every object a of $\mathcal{A}$, let $\theta_a : F(a) \to G(a)$ be an isomorphism in $\mathcal{B}$. For every ordered pair (a, a') of objects of $\mathcal{A}$, denote by $G_{a,a'}$ the unique set map,

$$G_{a,a'} : \text{Hom}_{\mathcal{A}}(a, a') \to \text{Hom}_{\mathcal{B}}(G(a), G(a')),$$

such that for every $u \in \text{Hom}_{\mathcal{A}}(a, a')$, the composite $G_{a,a'}(u) \circ \theta_a$ equals $\theta_{a'} \circ F_{a,a'}(u)$. The rule associating to every object a of $\mathcal{A}$ the object $G(a)$ of $\mathcal{B}$ and associating to every ordered pair (a, a') of objects of $\mathcal{A}$ the set map $G_{a,a'}$ is a covariant functor $G : \mathcal{A} \to \mathcal{B}$, and the rule associating to every object a of $\mathcal{A}$ the isomorphism θ_a in $\mathcal{B}$ is a natural equivalence between F and G. In this sense, a rule that covariantly associates to every object of $\mathcal{A}$ an object of $\mathcal{B}$ only up to unique isomorphism in $\mathcal{B}$ defines a "natural equivalence class" of covariant functors.

Example 5.10. As an explicit example of the preceding, let $R : \mathcal{B} \to \mathcal{A}$ be a fully faithful, essentially surjective covariant functor, i.e., an equivalence of categories. Also assume that $\mathcal{A}$ is strictly small. For every object a of $\mathcal{A}$, since R is essentially surjective, there exists an object b of and an isomorphism, $a \to R(b)$. Using the Axiom of Choice, let $b = L(a)$ and $\theta_a : a \to R(L(a))$ be such a choice of object and isomorphism for every object a of $\mathcal{A}$. For every ordered pair (a, a') of objects, since R is fully faithful, there exists a unique bijection of sets,

$$L_{a,a'} : \text{Hom}_{\mathcal{A}}(a, a') \to \text{Hom}_{\mathcal{B}}(R(a), R(a')), \quad u \mapsto v = L_{a,a'}(u)$$

such that the composition $R(v) \circ \theta_a$ equals $\theta_{a'} \circ u$ for every u in $\text{Hom}_{\mathcal{A}}(a, a')$. This defines a covariant functor $L : \mathcal{A} \to \mathcal{B}$ and a natural equivalence $\theta : \text{Id}_{\mathcal{A}} \Rightarrow R \circ L$. Since R is fully faithful, also L is fully faithful.

Again using that R is fully faithful, there is a unique natural equivalence $\eta : L \circ R \Rightarrow \text{Id}_{\mathcal{B}}$ such that the R-pullback $R^*\eta$ equals the inverse natural isomorphism of the R-pushforward $R_*\theta$. In particular, L is essentially surjective. Thus, L is also an equivalence of categories. For a given equivalence R from a category $\mathcal{A}$ to a strictly small category $\mathcal{B}$, the extended datum of functors and natural transformations, (L, R, θ, η) as above, is unique up to unique natural equivalence in R.

6 Adjoint Pairs of Functors

Let $\mathcal{A}$ and $\mathcal{B}$ be categories.

Definition 6.1. An **adjoint pair** of (covariant) functors between $\mathcal{A}$ and $\mathcal{B}$ is a pair of (covariant) functors,

$$L : \mathcal{A} \to \mathcal{B}, \ R : \mathcal{B} \to \mathcal{A},$$

be (covariant) functors, and a pair of natural transformations of functors,

$$\theta : \mathrm{Id}_{\mathcal{A}} \Rightarrow RL, \ \theta(a) : a \to R(L(a)),$$

$$\eta : LR \Rightarrow \mathrm{Id}_{\mathcal{B}}, \ \eta(b) : L(R(b)) \to b,$$

such that the following compositions of natural transformations equal Id_R, resp. Id_L,

$$(*_R) : R \overset{\theta \circ R}{\Rightarrow} RLR \overset{R \circ \eta}{\Rightarrow} R,$$

$$(*_L) : L \overset{L \circ \theta}{\Rightarrow} LRL \overset{\eta \circ L}{\Rightarrow} R.$$

For every object a of $\mathcal{A}$ and for every object b of $\mathcal{B}$, define set maps,

$$H_R^L(a,b) : \mathrm{Hom}_{\mathcal{B}}(L(a), b) \to \mathrm{Hom}_{\mathcal{A}}(a, R(b)),$$

$$(L(a) \xrightarrow{\phi} b) \mapsto \left(a \xrightarrow{\theta(a)} R(L(a)) \xrightarrow{R(\phi)} R(b)\right),$$

and

$$H_L^R(a,b) : \mathrm{Hom}_{\mathcal{A}}(a, R(b)) \to \mathrm{Hom}_{\mathcal{B}}(L(a), b),$$

$$(a \xrightarrow{\psi} R(b)) \mapsto \left(L(a) \xrightarrow{L(\psi)} L(R(b)) \xrightarrow{\eta(b)} b\right).$$

Adjoint Pairs Exercise.

(i) For L, R, θ and η as above, the conditions $(*_R)$ and $(*_L)$ hold if and only if for every object a of $\mathcal{A}$ and every object b of $\mathcal{B}$, $H_R^L(a,b)$ and $H_L^R(a,b)$ are inverse bijections.

(ii) Prove that both $H_R^L(a,b)$ and $H_L^R(a,b)$ are binatural in a and b.

(iii) For functors L and R, and for binatural inverse bijections $H_R^L(a,b)$ and $H_L^R(a,b)$ between the bifunctors

$$\mathrm{Hom}_{\mathcal{B}}(L(a), b), \mathrm{Hom}_{\mathcal{A}}(a, R(b)) : \mathcal{A} \times \mathcal{B} \to \mathbf{Sets},$$

prove that there exist unique θ and η extending L and R to an adjoint pair such that H^L_R and H^R_L agree with the binatural inverse bijections defined above.

(iv) Let (L, R, θ, η) be an adjoint pair. Let a (covariant) functor

$$\widetilde{R} : \mathcal{B} \to \mathcal{A},$$

and natural transformations,

$$\widetilde{\theta} : \mathrm{Id}_{\mathcal{A}} \Rightarrow \widetilde{R} \circ L, \widetilde{\eta} : L \circ \widetilde{R} \Rightarrow \mathrm{Id}_{\mathcal{B}},$$

be natural transformations such that $(L, \widetilde{R}, \widetilde{\theta}, \widetilde{\eta})$ is also an adjoint pair. For every object b of B, define $I(b)$ in $\mathrm{Hom}_{\mathcal{B}}(R(b), \widetilde{R}(b))$ to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathcal{B}}(b, b) \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(\theta(b), b)} \mathrm{Hom}_{\mathcal{B}}(L(R(b)), b) \xrightarrow{H^{\widetilde{R}}_L(R(b), b)} \mathrm{Hom}_{\mathcal{B}}(R(b), \widetilde{R}(b)).$$

Similarly, define $J(b)$ in $\mathrm{Hom}_{\mathcal{B}}(\widetilde{R}(b), R(b))$, to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathcal{B}}(b, b) \xrightarrow{\mathrm{Hom}_{\mathcal{B}}(\widetilde{\theta}(b), b)} \mathrm{Hom}_{\mathcal{B}}(L(\widetilde{R}(b)), b) \xrightarrow{H^{R}_L(\widetilde{R}(b), b)} \mathrm{Hom}_{\mathcal{B}}(\widetilde{R}(b), R(b)).$$

Prove that I and J are the unique natural transformations of functors,

$$I : R \Rightarrow \widetilde{R}, \ J : \widetilde{R} \Rightarrow R,$$

such that $\widetilde{\theta}$ equals $(I \circ L) \circ \theta$, θ equals $(J \circ L) \circ \widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta \circ (L \circ I)$, and η equals $\widetilde{\eta} \circ (L \circ J)$. Moreover, prove that I and J are inverse natural isomorphisms. In this sense, every extension of a functor L to an adjoint pair (L, R, θ, η) is unique up to unique natural isomorphisms (I, J). Formulate and prove the symmetric statement for all extensions of a functor R to an adjoint pair (L, R, θ, η).

(v) For every adjoint pair (L, R, θ, η), prove that also $(R^{\mathrm{opp}}, L^{\mathrm{opp}}, \eta^{\mathrm{opp}}, \theta^{\mathrm{opp}})$ is an adjoint pair.

(vi) Formulate the corresponding notions of adjoint pairs when L and R are contravariant functors (just replace one of the categories by its opposite category).

Exercise on Composition of Adjoint Pairs. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let

$$L' : \mathcal{A} \to \mathcal{B}, R' : \mathcal{B} \to \mathcal{A},$$

be (covariant) functors, and let

$$\theta' : \mathrm{Id}_{\mathcal{A}} \Rightarrow R'L', \ \eta' : L'R' \Rightarrow \mathrm{Id}_{\mathcal{B}},$$

be natural transformations that are an adjoint pair of functors. Also let

$$L'' : \mathcal{B} \to \mathcal{C}, R'' : \mathcal{C} \to \mathcal{B},$$

be (covariant) functors, and let

$$\theta'' : \mathrm{Id}_{\mathcal{B}} \Rightarrow R''L'',\ \eta'' : L''R'' \Rightarrow \mathrm{Id}_{\mathcal{C}},$$

be natural transformations that are an adjoint pair of functors. Define functors

$$L : \mathcal{A} \to \mathcal{C},\ R : \mathcal{C} \to \mathcal{A}$$

by $L = L'' \circ L'$, $R = R' \circ R''$. Define the natural transformation,

$$\theta : \mathrm{Id}_{\mathcal{A}} \Rightarrow R \circ L,$$

to be the composition of natural transformations,

$$\mathrm{Id}_{\mathcal{A}} \overset{\theta'}{\Rightarrow} R' \circ L' \overset{R'\circ\theta''\circ L'}{\Rightarrow} R' \circ R'' \circ L'' \circ L'.$$

Similarly, define the natural transformation,

$$\eta : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{C}},$$

to be the composition of natural transformations,

$$L'' \circ L' \circ R' \circ R'' \overset{L''\circ\eta'\circ R''}{\Rightarrow} L'' \circ R'' \overset{\eta''}{\Rightarrow} \mathrm{Id}_{\mathcal{C}}.$$

Prove that L, R, θ and η form an adjoint pair of functors. This is the **composition** of (L', R', θ', η') and $(L'', R'', \theta'', \eta'')$. If $\mathcal{A}$ equals $\mathcal{B}$, if L' and R' are the identity functors, and if θ' and η' are the identity natural transformations, prove that (L, R, θ, η) equals $(L'', R'', \theta'', \eta'')$. Similarly, if $\mathcal{B}$ equals $\mathcal{C}$, if L'' and R'' are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that (L, R, θ, η) equals (L', R', θ', η'). Finally, prove that composition of three adjoint pairs is associative.

7 Adjoint Pairs of Partially Ordered Sets

Partially Ordered Sets Exercise. Let $(S, \leq)$ and $(T, \leq)$ be partially ordered sets, and consider the associated categories. For an order-preserving function,

$$L : (S, \leq) \to (T, \leq),$$

prove that there exists an order-preserving function,

$$R : (T, \leq) \to (S, \leq)$$

extending (uniquely) to an adjoint pair of functors (L, R, θ, η) if and only if for every element t of T, there exists an element s of S (necessarily unique) such that

$$L^{-1}\{\tau \in T | \tau \leq t\} = \{\sigma \in S | \sigma \leq s\}.$$

In particular, conclude that L is injective and strict, i.e., the associated functor is fully faithful. Formulate and prove a similar criterion for an order-preserving function R from $(T, \leq)$ to $(S, \leq)$ to admit a left adjoint.

8 Adjoint Pair between a Category and its Pointed Category

Definition 8.1. A **pointed set** is an ordered pair (S, s) of a set S and an element s of the set S. For pointed sets (S, s) and (S', s'), the **set of morphisms of pointed sets** is the subset of $\mathrm{Hom}_{\mathbf{Sets}}(S, S')$ of set functions that map s to s'.

Notation 8.2. For every set S, denote by $\overline{S}$ the subset of the power set $\mathcal{P}(S)$ whose elements are $\{S\}$ and all singleton sets. Thus, $\overline{S}$ contains the image of the set function $\theta_S : S \to \mathcal{P}(S)$ from Example 5.6. For every set function $u : S \to S'$, define $\overline{u} : \overline{S} \to \overline{S'}$ to be the unique set function that maps $\{S\}$ to $\{S'\}$ and such that $\overline{u} \circ \theta_S$ equals $\theta_{S'} \circ u$. For every pointed set (S, s), define $\eta_{(S,s)} : (\overline{S}, \{S\}) \to (S, s)$ to be the unique function of pointed sets such that $\eta_{(S,s)} \circ \theta_S$ equals the identity function on S.

Pointed Sets Exercise.

(i) Prove that the rules above define a category **PtdSets** of pointed sets together with a faithful functor **PtdSets** $\to$ **Sets** associating to every pointed set (S, s) the set S and restricting to the inclusion from the set of morphisms of pointed sets from (S, s) to (S', s') inside the set of all set functions from S to s'. This is the **forgetful functor**.

(ii) Prove that the rule associating to every set S the ordered pair $(\overline{S}, \{S\})$ and associating to every set function $u : S \to S'$ the set function $\overline{u}$ defines a faithful functor from **Sets** to **PtdSets**.

(iii) Prove that the rule associating to every set S the set function $\theta_S : S \to \overline{S}$ defines a natural transformation from the identity functor on **Sets** to the composition of the above functors, **Sets** $\to$ **PtdSets** $\to$ **Sets**.

(iv) Prove that the rule associating to every pointed set (S, s) the set function $\eta_{(S,s)} : (\overline{S}, \{S\}) \to (S, s)$ is a natural transformation to the identity functor on **PtdSets** from the composition of the above functors **PtdSets** $\to$ **Sets** $\to$ **PtdSets**.

(v) Prove that these functors and natural transformations define an adjoint pair of functors.

Semigroups and Monoids Exercise. Modify the construction of the previous exercise to construct an adjoint pair of functors between **Semigroups** and **Monoids** whose right adjoint functor is the (faithful) forgetful functor from **Monoids** to **Semigroups** that "forgets" the specified identity element of the monoid (since identity elements in a monoid are unique, this functor is faithful).

Definition 8.3. A category is a **category with an initial object**, respectively a **category with a terminal object**, a **pointed category**, if it has an initial object, resp. if it has a terminal object, it has an object that is simultaneously an initial object and a terminal object, i.e., if it has a zero object. A functor between categories that both have an initial object, respectively a terminal object, a zero object, is a **initial preserving**, resp. **terminal preserving**, a **pointed functor**, if it maps each initial object to an initial object, resp. if it maps each terminal object to a terminal object, resp. if it maps each zero object to a zero object.

Definition 8.4. A **trivial category** is a pointed category such that every object is a zero object (i.e., there are objects, and every Hom set is a singleton set). A **terminal category** is a trivial category that has a unique object; every object of a trivial category gives a skeletal subcategory that is a terminal category.

Definition 8.5. For every category $\mathcal{C}$, for every set 0, the **associated category** $\mathcal{C}_{0,\mathbf{init}}$ **with initial object** 0 is the category whose objects consist of 0 together with ordered pairs $(A,0)$ for all objects A of $\mathcal{C}$. For every object of $\mathcal{C}_{\text{init}}$, the set of morphisms from 0 to that object is a singleton set. For every pair of objects A and B of $\mathcal{C}$, the set of morphisms of $\mathcal{C}_{0,\text{init}}$ from $(A,0)$ to $(B,0)$ is the set of morphisms of $\mathcal{C}$ from A to B. For every object A of $\mathcal{C}$, the set of morphisms in $\mathcal{C}_{0,\text{init}}$ from $(A,0)$ to 0 is the empty set. There is a rule $F_{\mathcal{C},0}$ that associates to every object A of $\mathcal{C}$ the object $(A,0)$ of $\mathcal{C}_{0,\text{init}}$ and, for every pair of objects A and B of $\mathcal{C}$, identifies the set of morphisms of $\mathcal{C}$ from A to B to the set of morphisms of $\mathcal{C}_{0,\text{init}}$ from $(A,0)$ to $(B,0)$. There is a unique composition rule on $\mathcal{C}_{0,\text{init}}$ that makes $\mathcal{C}_{0,\text{init}}$ a category in such a way that $F_{\mathcal{C},0}$ is a fully faithful functor.

Adjointness property of the associated category with initial object. Show that the object 0 of $\mathcal{C}_{0,\text{init}}$ is an initial object. Show that for every functor $G:\mathcal{C}\to\mathcal{B}$ to a category $\mathcal{B}$ and for every initial object b of $\mathcal{B}$, there exists a unique functor $G_{b,0,\text{init}}:\mathcal{C}_{\text{init}}\to\mathcal{B}$ that is initial preserving, that sends the initial object 0 of $\mathcal{C}_{0,\text{init}}$ to b, and such that $G_{b,0,\text{init}}\circ F$ equals G. Show that for every initial object b' of $\mathcal{B}$, there is a unique natural equivalence $G_{b',b,0,\text{init}}:G_{b,0,\text{init}}\Rightarrow G_{b',0,\text{init}}$ such that $G_{b',b,0,\text{init}}\circ F$ equals the identity natural equivalence of G to itself. In this sense, $(-)_{0,\text{init}}$ is a 2-functor from the 2-category of categories to the 2-category of categories with initial objects with morphisms being natural equivalence classes of initial preserving functors, and $(-)_{\text{init}}$ is "left adjoint" to the faithful (but not full) functor from the 2-category of categories with initial objects to the 2-category of categories (not necessarily having an initial object).

Associated category with a terminal object. For a category $\mathcal{C}$, define $\mathcal{C}_{0,\text{term}}$ to be the opposite category of the associated category with initial object of the opposite category $\mathcal{C}^{\text{opp}}$, i.e., $((\mathcal{C}^{\text{opp}})_{0,\text{init}})^{\text{opp}}$. Formulate the analogues of the above for the associated functor $F_{0,\text{term}}:\mathcal{C}\to\mathcal{C}_{0,\text{term}}$.

Definition 8.6. For every category $\mathcal{C}$ that has a terminal object, for every terminal object 0, the **associated category** $\mathcal{C}_0$ **with final object** $(0,\mathbf{Id}_0)$ is the category whose objects are all ordered pairs (A,f) of an object A of $\mathcal{C}$ and a morphism $f:0\to A$ of $\mathcal{C}$. For every pair of such ordered pairs, (A,f) and (A',f'), the set of morphisms of $\mathcal{C}_0$ from (A,f) to (A',f') is the set of all morphisms of $\mathcal{C}$ $g:A\to A'$ such that $g\circ f$ equals f'. There is a rule $\Phi_{\mathcal{C},0}$ that associates to every object (A,f) of $\mathcal{C}_0$ the object A of $\mathcal{C}$ and that associates to every morphism of $\mathcal{C}_0$, $g:(A,f)\to(A',f')$, the morphism $g:A\to A'$ of $\mathcal{C}$. There is a unique composition rule on $\mathcal{C}_0$ that makes $\mathcal{C}_0$ a category in such a way that $\Phi_{\mathcal{C},0}$ is a faithful functor (usually not full).

Adjointness property of the associated category with zero object. Show that $(0,\text{Id}_0)$ is a zero object of $\mathcal{C}_0$. Show that for every terminal-preserving functor $G:\mathcal{B}\to\mathcal{C}$ from a category with a zero object b to a category with a terminal object 0, there exists a unique zero-preserving functor $G_{0,b}:\mathcal{B}\to\mathcal{C}_0$ such that $\Phi_{\mathcal{C},0}\circ G_{0,b}$ equals G. In this sense, the rule associating to a category with a terminal object $\mathcal{C}$ the category with zero object $\mathcal{C}_0$ is right adjoint to the fully faithful 2-functor from the 2-category of categories with zero object and zero-preserving functors to the 2-category of categories with terminal object and terminal-preserving functors.

9 Adjoint Pairs of Free Objects

Definition 9.1. A **concrete category** is a category, $\mathcal{A}$, together with a faithful functor, $R : \mathcal{A} \to$ **Sets**, the **forgetful functor** of the concreted category. A left adjoint of R is a **free functor** for the specified concrete category. For concrete categories $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$, a **functor of concrete categories** is a functor $F : \mathcal{A} \to \mathcal{A}'$ together with a natural equivalence $\theta : R \Rightarrow R' \circ F$, cf. the articles of Porst.

Remark 9.2. If there exists a free functor L for R, then the natural equivalence θ in a functor of concrete categories is uniquely determined by its value on the object $L(\{*\})$ for any singleton set $\{*\}$. for a given functor $F : \mathcal{A} \to \mathcal{A}'$, there is at most one natural equivalence θ such that (R, θ) is a functor of concrete categories. Thus, there is a unique concrete equivalence of the concrete category of sets extending the identity functor, but the extensions of the identity functor on the concrete category of groups has two elements (the identity extension and the extension given by group inversion).

Notation 9.3. For every nonnegative integer n, denote by $[1, n]$ the set $\{k \in \mathbb{Z}_{>0} | k \leq n\}$, which has precisely n elements. For every ordered pair (n', n'') of nonnegative integers, denote by $q'_{n',n''}$ and $q''_{n',n''}$ the following set maps,

$$q'_{n',n''} : [n'] \to [n' + n''], \quad k \mapsto k,$$
$$q''_{n',n''} : [n''] \to [n' + n''], \quad k \mapsto n' + k.$$

For every set Σ and for every ordered pair of set functions,

$$f' : [n'] \to \Sigma, \quad f'' : [n''] \to \Sigma,$$

denote by $m_{\Sigma,n',n''}(f', f'')$ the unique set function

$$f : [n' + n''] \to \Sigma, \quad f \circ q'_{n',n''} = f', \quad f \circ q''_{n',n''} = f''.$$

Denote the unique set function $[0] \to \Sigma$ by 0_Σ. For every element $\sigma \in \Sigma$, denote by $\iota_{\Sigma,\sigma}$ the unique set function $[1] \to \Sigma$ with image $\{\sigma\}$.

Notation 9.4. For every set Σ, denote by $F(\Sigma)$ the set of all ordered pairs (n, f) of an integer $n \geq 0$ and a set map $f : [n] \to \Sigma$. For every set function $u : \Sigma \to \Pi$, denote by $F(u) : F(\Sigma) \to F(\Pi)$ the set function $(n, f) \mapsto (n, f \circ u)$. Denote by $\mathrm{pr}_{\Sigma,1} : F(\Sigma) \to \mathbb{Z}_{\geq 0}$ the set map that sends (n, f) to n. Denote by m_Σ the following binary operation,

$$m_\Sigma : F(\Sigma) \times F(\Sigma) \to F(\Sigma), \quad ((n', f'), (n'', f'')) \mapsto (n' + n'', m_{\Sigma,n',n''}(f', f'')).$$

Denote by ι_Σ the following set map,

$$\iota_\Sigma : \Sigma \to F(\Sigma), \quad \sigma \mapsto ([1], \iota_{\Sigma,\sigma}).$$

Free Monoids Exercise.

(i) Prove that the rule associating to every monoid (G, m, e) the set G and associating to every monoid morphism the same set map defines a faithful functor **Monoids** $\to$ **Sets**. This is the forgetful functor of the concrete category of monoids.

(ii) For every set Σ, prove that $(F(\Sigma), m_\Sigma, ([0], 0_\Sigma))$ is a monoid. For this monoid structure, for every set map $u : \Sigma \to \Pi$, prove that $F(u)$ is a monoid morphism. Prove that this defines a covariant functor **Sets** $\to$ **Monoids**.

(iii) Prove that the rule associating to every set Σ the set function ι_Σ is a natural transformation from the identity functor on **Sets** to the composition of the two functors above, **Sets** $\to$ **Monoids** $\to$ **Sets**.

(iv) For every monoid (G, m, e) and for every set function $j : \Sigma \to G$, use induction on the integer $n \geq 0$ to prove that there exists a unique morphism of monoids,

$$\widetilde{j} : F(\Sigma) \to G,$$

such that $\widetilde{j} \circ i_\Sigma$ equals j.

(v) For every monoid (G, m, e), for the identity set map $\mathrm{Id}_G : G \to G$, prove that the rule associating to (G, m, e) the monoid morphism $\widetilde{\mathrm{Id}}_G : F(G) \to G$ is a natural transformation to the identity functor on **Monoids** from the composition of the two functors above, **Monoids** $\to$ **Sets** $\to$ **Monoids**.

(vi) Check that these functors and natural transformations define an adjoint pair of functors. The monoid $F(\Sigma)$ is the **free monoid** on Σ.

(vii) Also check that for the functor **Sets** $\to$ **Monoids** that associates to every set the additive monoid $\mathbb{Z}_{\geq 0}$ and associates to every set function the identity morphism of $\mathbb{Z}_{\geq 0}$, the rule associating to every set Σ the monoid morphism $\mathrm{pr}_{1,\Sigma} : F(\Sigma) \to \mathbb{Z}_{\geq 0}$ is a natural transformation from the free monoid functor to this functor. Also, check that this equals the composition of the free monoid functor with the natural transformation from the identity functor on **Sets** to the "constant" functor from **Sets** to itself associating to every set the singleton $\{1\}$ and associating to every set function the identity set function on $\{1\}$ (since this singleton is a final object in **Sets**, there is a unique natural transformation from the identity functor to this constant functor).

Notation 9.5. For every set Σ, denote by $F_{>0}(\Sigma) \subset F(\Sigma)$ the inverse image under $\mathrm{pr}_{\Sigma,1}$ of the subset $\mathbb{Z}_{>0} \subset \mathbb{Z}_{\geq 0}$. For every set function $u : \Sigma \to \Pi$, define $F_{>0}(u)$ to be the restriction of $F(u)$ to $F_{>0}(\Sigma)$, which is a set function with image contained in $F_{>0}(\Pi)$.

Free Semigroups Exercise.

(i) Since $\mathbb{Z}_{>0}$ is a subsemigroup of $\mathbb{Z}_{\geq 0}$ (although not a submonoid), check that also $F_{>0}(\Sigma)$ is a subsemigroup of $F(\Sigma)$ for every set.

(ii) Also check that $F_{>0}(u)$ is a morphism of semigroups for every set function $u : \Sigma \to \Pi$.

(iii) Check that these rules define a functor from **Sets** to **Semigroups**. Check that the natural transformations of the previous exercise modify to define an adjoint pair of functors between **Sets** and **Semigroups** whose right adjoint functor is the forgetful functor.

(iv) Double-check that the composite of this adjoint pair with the adjoint pair between **Semigroups** and **Monoids** is naturally equivalent to the adjoint pair between **Sets** and **Monoids** from the previous exercise.

Notation 9.6. For every set Σ, denote the Cartesian product $\Sigma \times \{+1\}$, respectively $\Sigma \times \{-1\}$, by Σ_+, resp. Σ_-, with the corresponding bijections,

$$j_{\Sigma,+} : \Sigma \to \Sigma_+, \quad j_{\Sigma,-} : \Sigma \to \Sigma_-, \quad j_{\Sigma,+}(\sigma) = (\sigma, +1), \quad j_{\Sigma,-}(\sigma) = (\sigma, -1).$$

For every set function $u : \Sigma \to \Pi$, denote by $u_+ \sqcup u_-$ the unique set function from $\Sigma_+ \sqcup \Sigma_-$ to $\Pi_+ \sqcup \Pi_-$ whose composition with $j_{\Sigma,+}$, resp. with $j_{\Sigma,-}$, equals $j_{\Pi,+} \circ u$, resp. equals $j_{\Pi,-} \circ u$. Denote by $\Lambda_\Sigma \subset F(\Sigma_+ \sqcup \Sigma_-) \times F(\Sigma_+ \sqcup \Sigma_-)$, the subset whose elements are the following ordered pairs,

$$(f \cdot (i \circ j_{\Sigma,+})(\sigma) \cdot (i \circ j_{\Sigma,-})(\sigma) \cdot g, f \cdot (i \circ j_{\Sigma,-})(\sigma) \cdot (i \circ j_{\Sigma,+})(\sigma) \cdot g), \quad f, g \in F(\Sigma_+ \sqcup \Sigma_-), \quad \sigma \in \Sigma.$$

Denote by $\sim_\Sigma$ to be the weakest equivalent relation on $F(\Sigma_+ \sqcup \Sigma_-)$ generated by the relation Λ_Σ. Denote the quotient by this equivalence relation by

$$q_\Sigma : F(\Sigma_+ \sqcup \Sigma_-) \to F_{\mathbf{Groups}}(\Sigma).$$

Denote the composition $q_\Sigma \circ i \circ j_{\Sigma,+}$ by

$$i_{\mathbf{Groups},\Sigma} : \Sigma \to F_{\mathbf{Groups}}(\Sigma).$$

Free Groups Exercise.

(i) For an equivalence relation $\sim$ on a semigroup (G, m) with quotient $q : G \to H$, check that there exists a semigroup structure on H for which q is a morphism of semigroups if and only if there exists a left act of G on H for which q is a morphism of left G-acts if and only if there exists a right act of G on H for which q is a morphism of right acts if and only if $\sim$ satisfies the following: for every $g, g', g'' \in G$, if $g \sim g'$, then also $g \cdot g'' \sim g' \cdot g''$ and also $g'' \cdot g' \sim g'' \cdot g$.

(ii) For a monoid (G, m, e), check that every surjective morphism of semigroups $u : G \to G'$ is a morphism of monoids. Conclude that for an equivalence relation $\sim$ on G, the quotient is a morphism of monoids if and only if it is a morphism of semigroups.

(iii) Check that the rule associating to each set Σ the monoid $F(\Sigma_+ \sqcup \Sigma_-)$ and associating to each set function $u : \Sigma \to \Pi$ the monoid morphism $F(u_+ \sqcup u_-)$ is a functor from **Sets** to **Monoids**. Check that the functions $i \circ j_{\Sigma,+}$ and $i \circ j_{\Sigma,-}$ are natural transformations from the identity functor on **Sets** to the composite of this functor with the forgetful functor **Monoids** $\to$ **Sets**. Check that the rule associating to every set Σ the set $F(\Sigma_+ \sqcup \Sigma_-) \times F(\Sigma_+ \sqcup \Sigma_-)$ and associating to every set function $u : \Sigma \to \Pi$ the set function

$$F(u_+ \sqcup u_-) \times F(u_+ \sqcup u_-) : F(\Sigma_+ \sqcup \Sigma_-) \times F(\Sigma_+ \sqcup \Sigma_-) \to F(\Pi_+ \sqcup \Pi_-) \times F(\Pi_+ \sqcup \Pi_-)$$

is a functor from **Sets** to itself. Check that this function sends Λ_Σ to Λ_Π. Conclude that the rule associating to every set Σ the subset Λ_Σ and associating to every set function $u : \Sigma \to \Pi$ the restriction of $F(u_+ \sqcup u_-) \times F(u_+ \sqcup u_-)$ is a subfunctor of the previous functor. Conclude that the rule associating to every set Σ the equivalence relation $\sim_\Sigma$ is also a subfunctor.

(iv) For every set Σ, check that the equivalence relation $\sim_\Sigma$ satisfies the condition necessary for the quotient map to be a monoid morphism. Conclude that there is a unique pair of a functor **Sets** $\to$ **Monoids** and a natural transformation to this functor from the free monoid functor $F(\Sigma_+ \sqcup \Sigma_-)$ associating to every set Σ the monoid $F_{\mathbf{Groups}}(\Sigma)$ and the quotient monoid morphism q_Σ.

(v) Check that each of the monoid generators $i(j_{\Sigma,+}(\sigma))$ and $i(j_{\Sigma,-}(\sigma))$ of the free monoid $F(\Sigma_+ \sqcup \Sigma_-)$ map under q_Σ to an invertible element of $F_{\mathbf{Groups}}(\Sigma)$. Conclude that the functor $F_{\mathbf{Groups}}$ from **Sets** to **Monoids** factors through the full subcategory **Groups** of **Monoids**. Thus, $F_{\mathbf{Groups}}$ is a functor from **Sets** to **Groups**.

(vi) Check that the rule associating to every set Σ the set function $i_{\mathbf{Groups},\Sigma}$ is a natural transformation from the identity functor to the composition of the forgetful functor with the functor above, **Sets** $\to$ **Groups** $\to$ **Sets**. Similarly, modify the definition of η_Σ to obtain a natural transformation from the composition **Groups** $\to$ **Sets** $\to$ **Groups** to the identity functor on **Groups**. Prove that these functors and natural transformations define an adjoint pair whose right adjoint functor is the (faithful) forgetful functor **Groups** $\to$ **Sets**. The group $F_{\mathbf{Groups}}(\Sigma)$ is the **free group on the set** Σ.

(vii) For every monoid (G, m, e), denote by $N_{(G,m,e)}$ the fiber over e of the natural transformation,

$$\widetilde{\mathrm{Id}}_G : F(G) \to G.$$

Denote by $N_{\mathbf{Groups},(G,m,e)}$ the normal subgroup of $F_{\mathbf{Groups}}(G)$ generated by the image under $q \circ F(j_{G,+})$ of $N_{(G,m,e)}$. Check that this is functorial in (G, m, e) and that the quotient group $F_{\mathbf{Groups}}(G)/N_{\mathbf{Groups},(G,m,e)}$ define a left adjoint functor to the (fully faithful) forgetful functor from **Groups** to **Monoids**. This left adjoint functor is the **group completion functor**. Double-check that the composite of the group completion functor with the free monoids functor is naturally equivalent to $F_{\mathbf{Groups}}$.

(viii) For categories $\mathcal{B}$, $\mathcal{C}$, for functors

$$L'' : \mathcal{B} \to \mathcal{C}, \;\; R'' : \mathcal{C} \to \mathcal{B},$$

and for natural transformations

$$\theta'' : \mathrm{Id}_{\mathcal{B}} \Rightarrow R'' \circ L'', \;\; \eta'' : L'' \circ R'' \Rightarrow \mathrm{Id}_{\mathcal{C}},$$

such that $(L'', R'', \theta'', \eta'')$ is an adjoint pair, the adjoint pair is *reflective* if R'' is fully faithful. In this case, prove that there exists a unique binatural transformation

$$\widetilde{H}^{L''}_{R''}(b, b') : \mathrm{Hom}_{\mathcal{C}}(L''(R''(b)), b') \to \mathrm{Hom}_{\mathcal{C}}(b, b'),$$

such that the composition with R'',

$$\mathrm{Hom}_{\mathcal{C}}(L''(R''(b)), b') \xrightarrow{\widetilde{H}^{L''}_{R''}(b,b')} \mathrm{Hom}_{\mathcal{C}}(b, b') \xrightarrow{R''} \mathrm{Hom}_{\mathcal{B}}(R''(b), R''(b')),$$

equals $H^{L''}_{R''}(R(b), b')$. In particular, taking $b' = L''(R''(b))$, denote the image of $\mathrm{Id}_{b'}$ by

$$\widetilde{\eta}''_b : b \to L''(R''(b)).$$

Prove that $\widetilde{\eta}''_b$ is an inverse to $\eta''_b : L''(R''(b)) \to b$. Thus, for a reflective adjoint pair, η'' is a natural isomorphism. Conversely, if η'' is a natural isomorphism, prove that the adjoint pair is reflective, i.e., R'' is fully faithful. In particular, for the group completion, conclude that the group completion of the monoid underlying a group is naturally isomorphic to that group.

Free Abelian Groups Exercise. Denote by

$$\Phi : \mathbb{Z}-\mathrm{mod} \to \mathbf{Groups}$$

the full subcategory of **Groups** whose objects are Abelian groups. For every group $(G, \cdot, e)$, denote by $[G, G]$ the normal subgroup of G generated by all commutators

$$[g, h] = g \cdot h \cdot g^{-1} \cdot h^{-1}$$

for pairs $g, h \in G$. Denote by

$$\theta_G : G \to L(G),$$

the group quotient associated to the normal subgroup $[G, G]$ of G. Prove that $L(G)$ is an Abelian group. Moreover, for every Abelian group $(A, \cdot, e)$, prove that the set map

$$H^L_\Phi : \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(L(G), A) \to \mathrm{Hom}_{\mathbf{Groups}}(G, \Phi(A)),\ v \mapsto v \circ \theta_G,$$

is a bijection. In particular, for every group homomorphism,

$$u : G \to G',$$

the composition $\theta_{G'} \circ u : G \to L(G')$ is a group homomorphism, and thus there exists a unique group homomorphism,

$$L(u) : L(G) \to L(G'),$$

such that $H^L_\Phi(L(u)) \circ \theta_G$ equals $\theta_{G'} \circ u$. Prove that the rule $G \mapsto L(G)$, $u \mapsto L(u)$ defines a functor,

$$L : \mathbf{Groups} \to \mathbb{Z}-\mathrm{mod}.$$

This functor is called *Abelianization*. Prove that $G \mapsto \theta_G$ is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Groups}} \Rightarrow \Phi \circ L.$$

For every Abelian group A, prove that $[A, A]$ is the identity subgroup, and thus the quotient homomorphism,

$$\theta_{\Phi(A)} : \Phi(A) \to \Phi(L(\Phi(A))),$$

is an isomorphism. Thus there exists a unique group homomorphism, just the inverse isomorphism of $\theta_{\Phi(A)}$,

$$\eta_A : L(\Phi(A)) \to A,$$

such that $\theta_{\Phi(A)} \circ \Phi(\eta_A)$ equals the $\mathrm{Id}_{\Phi(A)}$. Prove that $A \mapsto \eta_A$ is a natural isomorphism,

$$\eta : L \circ \Phi \to \mathrm{Id}_{\mathbb{Z}-\mathrm{mod}}.$$

Prove that (L, Φ, θ, η) is an adjoint pair.

Factorization Exercise. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let

$$R' : \mathcal{B} \to \mathcal{A},\ R'' : \mathcal{C} \to \mathcal{B},$$

be fully faithful functors. Denote the composition $R' \circ R''$ by

$$R : \mathcal{C} \to \mathcal{A}.$$

(i) If there exist extensions to reflective adjoint pairs (L', R', θ', η'), $(L'', R'', \theta'', \eta'')$, prove that there is also an extension to a reflective adjoint pair (L, R, θ, η).

(ii) If there exists an extension of R to a reflective adjoint pair (L, R, θ, η), prove that there exists an extension $(L'', R'', \theta'', \eta'')$. Give an example demonstrating that R' need not extend to a reflective adjoint pair (for instance, consider the full subcategory of Abelian groups in the full subcategory of solvable groups in the category of all groups).

(iii) A monoid $(G, \cdot, e)$ is called **left cancellative**, resp. **right cancellative**, if for every f, g, h in G, if $f \cdot g$ equals $f \cdot h$, resp. if $g \cdot f$ equals $h \cdot f$, then g equals h. A monoid is **cancellative** if it is both left cancellative and right cancellative. A monoid is **commutative** if for every $f, g \in G$, $f \cdot g$ equals $g \cdot f$. A commutative monoid is left cancellative if and only if it is right cancellative if and only if it is cancellative. Denote by

$$\textbf{LCanMonoids}, \textbf{RCanMonoids}, \textbf{CanMonoids}, \textbf{CommMonoids}, \textbf{CommCanMonoids} \subseteq \textbf{Monoids}$$

the full subcategories of the category of all monoids whose objects are left cancellative monoids, resp. right cancellative monoids, cancellative monoids, commutative monoids, commutative cancellative monoids. In each of these cases, prove that the fully faithful inclusion functor R extends to a reflective adjoint pair. Use (ii) to conclude that for every inclusion functor among the full subcategories listed above, there is an extension of the inclusion functor to a reflective adjoint pair.

(iv) In particular, prove that the group completion adjoint pair

$$(L : \textbf{Monoids} \to \textbf{Groups}, R : \textbf{Groups} \to \textbf{Monoids}, \theta, \eta)$$

factors as the composition of the reflective adjoint pair

$$(L' : \mathbf{Monoids} \to \mathbf{CanMonoids}, R' : \mathbf{CanMonoids} \to \mathbf{Monoids}, \theta', \eta'),$$

and the restriction to **CanMonoids** of the group completion adjoint pair

$$(L'' = L \circ R', R'', \theta'', \eta'').$$

Similarly, prove that the composition of the Abelianization functor and the group completion functor

$$(L : \mathbf{Monoids} \to \mathbb{Z} - \text{mod}, R : \mathbb{Z} - \text{mod} \to \mathbf{Monoids}, \theta, \eta),$$

factors through the reflection to the full subcategory of commutative, cancellative monoids,

$$(L' : \mathbf{Monoids} \to \mathbf{CommCanMonoids}, R' : \mathbf{CommCanMonoids} \to \mathbf{Monoids}, \theta', \eta').$$

Adjointness of Tensor and Hom Exercise. Let A and B be unital, associative rings, and let $\phi : A \to B$ be a morphism of unital, associative rings.

(i) For every left B-module,

$$(N, m_{B,N} : B \times N \to N),$$

prove that the composition

$$A \times N \xrightarrow{\phi \times \mathrm{Id}_N} B \times N \xrightarrow{m_{B,N}} N,$$

makes the datum

$$(N, m_{B,N} \circ (\phi \times \mathrm{Id}_N) : A \times N \to N),$$

an R-module. For every morphism of left B-modules,

$$u : (N, m_{B,N}) \to (N', m_{B,N'}),$$

prove that also

$$u : (N, m_{B,N} \circ (\phi \times \mathrm{Id}_N)) \to (N', m_{B,N'} \circ (\phi \times \mathrm{Id}_{N'}))$$

is a morphism of left A-modules. Altogether, prove that the association $(N, m_{B,N}) \mapsto (N, m_{B,N} \circ (\phi \times \mathrm{Id}_N))$ and $u \mapsto u$ is a faithful functor

$$R_\phi : B - \text{mod} \to A - \text{mod}.$$

In particular, in the usual manner, for every unital, associative ring C and for every $B-C$-bimodule N, prove that $R_\phi(N)$ is naturally an $A - C$-bimodule.

(ii) Formulate and prove the analogous results for right modules, giving a faithful functor

$$R^\phi : \text{mod} - B \to \text{mod} - A.$$

For every $C - B$-bimodule N, prove that $R^\phi(N)$ is naturally a $C - A$-bimodule. In particular for the $B - B$-bimodule $N = B$, $R^\phi(B)$ is naturally a $B - A$-bimodule.

For every left A-module M, denote $L_\phi(M) = R^\phi(B) \otimes_A M$. For every morphism of left A-modules,

$$u : M \to M',$$

denote by $L_\phi(u) = \mathrm{Id}_{R^\phi(B)} \otimes u$,

$$L_\phi(u) : L_\phi(M) \to L_\phi(M'),$$

the associated morphism of left B-modules. Prove that the associations $M \mapsto L_\phi(M)$ and $u \mapsto L_\phi(u)$ define a functor

$$L_\phi : A - \mathrm{mod} \to B - \mathrm{mod}.$$

(iv) Denote by 1_B the multiplicative unit in B. For every left A-module M, prove that the composition

$$M \xrightarrow{1_B \times \mathrm{Id}_M} B \times M \xrightarrow{\beta_{B,M}} B \otimes_A M,$$

is a morphism of left A-modules,

$$\theta_M : M \to R_\phi(L_\phi(M)),$$

i.e., for every $a \in A$ and for every $m \in M$,

$$\beta_{B,M}(1_B, a \cdot m) = \beta_{B,M}(1_B \cdot \phi(a), m) = \beta_{B,M}(\phi(a) \cdot 1_B, m).$$

Prove that the association $M \mapsto \theta_M$ defines a natural transformation

$$\theta : \mathrm{Id}_{A-\mathrm{mod}} \Rightarrow R_\phi \circ L_\phi.$$

(v) For every left B-module $(N, m_{B,N})$, for the induced right A-module structure on $R^\phi(B)$ and left A-module structure on N, prove that

$$m_{B,N} : B \times N \to N$$

is A-bilinear, i.e., for every $a \in A$, for every $b \in B$, and for every $n \in N$,

$$m_{B,N}(b, \phi(a) \cdot n) = m_{B,N}(b \cdot \phi(a), n).$$

Thus, by the universal property of tensor product, there exists a unique homomorphism of Abelian groups,

$$m_N : B \otimes_A N \to N,$$

such that $m_N \circ \beta_{B,N}$ equals $m_{B,N}$. Prove that m_N is a morphism of left B-modules, i.e., for every $b, b' \in B$ and for every $n \in N$,

$$m_N(b \cdot \beta_{B,N}(b', n)) = m_N(\beta_{B,N}(b \cdot b', n)) = m_{B,N}(b \cdot b', n) = m_{B,N}(b, m_{B,N}(b', n)).$$

Prove that the association $N \mapsto m_N$ defines a natural transformation

$$m : R_\phi \circ L_\phi \Rightarrow \mathrm{Id}_{B-\mathrm{mod}}.$$

(vi) Prove that $(L_\phi, R_\phi, \theta, m)$ is an adjoint pair of functors. In particular, even though R_ϕ is faithful, the natural transformation m is typically not a natural isomorphism. Conclude that one cannot weaken the definition of reflective adjoint pair from "fully faithful" to "faithful".

(vii) Prove the analogues of the above for right modules. Also, taking A to be $\mathbb{Z}$, and taking $\phi : \mathbb{Z} \to B$ to be the unique ring homomorphism, obtain an adjoint pair

$$(L'' : \mathbb{Z} - \text{mod} \to B - \text{mod}, R'' : B - \text{mod} \to \mathbb{Z} - \text{mod}, \theta'', \eta'')$$

whose composition with the adjoint pair

$$(L' : \mathbf{CommCanMonoids} \to \mathbb{Z} - \text{mod}, R' : \mathbb{Z} - \text{mod} \to \mathbf{CommCanMonoids}, \theta', \eta')$$

is an adjoint pair (L, R, θ, η) extending the forgetful functor

$$R : B - \text{mod} \to \mathbf{CommCanMonoids}.$$

Composing this adjoint pair further with the other adjoint pairs above gives, in particular, an adjoint pair (F, Φ, i, η) extending the forgetful functor

$$\Phi : B - \text{mod} \to \mathbf{Sets}.$$

The functor $F : \mathbf{Set} \to B - \text{mod}$ and the natural transformation i is called the "free B-module". Use the usual functorial properties to conclude that F naturally maps to the category of $B - B$-bimodules.

Free Central A-algebras and Free Commutative Central A-algebras Exercise. Let A be an associative, unital ring that is commutative. Recall that a central A**-algebra** is a pair (B, ϕ) of an associative, unital ring B and a morphism of associative, unital rings, $\phi : A \to B$, such that for every $a \in A$ and every $b \in B$, $\phi(a) \cdot b$ equals $b \cdot \phi(a)$, i.e., $\phi(A)$ is contained in the center of B. In particular, the identity map

$$\text{Id}_B : R^\phi(B) \to R_\phi(B),$$

is an isomorphism of $A - A$-bimodules making B into a left-right A-module.

For central A-algebras (B, ϕ) and (B', ϕ'), a morphism of central A-algebras is a morphism of associative, unital rings, $\psi : B \to B'$, such that $\psi \circ \phi$ equals ϕ'. In particular, ψ is a morphism of left-right A-modules.

(i) Prove that the usual composition and the usual identity maps define a faithful (but not full!) subcategory

$$R : A - \mathbf{algebra} \to A - \mathbf{mod}$$

whose objects are central A-algebras and whose morphisms are morphisms of central A-algebras. The rest of this problem extends this to an adjoint pair that is a composition of two other (more elementary) adjoint pairs.

(ii) Let $n \geq 2$ be an integer. Let $M_1, \dots, M_n$ be (left-right) A-modules. For every A-module U, a map

$$\gamma : M_1 \times \cdots \times M_n \to U,$$

is an n-A-**multilinear** map if for every $i = 1, \dots, n$, for every choice of

$$\overline{m}_i = (m_1, \dots, m_{i-1}, m_{i+1}, \dots, m_n) \in M_1 \times \cdots \times M_{i-1} \times M_{i+1} \times \cdots \times M_n,$$

the induced map

$$\gamma_{\overline{m}_i} : M_i \to U,\ m_i \mapsto \gamma(m_1, \dots, m_{i-1}, m_i, m_{i+1}, \dots, m_n),$$

is a morphism of A-modules. Prove that there exists a pair $(T(M_1, \dots, M_n), \beta_{M_1,\dots,M_n})$ of an A-module $T(M_1, \dots, M_n)$ and an n-A-multilinear map

$$\beta_{M_1,\dots,M_N} : M_1 \times \cdots \times M_n \to T(M_1, \dots, M_n),$$

such that for every n-A-multilinear map γ as above, there exists a unique A-module homomorphism,

$$u : T(M_1, \dots, M_n) \to U,$$

such that $u \circ \beta_{M_1,\dots,M_n}$ equals γ. For $n = 3$, prove that β_{M_1,M_2,M_3} factors through

$$\beta_{M_1,M_2} \times \mathrm{Id}_{M_3} : M_1 \times M_2 \times M_3 \to (M_1 \otimes_A M_2) \times M_3.$$

Prove that the induced map

$$\beta_{M_1 \otimes M_2, M_3} : (M_1 \otimes_A M_2) \times M_3 \to T(M_1, M_2, M_3),$$

is A-bilinear. Conclude that there exists a unique A-module homomorphism,

$$u : (M_1 \otimes_A M_2) \otimes_A M_3 \to T(M_1, M_2, M_3).$$

Prove that this is an isomorphism of A-modules. Similarly, prove that there is a natural isomorphism of A-modules,

$$M_1 \otimes_A (M_2 \otimes_A M_3) \to T(M_1, M_2, M_3).$$

Conclude that there is a natural isomorphism of A-modules,

$$(M_1 \otimes_A M_2) \otimes_A M_3 \cong M_1 \otimes_A (M_2 \otimes_A M_3),$$

i.e., tensor product is associative for A-modules. Iterate this to conclude that there are natural isomorphisms between all the different interpretations of $M_1 \otimes_A \cdots \otimes_A M_n$, and each of these is naturally isomorphic to $T(M_1, \dots, M_n)$. (All of this is also true in the case of M_i that are $A_{i-1} - A_i$-bimodules with n-$(A_i)_i$-multilinearity defined appropriately.)

(iii) Let B be an A-algebra. A $\mathbb{Z}_+$-grading of B is a direct sum decomposition as an A-module,

$$B = \oplus_{n \geq 0} B_n,$$

such that for every pair of integers $n, p \geq 0$, the restriction to the summands B_n and B_p of the multiplication map,

$$m_B : B_n \times B_p \to B$$

factors through B_{n+p}. The induced A-bilinear map is denoted

$$m_{B,n,p} : B_n \times B_p \to B_{n+p}.$$

In particular, notice that this means that B_0 is an A-subalgebra of B, and every direct summand B_n is a $B_0 - B_0$-bimodule. Finally, for every triple of integers $n, p, r \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} B_n \times B_p \times B_r & \xrightarrow{m_{B,n,p} \times \mathrm{Id}_{B_r}} & B_{n+p} \times B_r \\ {\scriptstyle \mathrm{Id}_{B_n} \times m_{B,p,r}} \downarrow & & \downarrow {\scriptstyle m_{B,n+p,r}} \\ B_n \times B_{p+r} & \xrightarrow[m_{B,n,p+r}]{} & B_{n+p+r} \end{array}$$

Prove that a $\mathbb{Z}_+$-graded A-algebra is equivalent to the data $((B_n)_{n \in \mathbb{Z}_+}, (m_{B,n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$ satisfying the conditions above.

(iv) For $\mathbb{Z}_+$-graded A-algebras $((B_n)_{n \in \mathbb{Z}_+}, (m_{B,n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$ and $((B'_n)_{n \in \mathbb{Z}_+}, (m_{B',n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$, a morphism of $\mathbb{Z}_+$-graded A-algebras is a morphism of A-algebras,

$$\psi : B \to B',$$

such that for every integer $n \geq 0$, $\psi(B_n)$ is contained in B'_n. The induced A-linear map is denoted

$$\psi_n : B_n \to B'_n.$$

In particular, ψ_0 is a morphism of A-algebras. Relative to ψ_0, every map ψ_n is a morphism of $B_0 - B_0$-bimodules. Finally, for every pair of integers $n, p \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} B_n \times B_p & \xrightarrow{\psi_n \times \psi_p} & B'_n \times B'_p \\ {\scriptstyle m_{B,n,p}} \downarrow & & \downarrow {\scriptstyle m_{B',n,p}} \\ B_{n+p} & \xrightarrow[\psi_{n+p}]{} & B'_{n+p} \end{array}$$

Prove that a morphism of $\mathbb{Z}_+$-graded A-algebras is equivalent to the data $(\psi_n)_{n \in \mathbb{Z}_+}$ satisfying the conditions above. Prove that composition of morphisms of $\mathbb{Z}_+$-graded A-algebras is a morphism of $\mathbb{Z}_+$-graded A-algebras. Prove that identity maps are morphisms of $\mathbb{Z}_+$-graded A-algebras. Conclude that there is a faithful (but not full!) subcategory,

$$L'' : \mathbb{Z}_+ - A - \textbf{algebra} \to A - \textbf{algebra},$$

whose objects are $\mathbb{Z}_+$=graded A-algebras and whose morphisms are morphisms of $\mathbb{Z}_+$-graded A-algebras. Prove that this extends to an adjoint pair $(L'', R'', \theta'', \eta'')$ where

$$R'' : A - \textbf{algebra} \to \mathbb{Z}_+ - A - \textbf{algebra},$$

associates to an associative, unital A-algebra (C, m_C) the $\mathbb{Z}_+$-graded A-algebra,

$$((C_n)_{n\in\mathbb{Z}_+}, (m_{n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+}) = ((C)_{n\in\mathbb{Z}_+}, (m)_{(n,p)}).$$

Thus C_0 equals C as an A-algebra, and the C_0-algebra $\oplus_n C_n$ is equivalent as a $\mathbb{Z}_+$-graded C-algebra to $C[t] = C \otimes_{\mathbb{Z}} \mathbb{Z}[t]$, where $\mathbb{Z}[t]$ is graded in the usual way.

(v) Let M be an A-module. For every integer $n \geq 1$, denote

$$T_A^n(M) = T(M_1, \dots, M_n) = M^{\otimes n} = M \otimes_A \cdots \otimes_A M,$$

with the universal n-A-multilinear map,

$$\beta_M^n : M^n \to T_A^n(M).$$

Similarly, denote $T_A^0(M) = A$. For every pair of integers $n, p \geq 0$, the composition,

$$M^n \times M^p \xrightarrow{=} M^{n+p} \xrightarrow{\beta^{n+p}} T_A^{n+p}(M),$$

is n-A-multilinear, resp. p-A-multilinear in the two arguments separately. Thus the composition factors as

$$M^n \times M^p \xrightarrow{\beta_M^n \times \beta_M^p} T_A^n(M) \times T_A^p(M) \xrightarrow{\mu_M^{n,p}} T_A^{n+p}(M),$$

where $\mu_M^{n,p}$ is A-bilinear. Finally, for every triple of integers $n, p, r \geq 0$, associativity of tensor products implies that the following diagram commutes,

$$\begin{array}{ccc} T_A^n(M) \times T_A^p(M) \times T_A^r(M) & \xrightarrow{\mu_M^{n,p} \times \mathrm{Id}_{T_A^r(M)}} & T_A^{n+p}(M) \times T_A^r(M) \\ \Big\downarrow{\scriptstyle \mathrm{Id}_{T_A^n(M)} \times \mu_M^{p,r}} & & \Big\downarrow{\scriptstyle \mu_M^{n+p,r}} \\ T_A^n(M) \times T_A^{p+r}(M) & \xrightarrow[\mu_M^{n,p+r}]{} & T_A^{n+p+r}(M) \end{array}.$$

Thus, the data $((T_A^n(M))_{m\in\mathbb{Z}_+}, (\mu_M^{n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$ defines a $\mathbb{Z}_+$-graded A-algebra, denoted $T_A(M)$ and called the *tensor algebra* associated to M. For every $\mathbb{Z}_+$-graded A-algebra

$$B = ((B_n)_{n\in\mathbb{Z}_+}, (m_{B,n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+}),$$

for every integer n, inductively define the A-module morphism

$$\eta'_{B,n} : T_A^n(B_1) \to B_n,$$

by $\eta'_{B,0} : A \to B_0$ is the usual A-algebra structure map ϕ, $\eta'_{B,1} : T_A^1(B_1) \to B_1$ is the usual identity morphism on B_1, and for every $n \geq 0$, assuming that $\eta'_{B,n}$ is defined,

$$\eta'_{B,n+1} : T_A^{n+1}(B_1) = B_1 \otimes_A T_A^n(B) \to B_{n+1},$$

is the unique A-module homomorphism whose composition with the universal A-bilinear map,

$$\beta_M : B_1 \times T_A^n(B) \to B_A \otimes_A T_A^n(B),$$

equals the A-bilinear composition

$$B_1 \times T_A^n(B_1) \xrightarrow{\mathrm{Id}_{B_1} \times \eta_{B,n}} B_1 \times B_n \xrightarrow{m_{B,1,n}} B_{n+1}.$$

Use associativity of tensor product (and induction) to prove that for every pair of integers $n, p \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} T_A^n(B_1) \times T_A^p(B_1) & \xrightarrow{\eta'_{B,n} \times \eta'_{B,p}} & B_n \times B_p \\ \mu_{B_1}^{n,p} \downarrow & & \downarrow m_{B,n,p} \\ T_A^{n+p}(B_1) & \xrightarrow[\eta'_{B,n+p}]{} & B_{n+p} \end{array}$$

Conclude that $(\eta'_{B,n})_{n \in \mathbb{Z}_+}$ is a morphism of $\mathbb{Z}_+$-graded A-algebras,

$$\eta'_B : T_A(B_1) \to B.$$

(vi) Denote by

$$R' : \mathbb{Z}_+ - A - \text{algebra} \to A - \text{mod}$$

the functor that associates to a $\mathbb{Z}_+$-graded A-algebra $((B_n)_{n \in \mathbb{Z}_+}, (m_{B,n,p})_{(n,p) \in \mathbb{Z}_+ \times \mathbb{Z}_+})$ the A-module B_1 and that associates to a morphism $(\psi_n)_{n \in \mathbb{Z}_+}$ of $\mathbb{Z}_+$-graded A-algebras the A-module ψ_1. For every A-module M, denote by

$$\theta'_M : M \to R'(T_A(M))$$

the identity morphism $M \to T_A^1(M)$. Prove that this defines an adjoint pair $(T_A, R', \theta', \eta')$. Composing with the adjoint pair $(L'', R'', \theta'', \eta'')$ gives an adjoint pair $(L'' \circ T_A, R, \theta, \eta)$ extending the faithful (but not full!) forgetful functor

$$R : A - \textbf{algebra} \to A - \textbf{mod}, \ B \mapsto B.$$

10 Adjoint Pairs for Lawvere Theories

Definition 10.1. For a concrete category $\mathcal{A}$ with its forgetful functor $R : \mathcal{A} \to \textbf{Sets}$, for a category $\mathcal{B}$, an **$\mathcal{A}$-object** of $\mathcal{B}$ is a triple (b, F, θ) of an object b of $\mathcal{B}$, a contravariant functor $F : \mathcal{B}^{\text{opp}} \to \mathcal{A}$, and a natural equivalence of set-valued contravariant functors on $\mathcal{B}$, $\theta : h_a \Rightarrow R \circ F$. The contravariant functor F is the **Yoneda contravariant functor** associated to the $\mathcal{A}$-object of $\mathcal{B}$. For $\mathcal{A}$-objects of $\mathcal{B}$, (b, F, θ) and (b', F', θ'), a **morphism** of $\mathcal{A}$-objects of $\mathcal{B}$ from the first triple to the second triple is a pair $(u : b \to b', v : F' \Rightarrow F)$ of a $\mathcal{B}$-morphism u and a natural transformation of contravariant functors v such that $(F \circ v) \circ \theta'$ equals $\theta \circ h_u$ as natural transformations from $h_{b'}$ to $R \circ F$. Composition is defined in the evident way, and the identity of (b, F, θ) is $(\mathrm{Id}_b, \mathrm{Id}_F)$.

Remark 10.2. Because R is faithful, for every $\mathcal{B}$-morphism $u : b \to b'$, there is at most one morphism (u, v) from the $\mathcal{A}$-object (b, F, θ) to the morphism (b', F', θ'). Thus, the rule associating to each morphism (u, v) of $\mathcal{A}$-objects of $\mathcal{B}$ the $\mathcal{B}$-morphism u gives an identification of the morphisms (u, v) with a subset of the set of $\mathcal{B}$-morphisms; in particular, the morphisms (u, v) from (b, F, θ) to (b', F', θ') form a set. Using axioms on inaccessible cardinals or Grothendieck universes, one can also deal with the foundational issues around the objects. Altogether, this gives a category of $\mathcal{A}$-objects of $\mathcal{B}$, denoted $\mathcal{A} - \mathcal{B}$, together with a covariant, faithful functor, $L - \mathcal{B} : \mathcal{A} - \mathcal{B} \to \mathcal{B}$, sending (b, F, θ) to b and sending (u, v) to u.

The Yoneda Functor of an $\mathcal{A}$-Object. Formulate and prove the analogue of Problem for the Yoneda contravariant functors associated to $\mathcal{A}$-objects of $\mathcal{B}$.

Definition 10.3. Assume now that $\mathcal{A}$ has a terminal object and all finite products. A **Lawvere theory** for $\mathcal{A}$ is a category T with a terminal object and all finite products together with an $\mathcal{A}$-object (x_T, F_T, θ_T) in T such that for every category $\mathcal{B}$ having a terminal object and all finite products, every $\mathcal{A}$-object of $\mathcal{B}$ is equivalent to the $\mathcal{A}$-object of $\mathcal{B}$ associated to (b_T, F_T, θ_T) for a functor $G_{(b,F,\theta)} : T \to \mathcal{B}$ that is unique up to natural equivalence and satisfying the following minimality condition: every object of T equals the n-fold self product of x_T, x_T^n, for some nonnegative integer n.

Lawvere Theory for a Concrete Category with a Free Functor. If there exists a left adjoint $L : \mathbf{Sets} \to \mathcal{A}$ of R, then show that there is a Lawvere theory whose underlying category T equals the opposite category of the full subcategory of $\mathcal{A}$ obtained by evaluating L on the sets $[1, n]$ from Notation 9.3. In particular, conclude that there exists a Lawvere theory for monoids, for semigroups, for groups, for Abelian groups, for central A-algebras, and for commutative central A-algebras. When a Lawvere theory exists, use this to give another solution of the previous problem.

11 Adjoint Pairs of Limits and Colimits

Limits and Colimits Exercise. Mostly we use the special cases of products and coproducts. The notation here is meant to emphasize the connection with operations on presheaves and sheaves such as formation of global sections, stalks, pushforward and inverse image. Let τ be a small category. Let $\mathcal{C}$ be a category. A *τ-family* in $\mathcal{C}$ is a (covariant) functor,

$$\mathcal{F} : \tau \to \mathcal{C}.$$

Precisely, for every object U of τ, $\mathcal{F}(U)$ is a specified object of $\mathcal{C}$. For every morphism of objects of τ, $r : U \to V$, $\mathcal{F}(r) : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of $\mathcal{C}$. Also, $\mathcal{F}(\mathrm{Id}_U)$ equals $\mathrm{Id}_{\mathcal{F}(U)}$. Finally, for every pair of morphisms of τ, $r : U \to V$ and $s : V \to W$, $\mathcal{F}(s) \circ \mathcal{F}(r)$ equals $\mathcal{F}(s \circ r)$.

For every pair $\mathcal{F}$, $\mathcal{G}$ of τ-families in $\mathcal{C}$, a *morphism* of τ-families from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation of functors, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$. For every object a of $\mathcal{C}$, denote by

$$\underline{a}_\tau : \tau \to \mathcal{C}$$

the functor that sends every object to a and that sends every morphism to Id_a. For every morphism in $\mathcal{C}$, $p : a \to b$, denote by

$$\underline{p}_\tau : \underline{a}_\tau \Rightarrow \underline{b}_\tau$$

the natural transformation that assigns to every object U of τ the morphism $p : a \to b$. Finally, for every object U of τ, denote

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U), \quad \Gamma(U, \theta) = \theta(U),$$

and for every morphism $r : U \to V$ of τ, denote

$$\Gamma(r, \mathcal{F}) = \mathcal{F}(r).$$

(i)(Functor Categories and Section Functors) For τ-families $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, and for morphisms of τ-families, $\theta : \mathcal{F} \to \mathcal{G}$ and $\eta : \mathcal{G} \to \mathcal{H}$, define the composition of θ and η to be the composite natural transformation $\eta \circ \theta : \mathcal{F} \to \mathcal{H}$. **Prove** that with this notion, there is a category $\mathbf{Fun}(\tau, \mathcal{C})$ whose objects are τ-families $\mathcal{F}$ and whose morphisms are natural transformations. **Prove** that

$$\underline{\star}_\tau : \mathcal{C} \to \mathbf{Fun}(\tau, \mathcal{C}), \ a \mapsto \underline{a}_\tau, \ p \mapsto \underline{p}_\tau,$$

is a functor that preserves monomorphisms, epimorphisms and isomorphisms. For every object U of τ, **prove** that

$$\Gamma(U, -) : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathcal{C}, \ \mathcal{F} \mapsto \Gamma(U, \mathcal{F}), \ \theta \mapsto \Gamma(U, \theta),$$

is a functor. For every morphism $r : U \to V$ of τ, **prove** that $\Gamma(r, -)$ is a natural transformation $\Gamma(U, -) \Rightarrow \Gamma(V, -)$.

(ii)(Adjointness of Constant / Diagonal Functors and the Global Sections Functor) If $\mathcal{C}$ has an initial object X, **prove** that $(\underline{\star}_\tau, \Gamma(X, -))$ extends to an adjoint pair of functors. More generally, a *limit* of a τ-family $\mathcal{F}$ (if it exists) is a natural transformation $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ that is final among all such natural transformations, i.e., for every natural transformation $\theta : \underline{b}_\tau \Rightarrow \mathcal{F}$, there exists a unique morphism $t : b \to a$ in $\mathcal{C}$ such that θ equals $\eta \circ \underline{t}_\tau$. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, for limits $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ and $\theta : \underline{b}_\tau \Rightarrow \mathcal{G}$, **prove** that there exists a unique morphism $f : a \to b$ such that $\theta \circ \underline{p}_\tau$ equals $\phi \circ \eta$. In particular, **prove** that if a limit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object a of $\mathcal{C}$, **prove** that the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a limit of $\underline{a}_\tau$.

(iii)(Adjointness of Constant / Diagonal Functors and Limits) For this part, assume that every τ-family has a limit; a category $\mathcal{C}$ is said to *have all limits* if for every small category τ and for every τ-family $\mathcal{F}$, there is a limit. Assume further that there is a rule Γ_τ that assigns to every $\mathcal{F}$ an object $\Gamma_\tau(\mathcal{F})$ and a natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \to \mathcal{F}$ that is a limit. (Typically such a rule follows from the "construction" of limits, but such a rule also follows from some form of the Axiom of Choice.) **Prove** that this extends uniquely to a functor,

$$\Gamma_\tau : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation of functors

$$\eta : \underline{*}_\tau \circ \Gamma_\tau \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

Moreover, **prove** that the rule sending every object a of $\mathcal{C}$ to the identity natural transformation θ_a is a natural transformation $\theta : \mathrm{Id}_\mathcal{C} \Rightarrow \Gamma_\tau \circ \underline{*}_\tau$. **Prove** that $(\underline{*}_\tau, \Gamma, \theta, \eta)$ is an adjoint pair of functors. In particular, the limit functor Γ_τ preserves monomorphisms and sends injective objects of $\mathbf{Fun}(\tau,\mathcal{C})$ to injective objects of $\mathcal{C}$.

(iiii)(Adjointness of Colimits and Constant / Diagonal Functors) If $\mathcal{C}$ has a final object O, **prove** that $(\Gamma(O,-), \underline{*}_\tau)$ extends to an adjoint pair of functors. More generally, a *colimit* of a τ-family $\mathcal{F}$ (if it exists) is a natural transformation $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ that is final among all such natural transformations, i.e., for every natural transformation $\eta : \mathcal{F} \Rightarrow \underline{b}_\tau$, there exists a unique morphism $h : a \to b$ in $\mathcal{C}$ such that $\underline{h}_\tau \circ \theta$ equals η. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, for colimits $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ and $\eta : \mathcal{G} \Rightarrow \underline{b}_\tau$, **prove** that there exists a unique morphism $f : a \to b$ such that $\underline{f}_\tau \circ \theta$ equals $\eta \circ \phi$. In particular, **prove** that if a colimit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object a of $\mathcal{C}$, **prove** that the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a colimit of $\underline{a}_\tau$. Finally, **repeat** the previous part for colimits in place of limits. Deduce that colimits (if they exist) preserve epimorphisms and projective objects.

(v)(Functoriality in the Source) Let $x : \sigma \to \tau$ be a functor of small categories. For every τ-family $\mathcal{F}$, define $\mathcal{F}_x$ to be the composite functor $\mathcal{F} \circ x$, which is a σ-family. For every morphism of τ-families, $\phi : \mathcal{F} \to \mathcal{G}$, define $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ to be $\phi \circ x$, which is a morphism of σ-families. **Prove** that this defines a functor

$$*_x : \mathbf{Fun}(\tau,\mathcal{C}) \to \mathbf{Fun}(\sigma,\mathcal{C}).$$

For the identity functor $\mathrm{Id}_\tau : \tau \to \tau$, **prove** that $*_{\mathrm{Id}_\tau}$ is the identity functor. For $y : \rho\ \sigma$ a functor of small categories, **prove** that $*_{x\circ y}$ is the composite $*_y \circ *_x$. In this sense, deduce that $*_x$ is a contravariant functor in x.

For two functors, $x, x_1 : \sigma \to \tau$ and for a natural transformation $n : x \Rightarrow x_1$, define $\mathcal{F}_n : \mathcal{F}_x \Rightarrow \mathcal{F}_{x_1}$ to be $\mathcal{F}(n(V)) : \mathcal{F}(x(V)) \to \mathcal{F}(x_1(V))$ for every object V of σ. **Prove** that $\mathcal{F}_n$ is a morphism of σ-families. For every morphism of τ-families, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $\phi_{x_1} \circ \mathcal{F}_n$ equals $\mathcal{G}_n \circ \phi_x$. In this sense, conclude that $*_n$ is a natural transformation $*_x \Rightarrow *_{x'}$. For the identity natural transformation $\mathrm{Id}_x : x \Rightarrow x$, **prove** that $*_{\mathrm{Id}_x}$ is the identity natural transformation of $*_x$. For a second natural transformation $m : x_1 \Rightarrow x_1$, **prove** that $\mathcal{F}_{m\circ n}$ equals $\mathcal{F}_m \circ \mathcal{F}_n$. In this sense, deduce that $*_x$ is also compatible with natural transformations. In particular, if (x, y, θ, η) is an adjoint pair of functors, **prove** that $(*_y, *_x, *_\theta, *_\eta)$ is an adjoint pair of functors.

(vi)(Fiber Categories) The following notion of *fiber category* is a special case of the notion of *2-fiber product* of functors of categories. Let $x : \sigma \to \tau$ be a functor; this is also called a *category over* τ. For every object U of τ, a *$\sigma_{x,U}$-object* is a pair $(V, r : x(V) \to U)$ of an object V of σ and a τ-isomorphism $r : x(V) \to U$. For two objects $\sigma_{x,U}$-objects (V, r) and (V', r') of $\sigma_{x,U}$, a *$\sigma_{x,U}$-morphism* from (V, r) to (V', r') is a morphism of σ, $s : V \to V'$, such that $r' \circ x(s)$ equals r. **Prove** that Id_V is a $\sigma_{x,U}$-morphism from (V, r) to itself; more generally, the $\sigma_{x,U}$-morphisms from (V,r) to (V,r) are precisely the σ-morphisms $s : V \to V$ such that $x(s)$ equals $\mathrm{Id}_{x(V)}$. For every pair of $\sigma_{x,U}$-morphisms, $s : (V,r) \to (V',r')$ and $s' : (V',r') \to (V'',r'')$, **prove** that $s' \circ s$ is a $\sigma_{x,U}$-morphism from (V,r) to (V'',r''). Conclude that these rules form a category, denoted $\sigma_{x,U}$. **Prove** that the rule $(V,r) \mapsto V$ and $s \mapsto s$ defines a faithful functor,

$$\Phi_{x,U} : \sigma_{x,U} \to \sigma,$$

and $r : x(V) \to U$ defines a natural isomorphism $\theta_{x,U} : x \circ \Phi_{x,U} \Rightarrow \underline{U}_{\sigma_{x,U}}$. Finally, for every category σ', for every functor $\Phi' : \sigma' \to \sigma$, and for every natural isomorphism $\theta' : x \circ \Phi' \Rightarrow \underline{U}_{\sigma'}$, **prove** that there exists a unique functor $F : \sigma' \to \sigma_{x,U}$ such that Φ' equals $\Phi_{x,U} \circ F$ and θ' equals $\theta_{x,U} \circ F$. In this sense, $(\Phi_{x,U}, \theta_{x,U})$ is final among pairs (Φ', θ') as above.

For every pair of functors $x, x_1 : \sigma \to \tau$, and for every natural *isomorphism* $n : x \Rightarrow x_1$, for every $\sigma_{x_1,U}$-object $(V, r_1 : x_1(V) \to U)$, **prove** that $(V, r_1 \circ n_V : x(V) \to U)$ is an object of $\sigma_{x,U}$. For every morphism in $\sigma_{x_1,U}$, $s : (V,r_1) \to (V',r_1')$, **prove** that s is also a morphism $(V, r_1 \circ n_V) \to (V', r_1' \circ n_{V'})$. Conclude that these rules define a functor,

$$\sigma_{n,U} : \sigma_{x_1,U} \to \sigma_{x,U}.$$

Prove that this functor is a *strict equivalence* of categories: it is a bijection on Hom sets (as for all equivalences), but it is also a bijection on objects (rather than merely being essentially surjective). **Prove** that $\sigma_{n,U}$ is functorial in n, i.e., for a second natural isomorphism $m : x_1 \Rightarrow x_2$, prove that $\sigma_{m \circ n,U}$ equals $\sigma_{n,U} \circ \sigma_{m,U}$.

For every pair of functors, $x : \sigma \to \tau$ and $y : \rho \to \tau$, and for every functor $z : \sigma \to \rho$ such that x equals $y \circ z$ equals x, for every $\sigma_{x,U}$-object (V,r), **prove** that $(z(V),r)$ is a $\rho_{y,U}$-object. For every $\sigma_{x,U}$-morphism $s : (V,r) \to (V',r')$, **prove** that $z(s)$ is a $\rho_{y,U}$-morphism $(z(V),r) \to (z(V'),r')$. **Prove** that $z(\mathrm{Id}_V)$ equals $\mathrm{Id}_{z(V)}$, and **prove** that z preserves composition. Conclude that these rules define a functor,

$$z_U : \sigma_{x,U} \to \rho_{y,U}.$$

Prove that this is functorial in z: $(\mathrm{Id}_\sigma)_U$ equals $\mathrm{Id}_{\sigma_{x,U}}$, and for a third functor $w : \pi \to \tau$ and functor $z' : \rho \to \pi$ such that y equals $w \circ z'$, then $(z' \circ z)_U$ equals $z'_U \circ z_U$. For an object (W, r_W) of $\rho_{y,U}$, for each object $((V,r_V), q : Z(V) \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, define the *associated* object of $\sigma_{z,W}$ to be (V,q). For an object $((V',r_{V'}), q' : Z(V') \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, for every morphism $s : (V,r_V) \to (V',r_{V'})$ such that q equals $q' \circ z(s)$, define the *associated* morphism of $\sigma_{z,W}$ to be s. **Prove** that this defines a functor

$$\widetilde{z}_{U,(W,r_W)} : (\sigma_{x,U})_{z_U,(W,r_W)} \to \sigma_{z,W}.$$

Prove that this functor is a strict equivalence of categories. **Prove** that this equivalence is functorial in z. Finally, for two functors $z, z_1 : \sigma \to \rho$ such that x equals both $y \circ z$ and $y \circ z_1$, and for a natural transformation $m : z \Rightarrow z_1$, for every object $(V, r : x(V) \to U)$ of $\sigma_{x,U}$, **prove** that m_V is a morphism in $\rho_{y,U}$ from $(z(V),r)$ to $(z_1(V),r)$. Moreover, for every morphism in $\sigma_{x,U}$, $s : (V,r) \to (V',r')$, **prove** that $m_{V'} \circ z(s)$ equals $z_1(s) \circ m_V$. Conclude that this rule is a natural transformation $m_U : z_U \Rightarrow (z_1)_U$. **Prove** that this is functorial in m. If m is a natural isomorphism, **prove** that also m_U is a natural isomorphism, and the strict equivalence $(m_U)_{(W,r_W)}$ is compatible with the strict equivalence m_W. Finally, **prove** that $m \mapsto m_U$ is compatible with precomposition and postcomposition of m with functors of categories over τ.

(vii)(Colimits and Limits along an Essentially Surjective Functor) Let $x : \sigma \to \tau$ be a functor of small categories. **Prove** that every fiber category $\sigma_{x,U}$ is small. Next, assume that x is *essentially surjective*, i.e., for every object U of τ, there exists a $\sigma_{x,U}$-object (V, r). Let $y : \tau \to \sigma$ be a functor, and let $\alpha : \text{Id}_\sigma \Rightarrow y \circ x$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\alpha_V) : x(V) \to x(y(x(V)))$ is an isomorphism and $(y(x(V)), x(\alpha_V)^{-1})$ is a final object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and α extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has a final object.) For every adjoint pair (x, y, α, β), also $(\star_y, \star_x, \star_\alpha, \star_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and α, yet let L_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $L_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \star_x \circ L_x(\mathcal{F}),$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\theta_{\mathcal{F},x,U} : \mathcal{F} \circ \Phi_{x,U} \Rightarrow L_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(L_x(\mathcal{F})(U), \theta_{\mathcal{F},x,U})$ is a colimit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation

$$\theta_x : \text{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})} \Rightarrow \star_x \circ L_x.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\text{Id}_{\mathcal{G}} : \mathcal{G} \circ x \circ \Phi_{x,U} \to \mathcal{G} \circ \underline{U}_{\sigma_{x,U}},$$

factors uniquely through a $\mathcal{C}$-morphism $L_x(\mathcal{G} \circ x)(U) \to \mathcal{G}(U)$. **Prove** that this defines a morphism $\eta_{\mathcal{G}} : L_x(\mathcal{G} \circ x) \to \mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that is a natural transformation,

$$\eta : L_x \circ \star_x \Rightarrow \text{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

Prove that $(L_x, \star_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a Γ^x and θ as above.)

Next, as above, let $x : \sigma \to \tau$ be a functor of small catgories that is essentially surjective. Let $y : \tau \to sigma$ be a functor, and let $\beta : y \circ x \Rightarrow \text{Id}_\sigma$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\beta_v) : x(y(x(V))) \to x(V)$ is an isomorphism and $(y(x(V)), x(\beta_v))$ is an initial object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and β extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has an initial object.) For every adjoint pair $(y, x, \alpha, beta)$ also $(\star_x, \star_y, \star_\alpha, \star_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and β, yet let R_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $R_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\eta_{\mathcal{F}} : \star_x \circ R_x(\mathcal{F}) \to \mathcal{F},$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\eta_{\mathcal{F},x,U} : R_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{F} \circ \Phi_{x,U},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(R_x(\mathcal{F})(U), \eta_{\mathcal{F},x,U})$ is a limit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$R_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation,

$$\eta : \star_x \circ R_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})}.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G} \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{G} \circ x \circ \Phi_{x,U},$$

factors uniquely through a $\mathcal{G}(U) \to \mathcal{C}$-morphism $R_x(\mathcal{G} \circ x)(U)$. **Prove** that this defines a morphism $\theta_{\mathcal{G}} : \mathcal{G} \to R_x(\mathcal{G} \circ x)$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that this is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow R_x \circ \star_x.$$

Prove that $(\star_x, R_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a R_x and η as above.)

(viii)(Adjoints Relative to a Full, Upper Subcategory) In a complementary direction to the previous case, let $x : \sigma \to \tau$ be an embedding of a full subcategory (thus, x is essentially surjective if and only if x is an equivalence of categories). In this case, the functor

$$\star_x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C})$$

is called *restriction*. Assume further that σ is *upper* (a la the theory of partially ordered sets) in the sense that every morphism of τ whose source is an object of σ also has target an object of σ. Assume that $\mathcal{C}$ has an initial object, $\odot$. Let $\mathcal{G}$ be a σ-family of objects of $\mathcal{C}$. Also, let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of σ-families. For every object U of τ, if U is an object of σ, then define ${}_x\mathcal{G}(U)$ to be $\mathcal{G}(U)$, and define ${}_x\phi(U)$ to be $\phi(U)$. For every object U of τ that is not an object of σ, define ${}_x\mathcal{G}(U)$ to be $\odot$, and define ${}_x\phi(U)$ to be $\mathrm{Id}_\odot$. For every morphism $r : U \to V$, if U is an object of σ, then r is a morphism of σ. In this case, define ${}_x\mathcal{G}(r)$ to be $\mathcal{G}(r)$. On the other hand, if U is not an object of σ, then $\mathcal{G}(U)$ is the initial object $\odot$. In this case, define ${}_x\mathcal{G}(r)$ to be the unique morphism ${}_x\mathcal{G}(U) \to {}_x\mathcal{G}(V)$. **Prove** that ${}_x\mathcal{G}$ is a τ-family of objects, i.e., the definitions above are compatible with composition of morphisms in τ and with identity morphisms. Also **prove** that ${}_x\phi$ is a morphism of τ-families. **Prove** that ${}_x\mathrm{Id}_{\mathcal{G}}$ equals $\mathrm{Id}_{{}_x\mathcal{G}}$. Also, for a second morphism of σ-families, $\psi : \mathcal{H} \to \mathcal{I}$, **prove** that ${}_x(\psi \circ \phi)$ equals ${}_x\psi \circ {}_x\phi$. Conclude that these rules form a functor,

$${}_x* : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}).$$

Prove that $({}_x*, *_x)$ extends to an adjoint pair of functors. In particular, conclude that $*_x$ preserves epimorphisms and ${}_x*$ preserves monomorphisms.

Next assume that $\mathcal{C}$ is an Abelian category that satisfies (AB3). For every τ-family $\mathcal{F}$, for every object U of τ, define $\theta_{\mathcal{F}}(U) : \mathcal{F}(U) \to {}^x\mathcal{F}(U)$ to be the cokernel of $\mathcal{F}(U)$ by the direct sum of the images of

$$\mathcal{F}(s) : \mathcal{F}(T) \to \mathcal{F}(U),$$

for all morphisms $s : T \to U$ with V *not* in σ (possibly empty, in which case $\theta_{\mathcal{F}}(U)$ is the identity on $\mathcal{F}(U)$). In particular, if U is not in σ, then ${}^x\mathcal{F}(U)$ is zero. For every morphism $r : U \to V$ in τ, **prove** that the composition $\theta_{\mathcal{F}}(V) \circ \mathcal{F}(r)$ equals ${}^x\mathcal{F}(r) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$${}^x\mathcal{F}(r) : {}^x\mathcal{F}(U) \to {}^x\mathcal{F}(V).$$

Prove that ${}^x\mathcal{F}(\mathrm{Id}_U)$ is the identity morphism of ${}^x\mathcal{F}(U)$. **Prove** that $r \mapsto {}^x\mathcal{F}(r)$ is compatible with composition in τ. Conclude that ${}^x\mathcal{F}$ is a τ-family, and $\theta_{\mathcal{F}}$ is a morphism of τ-families. For every morphism $\phi : \mathcal{F} \to \mathcal{E}$ of τ-families, for every object U of τ, **prove** that $\theta_{\mathcal{E}}(U) \circ \phi(U)$ equals ${}^x\phi(U) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$${}^x\phi(U) : {}^x\mathcal{F}(U) \to {}^x\mathcal{E}(U).$$

Prove that the rule $U \mapsto {}^x\phi(U)$ is a morphism of τ-families. **Prove** that ${}^x\mathrm{Id}_{\mathcal{F}}$ is the identity on ${}^x\mathcal{F}$. Also **prove** that $\phi \mapsto {}^x\phi$ is compatible with composition. Conclude that these rules define a functor

$${}^x* : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}).$$

Prove that the rule $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is a natural transformation $\mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow {}^x*$. **Prove** that the natural morphism of τ-families,

$${}^x\mathcal{F} \to {}_x(({}^x\mathcal{F})_x),$$

is an isomorphism. Conclude that there exists a unique functor,

$$*^x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}),$$

and a natural isomorphism ${}^x* \Rightarrow {}_x(*^x)$. **Prove** that $(*^x, {}_x*, \theta)$ extends to an adjoint pair of functors. In particular, conclude that ${}_x*$ preserves epimorphisms and $*^x$ preserves monomorphisms.

Finally, drop the assumption that $\mathcal{C}$ has an initial object, but assume that σ is upper, assume that σ has an initial object, W_σ, and assume that there is a functor

$$y : \tau \to \sigma$$

and a natural transformation $\theta : \mathrm{Id}_\tau \Rightarrow x \circ y$, such that for every object U of τ, the unique morphism $W_\sigma \to y(U)$ and the morphism $\theta_U : U \to y(U)$ make $y(U)$ into a coproduct of W_σ and U in τ. For simplicity, for every object U of σ, assume that $\theta_U : U \to y(U)$ is the identity Id_U (rather than merely being an isomorphism), and for every morphism $r : U \to V$ in σ, assume that $y(r)$ equals r. Thus, for every object V of σ, the identity morphism $y(V) \to V$ defines a natural transformation $\eta : y \circ x \Rightarrow \mathrm{Id}_\sigma$. **Prove** that (y, x, θ, η) is an adjoint pair of functors. Conclude that $(*_x, *_y, *_\theta, *_\eta)$ is an adjoint pair of functors. In particular, conclude that $*_x$ preserves monomorphisms and $*_y$ preserves epimorphisms.

(ix)(Compatibility of Limits and Colimits with Functors) Denote by 0 the "singleton category" 0 with a single object and a single morphism. **Prove** that $\Gamma(0, -)$ is an equivalence of categories. For an arbitrary category τ, for the unique natural transformation $\hat{\tau} : \tau \to 0$, **prove** that $*_{\hat{\tau}}$ equals the composite $\underline{*}_\tau \circ \Gamma(0, -)$ so that $\underline{*}_\tau$ is an example of this construction. In particular, for every functor $x : \sigma \to \tau$, **prove** that $(\underline{a}_\tau)_x$ equals $\underline{a}_\sigma$. If $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ is a limit of a τ-family $\mathcal{F}$, and if $\theta : \underline{b}_\sigma \Rightarrow \mathcal{F}_x$ is a limit of the associated σ-family $\mathcal{F}_x$, then **prove** that there is a unique morphism $h : a \to b$ in $\mathcal{C}$ such that η_x equals $\theta \circ \underline{p}_\sigma$. If there are right adjoints Γ_τ of $\underline{*}_\tau$ and Γ_σ of $\underline{*}_\sigma$, conclude that there exists a unique natural transformation

$$\Gamma_x : \Gamma_\tau \Rightarrow \Gamma_\sigma \circ *_x$$

so that $\eta_{\mathcal{F}_x} \circ \underline{\Gamma_x(\mathcal{F})}_\sigma$ equals $(\eta_{\mathcal{F}})_x$. **Repeat** this construction for colimits.

(x)(Limits / Colimits of a Concrete Category) Let σ be a small category in which the only morphisms are identity morphisms: identify σ with the underlying set of objects. Let $\mathcal{C}$ be the category **Sets**. For every σ-family $\mathcal{F}$, **prove** that the rule

$$\Gamma_\sigma(\mathcal{F}) := \prod_{U \in \Sigma} \Gamma(U, \mathcal{F})$$

together with the morphism

$$\eta_{\mathcal{F}} : \underline{\Gamma_\sigma(\mathcal{F})}_\sigma \Rightarrow \mathcal{F},$$
$$\eta_{\mathcal{F}}(V) = \mathrm{pr}_V : \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}),$$

is a limit of $\mathcal{F}$. Next, for every small category τ, define σ to be the category with the same objects as τ, but with the only morphisms being identity morphisms. Define $x : \sigma \to \tau$ to be the unique functor that sends every object to itself. Define $\Gamma_\tau(\mathcal{F})$ to be the subobject of $\Gamma_\sigma(\mathcal{F}_x)$ of data $(f_U)_{U \in \Sigma}$ such that for every morphism $r : U \to V$, $\mathcal{F}(r)$ maps f_U to f_V. **Prove** that with this definition, there exists a unique natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \Rightarrow \mathcal{F}$ such that the natural transformation $\underline{\Gamma_\tau(\mathcal{F})}_\sigma \Rightarrow \underline{\Gamma_\sigma(\mathcal{F}_x)} \Rightarrow \mathcal{F}_x$ equals $(\eta_{\mathcal{F}})_x$. **Prove** that $\eta_{\mathcal{F}}$ is a limit of $\mathcal{F}$. Conclude that **Sets** has all small limits. Similarly, for associative, unital rings R and S, **prove** that the forgetful functor

$$\Phi : R - S - \mathrm{mod} \to \mathbf{Sets}$$

sends products to products. Let $\mathcal{F}$ be a τ-family of $R - S$-modules. **Prove** that the defining relations for $\Gamma_\tau(\Phi \circ \mathcal{F})$ as a subset of $\Gamma_\sigma(\Phi \circ \mathcal{F})$ are the simultaneous kernels of $R - S$-module homomorphisms. Conclude that there is a natural $R-S$-module structure on $\Gamma_\tau(\Phi\circ\mathcal{F})$, and use this to **prove** that $R-S$-mod has all limits.

(xi)(Functoriality in the Target) For every functor of categories,

$$H:\mathcal{C}\to\mathcal{D},$$

for every τ-family $\mathcal{F}$ in $\mathcal{C}$, **prove** that $H\circ\mathcal{F}$ is a τ-family in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi:\mathcal{F}\Rightarrow\mathcal{G}$, **prove** that $H\circ\phi$ is a morphism of τ-families in $\mathcal{D}$. **Prove** that this defines a functor

$$H_\tau:\mathbf{Fun}(\tau,\mathcal{C})\to\mathbf{Fun}(\tau,\mathcal{D}).$$

For the identity functor $\mathrm{Id}_{\mathcal{C}}$, **prove** that $(\mathrm{Id}_{\mathcal{C}})_\tau$ is the identity functor. For $I:\mathcal{D}\to\mathcal{E}$ a functor of categories, **prove** that $(I\circ H)_\tau$ is the composite $I_\tau\circ H_\tau$. In this sense, deduce that H_τ is functorial in H.

For two functors, $H,I:\mathcal{C}\to\mathcal{D}$, and for a natural transformation $N:H\Rightarrow I$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, define $N_\tau(\mathcal{F})$ to be

$$N\circ\mathcal{F}:H\circ\mathcal{F}\Rightarrow I\circ\mathcal{F}.$$

Prove that $N_\tau(\mathcal{F})$ is a morphism of τ-families in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi:\mathcal{F}\to\mathcal{G}$, **prove** that $N_\tau(\mathcal{G})\circ H_\tau(\phi)$ equals $I_\tau(\phi)\circ N_\tau(\mathcal{F})$. In this sense, conclude that N_τ is a natural transformation $H_\tau\Rightarrow I_\tau$. For the identity natural transformation $\mathrm{Id}_H:H\Rightarrow H$, **prove** that $(\mathrm{Id}_H)_\tau$ is the identity natural transformation of H_τ. For a second natural transformation $M:I\Rightarrow J$, **prove** that $(M\circ N)_\tau$ equals $M_\tau\circ N_\tau$. In this sense, deduce that $(-)_\tau$ is also compatible with natural transformations.

(xii)(Reductions of Limits to Finite Systems for Concrete Categories) A category is *cofiltering* if for every pair of objects U and V there exists a pair of morphisms, $r:W\to U$ and $s:W\to V$, and for every pair of morphisms, $r,s:V\to U$, there exists a morphism $t:W\to V$ such that $r\circ t$ equals $s\circ t$ (both of these are automatic if the category has an initial object X). Assume that the category $\mathcal{C}$ has limits for all categories τ with finitely many objects, and also for all small cofiltering categories. For an arbitrary small category τ, define $\widehat{\tau}$ to be the small category whose objects are finite full subcategories σ of τ, and whose morphisms are inclusions of subcategories, $\rho\subset\sigma$, of τ. **Prove** that $\widehat{\tau}$ is cofiltering. Let $\mathcal{F}$ be a τ-family in $\mathcal{C}$. For every finite full subcategory $\sigma\subset\tau$, denote by $\mathcal{F}_\sigma$ the restriction as in (f) above. By hypothesis, there is a limit $\eta_\sigma:\underline{\widehat{\mathcal{F}}(\sigma)}_\sigma\Rightarrow\mathcal{F}_\sigma$. Moreover, by (g), for every inclusion of full subcategories $\rho\subset\sigma$, there is a natural morphism in $\mathcal{C}$, $\widehat{\mathcal{F}}(\rho)\to\widehat{\mathcal{F}}(\sigma)$, and this is functorial. Conclude that $\widehat{\mathcal{F}}$ is a $\widehat{\tau}$-family in $\mathcal{C}$. Since $\widehat{\tau}$ is filtering, there is a limit

$$\eta_{\widehat{\mathcal{F}}}:\underline{a}_{\widehat{\tau}}\Rightarrow\widehat{\mathcal{F}}.$$

Prove that this defines a limit $\eta_{\mathcal{F}}\underline{a}_\tau\Rightarrow\mathcal{F}$.

Finally, use this to **prove** that limits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

(xiii)(bis, Colimits) Repeat the steps above for colimits in place of limits. Use this to **prove** that colimits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

Practice with Limits and Colimits Exercise. In each of the following cases, say whether the given category (a) has an initial object, (b) has a final object, (c) has a zero object, (d) has finite products, (e) has finite coproducts, (f) has arbitrary products, (g) has arbitrary coproducts, (h) has arbitrary limits (sometimes called *inverse limits*), (i) has arbitrary colimits (sometimes called *direct limits*), (j) coproducts / filtering colimits preserve monomorphisms, (k) products / cofiltering limits preserve epimorphisms.

(i) The category **Sets** whose objects are sets, whose morphisms are set maps, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(ii) The opposite category $\mathbf{Sets}^{\text{opp}}$.

(iii) For a given set S, the category whose objects are elements of the set, and where the only morphisms are the identity morphisms from an element to that same element. What if the set is the empty set? What if the set is a singleton set?

(iv) For a partially ordered set $(S, \leq)$, the category whose objects are elements of S, and where the Hom set between two elements x, y of S is a singleton set if $x \leq y$ and empty otherwise. What if the partially ordered set $(S, \leq)$ is a **lattice**, i.e., every finite subset (resp. arbitrary subset) has a least upper bound and has a greatest lower bound?

(v) For a monoid $(M, \cdot, 1)$, the category with only one object whose Hom set, with its natural composition and identity, is $(M, \cdot, 1)$. What is M equals $\{1\}$?

(vi) For a monoid $(M, \cdot, 1)$ and an action of that monoid on a set, $\rho : M \times S \to S$, the category whose objects are the elements of S, and where the Hom set from x to y is the subset $M_{x,y} = \{m \in M | m \cdot x = y\}$. What if the action is both transitive and faithful, i.e., S equals M with its left regular representation?

(vii) The category **PtdSets** whose objects are pairs (S, s_0) of a set S and a specified element s_0 of S, i.e., *pointed sets*, whose morphisms are set maps that send the specified point of the domain to the specified point of the target, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(viii) The category **Monoids** whose objects are monoids, whose morphisms are homomorphisms of monoids, whose composition is sual composition, and whose identity morphisms are usual identity maps.

(ix) For a specified monoid $(M,\cdot,1)$, the category whose objects are pairs (S,ρ) of a set S and an action $\rho: M\times S \to S$ of M on S, whose morphisms are set maps compatible with the action, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(x) The full subcategory **Groups** of **Monoids** whose objects are groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xi) The full subcategory $\mathbb{Z}-\text{mod}$ of **Groups** whose objects are Abelian groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xii) The full subcategory **FiniteGroups** of **Groups** whose objects are finite groups. Are coproducts, resp. products, in the subcategory also coproducts, resp. products, in the larger category **Groups**? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xiii) The full subcategory $\mathbb{Z}-\text{mod}_{\text{tor}}$ of $\mathbb{Z}-\text{mod}$ consisting of torsion Abelian groups, i.e., every element has finite order (allowed to vary from element to element). Are coproducts, resp. products, preserved by the inclusion functor? Are monomorphisms, resp. epimorphisms preserved?

(xiv) The category **Rings** whose objects are associative, unital rings, whose morphisms are homomorphisms of rings (preserving the multiplicative identity), whose composition is the usual composition, and whose identity morphisms are the usual identity maps. **Hint.** For the coproduct of two associative, unital rings $(R',+,0,\cdot',1')$ and $(R'',+,0,\cdot'',1'')$, first form the coproduct $R'\oplus R''$ of $(R',+,0)$ and $(R'',+,0)$ as a $\mathbb{Z}$-module, then form the total tensor product ring $T^{\bullet}_{\mathbb{Z}}(R'\oplus R'')$ as in the previous problem set. For the two natural maps $q': R' \hookrightarrow T^1_{\mathbb{Z}}(R'\oplus R'')$ and $q'': R'' \hookrightarrow T^1_{\mathbb{Z}}(R'\oplus R'')$ form the left-right ideal $I\subset T^{\bullet}_{\mathbb{Z}}(R'\oplus R'')$ generated by $q'(1')-1$, $q''(1'')-1$, $q'(r'\cdot' s')-q'(r')\cdot q'(s')$, and $q''(r''\cdot'' s'')-q''(r'')\cdot q''(s'')$ for all elements $r',s'\in R'$ and $r'',s''\in R''$. Define

$$p: T^1_{\mathbb{Z}}(R'\oplus R'') \to R,$$

to be the quotient by I. Prove that $p\circ q': R'\to R$ and $p\circ q'': R''\to R$ are ring homomorphisms that make R into a coproduct of R' and R''.

(xv) The full subcategory **CommRings** of **Rings** whose objects are commutative, unital rings. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvi) The full subcategory **NilCommRings** of **CommRings** whose objects are commutative, unital rings such that every noninvertible element is nilpotent. Does the inclusion functor preserve coproducts, resp. products? (Be careful about products!) Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvii) Let R and S be associative, unital rings. Let $R-\text{mod}$, resp. $\text{mod}-S$, $R-S-\text{mod}$, be the category of left R-modules, resp. right S-modules, $R-S$-bimodules. Does the inclusion functor from $R-S-\text{mod}$ to $R-\text{mod}$, resp. to $\text{mod}-S$, preserve coproduct, products, monomorphisms and epimorphisms?

(xviii) Let $(I, \le)$ be a partially ordered set. Let $\mathcal{C}$ be a category. An $(I, \le)$-*system* in $\mathcal{C}$ is a datum

$$c = ((c_i)_{i \in I}, (f_{i,j})_{(i,j) \in I \times I, i \le j})$$

where every c_i is an object of $\mathcal{C}$, where for every pair $(i,j) \in I \times I$ with $i \le j$, $c_{i,j}$ is an element of $\mathrm{Hom}_{\mathcal{C}}(c_i, c_j)$, and satisfying the following conditions: (a) for every $i \in I$, $c_{i,i}$ equals Id_{c_i}, and (b) for every triple $(i,j,k) \in I$ with $i \le j$ and $j \le k$, $c_{j,k} \circ c_{i,j}$ equals $c_{i,k}$. For every pair of $(I, \le)$-systems in $\mathcal{C}$, $c = ((c_i)_{i \in I}, (c_{i,j})_{i \le j})$ and $c' = ((c'_i)_{i \in I}, (c'_{i,j})_{i \le j})$, a morphism $g : c \to c'$ is defined to be a datum $(g_i)_{i \in I}$ of morphisms $g_i \in \mathrm{Hom}_{\mathcal{C}}(c_i, c'_i)$ such that for every $(i,j) \in I \times I$ with $i \le j$, $g_j \circ c_{i,j}$ equals $c'_{i,j} \circ g_i$. Composition of morphisms g and g' is componentwise $g'_i \circ g_i$, and identities are $\mathrm{Id}_c = (\mathrm{Id}_{c_i})_{i \in I}$. This category is $\mathrm{Fun}((I, \le), \mathcal{C})$, and is sometimes referred to as the category of $(I, \le)$-*presheaves*. Assuming $\mathcal{C}$ has finite coproducts, resp. finite products, arbitrary coproducts, arbitrary products, a zero object, kernels, cokernels, etc., what can you say about $\mathrm{Fun}((I, \le), \mathcal{C})$?

(xix) Let $\mathcal{C}$ be a category that has arbitrary products. Let $(I, \le)$ be a partially ordered set whose associated category as in (iv) has finite coproducts and has arbitrary products. The main example is when $I = \mathfrak{U}$ is the collection of all open subsets U of a topology on a set X, and where $U \le V$ if $U \supseteq V$. Then coproduct is intersection and product is union. Motivated by this case, an *covering* of an element i of I is a collection $\underline{j} = (j_\alpha)_{\alpha \in A}$ of elements j_α of I such that for every α, $i \le j_\alpha$, and such that i is the product of $(j_\alpha)_{\alpha \in A}$ in the sense of (iv). In this case, for every $(\alpha, \beta) \in A \times A$, define $j_{\alpha,\beta}$ to be the element of I such that $j_\alpha \le j_{\alpha,\beta}$, such that $j_\beta \le j_{\alpha,\beta}$, and such that $j_{\alpha,\beta}$ is a coprodcut of (j_α, j_β). An $(I, \le)$-presheaf $c = ((c_i)_{i \in I}, (c_{i,j})_{i \le j})$ is an $(I, \le)$-*sheaf* if for every element i of I and for every covering $\underline{j} = (j_\alpha)_{\alpha \in A}$, the following diagram in $\mathcal{C}$ is *exact* in a sense to be made precise,

$$c_i \xrightarrow{q} \prod_{\alpha \in A} c_{j_\alpha} \overset{p'}{\rightrightarrows} p'' \prod_{(\alpha,\beta) \in A \times A} c_{j_{\alpha,\beta}}.$$

For every $\alpha \in A$, the factor of q,

$$\mathrm{pr}_\alpha \circ q : c_i \to c_{j_\alpha},$$

is defined to be c_{i,j_α}. For every $(\alpha, \beta) \in A \times A$, the factor of p',

$$\mathrm{pr}_{\alpha,\beta} \circ p' : \prod_{\gamma \in A} c_{j_\gamma} \to c_{j_{\alpha,\beta}},$$

is defined to be $c_{j_\alpha, j_{\alpha,\beta}} \circ \mathrm{pr}_\alpha$. Similarly, $\mathrm{pr}_{\alpha,\beta} \circ p''$ is defined to be $c_{j_\beta, j_{\alpha,\beta}} \circ \mathrm{pr}_\beta$. The diagram above is *exact* in the sense that q is a monomorphism in $\mathcal{C}$ and q is a fiber product in $\mathcal{C}$ of the pair of morphisms (p', p''). The category of $(I, \le)$ is the full subcategoryof the category of $(I, \le)$-presheaves whose objects are $(I, \le)$-sheaves. Does this subcategory have coproducts, products, etc.? Does the inclusion functor preserve coproducts, resp. products, monomorphisms, epimorphisms? Before considering the general case, it is probably best to first consider the case that $\mathcal{C}$ is $\mathbb{Z}-\mathrm{mod}$, and then consider the case that $\mathcal{C}$ is **Sets**.

12 Adjoint Pairs and Yoneda Functors

Adjoint Pairs and Representable Functors. Let $\mathcal{A}$ be a category, and let $\mathcal{B}$ be a strictly small category. Let $L : \mathcal{A} \to \mathcal{B}$ be a covariant functor. For every object b of $\mathcal{B}$, assume that the following contravariant functor from $\mathcal{A}$ to **Sets** is representable,

$$\mathrm{Hom}_{\mathcal{B}}(L(-), b) : \mathcal{A}^{\mathrm{opp}} \to \mathbf{Sets}.$$

Prove that there exists an adjoint pair (L, R, θ, η). Using the opposite adjoint pair $(R^{\mathrm{opp}}, L^{\mathrm{opp}}, \eta^{\mathrm{opp}}, \theta^{\mathrm{opp}})$, formulate and prove the analogous result for a contravariant functor R from a category $\mathcal{A}$ to a strictly small category $\mathcal{B}$.

The Yoneda Functor as an Adjoint Functor. Let $\mathcal{A}$ be a strictly small category, so that there is a well-defined category $\mathbf{Sets}^{\mathcal{A}}$ of set-valued covariant functors from $\mathcal{A}$ with natural transformations as morphisms (independent of axioms on inaccessible cardinals or Grothendieck universes). As in Example , for every ordered pair (a, a') of objects of $\mathcal{A}$, composition in $\mathcal{A}$ enriches the set $H^a_{a'} := \mathrm{Hom}_{\mathcal{A}}(a, a')$ with an $H^{a'}_{a'} - H^a_a$-action. For every set S together with a right H^a_a-action, define $H^{S,a}_{a'}$ to be the set of right H^a_a-equivariant maps from S to $H^a_{a'}$,

$$H^{S,a}_{a'} = \mathrm{Hom}_{\mathbf{Set}-H^a_a}(S, H^a_{a'}).$$

This is compatible with postcomposition by $\mathcal{A}$-morphisms in $H^{a'}_{a''}$. Altogether, this defines a covariant, set-valued functor,

$$h^{S,a} : \mathcal{A} \to \mathbf{Sets}, \quad h^{S,a}(a') = H^{S,a}_{a'},$$

the **Yoneda functor** of a and S. Prove that the rule that associates to a set with right H^a_a-action the covariant functor $h^{S,a}$ is itself a functor,

$$h^{-,a} : \mathbf{Sets} - H^a_a \to \mathbf{Sets}^{\mathcal{A}}.$$

Conversely, for every set-valued functor F on $\mathcal{A}$, the set $F(a)$ is enriched with a right H^a_a-action. Prove that the rule associating to each set-valued functor F on $\mathcal{A}$ the set $F(a)$ with its right H^a_a-action is itself a functor,

$$-(a) : \mathbf{Sets}^{\mathcal{A}} \to \mathbf{Sets} - H^a_a.$$

Prove that these two functors are adjoint, i.e., there is a binatural bijection

$$\mathbf{Hom}_{\mathbf{Sets}-H^a_a}(S, F(a)) \cong \mathbf{Hom}_{\mathbf{Sets}^{\mathcal{A}}}(h^{S,a}, F).$$

In particular, when S equals H^a_a with its right regular action this gives the usual Yoneda bijection,

$$F(a) \cong \mathbf{Hom}_{\mathbf{Sets}^{\mathcal{A}}}(h^a, F).$$

Specializing further, when F equals the Yoneda functor $h^{a'}$, this gives a binatural bijection,

$$H^a_{a'} \cong \mathrm{Hom}_{\mathbf{Sets}^{\mathcal{A}}}(h^a, h^{a'}).$$

Deduce that the rule,

$$h : \mathcal{A} \to \mathbf{Sets}^{\mathcal{A}}, \quad a \mapsto h^a,$$

is an equivalence of the category $\mathcal{A}$ with a full subcategory of the functor category $\mathbf{Sets}^{\mathcal{A}}$. Formulate and prove the analogous result for the contravariant Yoneda functors. Finally, if you know the axioms about inaccessible cardinals or the notion of Grothendieck universes, formulate a version of this for categories that are not necessarily strictly small.

13 Preservation of Exactness by Adjoint Additive Functors

Exactness and adjoint pairs. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let (L, R, θ, η) be an adjoint pair of additive functors

$$L : \mathcal{A} \to \mathcal{B}, \ R : \mathcal{B} \to \mathcal{A}.$$

(a) For every short exact sequence in $\mathcal{A}$,

$$\Sigma : \ 0 \longrightarrow A' \xrightarrow{q_A} A \xrightarrow{p_A} A'' \longrightarrow 0,$$

for every object B in $\mathcal{B}$, prove that the induced morphism of Abelian groups,

$$\mathrm{Hom}_{\mathcal{A}}(p_A, R(B)) : \mathrm{Hom}_{\mathcal{A}}(A'', R(B)) \to \mathrm{Hom}_{\mathcal{A}}(A, R(B)),$$

is a monomorphism. Conclude that also the associated morphism of Abelian groups,

$$\mathrm{Hom}_{\mathcal{B}}(L(p_A), B) : \mathrm{Hom}_{\mathcal{B}}(L(A''), B) \to \mathrm{Hom}_{\mathcal{B}}(L(A), B),$$

is a monomorphism. In the special case that B equals $\mathrm{Coker}(L(p_A))$, use this to conclude that B must be a zero object. Conclude that R preserves epimorphisms.

(b) Prove that the following induced diagram of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{A}}(A'', R(B)) \xrightarrow{p_A^*} \mathrm{Hom}_{\mathcal{A}}(A, R(B)) \xrightarrow{q_A^*} \mathrm{Hom}_{\mathcal{A}}(A', R(B)).$$

Conclude that also the following associated diagram of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{B}}(L(A''), B) \xrightarrow{p_A^*} \mathrm{Hom}_{\mathcal{B}}(L(A), B) \xrightarrow{q_A^*} \mathrm{Hom}_{\mathcal{B}}(L(A'), B).$$

In the special case that B equals $\mathrm{Coker}(L(q_A))$, conclude that the induced epimorphism $B \to L(A'')$ is split. Conclude that L is half-exact, hence right exact.

(c) Use similar arguments, or opposite categories, to conclude that also R is left exact.

(d) In case R is exact (not just left exact), prove that for every projective object P of $\mathcal{A}$, also $L(P)$ is a projective object of $\mathcal{B}$. Similarly, if L is exact (not just right exact), prove that for every injective object I of $\mathcal{A}$, also $R(I)$ is an injective object of $\mathcal{A}$.

14 Derived Functors as Adjoint Pairs

Problem 0.(The Cochain Functor of an Additive Functor) Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Denote by $\mathrm{Ch}(\mathcal{A})$, respectively $\mathrm{Ch}(\mathcal{B})$, the associated Abelian category of cochain complexes of objects of $\mathcal{A}$, resp. of objects of $\mathcal{B}$.

Let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor. There is an induced additive functor,

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$$

that associates to a cochain complex

$$A^\bullet = ((A^n)_{n\in\mathbb{Z}}, (d_A^n : A^n \to A^{n+1})_{n\in\mathbb{Z}}),$$

in $\mathcal{A}$ the cochain complex

$$F(A^\bullet) = ((F(A^n))_{n\in\mathbb{Z}}, (F(d_A^n) : F(A^n) \to F(A^{n+1}))_{n\in\mathbb{Z}}).$$

(a) Prove that F is half-exact, resp. left exact, right exact, exact, if and only if $\mathrm{Ch}(F)$ is half-exact, resp. left exact, right exact, exact.

(b) Prove that the functor $\mathrm{Ch}(F)$ induces natural transformations,

$$\theta^n_{B,F} : B^n \circ \mathrm{Ch}(F) \Rightarrow F \circ B^n,\ \theta^n_{F,Z} : F \circ Z^n \Rightarrow Z^n \circ \mathrm{Ch}(F).$$

Thus, for the functor $\overline{A}^n = A^n/B^n(A^\bullet)$, there is also an induced natural transformation,

$$\theta_{\overline{\cdot},F} : \overline{\cdot}^n \circ \mathrm{Ch}(F) \Rightarrow F \circ \overline{\cdot}^n.$$

(c) Assume now that F is right exact (half-exact and preserves epimorphisms). Denote by

$$p^n : Z^n \Rightarrow H^n,$$

the usual natural transformation of functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$. Conclude the existence of a unique natural transformation

$$\theta^n_{F,H} : F \circ H^n \Rightarrow H^n \circ \mathrm{Ch}(F),$$

such that for every $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, the following diagram commutes,

$$\begin{array}{ccc} F(Z^n(A^\bullet)) & \xrightarrow{F(p^n)} & F(H^n(A^\bullet)) \\ \Big\downarrow{\scriptstyle \theta^n_{F,Z}(A^\bullet)} & & \Big\downarrow{\scriptstyle \theta^n_{F,H}(A^\bullet)} \\ Z^n(\mathrm{Ch}(F)(A^\bullet)) & \xrightarrow[p^n]{} & H^n(\mathrm{Ch}(F)(A^\bullet)) \end{array}.$$

Finally, for every short exact sequence in $\text{Ch}(\mathcal{A})$,

$$\Sigma: \ 0 \longrightarrow K^\bullet \xrightarrow{u^\bullet} A^\bullet \xrightarrow{v^\bullet} 0,$$

such that also $F(\Sigma)$ is a short exact sequence in $\text{Ch}(\mathcal{B})$ (this holds, for instance, if Σ is term-by-term split), prove that the following diagram commutes,

$$\begin{array}{ccc} F(H^n(Q^\bullet)) & \xrightarrow{F(\delta^n_\Sigma)} & F(H^{n+1}(K^\bullet)) \\ {\scriptstyle \theta^n_{F,H}(Q^\bullet)}\downarrow & & \downarrow{\scriptstyle \theta^{n+1}_{F,H}(K^\bullet)} \\ H^n(F(Q^\bullet)) & \xrightarrow[\delta^n_{F(\Sigma)}]{} & > H^{n+1}(F(K^\bullet)) \end{array}.$$

(d) Assume now that F is left exact (half-exact and preserves monomorphisms). Denote by

$$q^n : H^n(A^\bullet) \Rightarrow \overline{A}^n = A^n/B^n(A^\bullet),$$

the usual natural transformation of functors $\text{Ch}(\mathcal{A}) \to \mathcal{A}$. Conclude the existence of a unique natural transformation

$$\theta^n_{H,F} : H^n \circ \text{Ch}(F) \Rightarrow F \circ H^n,$$

such that for every $A^\bullet$ in $\text{Ch}(\mathcal{A})$, the following diagram commutes,

$$\begin{array}{ccc} H^n(\text{Ch}(F)(A^\bullet)) & \xrightarrow{q^n} & \overline{\text{Ch}(F)(A^\bullet)}^n \\ {\scriptstyle \theta^n_{H,F}(A^\bullet)}\downarrow & & \downarrow{\scriptstyle \theta^n_{\cdot,F}(A^\bullet)} \\ \overline{\text{Ch}(F)(A^\bullet)}^n & \xrightarrow[F(q^n)]{} & F(\overline{A}^n) \end{array}.$$

Finally, for every short exact sequence in $\text{Ch}(\mathcal{A})$,

$$\Sigma: \ 0 \longrightarrow K^\bullet \xrightarrow{u^\bullet} A^\bullet \xrightarrow{v^\bullet} 0,$$

such that also $F(\Sigma)$ is a short exact sequence in $\text{Ch}(\mathcal{B})$ (this holds, for instance, if Σ is term-by-term split), prove that the following diagram commutes,

$$\begin{array}{ccc} H^n(F(Q^\bullet)) & \xrightarrow{\delta^n_{F(\Sigma)}} & H^{n+1}(F(K^\bullet)) \\ {\scriptstyle \theta^n_{H,F}(Q^\bullet)}\downarrow & & \downarrow{\scriptstyle \theta^{n+1}_{H,F}(K^\bullet)} \\ F(H^n(Q^\bullet)) & \xrightarrow[F(\delta^n_\Sigma)]{} & F(H^{n+1}(K^\bullet)) \end{array}$$

Preservation of Direct Sums Exercise. Let $\mathcal{A}$ be an additive category. Let A_1 and A_2 be objects of $\mathcal{A}$. Let $(q_1 : A_1 \to A, q_2 : A_2 \to A)$ be a coproduct (direct sum) in $\mathcal{A}$. Define $p_1 : A \to A_1$

to be the unique morphism in $\mathcal{A}$ such that $p_1 \circ q_1$ equals Id_{A_1} and $p_1 \circ q_2$ is zero. Similarly define $p_2 : A \to A_2$ to be the unique morphism in $\mathcal{A}$ such that $p_2 \circ q_1$ is zero and $p_2 \circ q_2$ equals Id_{A_2}. Prove that $q_1 \circ p_1 + q_2 \circ p_2$ equals Id_A both compose with q_i to equal q_i, and thus both are equal. Conclude that $(p_1 : A \to A_1, p_2 : A \to A_2)$ is a product in $\mathcal{A}$.

Now let $\mathcal{B}$ be a second additive category, and let

$$F : \mathcal{A} \to \mathcal{B}$$

be an additive functor. Define $B_i = F(A_i)$ and $B = F(A)$. Prove that $F(p_i) \circ F(q_j)$ equals Id_{B_i} if $j = i$ and equals 0 otherwise. Also prove that Id_B equals $F(q_1) \circ F(p_1) + F(q_2) \circ F(p_2)$. Conclude that both $(F(q_1) : B_1 \to B, F(q_2) : B_2 \to B)$ is a coproduct in $\mathcal{B}$ and $(F(p_1) : B \to B_1, F(p_2) : B \to B_2)$ is a product in $\mathcal{B}$. Hence, additive functors preserve direct sums. In particular, additive functors send split exact sequences to split exact sequences.

Preservation of Homotopies Exercise. Let $\mathcal{A}$ be an Abelian category. Let $A^\bullet$ and $C^\bullet$ be cochain complexes in $\mathrm{Ch}(\mathcal{A})$. Let $f^\bullet : A^\bullet \to C^\bullet$ be a cochain morphism. A *homotopy* from $f^\bullet$ to 0 is a sequence $(s^n : A^n \to C^{n-1})_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$,

$$f^n = d_C^{n-1} \circ s^n + s^{n+1} \circ d_A^n.$$

In this case, $f^\bullet$ is called *homotopic* to 0 or *null homotopic*. Cochain morphisms $g^\bullet, h^\bullet : A^\bullet \to C^\bullet$ are *homotopic* if $f^\bullet = g^\bullet - h^\bullet$ is homotopic to 0.

(a) Prove that the null homotopic cochain morphisms form an Abelian subgroup of $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A^\bullet, C^\bullet)$. Moreover, prove that the precomposition or postcomposition of a null homotopic cochain morphism with an arbitrary cochain morphism is again null homotopic (the null homotopic cochain morphisms form a "left-right ideal" with respect to composition).

(b) If $f^\bullet$ is homotopic to 0, prove that for every $n \in \mathbb{Z}$, the induced morphism,

$$H^n(f^\bullet) : H^n(A^\bullet) \to H^n(C^\bullet),$$

is the zero morphism. In particular, if $\mathrm{Id}_{A^\bullet}$ is homotopic to 0, conclude that every $H^n(A^\bullet)$ is a zero object.

(c) For a short exact sequence in $\mathcal{A}$

$$\Sigma : \ 0 \longrightarrow K \xrightarrow{\ q\ } A \xrightarrow{\ p\ } Q \longrightarrow 0,$$

considered as a cochain complex $A^\bullet$ in $\mathcal{A}$ concentrated in degrees −1, 0, 1, prove that a homotopy from $\mathrm{Id}_{A^\bullet}$ to 0 is the same thing as a splitting of the short exact sequence.

(d) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}).$$

If F is half-exact, resp. left exact, right exact, exact, prove that also $\mathrm{Ch}(F)$ is half-exact, resp. left exact, right exact, exact. Prove that $\mathrm{Ch}(F)$ preserves homotopies. In particular, if $g^\bullet$ and $h^\bullet$ are homotopic in $\mathrm{Ch}(\mathcal{A})$, then for every integer $n \in \mathbb{Z}$, $H^n(\mathrm{Ch}(F)(g^\bullet))$ equals $H^n(\mathrm{Ch}(F)(h^\bullet))$.

Preservation of Translation Exercise. Let $\mathcal{A}$ be an Abelian category. For every integer m, for every cochain complex $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, define $T^m(A^\bullet) = A^\bullet[m]$ to be the cochain complex such that $T^m(A^\bullet)^n = A^{m+n}$, and with differential

$$d^n_{T^m(A^\bullet)} : T^m(A^\bullet)^n \to T^m(A^\bullet)^{n+1}$$

equal to $(-1)^m d^{m+n}_{A^\bullet}$. For every cochain morphism

$$f^\bullet : A^\bullet \to C^\bullet,$$

define

$$T^m(f^\bullet)^n : T^m(A^\bullet)^n \to T^m(C^\bullet)^n$$

to be f^{m+n}. Finally, for every homotopy $s^\bullet$ from $g^\bullet - h^\bullet$ to 0, define

$$T^m(s^\bullet)^n = (-1)^m s^{m+n}.$$

(a) Prove that $T^m : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ is an additive functor that is exact. Prove that T^0 is the identity functor. Also prove that $T^m \circ T^\ell$ equals $T^{m+\ell}$. Prove that not only are T^m and T^{-m} inverse functors, but also (T^m, T^{-m}) is an adjoint pair of functors (which implies that also (T^{-m}, T^m) is an adjoint pair). Finally, if $s^\bullet$ is a homotopy from $g^\bullet - h^\bullet$ to 0, prove that $T^m(s^\bullet)$ is a homotopy from $T^m(g^\bullet) - T^m(h^\bullet)$ to 0.

(b) Via the identification $T^m(A^\bullet)^n = A^{m+n}$, prove that the subfunctor $Z^n(T^m(A^\bullet))$ is naturally identified with $Z^{m+n}(A^\bullet)$. Similarly, prove that the subfunctor $B^n(T^m(A^\bullet))$ is naturally identified with $B^{m+n}(A^\bullet)$. Thus, show that the epimorphism $(T^m(A^\bullet))^n \to \overline{T^m(A^\bullet)}^n$ is identified with the epimorphism $A^{m+n} \to \overline{A}^{m+n}$. Finally, use these natural equivalences to deduce a natural equivalence of half-exact, additive functors $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$,

$$\iota^{m,n} : H^{m+n} \Rightarrow H^n \circ T^m.$$

(c) For a short exact sequence in $\mathrm{Ch}(\mathcal{A})$,

$$\Sigma : \ K^\bullet \xrightarrow{q^\bullet} A^\bullet \xrightarrow{p^\bullet} Q^\bullet \longrightarrow 0,$$

for the associated short exact sequence,

$$\Sigma[+1] = T(\Sigma) : \ T(K^\bullet) \xrightarrow{T(q^\bullet)} T(A^\bullet) \xrightarrow{T(p^\bullet)} T(Q^\bullet) \longrightarrow 0,$$

prove that the following diagram commutes,

$$\begin{array}{ccc} H^{n+1}(Q^\bullet) & \xrightarrow{-\delta_\Sigma^{n+1}} & H^{n+1}(K^\bullet) \\ {\scriptstyle \iota^n(Q^\bullet)}\downarrow & & \downarrow{\scriptstyle \iota^{n+1}(K^\bullet)} \\ H^n(T(Q^\bullet)) & \xrightarrow[\delta^n_{T(\Sigma)}]{} & H^{n+1}(T(K^\bullet)) \end{array}$$

Iterate this to prove that for every $m \in \mathbb{Z}$, $\delta^n_{\Sigma[m]}$ is identified with $(-1)^m \delta_\Sigma^{n+m}$.

(d) For every integer m, define

$$e_{\geq m} : \mathrm{Ch}^{\geq m}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$$

to be the full additive subcategory whose objects are complexes $A^\bullet$ such that for every $n < m$, A^n is a zero object. (From here on, writing $A = 0$ for an object A means "A is a zero object".) Check that $\mathrm{Ch}^{\geq m}(\mathcal{A})$ is an Abelian category, and $e_{\geq m}$ is an exact functor. For every integer m, define the "brutal truncation"

$$\sigma_{\geq m} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}^{\geq m}(\mathcal{A}),$$

to be the additive functor such that for every object $A^\bullet$

$$(\sigma_{\geq m} A^\bullet)^n = \begin{cases} A^n, & n \geq m \\ 0, & n < m \end{cases}$$

and for every morphism $u^\bullet : A^\bullet \to C^\bullet$,

$$(\sigma_{\geq m} f^\bullet)^n = \begin{cases} f^n, & n \geq m, \\ 0, & n < m \end{cases}$$

Check that $\sigma_{\geq m}$ is exact and is right adjoint to $e_{\geq m}$. For the natural transformation,

$$\eta_{\geq m} : e_{\geq m} \circ \sigma_{\geq m} \Rightarrow \mathrm{Id}_{\mathrm{Ch}(\mathcal{A})},$$

check that the induced natural transformation,

$$\overline{\eta_{\geq m}(A^\bullet)^n} : \overline{(\sigma_{\geq m}(A))^n} \to \overline{A^n},$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$A^m \twoheadrightarrow \overline{A^m}.$$

Check that the induced natural transformation

$$Z^n(\eta_{\geq m}(A^\bullet)) : Z^n(\sigma_{\geq m}(A^\bullet)) \to Z^n(A^\bullet),$$

is zero for $n < m$, and it is the identity for $n \geq m$. Check that the induced natural transformation,

$$B^n(\eta_{\geq m}(A^\bullet)) : B^n(\sigma_{\geq m}(A^\bullet)) \to B^n(A^\bullet),$$

is zero for $n \le m$, and it is the identity for $n > m$. Check that the induced natural transformation,

$$H^n(\eta_{\ge m}(A^\bullet)) : H^n(\sigma_{\ge m}(A^\bullet)) \to H^n(A^\bullet),$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$Z^m(A^\bullet) \twoheadrightarrow H^n(A^\bullet).$$

Check that for every integer ℓ, there is a unique (exact) equivalence of categories,

$$T^\ell_m : \mathrm{Ch}^{\ge m}(\mathcal{A}) \to \mathrm{Ch}^{\ge \ell+m}(\mathcal{A}),$$

such that $T^\ell_m \circ \sigma_{\ge m}$ equals $\sigma_{\ge \ell+m} \circ T^\ell$, and T^ℓ_m. Check that $(T^\ell_m, T^{-\ell}_{\ell+m})$ is an adjoint pair of functors, so that also $(T^{-\ell}_{\ell+m}, T^\ell_m)$ is an adjoint pair of functors.

(d)bis Similarly, define the "good truncation"

$$\tau_{\ge m} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}^{\ge m}(\mathcal{A}),$$

to be the additive functor such that for every object $A^\bullet$

$$(\tau_{\ge m} A^\bullet)^n = \begin{cases} A^n, & n > m, \\ \overline{A^m}, & n = m, \\ 0, & n < m \end{cases}$$

and for every morphism $u^\bullet : A^\bullet \to C^\bullet$,

$$(\tau_{\ge m} f^\bullet)^n = \begin{cases} f^n, & n > m, \\ \overline{f^m}, & n = m, \\ 0, & n < m \end{cases}$$

Check that τ_m is right exact and is left adjoint to $e_{\ge m}$. For the natural transformation

$$\theta_m : \mathrm{Id}_{\mathrm{Ch}(\mathcal{A})} \Rightarrow e_m \circ \tau_{\ge m},$$

check that the induced morphism,

$$Z^n(\theta_{A^\bullet}) : Z^n(A^\bullet) \to Z^n(\tau_{\ge m}(A^\bullet)),$$

is zero for $n < m$, is the identity for $n > m$, and for $n = m$ it is the epimorphism,

$$Z^n(A^\bullet) \to H^n(A^\bullet).$$

Check that the induced natural transformation,

$$B^n(\theta_{A^\bullet}) : B^n(A^\bullet) \to B^n(\tau_{\ge m}(A^\bullet)),$$

is zero for $n \leq m$, and it is the identity for $n > m$. Check that the induced natural transformation,

$$\overline{\theta_{A^\bullet}}^n : \overline{A}^n \to \overline{\tau_{\geq m}(A^\bullet)}^n$$

is zero for $n < m$, and it is the identity for $n \geq m$. Check that the induced natural transformation,

$$H^n(\theta_{A^\bullet}) : H^n(A^\bullet) \to H^n(\tau_{\geq m}(A^\bullet)),$$

is zero for $n < m$, and it is the identity for $n \geq m$.

Finally, e.g., using the opposite category, formulate and prove the corresponding results for the full embedding,

$$e_{\leq m} : \mathrm{Ch}^{\leq m}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A}),$$

whose objects are complexes $A^\bullet$ such that A^n is a zero object for all $n > m$. In particular, note that although the sequence of brutal truncations,

$$0 \longrightarrow \sigma_{\geq m}(A^\bullet) \xrightarrow{\eta_{\geq m}(A^\bullet)} A^\bullet \xrightarrow{\theta_{\leq m-1}(A^\bullet)} \sigma_{\leq m-1}(A^\bullet) \longrightarrow 0$$

is exact, the corresponding morphisms of good truncations,

$$\mathrm{Ker}(\theta_{\geq m}(A^\bullet)) \hookrightarrow \tau_{\leq m}(A^\bullet), \quad \tau_{\geq m}(A^\bullet) \twoheadrightarrow \mathrm{Coker}(\eta_{\leq m}(A^\bullet)),$$

are not isomorphisms; in the first case the cokernel is $H^m(A^\bullet)[m]$, and in the second case the kernel is $H^m(A^\bullet)[m]$. However, check that the natural morphisms,

$$\tau_{\leq m-1}(A^\bullet) \xrightarrow{\eta_{\leq m-1}} \mathrm{Ker}(\theta_{\geq m}(A^\bullet),$$

$$\mathrm{Coker}(\eta_{\leq m-1}(A^\bullet)) \xrightarrow{\theta_{\geq m}} \tau_{\geq m}(A^\bullet),$$

are quasi-isomorphisms. (One reference slightly misstates this, claiming that the morphisms are isomorphisms, which is "morally" correct after passing to the derived category.)

(e) Beginning with the cohomological δ-functor (in all degrees) $\mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$,

$$H^\bullet = ((H^n)_{n\in\mathbb{Z}}, (\delta^n)_{n\in\mathbb{Z}}),$$

the associated cohomological δ-functor,

$$H^\bullet \circ T^\ell = ((H^n \circ T^\ell)_{n\in}, (\delta^n \circ T^\ell)_{n\in\mathbb{Z}}),$$

the cohomological δ-functor

$$H^{\bullet+\ell} = ((H^{n+\ell})_{n\in\mathbb{Z}}, (\delta^{n+\ell})_{n\in\mathbb{Z}}),$$

and the equivalence,

$$\iota^{\ell,0} : H^\ell \Rightarrow H^0 \circ T^\ell,$$

prove that there exists a unique natural transformation of cohomological δ-functors,

$$\theta_\ell : H^{\bullet+\ell} \Rightarrow H^\bullet \circ T^\ell, \ \ (\theta_\ell^n : H^{n+\ell} \Rightarrow H^n \circ T^\ell)_{n\in\mathbb{Z}},$$

and that $\theta_\ell^n = (-1)^{n\ell}\iota^{\ell,n}$.

(e)bis The truncation $\tau_{\geq m}H^\bullet$ in degrees $\geq m$ is obtained by replacing H^m by the subfunctor Z^m. Check that θ_ℓ restricts to a natural transformation $\tau_{\geq \ell+m}H^{\bullet+\ell} \to \tau_{\geq m}H^\bullet \circ T^\ell$. Assuming that $\tau_{\geq m}H^\bullet$ is a universal cohomological δ-functor in degrees $\geq m$, conclude that also $\tau_{\geq \ell+m}H^\bullet$ is a universal cohomological δ-functor in degrees $\geq \ell + m$. Also, formulate and prove the corresponding result for the universal δ-functors $\tau_{\leq 0}H^\bullet$ and $\tau_{\leq m}H^\bullet$.

(f) Let $\mathcal{B}$ be an Abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. This induces an additive functor

$$\text{Ch}(F) : \text{Ch}(\mathcal{A}) \to \text{Ch}(\mathcal{B}).$$

Prove that $\text{Ch}(F) \circ T_{\mathcal{A}}$ equals $T_{\mathcal{B}} \circ \text{Ch}(F)$.

Compatibility with Automorphisms Exercise. Let $\mathcal{A}$ be an Abelian category. Let

$$\Sigma : \ 0 \longrightarrow K^\bullet \xrightarrow{q^\bullet} A^\bullet \xrightarrow{p^\bullet} Q^\bullet \longrightarrow 0$$

be a short exact sequence in $\text{Ch}(\mathcal{A})$. Let

$$u^\bullet : K^\bullet \to K^\bullet, \ \ v^\bullet : Q^\bullet \to Q^\bullet$$

be isomorphisms in $\text{Ch}(\mathcal{A})$.

(a) Prove that the following sequence is a short exact sequence,

$$\Sigma_{u^\bullet,v^\bullet} : \ 0 \longrightarrow K^\bullet \xrightarrow{q^\bullet\circ u^\bullet} A^\bullet \xrightarrow{v^\bullet\circ p^\bullet} Q^\bullet \longrightarrow 0.$$

(b) Prove that the following diagrams are commutative diagrams.

$$\begin{array}{cccccccccc}
\Sigma_{u^\bullet,\text{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0 \\
\widetilde{u}\downarrow & & & u^\bullet\downarrow & & \downarrow\text{Id}_A & & \downarrow\text{Id}_Q & & , \\
\Sigma_{\text{Id}_K,\text{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0
\end{array}$$

$$\begin{array}{cccccccccc}
\Sigma_{u^\bullet,\text{Id}_Q} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{p^\bullet} & Q^\bullet & \longrightarrow & 0 \\
\widetilde{v}\downarrow & & & \text{Id}_K\downarrow & & \downarrow\text{Id}_A & & \downarrow v^\bullet & & . \\
\Sigma_{u^\bullet,v^\bullet} : & 0 & \longrightarrow & K^\bullet & \xrightarrow{q^\bullet\circ u^\bullet} & A^\bullet & \xrightarrow{v^\bullet\circ p^\bullet} & Q^\bullet & \longrightarrow & 0
\end{array}$$

(c) Use the commutative diagram of long exact sequences associated to a commutative diagrams of short exact sequences to prove that

$$\delta^n_{\Sigma} = H^{n+1}(u^\bullet) \circ \delta^n_{\Sigma_{u^\bullet,v^\bullet}} \circ H^n(v^\bullet),$$

for every integer n.

Compatibility with Natural Transformations of Additive Functors. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories.

(a) For additive functors,

$$F, G : \mathcal{A} \to \mathcal{B},$$

let

$$\alpha : F \Rightarrow G,$$

be a natural transformation. For every cochain complex $A^\bullet$ in $\text{Ch}(\mathcal{A})$, prove that

$$(\alpha_{A^n} : F(A^n) \to G(A^n))_{n\in\mathbb{Z}}$$

is a morphism of cochain complexes in $\text{Ch}(\mathcal{B})$,

$$\text{Ch}(\alpha)(A^\bullet) : \text{Ch}(F)(A^\bullet) \to \text{Ch}(G)(A^\bullet).$$

(b) Prove that the rule $A^\bullet \mapsto \text{Ch}(\alpha)(A^\bullet)$ is a natural transformation

$$\text{Ch}(\alpha) : \text{Ch}(F) \Rightarrow \text{Ch}(G).$$

Moreover, for every morphism $u^\bullet : C^\bullet \to A^\bullet$ in $\text{Ch}(\mathcal{A})$, and for every homotopy $(s^n : C^n \to A^{n-1})_{n\in\mathbb{Z}}$ from $u^\bullet$ to 0, prove that also $\text{Ch}(\alpha)(A^\bullet) \circ \text{Ch}(F)(s^\bullet)$ equals $\text{Ch}(G)(s^\bullet) \circ \text{Ch}(\alpha)(C^\bullet)$.

(c) For the identity natural transformation $\text{Id}_F : F \Rightarrow F$, prove that $\text{Ch}(\text{Id}_F)$ is the identity natural transformation $\text{Ch}(F) \Rightarrow \text{Ch}(F)$. Also, for every pair of natural transformations of additive functors $\mathcal{A} \to \mathcal{B}$,

$$\alpha : F \Rightarrow G, \quad \beta : E \Rightarrow F,$$

for the composite natural transformation $\alpha \circ \beta$, prove that $\text{Ch}(\alpha \circ \beta)$ equals $\text{Ch}(\alpha) \circ \text{Ch}(\beta)$. In this sense, Ch is a "functor" from the "2-category" of Abelian categories to the "2-category" of Abelian categories.

Derived Functors as Adjoint Pairs Exercise. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Assume that $\mathcal{A}$ has enough injective objects. Thus, every object A admits an injective resolution in $\text{Ch}(\mathcal{A})$,

$$\begin{array}{rccccccccc}
A[0]: & \dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \\
{\scriptstyle \epsilon_A}\downarrow & & & \downarrow & & {\scriptstyle \epsilon}\downarrow & & \downarrow & & \\
I_A^\bullet: & \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{d_I^0} & I^1 & \xrightarrow{d_I^1} & \dots
\end{array},$$

which is functorial up to null homotopies (in particular, any two injective resolutions are homotopy equivalent). Moreover, for every short exact sequence in $\mathcal{A}$,

$$\Sigma:\ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

there exists a diagram of injective resolutions with rows being short exact sequences in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccccc}
\Sigma[0]: & 0 & \longrightarrow & K[0] & \xrightarrow{q[0]} & A[0] & \xrightarrow{p[0]} & Q[0] & \longrightarrow & 0 \\
{\scriptstyle \epsilon_\Sigma}\downarrow & & & {\scriptstyle \epsilon_K}\downarrow & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_Q} & & \\
I_\Sigma: & 0 & \longrightarrow & I_K^\bullet & \xrightarrow{q^\bullet} & I_A^\bullet & \xrightarrow{p^\bullet} & I_Q^\bullet & \longrightarrow & 0
\end{array}$$

whose associated short exact sequences in $\mathcal{A}$,

$$I_\Sigma^n:\ 0 \longrightarrow I_K^n \xrightarrow{q^n} I_A^n \xrightarrow{p^n} I_Q^n \longrightarrow 0,$$

are automatically split. Moreover, this diagram of injective resolutions is functorial up to homotopy, i.e., for every commutative diagram of short exact sequences in $\mathcal{A}$,

$$\begin{array}{ccccccccccc}
\Sigma: & 0 & \longrightarrow & K & \xrightarrow{q} & A & \xrightarrow{p} & Q & \longrightarrow & 0 \\
{\scriptstyle u}\downarrow & & & {\scriptstyle u_K}\downarrow & & \downarrow{\scriptstyle u_A} & & \downarrow{\scriptstyle u_Q} & & \\
\widetilde{\Sigma}: & 0 & \longrightarrow & \widetilde{K} & \xrightarrow{\widetilde{q}} & \widetilde{A} & \xrightarrow{\widetilde{p}} & \widetilde{Q} & \longrightarrow & 0
\end{array},$$

there exists a commutative diagram in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccccc}
I_\Sigma: & 0 & \longrightarrow & I_K & \xrightarrow{q^\bullet} & I_A & \xrightarrow{p^\bullet} & I_Q & \longrightarrow & 0 \\
{\scriptstyle u^\bullet}\downarrow & & & {\scriptstyle u_K^\bullet}\downarrow & & \downarrow{\scriptstyle u_A^\bullet} & & \downarrow{\scriptstyle u_Q^\bullet} & & \\
I_{\widetilde{\Sigma}}: & 0 & \longrightarrow & I_{\widetilde{K}} & \xrightarrow{\widetilde{q}^\bullet} & I_{\widetilde{A}} & \xrightarrow{\widetilde{p}^\bullet} & I_{\widetilde{Q}} & \longrightarrow & 0
\end{array}$$

compatible with the morphisms ϵ_-, and the cochain morphisms $u^\bullet$ making all diagrams commute are unique up to homotopy.

As proved in lecture, there is an associated cohomological δ-functor in degrees ≥ 0, $R^\bullet F$, with

$$R^n F : \mathcal{A} \to \mathcal{B}, \quad R^n F(A) = H^n(\mathrm{Ch}(F)(A^\bullet)).$$

For every short exact sequence in $\mathcal{A}$,

$$\Sigma:\ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

the corresponding complex in $\mathcal{B}$, $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(F)(I_\Sigma):\ 0 \longrightarrow \mathrm{Ch}(F)(I_K^\bullet) \xrightarrow{\mathrm{Ch}(F)(q^\bullet)} \mathrm{Ch}(F)(I_A^\bullet) \xrightarrow{\mathrm{Ch}(F)(p^\bullet)} \mathrm{Ch}(F)(I_Q^\bullet) \longrightarrow 0,$$

has associated complexes in $\mathcal{B}$,

$$\mathrm{Ch}(F)(I_\Sigma)^n : \ 0 \longrightarrow F(I_K^n) \xrightarrow{F(q^n)} F(I_A^n) \xrightarrow{F(p^n)} F(I_Q^n) \longrightarrow 0,$$

being split exact sequences (since the additive functor F preserves split exact sequences), and hence $\mathrm{Ch}(F)(I_\Sigma)$ is a short exact sequence in $\mathcal{B}$. The maps $\delta^n_{R^\bullet F,\Sigma}$ are the connecting maps determined by the Snake Lemma for this short exact sequence,

$$\delta^n_{\mathrm{Ch}(F)(I_\Sigma)} : H^n(\mathrm{Ch}(F)(I_Q^\bullet)) \to H^{n+1}(\mathrm{Ch}(F)(I_K^\bullet)).$$

Associated to ϵ, there are morphisms in $\mathcal{B}$

$$F(\epsilon_A) : F(A) \to R^0F(A).$$

(a) Let $G : \mathcal{A} \to \mathcal{B}$ be an additive functor. Let

$$\alpha : F \Rightarrow G,$$

be a natural transformation. For every object A of $\mathcal{A}$ and for every injective resolution $\epsilon : A[0] \to I_A^\bullet$, there is an induced morphism in $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(\alpha)(I_A^\bullet) : \mathrm{Ch}(F)(I_A^\bullet) \to \mathrm{Ch}(G)(I_A^\bullet).$$

This induces morphisms,

$$R^n\alpha(A) : R^nF(A) \to R^nG(A),$$

given by,

$$H^n(\mathrm{Ch}(\alpha)(I_A^\bullet)) : H^n(\mathrm{Ch}(F)(I_A^\bullet)) \to H^n(\mathrm{Ch}(G)(I_A^\bullet)).$$

For every n, prove that $A \mapsto R^n\alpha(A)$ defines a natural transformation

$$R^n\alpha : R^nF \Rightarrow R^nG.$$

Moreover, prove that this natural transformation is a morphism of δ-functors, i.e., for every short exact sequence,

$$\Sigma : \ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

for every integer n, the following diagram commutes,

$$\begin{array}{ccc} R^nF(Q) & \xrightarrow{\delta^n_{R^\bullet F,\Sigma}} & R^{n+1}F(K) \\ \downarrow{\scriptstyle R^n\alpha(Q)} & & \downarrow{\scriptstyle R^{n+1}\alpha(K)} \\ R^nG(Q) & \xrightarrow[\delta^n_{R^\bullet G,\Sigma}]{} & R^{n+1}G(K) \end{array}$$

(b) Prove that the morphisms $F(\epsilon_A)$ form a natural transformation, $\rho_F : F \to R^0F$.

(c) Prove that R^0F is a left-exact functor. Assuming that F is left-exact, prove that ρ_F is a natural equivalence of funcors. In particular, conclude that $\rho_{R^0F} : R^0F \to R^0(R^0F)$ is a natural equivalence of functors.

(d) For every half-exact functor,

$$G : \mathcal{A} \to \mathcal{B},$$

and for every natural transformation,

$$\gamma : F \Rightarrow G,$$

prove that the two natural transformations,

$$R^0\gamma \circ \rho_F, \rho_G \circ \gamma : F \Rightarrow R^0G,$$

are equal. In particular, if G is left-exact, so that ρ_G is a natural equivalence, conclude that there exists a unique natural transformation,

$$\widetilde{\gamma} : R^0F \Rightarrow G,$$

such that γ equals $\widetilde{\gamma} \circ \rho_F$.

(e) Now assume that $\mathcal{A}$ and $\mathcal{B}$ are small Abelian categories. Thus, functors from $\mathcal{A}$ to $\mathcal{B}$ are well-defined in the usual axiomatization of set theory. Let $\text{Fun}(\mathcal{A}, \mathcal{B})$ be the category whose objects are functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are natural transformations of functors. Let $\text{AddFun}(\mathcal{A}, \mathcal{B})$ be the full subcategory of additive functors. Let

$$e : \text{LExactFun}(\mathcal{A}, \mathcal{B}) \to \text{AddFun}(\mathcal{A}, \mathcal{B}),$$

be the full subcategory whose objects are left-exact additive functors from $\mathcal{A}$ to $\mathcal{B}$. Prove that the rule associating to F the left-exact functor R^0F and associating to every natural transformation $\alpha : F \Rightarrow G$ the natural transformation $R^0\alpha : R^0F \Rightarrow R^0G$ is a left adjoint to e.

(f) With the same hypotheses as above, denote by $\text{Fun}_\delta^{\geq 0}(\mathcal{A}, \mathcal{B})$ the category whose objects are cohomological δ-functors from $\mathcal{A}$ to $\mathcal{B}$ concentrated in degrees ≥ 0,

$$T^\bullet = ((T^n : \mathcal{A} \to \mathcal{B})_{n \in \mathbb{Z}}, (\delta_T^n)_{n \in \mathbb{Z}}),$$

and whose morphisms are natural transformations of δ-functors,

$$\alpha^\bullet : S^\bullet \to T^\bullet, \quad (\alpha^n : S^n \Rightarrow T^n)_{n \in \mathbb{Z}}.$$

Denote by

$$(-)^0 : \text{Fun}_\delta^{\geq 0}(\mathcal{A}, \mathcal{B}) \to \text{LExactFun}(\mathcal{A}, \mathcal{B}),$$

the functor that associates to every cohomological δ-functor, $T^\bullet$, the functor, T^0, and that associates to every natural transformation of cohomological δ-functors, $u^\bullet : S^\bullet \to T^\bullet$, the natural transformation $u^0 : S^0 \to T^0$. Denote by

$$R : \text{LExactFun}(\mathcal{A}, \mathcal{B}) \to \text{Fun}_\delta^{\geq 0}(\mathcal{A}, \mathcal{B}),$$

the functor that associates to every left-exact functor, F, the cohomological δ-functor, $R^\bullet F$, and that associates to the natural transformation, $\alpha : F \Rightarrow G$, the natural transformation of cohomological δ-functors, $R^\bullet\alpha : R^\bullet F \Rightarrow R^\bullet G$. Prove that R is left adjoint to $(-)^0$.

(g) In particular, for $n > 0$, prove that $R^0(R^nF)$ is the zero functor. Thus, for every $m \geq n$, $R^m(R^nF)$ is the zero functor.

Right Derived Functors and Filtering Colimits Exercise. Let $\mathcal{B}$ be a cocomplete Abelian category satisfying Grothendieck's condition (AB5). Let I be a small filtering category. Let $C^\bullet : I \to \mathrm{Ch}^\bullet(\mathcal{B})$ be a functor.

(a) For every $n \in \mathbb{Z}$, prove that the natural $\mathcal{B}$-morphism,

$$\operatorname*{colim}_{i\in I} H^n(C^\bullet(i)) \to H^n(\operatorname*{colim}_{i\in I} C^\bullet(i)),$$

is an isomorphism. **Prove** that this extends to a natural isomorphism of cohomological δ-functors. This is "commutation of cohomology with filtered colimits".

(b) Let $\mathcal{A}$ be an Abelian category with enough injective objects. Let $F : I \times \mathcal{A} \to \mathcal{B}$ be a bifunctor such that for every object i of I, the functor $F_i : \mathcal{A} \to \mathcal{B}$ is additive and left-exact. Prove that $F_\infty(-) := \operatorname{colim}_{i\in I} F_i(-)$ also forms an additive functor that is left-exact. Also prove that the natural map

$$\operatorname*{colim}_{i\in I} R^n(F_i) \to R^n(F_\infty)$$

is an isomorphism. This is "commutation of right derived functors with filtered colimits".

15 Constructing Injectives via Adjoint Pairs

Projective / Injective Objects and Adjoint Pairs Exercise. Recall that for a category $\mathcal{C}$, for every object X of $\mathcal{C}$, there is a covariant Yoneda functor,

$$h^X : \mathcal{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(X, B),$$

and for every object Y of $\mathcal{C}$, there is a contravariant Yoneda functor,

$$h_Y : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Sets}, \quad A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, Y).$$

An object X of $\mathcal{C}$ is **projective** if the Yoneda functor h^X sends epimorphisms to epimorphisms. An object Y of $\mathcal{C}$ is **injective** if the Yoneda functor h_Y sends monomorphisms to epimorphisms. The category has **enough projectives** if for every object B there exists a projective object X and an epimorphism $X \to B$. The category has **enough injectives** if for every object A there exists an injective object Y and a monomorphism from A to Y.

(a) Check that this notion agrees with the usual definition of projective and injective for objects in an Abelian category.

(b) For the category **Sets**, assuming the Axioms of Choice, prove that every object is both projective and injective. Deduce the same for the opposite category, $\mathbf{Sets}^{\text{opp}}$.

(c) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let (L, R, θ, η) be an adjoint pair of covariant functors,

$$L : \mathcal{C} \to \mathcal{D}, \quad R : \mathcal{D} \to \mathcal{C}.$$

For every object d of $\mathcal{D}$, prove that

$$\eta(d) : L(R(d)) \to d,$$

is an epimorphism. For every object c of $\mathcal{C}$, prove that

$$\theta : c \to R(L(c)),$$

is a monomorphism. Thus, if every $L(R(d))$ is a projective object, then $\mathcal{C}$ has enough projective objects. Similarly, if every $R(L(c))$ is an injective object, then $\mathcal{C}$ has enough injective objects.

(d) Assuming that R sends epimorphisms to epimorphisms, prove that L sends projective objects of $\mathcal{C}$ to projective objects of $\mathcal{D}$. Thus, if every object of $\mathcal{C}$ is projective, conclude that $\mathcal{D}$ has enough projective objects. More generally, assume further that R is **faithful**, i.e., R sends distinct morphisms to distinct morphisms. Then conclude for every epimorphism $X \to R(D)$ in $\mathcal{C}$, the associated morphism $L(X) \to D$ in $\mathcal{D}$ is an epimorphism. Thus, if $\mathcal{C}$ has enough projective objects, also $\mathcal{D}$ has enough projective objects.

Similarly, assuming that L sends monomorphisms to monomorphisms, prove that R sends injective objects of $\mathcal{D}$ to injective objects of $\mathcal{C}$. Thus, if every object of $\mathcal{D}$ is injective, conclude that there are enough injective objects of $\mathcal{C}$. More generally, assume further that L is faithful. Then conclude for every monomorphism $L(C) \to Y$ in $\mathcal{D}$, the associated morphism $C \to R(Y)$ in $\mathcal{C}$ is a monomorphism. Thus, if $\mathcal{D}$ has enough injective objects, also $\mathcal{C}$ has enough injective objects.

(e) Let S and T be associative, unital algebras. Let $\mathcal{C}$ be the category **Sets**. Let $\mathcal{D}$ be the category $S - T - \text{mod}$ of $S - T$-bimodules. Let

$$R : S - T - \text{mod} \to \mathbf{Sets}$$

be the forgetful functor that sends every $S - T$-bimodule to the underlying set of elements of the bimodule. Prove that R sends epimorphisms to epimorphisms and R is faithful. Prove that there exists a left adjoint functor,

$$L : \mathbf{Sets} \to S - T - \text{mod},$$

that sends every set Σ to the corresponding $S - T$-bimodule, $L(\Sigma)$ of functions $f : \Sigma \to S \otimes_{\mathbb{Z}} T$ that are zero except on finitely many elements of Σ. Since **Sets** has enough projective objects (in fact every object is projective), conclude that $S - T - \text{mod}$ has enough projective objects.

(e) Let S, T and U be associative, unital rings. Let B be a $T - U$-bimodule. Let $\mathcal{C}$ be the Abelian category of $S - T$-bimodules, let $\mathcal{D}$ be the Abelian category of $S - U$-bimodules, let L be the exact, additive functor,

$$L : S - T - \text{mod} \to S - U - \text{mod}, \quad L(A) = A \otimes_T B,$$

and let R be the right adjoint functor,

$$R : S-U-\text{mod} \to S-T-\text{mod}, \quad R(C) = \text{Hom}_{\text{mod}-U}(B, C).$$

Prove that if B is a flat (left) T-module, resp. a faithfully flat (left) T-module, then L sends monomorphisms to monomorphisms, resp. L sends monomorphism to monomorphisms and is faithful. Conclude, then, that R sends injective objects of $S-U-\text{mod}$ to injective objects of $S-T-\text{mod}$, resp. if $S-U-\text{mod}$ has enough injective objects then also $S-T-\text{mod}$ has enough injective objects.

(f) Continuing as above, for every ring homomorphism $U \to T$, prove that the induced $T-U$-module structure on T is faithfully flat as a left T-module. Thus, given rings Λ and Π, define $S = \Lambda$, define $T = \Pi$, and define U to be $\mathbb{Z}$ with its unique ring homomorphism to T. Conclude that if there exist enough injective objects in $\Lambda-\text{mod}$, then there exist enough injective objects in $\Lambda-\Pi-\text{mod}$.

(g) For the next step, define T and U to be Λ, define B to be Λ as a left-right T-module, and define S to be $\mathbb{Z}$. Conclude that if there are enough injective $\mathbb{Z}$-modules, then there are enough injective Λ-modules, and hence there are enough injective $\Lambda-\Pi$-bimodules. Thus, to prove that there are enough $\Lambda-\Pi$-bimodules, it is enough to prove that there are enough $\mathbb{Z}$-modules.

Enough Injective Modules Exercise. Let $\mathcal{A}$ be an Abelian category that has all small products. An object Y of $\mathcal{A}$ is an **injective cogenerator** if Y is injective and for every pair of distinct morphisms,

$$u, v : A' \to A,$$

in $\mathcal{A}$, there exists a morphism $w : A \to Y$ such that $w \circ u$ and $w \circ v$ are also distinct.

(a) Let $\mathcal{C}$ be the category $\textbf{Sets}^{\text{opp}}$. For an object Y of $\mathcal{A}$, define L to be the Yoneda functor

$$h_Y : \mathcal{A} \to \textbf{Sets}^{\text{opp}}, \quad h_Y(A) = \text{Hom}_{\mathcal{A}}(A, Y).$$

Similarly, define the functor,

$$R : \textbf{Sets}^{\text{opp}} \to \mathcal{A}, \quad L(\Sigma) = \text{"Hom}_{\textbf{Sets}}(\Sigma, Y)\text{"},$$

that sends every set Σ to the object $R(\Sigma)$ in $\mathcal{A}$ that is the direct product of copies of Y indexed by elements of Σ. Prove that L and R are an adjoint pair of functors.

(b) Assuming that $\mathcal{A}$ has an injective cogenerator Y, prove that L sends monomorphisms to monomorphisms, and prove that L is faithful. Conclude that $\mathcal{A}$ has enough injective objects.

(c) Now let S be an associative, unital ring (it suffices to consider the special case that S is $\mathbb{Z}$). Let $\mathcal{A}$ be $\text{mod}-S$. Use the Axiom of Choice to prove Baer's criterion: a right S-module Y is injective if and only if for every right ideal J of S, the induced set map

$$\text{Hom}_{\text{mod}-S}(S, Y) \to \text{Hom}_{\text{mod}-S}(J, Y)$$

is surjective. In particular, if S is a principal ideal domain, conclude that Y is injective if and only if Y is divisible.

(d) Finally, defining S to be $\mathbb{Z}$, conclude that $Y = \mathbb{Q}/\mathbb{Z}$ is injective, since it is divisible. Finally, for every Abelian group A and for every nonzero element a of A, conclude that there is a nonzero $\mathbb{Z}$-module homomorphism $\mathbb{Z}\cdot a \to \mathbb{Q}/\mathbb{Z}$. Thus, for every pair of elements $a', a'' \in A$ such that $a = a' - a''$ is nonzero, conclude that there exists a $\mathbb{Z}$-module homomorphisms $w : A \to \mathbb{Q}/\mathbb{Z}$ such that $w(a') - w(a'')$ is nonzero. Conclude that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathbb{Z}$. Thus $\text{mod} - \mathbb{Z}$ has enough injective objects. Thus, for every pair of associative, unital rings Λ, Π, the Abelian category $\Lambda - \Pi - \text{mod}$ has enough injective objects.

Enough Injectives / Projectives in the Cochain Category Exercise. Let S be an associative, unital ring. Prove that $\text{Ch}^{\geq 0}(S)$ has enough injective objects, and prove that $\text{Ch}^{\leq 0}(S)$ has enough projective objects.

16 The Koszul Complex via Adjoint Pairs

Exterior Algebra CDGA as an Adjoint Pair Exercise. Let R be a commutative, unital ring. An *associative, unital, graded commutative R-algebra* (with homological indexing) is a triple

$$A_\bullet = ((A_n)_{n\in\mathbb{Z}}, (m_{p,q} : A_p \times A_q \to A_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0))$$

of a sequence $(A_n)_{n\in\mathbb{Z}}$ of R-modules, of a sequence $(m_{p,q})_{p,q\in\mathbb{Z}}$ of R-bilinear maps, and an R-module morphism ϵ such that the following hold.

(i) For the associated R-module $A = \bigoplus_{n\in\mathbb{Z}} A_n$ and the induced morphism $m : A \times A \to A$ whose restriction to each $A_p \times A_q$ equals $m_{p,q}$, $(A, m, \epsilon(1))$ is an associative, unital, R-algebra.

(ii) For every $p, q \in \mathbb{Z}$, for every $a_p \in A_p$ and for every $a_q \in A_q$, $m_{q,p}(a_q, a_p)$ equals $(-1)^{pq} m_{p,q}(a_p, a_q)$.

(a) Prove that the R-submodules of A,

$$A_{\geq 0} = \bigoplus_{n\geq 0} A_n, \quad A_{\leq 0} = \bigoplus_{n\leq 0} A_n,$$

are both associative, unital R-subalgebras. Moreover, prove that the R-submodule,

$$A_{>0} = \bigoplus_{n>0} A_n, \text{ resp. } A_{<0} = \bigoplus_{n<0} A_n,$$

is a left-right ideal in $A_{\geq 0}$, resp. in $A_{\leq 0}$.

(b) For associative, unital, graded commutative R-algebras $A_\bullet$ and $B_\bullet$, a graded homomorphism of R-algebras is a collection

$$f_\bullet = (f_n : A_n \to B_n)_{n\geq 0}$$

such that for the unique R-module homomorphism $f : A \to B$ whose restriction to every A_n equals f_n, f is an R-algebra homomorphism. Prove that such $f_\bullet$ is uniquely reconstructed from the homomorphism f. Prove that Id_A comes from a unique graded homomorphism $\text{Id}_{A_\bullet}$. Prove that for a graded homomorphism of R-algebras, $g_\bullet : B_\bullet \to C_\bullet$, the composition $g \circ f$ arises from a unique graded homomorphism of R-algebras, $A_\bullet \to C_\bullet$. Using this to define composition of homomorphisms of graded R-algebras, prove that composition is associative and the identity morphisms abe are left-right identities for composition. Conclude that these notions form a category $R - \mathrm{GrComm}$ of associative, unital, graded commutative R-algebras. Prove that the rule $A_\bullet \mapsto A$, $f_\bullet \mapsto f$ defines a faithful functor

$$R - \mathrm{GrComm} \to R - \mathrm{Algebra}.$$

Give an example showing that this functor is not typically full.

(c) Let $A_\bullet$ be an associative, unital, graded commutative R-algebra. Prove that R is commutative (in the usual sense) if and only if A_n is a zero module for every even integer n. Denote by $R - \mathrm{Comm}$ the category of associative, unital R-algebras S that are commutative. Denote by $\mathbb{Z} - R - \mathrm{Comm}$ the faithful (but not full) subcategory whose objects are triples,

$$S_\bullet = ((S_n)_{n\in\mathbb{Z}}, (m_{p,q} : S_p \times S_q \to S_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to S_0))$$

as above, but such that the multiplication is commutative rather than graded commutative, i.e., $m_{q,p}(s_q, s_p) = m_{p,q}(s_p, s_q)$. Prove that there is a functor,

$$v_{\mathrm{even}} : R - \mathrm{GrComm} \to \mathbb{Z} - R - \mathrm{Comm},$$

$((A_n)_{n\in\mathbb{Z}}, (m_{p,q} : A_p \times A_q \to A_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0)) \mapsto ((A_{2n})_{n\in\mathbb{Z}}, (m_{2p,2q} : A_{2p} \times A_{2q} \to A_{2(p+q)})_{p,q\in\mathbb{Z}}, (\epsilon : R \to A_0 =$

and $f_\bullet : A_\bullet \to B_\bullet$ maps to $v_{\mathrm{ev}}(f) = (f_{2n})_{n\in\mathbb{Z}}$. Also prove that there is a left adjoint to v_{even},

$$w_{\mathrm{even}} : \mathbb{Z} - R - \mathrm{Comm} \to R - \mathrm{GrComm},$$

where $w_{\mathrm{even}}(S_\bullet)_{2n}$ equals S_n, where $w_{\mathrm{even}}(S_\bullet)_p$ is the zero module for every odd p, where

$$A_{2p} \times A_{2q} \to A_{2(p+q)}$$

is $m_{p,q}$ for $S_\bullet$, and where $R \to A_0$ is $\epsilon : R \to S_0$. For a morphism $f_\bullet : S_\bullet \to T_\bullet$ in $\mathbb{Z} - R - \mathrm{Comm}$, $w_{\mathrm{even}}(f_\bullet)$ is the unique morphism whose component in degree $2n$ equals f_n for every $n \in \mathbb{Z}$.

(d) Let e be an odd integer. For every associative, unital, graded commutative R-algebra $A_\bullet$ define $v_e(A_\bullet)$ to be the collection

$$((A_{ne})_{n\in\mathbb{Z}}, (m_{pe,qe} : A_{pe} \times A_{qe} \to A_{(p+q)e})_{p,q\in\mathbb{Z}}, \epsilon : R \to A_0 = A_{0e}).$$

Prove that $v_e(A_\bullet)$ is again an associative, unital, graded commutative R-algebra. For every morphism of associative, unital, graded commutative R-algebras, $f_\bullet : A_\bullet \to B_\bullet$, the collection $v_e(f_\bullet) = (f_{ne})_{n\in\mathbb{Z}}$ is a morphism of associative, unival, graded commutative R-algebras, $v_e(A_\bullet) \to v_e(B_\bullet)$. Prove that this defines a functor,

$$v_e : R - \mathrm{GrComm} \to R - \mathrm{GrComm}.$$

This is sometimes called the *Veronese functor* (it is closely related to the Veronese morphism of projective spaces). If e is positive, prove that the induced morphism $v_e(A_{\ge 0}) \to v_e(A_\bullet)$, resp. $v_e(A_{\le 0}) \to v_e(A_\bullet)$, is an isomorphism to $(v_e(A_\bullet))_{\ge 0}$, resp. to $(v_e(A_\bullet))_{\le 0}$. Similarly, if e is negative (e.g., if e equals -1), this defines an isomorphism to $(v_e(A_\bullet))_{\le 0}$, resp. to $(v_e(A_\bullet))_{\ge 0}$. Prove that v_1 is the identity functor. For odd integers d and e, construct a natural isomorphism of functors,

$$v_{d,e} : v_d \circ v_e \Rightarrow v_{de},$$

prove that $v_{d,1}$ and $v_{1,e}$ are identity natural transformations, and prove that these natural isomorphisms are associative: $v_{de,f} \circ (v_{d,e} \circ v_f)$ equals $v_{d,ef} \circ (v_d \circ v_{e,f})$ for all odd integers d, e and f.

(e) For every associative, unital, graded commutative R-algebra $A_\bullet$, for every odd integer e, define

$$w_e : R - \text{GrComm} \to R - \text{GrComm},$$

where $w_e(A_\bullet)_{ne}$ equals A_n for every integer n, and where $w_e(A_\bullet)_m$ is a zero module if e does not m. For every morphism $f_\bullet : A_\bullet \to B_\bullet$, define $w_e(f_\bullet)$ to the be the unique morphism whose component in degree en equals f_n for every $n \in \mathbb{Z}$. Prove that w_e is a functor. For the natural isomorphism,

$$\theta_e(A_\bullet) : A_\bullet \to v_e(w_e(A_\bullet)), (A_n \xrightarrow{=} A_n)_{n\in\mathbb{Z}}$$

and the natural monomorphisms

$$\eta_e(B_\bullet) : w_e(v_e(B_\bullet)) \to B_\bullet, (B_{ne} \xrightarrow{=} B_{ne})_{n\in\mathbb{Z}},$$

prove that $(w_e, v_e, \theta_e, \eta_e)$ is an adjoint pair.

(f) For every integer $n \ge 0$, recall from Problem 5(iv) of Problem Set 1, that there is a functor,

$$\bigwedge_R^n : R - \text{mod} \to R - \text{mod},\ M \mapsto \bigwedge_R^n(M).$$

In particular, there is a natural isomorphism

$$\epsilon(M) : R \to \bigwedge_R^0(M),$$

and there is a natural isomorphism,

$$\theta(M) : M \to \bigwedge_R^1(M).$$

By convention, for every integer $n < 0$, define $\bigwedge_R^n(M)$ to be the zero module. For every pair of integers $q, r \ge 0$, prove that the natural R-bilinear map

$$\otimes : M^{\otimes q} \times M^{\otimes r} \to M^{\otimes(q+r)},\ ((m_1 \otimes \cdots \otimes m_q), (m_1' \otimes \cdots \otimes m_r')) \mapsto m_1 \otimes \dots m_q \otimes m_1' \otimes \cdots \otimes m_r',$$

factors uniquely through an R-bilinear map,

$$\wedge : \bigwedge_R^q(M) \times \bigwedge_R^r(M) \to \bigwedge_R^{q+r}(M).$$

Prove that $\bigwedge_R^\bullet(M)$ is an associative, unital, graded commutative R-algebra. For every R-module homomorphism $\phi : M \to N$, prove that the associated R-module homomorphisms,

$$\bigwedge_R^n(\phi) : \bigwedge_R^n(M) \to \bigwedge_R^n(N),$$

define a morpism of associative, unital, graded commutative R-algebras,

$$\bigwedge_R^\bullet(\phi) : \bigwedge_R^\bullet(M) \to \bigwedge_R^\bullet(N).$$

Prove that for every R-module homomorphism $\psi : N \to P$, $\bigwedge_R^\bullet(\psi \circ \phi)$ equals $\bigwedge_R^\bullet(\psi) \circ \bigwedge_R^\bullet(\phi)$. Also prove that $\bigwedge_R^\bullet(\mathrm{Id}_M)$ is the identity morphism of $\bigwedge_R^\bullet(M)$.

(g) An associative, unital, graded commutative R-algebra $A_\bullet$ is (strictly) 0-*connected*, resp. *weakly* 0-*connected*, if the inclusion $A_{\geq 0} \to A$ is an isomorphism and the R-module homorphism ϵ is an isomorphism, resp. an epimorphism. If R is a field, prove that every weakly 0-connected algebra is strictly 0-connected. Denote by

$$R - \mathrm{GrComm}_{\geq 0}, \text{ resp. } R - \mathrm{GrComm}'_{\geq 0}$$

the full subcategory of $R - \mathrm{GrComm}$ whose objects are the 0-connected algebras, resp. the weakly 0-connected algebras. Prove that v_{even} restricts to a functor,

$$R - \mathrm{GrComm}_{\geq 0} \to \mathbb{Z}_+ - R - \mathrm{Comm},$$

where $\mathbb{Z}_+ - R - \mathrm{Comm}$ is the full subcategory of $\mathbb{Z} - R - \mathrm{Comm}$ of algebras graded in nonnegative degrees such that $R \to S_0$ is an isomorphism. For e an odd positive integer, prove that v_e and w_e restrict to an adjoint pair of functors,

$$v_e : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{GrComm}_{\geq 0},$$

$$w_e : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{GrComm}_{\geq 0}.$$

For every odd positive integer e, define a functor

$$\Phi_e : R - \mathrm{GrComm}_{\geq 0} \to R - \mathrm{mod},$$

that sends $A_\bullet$ to A_e and sends $f_\bullet$ to f_e. Of course, for every odd positive integer d, $\Phi_e \circ v_d$ is naturally isomorphic to Φ_{de} and $\Phi_{de} \circ w_d$ is Φ_e. By the previous part, there is a functor

$$\bigwedge_R^\bullet : R - \mathrm{mod} \to R - \mathrm{GrComm}_{\geq 0}$$

that sends every module M to the 0-connected, associative, unital, graded commutative R-algebra $(\bigwedge_R^n(M))_{n\geq 0}$. Moreover, there is a natural transformation,

$$\theta : \mathrm{Id}_{R-\mathrm{mod}} \Rightarrow \Phi_1 \circ \bigwedge_R^{\bullet}.$$

Prove that this extends uniquely to an adjoint pair of functors

$$(\bigwedge_R^{\bullet}, \Phi_1, \theta, \eta).$$

Using the natural isomorphisms $\Phi_e \circ v_d = \Phi_{de}$ and $\Phi_{de} \circ w_d = \Phi_e$, prove that there is also an adjoint pair of functors

$$(w_e \circ \bigwedge_R^{\bullet}, \Phi_e, \theta, \eta_e).$$

The Koszul Complex CDGA as an Adjoint Pair. Let R be a commutative, unital ring. A (homological, unital, associative, graded commutative) *differential graded R-algebra* is a pair

$$((C_n)_{n\in\mathbb{Z}}, (\wedge : C_p \times C_q \to C_{p+q})_{p,q\in\mathbb{Z}}, (\epsilon : R \to C_0), (d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}),$$

of an associative, unital, graded commutative R-algebra $C_\bullet$ together with R-linear morphisms $(d_n)_{n\in\mathbb{Z}}$ such that $d_{n-1} \circ d_n$ equals 0 for every $n \in \mathbb{Z}$, and that satisfies the graded Leibniz identity,

$$d_{p+q}(c_p \wedge c_q) = d_p(c_p) \wedge c_q + (-1)^p c_p \wedge d_q(c_q),$$

for every $p, q \in \mathbb{Z}$, for every $c_p \in C_p$, and for every $c_q \in C_q$. A *morphism* of differential graded R-algebras,

$$\phi_\bullet : C_\bullet \to A_\bullet,$$

is a morphism $\phi_\bullet = (\phi_n)_{n\in\mathbb{Z}}$ that is simultaneously a morphism of chain complexes of R-modules and a morphism of associative, unital, graded commutative R-algebras.

(a) For morphisms of differential graded R-algebras, $\phi_\bullet : C_\bullet \to A_\bullet$, $\psi_\bullet : D_\bullet \to C_\bullet$, prove that the composition of $\psi_\bullet \circ \phi_\bullet$ of graded R-modules is both a morphism of chain complexes of R-modules and a morphism of associative, unital, graded commutative R-algebras. Thus, it is a composition of morphisms of differential graded R-algebras. With this composition, prove that this defines a category $R - \mathrm{CDGA}$ of differential graded R-algebras.

(b) For every associative, unital, graded commutative R-algebra $A_\bullet$, for every integer n, define $d_{E(A),n} : A_n \to A_{n-1}$ to be the zero morphism. Prove that this gives a differential graded R-algebra, denoted $E(A_\bullet)$. For every morphism $f_\bullet : A_\bullet \to B_\bullet$ of associative, unital, graded commutative R-algebras, prove that $f_\bullet : E(A_\bullet) \to E(B_\bullet)$ is a morphism of differential graded R-algebras, denoted $E(f_\bullet)$. Prove that this defines a functor

$$E : R - \mathrm{GrComm} \to R - \mathrm{CDGA}.$$

For every differential graded R-algebra $C_\bullet$, prove that the subcomplex $Z_\bullet(C_\bullet)$ is a differential graded R-subalgebra with zero differential, and the inclusion,

$$\eta(C_\bullet) : E(Z_\bullet(C_\bullet)) \to C_\bullet,$$

is a morphism of differential graded R-algebras. Also, for every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of differential graded R-algebras, prove that the induced morphism $Z_\bullet(f_\bullet) : Z_\bullet(C_\bullet) \to Z_\bullet(D_\bullet)$ is a morphism of associative, unital, graded commutative R-algebras. Prove that this defines a functor

$$Z_\bullet : R-\text{CDGA} \to R-\text{GrComm}.$$

For every associative, unital, graded commutative R-algebra $A_\bullet$, the inclusion $Z_\bullet(E(A_\bullet)) \to E(A_\bullet)$ is just the identity map, whose inverse,

$$\theta(A_\bullet) : A_\bullet \to Z_\bullet(E(A_\bullet)),$$

is an isomorphism. Prove that $(E, Z_\bullet, \theta, \eta)$ is an adjoint pair of functors. Finally, prove that the subcomplex $B_\bullet(C_\bullet) \subset Z_\bullet(C_\bullet)$ is a left-right ideal in the associative, unital, graded commutative R-algebra $Z_\bullet(C_\bullet)$. Conclude that there is a unique structure of associative, unital, graded commutative R-algebra on the cokernel $H_\bullet(C_\bullet)$ such that the quotient morphism $Z_\bullet(C_\bullet) \to H_\bullet(C_\bullet)$ is a morphism of differential graded R-algebras. Prove that altogether this defines a functor,

$$H : R-\text{CDGA} \to R-\text{GrComm}.$$

(c) A differential graded R-algebra $C_\bullet$ is (strictly) 0-*connected*, resp. *weakly* 0-*connected*, if the underlying associative, unital, graded commutative R-algebra is 0-connected, resp. weakly 0-connected. Denote by $R-\text{CDGA}_{\geq 0}$, resp. $R-\text{CDGA}'_{\geq 0}$, the full subcategory of $R-\text{CDGA}$ whose objects are the 0-connected differential graded R-algebras, resp. those that are weakly 0-connected. Prove that the functors above restrict to functors,

$$E : R-\text{GrComm}_{\geq 0} \to R-\text{CDGA}_{\geq 0},$$

$$Z_\bullet : R-\text{CDGA}_{\geq 0} \to R-\text{GrComm}_{\geq 0},$$

such that (E, Z, θ, η) is still an adjoint pair. Similarly, show that H restricts to a functor

$$H : R-\text{CDGA}_{\geq 0} \to R-\text{GrComm}'_{\geq 0}.$$

(d) Denote by $R-\text{CDGA}_{[0,1]}$ the full subcategory of $R-\text{CDGA}_{\geq 0}$ whose objects are 0-connected differential graded R-algebras $C_\bullet$ such that C_n is a zero object for $n > 1$. Prove that every such object is uniquely determined by the data of an R-module C_1 and an R-module homomorphism $d_{C,1} : C_1 \to C_0 = R$, and conversely such data uniquely determine an object of $R-\text{CDGA}_{[0,1]}$. Prove that for such algebras $C_\bullet$ and $D_\bullet$, every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of differential graded R-algebras is uniquely determined by an R-module homomorphism $\phi_1 : C_1 \to D_1$ such that $d_{D,1} \circ \phi_1$ equals $d_{C,1}$, and conversely, such an R-module homomorphism uniquely determines a morphism of differential graded R-algebras. Conclude that there is a functor

$$\sigma_{[0,1]} : R-\mathrm{CDGA}_{\geq 0} \to R-\mathrm{CDGA}_{[0,1]},$$

that associates to every 0-connected differential graded R-algebra $C_\bullet$ the algebra $\sigma_{[0,1]}(C_\bullet)$ uniquely determined by the R-module homomorphism $d_{C,1} : C_1 \to C_0 = R$ and that associates to every morphism $\phi_\bullet : C_\bullet \to D_\bullet$ of 0-connected differential graded R-algebras the morphism,

$$\sigma_{[0,1]}(\phi_\bullet) : \sigma_{[0,1]}(C_\bullet) \to \sigma_{[0,1]}(D_\bullet),$$

uniquely determined by the morphism $\phi_1 : C_1 \to D_1$.

(e) For every R-module M and for every R-module homomorphism $\phi : M \to R$, prove that there exists a unique sequence of R-module homomorphisms,

$$(d_{M,\phi,n} : \bigwedge_R^n(M) \to \bigwedge_R^{n-1}(M))_{n>0},$$

such that d_1 equals ϕ and such that $(\bigwedge_R^\bullet(M), d_{M,\phi})$ is a 0-connected differential graded R-algebra. It may be convenient to begin with the case of a free R-module P and a morphism $\psi : P \to R$, in which case every $\bigwedge_R^n(P)$ is also free and the R-module homomorphisms d_n is uniquely determined by its restriction to a convenient basis. Given a presentation $M = P/K$ such that ψ factors uniquely through $\phi : M \to R$, prove that the associative, unital, graded commutative R-algebra $\bigwedge_R^\bullet(M)$ is the quotient of $\bigwedge_R^\bullet(P)$ by the left-right ideal generated by $K \subset P = \bigwedge_R^1(P)$. Also prove that $d_{P,\psi}$ maps this ideal to itself, i.e., the ideal is differentially-closed. Conclude that there is a unique structure of differential graded algebra on the quotient $\bigwedge_R^\bullet(M)$ such that the quotient map is a morphism of differential graded R-algebras.

(f) Prove that the construction of the previous part defines a functor,

$$\bigwedge_R^\bullet : R-\mathrm{CDGA}_{[0,1]} \to R-\mathrm{CDGA}_{\geq 0}.$$

Prove that for every object $(\phi : M \to R)$ of $R-\mathrm{CDGA}_{[0,1]}$, the morphism

$$\theta(M,\phi) : M \xrightarrow{=} \bigwedge_R^1(M)$$

is a natural isomorphism

$$\theta : \mathrm{Id}_{R-\mathrm{CDGA}_{[0,1]}} \Rightarrow \sigma_{[0,1]} \circ \bigwedge_R^\bullet.$$

Similarly, for every object 0-connected differential graded R-algebra $C_\bullet$, prove that the natural transformation from Problem 10(g),

$$\eta(C_\bullet) : \bigwedge_R^\bullet(C_1) \to C_\bullet,$$

is compatible with the differential on $\bigwedge_R^\bullet(C_1)$ induced by $d_{C,1}: C_1 \to C_0 = R$, i.e., $\eta(C_\bullet)$ is a natural transformation,

$$\eta : \bigwedge_R^\bullet \circ \sigma_{[0,1]} \to \mathrm{Id}_{R-\mathrm{CDGA}_{\geq 0}}.$$

Conclude that $(\bigwedge_R^\bullet, \sigma_{[0,1]}, \theta, \eta)$ is an adjoint pair of functors. For every $\phi : M \to R$ in $R-\mathrm{CDGA}_{[0,1]}$, the associated 0-complete differential graded R-algebra structure on $\bigwedge_R^\bullet(M)$ is called the *Koszul algebra* of $\phi : M \to R$ and denoted $K_\bullet(M, \phi)$.

(g) For every R-module M, and for every R-submodule M' of M, denote by $F^1 \subset \bigwedge_R^\bullet(M)$ the left-right ideal generated by $M' \subset M = \bigwedge_R^1(M)$. For every integer $n \leq 0$, denote by $F^n \subset \bigwedge_R^\bullet(M)$ the entire algebra. For every integer $n \geq 1$, denote by F^n the left-right ideal of $\bigwedge_R^\bullet(M)$ generated by the n-fold self-product $F^1 \cdot \dots \cdot F^1$. For every pair of nonnegative integers p, q, prove that the ideal $F^p \cdot F^q$ equals F^{p+q}. In particular, prove that there is a natural epimorphism,

$$\bigwedge_R^p(F_1^1) \otimes_R \bigwedge_R^q(M) \to F_{p+q}^p.$$

Denote the quotient M/M' by M'', and denote by Σ the short exact sequence,

$$\Sigma : \ 0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0.$$

For every nonnegative integer q, prove that the R-module morphism,

$$\bigwedge_R^q(v) : \bigwedge_R^q(M) \to \bigwedge_R^q(M''),$$

is an epimorphism whose kernel equals F_q^1. Conclude that the composite epimorphism

$$\bigwedge_R^p(M') \otimes_R \bigwedge_R^q(M) \to F_{p+q}^p \to F_{p+q}^p / F_{p+q}^{p+1}$$

factors uniquely through an R-module epimorphism

$$c_{\Sigma,p,q} : \bigwedge_R^p(M') \otimes_R \bigwedge_R^q(M'') \to F_{p+q}^p / F_{p+q}^{p+1}.$$

In case there exists a splitting of Σ, prove that every epimorphism $c_{\Sigma,p,q}$ is an isomorphism. On the other hand, find an example where Σ is not split and some morphism $c_{\Sigma,p,q}$ is not a monomorphism (there exist such examples for $R = \mathbb{C}[x,y]$).

(h) Continuing the previous problem, assume that M'' is isomorphic to R as an R-module (or, more generally, projective of constant rank 1), so that Σ is split. For every nonnegative integer p, conclude that there exists a short exact sequence,

$$\Sigma_{p,1} : \ 0 \longrightarrow \bigwedge_R^{p+1}(M') \xrightarrow{\bigwedge_R^{p+1}(u)} \bigwedge_R^{p+1}(M) \xrightarrow{c_{\Sigma,p,1}^{-1}} \bigwedge_R^p(M') \otimes_R M'' \longrightarrow 0,$$

that is split. Check that this is compatible with the product structure and, thus, defines a short exact sequence of graded (left) $\bigwedge_R^\bullet(M)$-modules,

$$\bigwedge_R^\bullet(\Sigma):\ 0 \longrightarrow \bigwedge_R^\bullet(M') \xrightarrow{\bigwedge_R^\bullet(u)} \bigwedge_R^\bullet(M) \xrightarrow{c_\Sigma^{-1}} \bigwedge_R^\bullet(M')\otimes_R M''[+1] \longrightarrow 0.$$

(i) Now, let $\phi : M \to R$ be an R-module homomorphism. Denote by $\phi' : M' \to R$ the restriction $\phi\circ u$. These morphisms define structures of differential graded R-algebra, $K_\bullet(M,\phi)$ on $\bigwedge_R^\bullet(M)$, and $K_\bullet(M',\phi')$ on $\bigwedge_R^\bullet(M')$. Moreover, the morphism $\bigwedge_R^\bullet(u)$ above is a morphism of differential graded R-modules,

$$K(u) : K_\bullet(M',\phi') \to K_\bullet(M,\phi).$$

Prove that the induced morphism

$$c_\Sigma^{-1} : K_\bullet(M,\phi) \to K_\bullet(M',\phi')\otimes_R M''[+1]$$

is a morphism of cochain complexes. Moreover, for a choice of splitting $s : M'' \to M$, for the induced morphism $\phi'' : M'' \to R$, $\phi'' = \phi\circ s$, for the induced morphism of cochain complexes,

$$\mathrm{Id}_{K_\bullet(M',\phi')}\otimes\phi'' : K_\bullet(M',\phi')\otimes_R M'' \to K_\bullet(M',\phi'),$$

prove that there is a unique commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc}
T_{\mathrm{Id}\otimes\phi''}: & 0 & \longrightarrow & K_\bullet(M',\phi') & \xrightarrow{q_{\mathrm{Id}\otimes\phi''}} & \mathrm{Cone}(\mathrm{Id}\otimes\phi'') & \xrightarrow{p_{\mathrm{Id}\otimes\phi''}} & K_\bullet(M',\phi')\otimes_R M''[+1] & \longrightarrow & 0 \\
\downarrow{\scriptstyle\widetilde{s}} & & & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow{\scriptstyle\widetilde{s}} & & \downarrow{\scriptstyle\mathrm{Id}} & & \\
K(\Sigma) & 0 & \longrightarrow & K_\bullet(M',\phi') & \xrightarrow{K_\bullet(u)} & K_\bullet(M,\phi) & \xrightarrow{c_\Sigma^{-1}} & K_\bullet(M',\phi')\otimes_R M'' & \longrightarrow & 0.
\end{array}$$

(j) With the same hypotheses as above, conclude that there is an exact sequence of homology (remember the shift $[+1]$ above is cohomological),

$$H_0(K_\bullet(M',\phi'))\otimes_R M'' \xrightarrow{\mathrm{Id}\otimes\phi''} H_0(K_\bullet(M',\phi')) \xrightarrow{K_0(u)} H_0(K_\bullet(M,\phi)) \to 0,$$

i.e., $H_0(K_\bullet(M,\phi)) \cong H_0(K_\bullet(M,\phi))/\phi(M'')\cdot H_0(K_\bullet(M,\phi))$ as a quotient algebra of R. Also, for every $n>0$, conclude the existence of a short exact sequence of Koszul homologies,

$$0 \to K_n(M',\phi')\otimes_R R/\mathrm{Im}(\phi'') \xrightarrow{\psi''} K_n(M,\phi) \to K_{n-1}(M',\phi';M'')_{\mathrm{Im}(\phi'')} \to 0,$$

where for every R-module N, $N_{\mathrm{Im}(\phi'')}$ denotes the submodule of elements that are annihilated by the ideal $\mathrm{Im}(\phi'')\subset R$. As graded modules over the associative, unital, graded commutative R-algebra $K_*(M',\phi') = H_*(K_\bullet(M',\phi'))$, this is a short exact sequence,

$$0 \to K_*(M',\phi')\otimes_R R/\mathrm{Im}(\phi'') \xrightarrow{\psi''} K_*(M,\phi) \to K_{*-1}(M',\phi';M'')_{\mathrm{Im}(\phi'')} \to 0,$$

As a special case, if $K_\bullet(M',\phi')$ is acyclic, and if the morphism

$$H_0(K_\bullet(M',\phi'))\otimes_R M'' \xrightarrow{\mathrm{Id}\otimes\phi''} H_0(K_\bullet(M',\phi'))$$

is injective, conclude that also $K_\bullet(M,\phi)$ is acyclic.

(k) Repeat this exercise for the cohomological Koszul complexes $K^\bullet(M,\phi)$.

17 Adjoint Pairs of Simplicial and Cosimplicial Objects

Constant Cosimplicial Objects and the Right Adjoint. Please read the basic definitions of cosimplicial objects in a category $\mathcal{C}$. In particular, for the small category Δ of totally ordered finite sets with nondecreasing morphisms, read the equivalent characterization of a (covariant) functor

$$C : \Delta \to \mathcal{C},$$

via the specification for every integer $r \geq 0$ of an object C^r of $\mathcal{C}$, the specification for every integer $r \geq 0$ and every integer $i = 0, \dots, r+1$, of a morphism,

$$\partial_r^i : C^r \to C^{r+1},$$

and the specification for every integer $r \geq 0$ and every integer $i = 0, \dots, r$, of a morphism,

$$\sigma_{r+1}^i : C^{r+1} \to C^r,$$

satisfying the *cosimplicial identities*: for every $r \geq 0$, for every $0 \leq i < j \leq r+2$,

$$\partial_{r+1}^j \circ \partial_r^i = \partial_{r+1}^i \circ \partial_r^{j-1},$$

for every $0 \leq i \leq j \leq r$,

$$\sigma_{r+1}^j \circ \sigma_{r+2}^i = \sigma_{r+1}^i \circ \sigma_{r+2}^{j+1},$$

and for every $0 \leq i \leq r+1$ and $0 \leq j \leq r$,

$$\sigma_{r+1}^j \circ \partial_r^i = \begin{cases} \partial_{r-1}^i \circ \sigma_r^{j-1}, & i < j, \\ \mathrm{Id}_{C^r}, & i = j, i = j+1, \\ \partial_{r-1}^{i-1} \circ \sigma_r^j, & i > j+1 \end{cases}$$

Moreover, for cosimplicial objects $C^\bullet = (C^r, \partial_r^i, \sigma_{r+1}^i)$ and $\widetilde{C}^\bullet = (\widetilde{C}^r, \widetilde{\partial}_r^i, \widetilde{\sigma}_{r+1}^i)$, read about the equivalent specification of a natural transformation $\alpha^\bullet : C^\bullet \to \widetilde{C}^\bullet$ as the specification for every integer $r \geq 0$ of a $\mathcal{C}$-morphism $\alpha^r : C^r \to \widetilde{C}^r$ such that for every r and i,

$$\widetilde{\partial}_r^i \circ \alpha^r = \alpha^{r+1} \circ \partial_r^i, \quad \widetilde{\sigma}_{r+1}^i \circ \alpha^{r+1} = \alpha^r \circ \sigma_{r+1}^i.$$

Finally, for every pair of morphisms of cosimplicial objects, $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$, a *cosimplicial homotopy* is a specification for every integer $r \geq 0$ and for every integer $i = 0, \dots, r$ of a $\mathcal{C}$-morphism,

$$g_{r+1}^i : C^{r+1} \to \widetilde{C}^r,$$

satisfying the following *cosimplicial homotopy identities*: for every $r \geq 0$,

$$g_{r+1}^0 \circ \partial_r^0 = \alpha^r, \quad g_{r+1}^r \circ \partial_r^{r+1} = \beta^r,$$

$$g^j_{r+1} \circ \partial^i_r = \begin{cases} \widetilde{\partial}^i_{r-1} \circ g^{j-1}_r, & 0 \le i < j \le r, \\ g^{i-1}_{r+1} \circ \partial^i_r, & 0 < i = j \le r, \\ \widetilde{\partial}^{i-1}_{r-1} \circ g^j_r, & 1 \le j+1 < i \le r+1. \end{cases}$$

$$g^j_r \circ \sigma^i_{r+1} = \begin{cases} \widetilde{\sigma}^i_r \circ g^{j+1}_{r+1}, & 0 \le i \le j \le r-1, \\ \widetilde{\sigma}^{i-1}_r \circ g^j_{r+1}, & 0 \le j < i \le r. \end{cases}$$

(a)(Constant Cosimplicial Objects) For every object C of $\mathcal{C}$, define $\mathrm{const}(C)$ to be the rule that associates to every integer $r \ge 0$ the object C of $\mathcal{C}$, and that associates to (r,i) the morphisms $\partial^i_r = \mathrm{Id}_C$, $\sigma^i_{r+1} = \mathrm{Id}_C$. **Prove** that $\mathrm{const}(C)$ is a cosimplicial object of $\mathcal{C}$. For every morphism of objects $\alpha : C \to \widetilde{C}$, **prove** that the specification for every integer $r \ge 0$ of the morphism $\alpha : C \to \widetilde{C}$ defines a morphism of cosimplicial objects,

$$\mathrm{const}(\alpha) : \mathrm{const}(C) \to \mathrm{const}(\widetilde{C}).$$

Prove that $\mathrm{const}(\mathrm{Id}_C)$ is the identity morphism of $\mathrm{const}(C)$. For a pair of morphisms, $\alpha : C \to \widetilde{C}$ and $\beta : \widetilde{C} \to \widehat{C}$, **prove** that $\mathrm{const}(\beta \circ \alpha)$ equals $\mathrm{const}(\beta) \circ \mathrm{const}(\alpha)$. Conclude that these rules define a functor

$$\mathrm{const} : \mathcal{C} \to \mathbf{Fun}(\Delta, \mathcal{C}).$$

Prove that this is functorial in $\mathcal{C}$, i.e., given a functor $F : \mathcal{C} \to \mathcal{D}$, for the associated functor,

$$\mathbf{Fun}(\Delta, F) : \mathbf{Fun}(\Delta, \mathcal{C}) \to \mathbf{Fun}(\Delta, \mathcal{D}),\ (C^r, \partial^i_r, \sigma^i_{r+1}) \mapsto (F(C^r), F(\partial^i_r), F(\sigma^i_{r+1})),$$

$\mathbf{Fun}(\Delta, F) \circ \mathrm{const}_{\mathcal{C}}$ strictly equals $\mathrm{const}_{\mathcal{D}} \circ F$.

(b)(Morphisms from a Constant Cosimplicial Object) For every integer $r \ge 1$ and for every pair of distinct morphisms $[0] \to [r]$, **prove** that there exists a unique Δ-morphism $F : [1] \to [r]$ such that the two morphisms are $F \circ \partial^0_0$ and $F \circ \partial^1_0$. Let $C^\bullet = (C^r, \partial^i_r, \sigma^i_{r+1})$ be a cosimplicial object in $\mathcal{C}$. For every object A of $\mathcal{C}$ and for every morphism, $\alpha^\bullet : \mathrm{const}(A) \to C^\bullet$, of cosimplicial objects, **prove** that $\alpha^0 : A \to C^0$ is a morphism such that $\partial^0_0 \circ \alpha^0$ equals $\partial^1_0 \circ \alpha^0$. **Prove** that the morphism $\alpha^\bullet$ is uniquely determined by α^0, i.e., for every $r \ge 0$, and for every morphism $f : [0] \to [r]$, $\alpha^r : A \to C^r$ equals $C(f) \circ \alpha^0$. Conversely, for every morphism $\alpha^0 : A \to C^0$ such that $\partial^0_0 \circ \alpha^0$ equals $\partial^1_0 \circ \alpha^0$, **prove** that the morphisms $\alpha^r := C(f) \circ \alpha^0$ are well-defined and define a morphism $\alpha^\bullet : \mathrm{const}(A) \to C^\bullet$ of cosimplicial objects. Conclude that the set map,

$$\mathrm{Hom}_{\mathbf{Fun}(\Delta,\mathcal{C})}(\mathrm{const}(A), C^\bullet) \to \{\alpha^0 \in \mathrm{Hom}_{\mathcal{C}}(A, C^0) | \partial^0_0 \circ \alpha^0 = \partial^1_0 \circ \alpha^0\},\ \alpha^\bullet \mapsto \alpha^0,$$

is a bijection. **Prove** that this bijection is natural in both A and in $C^\bullet$. In particular, conclude that the functor,

$$\mathrm{const} : \mathcal{C} \to \mathbf{Fun}(\Delta, \mathcal{C}),$$

is fully faithful. Finally, for every pair of morphisms, $\alpha^0, \beta^0 : A \to C^0$ equalizing ∂^0_0 and ∂^1_0, **prove** that there exists a cosimplicial homotopy $g^i_{r+1} : A \to C^r$ from $\alpha^\bullet$ to $\beta^\bullet$ if and only if β^0 equals α^0, and in this case there is a unique cosimplicial homotopy given by $g^i_{r+1} = \alpha^r = \beta^r$.

(c)(Equalizers in Cartesian Categories) Let $\Delta_{\le 1}$ be the category of totally ordered sets of cardinality ≤ 1. Prove that a functor $C^\bullet : \Delta_{\le 1} \to \mathcal{C}$ is equivalent to the data of a pair of objects C^0, C^1, a pair of morphisms $\partial_0^0, \partial_0^1 : C^0 \to C^1$, and a morphism $\sigma_1^0 : C^1 \to C^0$ such that $\sigma_1^0 \circ \partial_0^0 = \sigma_1^0 \circ \partial_0^1 = \text{Id}_{C^0}$. Let,

$$Z^0 : \mathbf{Fun}(\Delta_{\le 1}, \mathcal{C}) \to \mathcal{C},$$

be a functor and let,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})},$$

be a natural transformation such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors $(\text{const}, Z^0, \theta, \eta)$. **Prove** that the natural transformation θ is a natural isomorphism. **Prove** that for every object $C^\bullet$ of $\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})$, the morphism $\eta_{C^\bullet} : Z^0(C^\bullet) \to C^0$ satisfies $\partial_0^0 \circ \eta_{C^\bullet} = \partial_0^1 \circ \eta_{C^\bullet}$ and is final among all such morphisms. **Prove** that if $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$ are two morphisms of cosimplicial objects, and if $(g_{r+1}^i : C^{r+1} \to \widetilde{C}^r)$ is a cosimplicial homotopy from $\alpha^\bullet$ to $\beta^\bullet$, then $Z^0(\alpha^\bullet)$ equals $Z^0(\beta^\bullet)$.

Assume that $\mathcal{C}$ has finite products. For every pair of objects N^0 and N^1 of $\mathcal{C}$ and for every pair of morphisms $d_0^0, d_0^1 : N^0 \to N^1$, define $C^0 = N^0$, define $C^1 = N^0 \times N^1$, define $\partial_0^0 = (\text{Id}_{C^0}, d_0^0)$, define $\partial_0^1 = (\text{Id}_{C^0}, d_0^1)$, and define $\sigma_1^0 = \text{pr}_{N^0}$. **Prove** that $C^\bullet$ is an object of $\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})$, and **prove** that $\eta_{C^\bullet} : Z^0(C^\bullet) \to C^0$ is an equalizer of $d_0^0, d_0^1 : N^0 \to N^1$. In particular, if $\mathcal{C}$ has both finite products and Z^0, **prove** that $\mathcal{C}$ has all equalizers of a pair of morphisms. For every pair of morphisms $f_0^0 : M_0^0 \to N^1$ and $f_0^1 : M_1^0 \to N^1$ in $\mathcal{C}$, for $N^0 = M_0^0 \times M_1^0$, and for $d_0^0 = f_0^0 \circ \text{pr}_{M_0^0}$ and $d_0^1 = f_0^1 \circ \text{pr}_{M_1^0}$, **prove** that the equalizer of $d_0^0, d_0^1 : N^0 \to N^1$ is a fiber product of f_0^0 and f_0^1. Conclude that $\mathcal{C}$ has all finite fiber products, i.e., $\mathcal{C}$ is a *Cartesian category*. Conversely, assuming that $\mathcal{C}$ is a Cartesian category, then, up to some form of the Axiom of Choice, prove that there exists a functor Z^0 and a natural transformation η such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors.

(d)(The Right Adjoint to the Constant Cosimplicial Object) Assume now that there exists a functor

$$Z^0 : \mathbf{Fun}(\Delta_{\le 1}, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\le 1}, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors. For every cosimplicial object $C^\bullet : \Delta \to \mathcal{C}$, for the equalizer $\eta : Z^0(C^\bullet) \to C^0$ of ∂_0^0 and ∂_0^1, use (b) above to prove that there exists a unique extension $\eta^\bullet : \text{const}(Z^0) \to C^\bullet$ of η to a morphism of cosimplicial objects of $\mathcal{C}$. **Prove** that this defines a functor,

$$Z^0 : \mathbf{Fun}(\Delta, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta^\bullet : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta^\bullet)$ extends uniquely to an adjoint pair of functors, $(\text{const}, Z^0, \eta^\bullet, \theta)$. **Prove** that θ is a natural isomorphism. **Prove** that if $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$ are two morphisms of cosimplicial objects, and if $(g_{r+1}^i : C^{r+1} \to \widetilde{C}^r)$ is a cosimplicial homotopy from $\alpha^\bullet$ to $\beta^\bullet$, then $Z^0(\alpha^\bullet)$ equals $Z^0(\beta^\bullet)$.

18 Topology Adjoint Pairs

Categories of Topologies on a Fixed Set Exercise. Recall from Problem 1(iv) on Problem Set 3, for every partially ordered set there is an associated category. For a set P, form the partially ordered set $\mathcal{P}(P)$ of subsets S of P. Then for objects S, S' of the category $\mathcal{P}(P)$, i.e., for subsets of P, the Hom set $\text{Hom}_{\mathcal{P}(P)}(S, S')$ is nonempty if and only if $S' \subset S$, in which case the Hom set is a singleton set. In particular, this category has arbitrary (inverse) limits, namely unions, and it has arbitrary colimits (direct limits), namely intersections. Moreover, it has a final object, $\varnothing$, and it has an initial object, P.

Now let X be a set, and let P be $\mathcal{P}(X)$, so that P is a lattice. Denote by Power_X the category from the previous paragraph. Thus, objects are subsets $S \subset \mathcal{P}(X)$, and there exists a morphism from S to S' if and only if $S' \subset S$, and then the morphism is unique. We say that S *refines* S'. There is a covariant functor

$$\cup : \mathcal{P}(P) \to P, \cup S = \{x \in X | \exists p \in S, x \in p\},$$

and a contravariant functor

$$\cap : \mathcal{P}(P)^{\text{opp}} \to P, \cap S = \{x \in X | \forall p \in S, x \in p\}.$$

By convention, $\cup \varnothing = \varnothing$ and $\cap \varnothing = X$.

A *topology* on X is a subset $\tau \subset \mathcal{P}(X)$ such that (i) $\varnothing \in \tau$ and $X \in \tau$, (ii) for every finite subset $S \subset \tau$, also $\cap S$ is in τ, and (iii) for every $S \subset \tau$ (possibly infinite), the set $\cup S$ is in τ. Denote by Top_X the full subcategory of Power_X whose objects are topologies on X. A *topological basis* on X is a subset $B \subset \mathcal{P}(X)$ such that for every finite subset S of B, the set $V = \cap S$ equals $\cup B_V$, where $B_V = \{U \in B : U \subset V\}$. Denote by Basis_X the full subcategory of Power_X whose objects are topological bases on X.

(a) Prove that Top_X is stable under colimits, i.e., for every collection of topologies, there is a topology that is refined by every topology in the collection and that refines every topology that is refined by every topology in the collection. **Prove** that Top_X is a full subcategory of Basis_X. For every topological basis B on X, define $\mathcal{T}(B)$ to consist of all elements $\cup S$ for $S \subset B$. **Prove** that $\mathcal{T}(B)$ is a topology on X. **Prove** that this uniquely extends to a functor

$$\mathcal{T} : \text{Basis}_X \to \text{Top}_X,$$

and **prove** that $\mathcal{T}$ is a right adjoint of the full embedding. Moreover, for every subset $S \subset \mathcal{P}(X)$, define $\mathcal{B}(S)$ to consist of all elements $\cap R$ for $R \subset S$ a *finite* subset. In particular, $\cap \varnothing = X$ is an element of $\mathcal{B}(S)$. **Prove** that $\mathcal{B}(S)$ is topological basis on X. **Prove** that this uniquely extends to a functor

$$\mathcal{B} : \text{Power}_X \to \text{Basis}_X,$$

and **prove** that $\mathcal{T} \circ \mathcal{B}$ is a right adjoint to the full embedding of Basis_X in Power_X.

(b) Prove that for every adjoint pair of functors, the left adjoint functor preserves colimits (direct limits), and the right adjoint functor preserves limits (inverse limits). Conclude that Top_X is stable under limits, i.e., for every collection of topologies, there is a topology that refines every topology in the collection and that is refined by every topology that refines every topology in the collection.

(c) Let $f : Y \to X$ be a set map. Denote by

$$\mathcal{P}^f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

the functor that associates to every subset S of X the preimage subset $f^{-1}(S)$ of Y, and denote by

$$\mathcal{P}_f : \mathcal{P}(Y) \to \mathcal{P}(X)$$

the functor that associates to every subset T of Y the image subset $f(T)$ of X. **Prove** that $(\mathcal{P}^f, \mathcal{P}_f)$ extends uniquely to an adjoint pair of functors. In particular, define

$$\text{Power}_f : \text{Power}_X \to \text{Power}_Y$$

to be $\mathcal{P}_{\mathcal{P}^f}$, i.e., for every subset $S \subset \mathcal{P}(X)$, $\text{Power}_f(S) \subset \mathcal{P}(Y)$ is the set of all subsets $f^{-1}(U) \subset Y$ for subsets $U \subset X$ that are in S. Similarly, define

$$\text{Power}^f : \text{Power}_Y \to \text{Power}_X,$$

to be $\mathcal{P}^{\mathcal{P}^f}$, i.e., for every subset $T \subset \mathcal{P}(Y)$, $\text{Power}^f(Y) \subset \mathcal{P}(X)$ is the set of all subsets $U \subset X$ such that the subset $f^{-1}(U) \subset Y$ is in T. **Prove** that $(\text{Power}^f, \text{Power}_f)$ extends uniquely to an adjoint pair of functors. **Prove** that Power_f and Power^f restrict to functors $\text{Top}_X \to \text{Top}_Y$. For a given topology σ on Y and τ on X, f is *continuous* with respect to σ and τ if σ refines $\text{Power}_f(\tau)$, i.e., for every τ-open subset U of X, also $f^{-1}(U)$ is σ-open in Y. For a given topology τ on X, for every topology σ on Y, σ refines $\text{Power}_f(\tau)$ if and only if f is continuous with respect to σ and τ. Similarly, for a given topology σ on Y, for every topology τ on X, $\text{Power}^f(\sigma)$ refines τ if and only if f is continuous with respect to σ and τ.

Adjoint Pair for the Category of Topological Spaces Exercise. A topological space is a pair (X, τ) of a set X and a topology τ on X. For topological spaces (X, τ) and (Y, σ), a *continuous map* is a function $f : X \to Y$ such that for every subset V of Y that is in σ, the inverse image subset $f^{-1}(V)$ of X is in τ, i.e., σ refines $\text{Power}_f(\tau)$ and τ is refined by $\text{Power}^f(\sigma)$.

(a) Prove that for every topological space (X, τ), the identity function $\text{Id}_X : X \to X$ is a continuous map from (X, τ) to (X, τ). For every pair of continuous maps $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$, **prove** that the composition $g \circ f : (X, \tau) \to (Z, \rho)$ is a continuous map. With this notion of composition of continuous map, check that the topological spaces and continuous maps form a category, **Top**.

(b) For every topological space (X, τ), define $\Phi(X)$ to be the set X. For every continuous map of topological spaces, $f : (X, \tau) \to (Y, \sigma)$, define $\Phi(f) : \Phi(X) \to \Phi(Y)$ to be $f : X \to Y$. **Prove** that this defines a covariant functor,

$$\Phi : \textbf{Top} \to \textbf{Sets}.$$

(c) For every set X, define $L(X) = (X, \mathcal{P}(X))$, i.e., every subset of X is open. **Prove** that $\mathcal{P}(X)$ satisfies the axioms for a topology on X. This is called the *discrete topology* on X. For every set map, $f : X \to Y$, **prove** that $f : (X, \mathcal{P}(X)) \to (Y, \mathcal{P}(Y))$ is a continuous map, denoted $L(f)$. **Prove** that this defines a functor,

$$L : \mathbf{Sets} \to \mathbf{Top}.$$

For every set X, define $\theta_X : X \to \Phi(L(X))$ to be the identity map on X. **Prove** that θ is a natural equivalence $\mathrm{Id}_{\mathbf{Sets}} \Rightarrow \Phi \circ L$. For every topological space (X, τ), **prove** that Id_X is a continuous map $(X, \mathcal{P}(X)) \to (X, \tau)$, denoted $\eta_{(X,\tau)}$. **Prove** that η is a natural transformation $L \circ \Phi \Rightarrow \mathrm{Id}_{\mathbf{Top}}$. **Prove** that (L, Φ, θ, η) is an adjoint pair of functors. In particular, Φ preserves monomorphisms and limits (inverse limits).

(d) For every set X, define $R(X) = (X, \{\varnothing, X\})$. **Prove** that $\{\varnothing, X\}$ satisfies the axioms for a topology on X. This is called the *indiscrete topology* on X. For every set map $f : X \to Y$, **prove** that $f : R(X) \to R(Y)$ is a continuous map, denoted $R(f)$. **Prove** that this defines a functor,

$$R : \mathbf{Sets} \to \mathbf{Top}.$$

For every set topological space (X, τ), **prove** that Id_X is a continous map $(X, \tau) \to R(\Phi(X, \tau))$, denoted $\alpha_{(X,\tau)}$. **Prove** that α is a natural transformation $\mathrm{Id}_{\mathbf{Top}} \Rightarrow R \circ \Phi$. For every set S, denote by $\beta_X : \Phi(R(X)) \to X$ the identity morphism. **Prove** that β is a natural equivalence $\Phi \circ R \Rightarrow \mathrm{Id}_{\mathbf{Sets}}$. **Prove** that (Φ, R, α, β) is an adjoint pair of functors. In particular, Φ preserves epimorphisms and colimits (direct limits).

(e) Use the method of Problem 0 to prove that **Top** has (small) limits and colimits. Finally, **prove** that the projective objects in **Top** are precisely the discrete topological spaces, and the injective objects in **Top** are precisely the nonempty indiscrete topological spaces.

Adjoint Pair of Direct Image and Inverse Image Presheaves. Let (X, τ_X) be a topological space. As above, consider τ_X as a category whose objects are open sets U of the topology, and where for open sets U and V, there is a unique morphism from U to V if $U \supseteq V$, and otherwise there is no morphism. Let $\mathcal{C}$ be a category. A **presheaf** on (X, τ_X) of objects of $\mathcal{C}$ is a functor,

$$A : \tau_X \to \mathcal{C},$$

i.e., a τ_X-family as in Problem 0. By Problem 0, the τ-families form a category $\mathbf{Fun}(\tau_X, \mathcal{C})$, called the category of presheaves of objects of $\mathcal{C}$. For every continuous map $f : (Y, \tau_Y) \to (X, \tau_X)$, define

$$f^{-1} : \tau_X \to \tau_Y,$$

as in Problem 1(c), i.e., $U \mapsto f^{-1}(U)$. The corresponding functor

$$*_{f^{-1}} : \mathbf{Fun}(\tau_Y, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C})$$

is called the *direct image functor* and is denoted f_*, i.e., for every presheaf $\mathcal{F}$ on (Y, τ_Y), $f_*\mathcal{F}$ is a presheaf on (X, τ_X) given by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.

(a) Denote by σ_f the category whose objects are pairs (U,V) of an object U of τ_X and an object V of τ_V such that V is contained in $f^{-1}(U)$. For objects (U,V) and (U',V'), there is a morphism from (U,V) to (U',V') if and only if there is a morphism $U \supseteq U'$ in τ_X and a morphism $V \supseteq V'$ in τ_Y, and in this case the morphism for (U,V) to (U',V') is unique. **Prove** that this is a category. **Prove** that the map on objects,

$$x : \sigma_f \to \tau_X,\ (U,V) \mapsto U,$$

extends uniquely to a functor that is essentially surjective (in fact strictly surjective on objects). **Prove** that the following maps on objects,

$$\ell x : \tau_X \to \sigma_f,\ U \mapsto (U, f^{-1}(U)),$$

$$rx : \tau_X \to \sigma_f,\ U \mapsto (U, \varnothing)$$

extend uniquely to functors, and **prove** that $(\ell x, x)$ and (x, rx) extend uniquely to adjoint functors, i.e., $(U, f^{-1}(U))$, resp. $(U,\varnothing)$, is the initial object, resp. final object, in the fiber category $(\sigma_f)_{x,U}$. **Prove** that the map on objects

$$y : \sigma_f \to \tau_Y,\ (U,V) \mapsto V$$

extends uniquely to a functor that is essentially surjective (in fact strictly surjective on objects). **Prove** that the following map on objects,

$$\ell y : \tau_Y \to \sigma_f,\ V \mapsto (X,V),$$

extends uniquely to a functor, and **prove** that $(\ell y, y)$ extends uniquely to an adjoint functor, i.e., (X,V) is the initial object in the fiber category $(\sigma_f)_{y,V}$. Prove that $y \circ \ell_x$ is the functor $f^{-1} : \tau_X \to \tau_Y$ from above. **Find** an example where y does not admit a right adjoint.

Assume now that $\mathcal{C}$ has colimits. Apply Problem 0(g) to conclude that there are adjoint pairs of functors $(\star_x, \star_{\ell x})$, $(\star_{rx}, \star_x)$, $(\star_y, \star_{\ell y})$, and $(L_y, \star_y)$. Compose these adjoint pairs to obtain an adjoint pair $(L_y \circ \star_x, \star_{\ell x} \circ \star_y)$. Also, by functoriality of $\star_z$ in z, $\star_{\ell x} \circ \star_y$ equals $\star_{y \circ \ell x}$, and this equals $\star_{f^{-1}}$. Thus, this is an adjoint pair $(L_y \circ \star_x, f_*)$. Unwind the defintions from Problem 0(g) to **check** that for every presheaf A on X and for every V an object of τ_Y, $L_y \circ \star_x(A)$ on V is the colimit over the fiber category $(\sigma_f)_{y,V}$ of all U an object of τ_X with $V \subseteq f^{-1}(U)$ of $A(U)$. The functor $L_y \circ \star_x$ is the *inverse image functor* for presheaves,

$$f^{-1} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_Y, \mathcal{C}).$$

Čech Cosimplicial Object of a Covering Exercise. Let (X,τ_X) be a topological space. For every object U of τ_X, **prove** that the topology τ_U on U associated to $i : U \to X$ via Problem 1(c) is a full, upper subcategory of τ_X that has an initial object $\odot = U$. For every U, an *open covering* of U is a set $\mathfrak{U}$ and a set map $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$ such that $\cup\text{Image}(\iota_{\mathfrak{U}})$ equals U. Define σ to be the category whose objects are pairs $(U,\mathfrak{U})$ of an open U in τ_X and an open covering $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$. For objects $(U,\mathfrak{U})$ and $(V,\mathfrak{V})$, a σ-morphism from $(U,\mathfrak{U})$ to $(V,\mathfrak{V})$ is a pair $U \supseteq V$ of a morphism in τ_X and a *refinement* $\phi : \mathfrak{U} \geq \mathfrak{V}$, i.e., a set function $\phi : \mathfrak{V} \to \mathfrak{U}$ such that for every V_0 in $\mathfrak{V}$, $\iota_{\mathfrak{U}}(\phi(V_0))$

contains $\iota_{\mathfrak{V}}(V_0)$. In particular, for every object $(U, \iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U)$ of σ, define $\mathfrak{V} = \text{Image}(\iota_{\mathfrak{U}})$ with its natural inclusion $\iota_{\mathfrak{V}} : \mathfrak{V} \hookrightarrow \tau_U$. Up to the Axiom of Choice, **prove** that there exists a refinement $\phi : (U, \mathfrak{U}) \geq (U, \mathfrak{V})$. Thus, the open coverings with ι a monomorphism are cofinal in the category σ.

(a)(Category of Open Coverings) For every pair of refinements, $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ and $\psi : (V, \mathfrak{V}) \geq (W, \mathfrak{W})$, **prove** that the composition $\phi \circ \psi : \mathfrak{W} \to \mathfrak{U}$ is a refinement, $\phi \circ \psi : (U, \mathfrak{U}) \to (W, \mathfrak{W})$. Also **prove** that $\text{Id}_{\mathfrak{U}} : \mathfrak{U} \to \mathfrak{U}$ is a refinement $(U, \mathfrak{U}) \to (U, \mathfrak{U})$. Conclude that these rules define a category σ whose objects are open coverings $(U, \mathfrak{U})$ of opens U in τ_X and whose morphisms are refinements. Define $x : \sigma \to \tau_X$ to be the rule that associates to every $(U, \mathfrak{U})$ the open U and that associates to every refinement $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ the inclusion $U \supseteq V$. **Prove** that this is a strictly surjective functor. **Prove** that the map on objects,

$$\ell x : \tau_X \to \sigma, \ U \mapsto (U, \{U\}),$$

extends uniquely to a functor, and **prove** that $(\ell x, x)$ extends uniquely to an adjoint pair of functors, i.e., $(U, \{U\})$ is the initial object in the fiber category $\sigma_{x,U}$. Typically x does not admit a right adjoint.

For every open covering $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$, for every integer $r \geq 0$, define the following set map,

$$\iota_{\mathfrak{U}^{r+1}} : \mathfrak{U}^{r+1} \to \tau_U, \ (U_0, U_1, \dots, U_r) \mapsto \iota_{\mathfrak{U}}(U_0) \cap \iota_{\mathfrak{U}}(U_1) \cap \dots \cap \iota_{\mathfrak{U}}(U_r).$$

Let $\mathcal{C}$ be a category, and let A be an $\mathcal{C}$-presheaf on (X, τ_X). Let $(U, \mathfrak{U})$ be an object of σ. Recall that for every object T of $\mathcal{C}$, there is a Yoneda functor,

$$h_T : \mathcal{C}^{\text{opp}} \to \textbf{Sets}, \ S \mapsto \textbf{Hom}_{\mathcal{C}}(S, T),$$

and this is covariant in T. For every integer $r \geq 0$, define

$$h_{A,\mathfrak{U},r} : \mathcal{C}^{\text{opp}} \to \textbf{Sets}, \ S \mapsto \prod_{(U_0,\dots,U_r) \in \mathfrak{U}^{r+1}} h_{A(\iota(U_0,\dots,U_r))}(S),$$

together with the projections,

$$\pi_{(U_0,\dots,U_r)} : h_{A,\mathfrak{U},r} \to h_{A(\iota(U_0,\dots,U_r))}.$$

For every integer $r \geq 0$, and for every integer $i = 0, \dots, r+1$, define

$$\partial_r^i : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{U},r+1},$$

to be the unique natural transformation such that for every $(U_0, \dots, U_{r+1}) \in \mathfrak{U}^{r+2}$, $\pi_{(U_0,\dots,U_{r+1})} \circ \partial_r^i$ equals the composition of the projection,

$$\pi_{(U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1})} : h_{A,\mathfrak{U},r} \to h_{A(\iota(U_0,\dots,U_{i-1},U_{i+1}\cap,\cap U_{r+1}))},$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota(U_0) \cap \dots \cap \iota(U_{i-1}) \cap \iota(U_{i+1}) \cap \dots \cap \iota(U_{r+1})) \to A(\iota(U_0) \cap \dots \cap \iota(U_{r+1})).$$

Similarly, for every $i = 0, \ldots, r$, define

$$\sigma^i_{r+1} : h_{A,\mathfrak{U},r+1} \to h_{A,\mathfrak{U},r}$$

to be the unique natural transformation such that for every $(U_0, \ldots, U_r) \in \mathfrak{U}^{r+1}$, $\pi_{(U_0,\ldots,U_{r+1})} \circ \sigma^i_{r+1}$ equals the projection $\pi_{(U_0,\ldots,U_{i-1},U_i,U_i,U_{i+1},\ldots,U_r)}$.

(b)(Cosimplicial Identities) **Prove** that these natural transformations satisfy the *cosimplicial identities*: for every $r \geq 0$, for every $0 \leq i < j \leq r+2$,

$$\partial^j_{r+1} \circ \partial^i_r = \partial^i_{r+1} \circ \partial^{j-1}_r,$$

for every $0 \leq i \leq j \leq r$,

$$\sigma^j_{r+1} \circ \sigma^i_{r+2} = \sigma^i_{r+1} \circ \sigma^{j+1}_{r+2},$$

and for every $0 \leq i \leq r+1$ and $0 \leq j \leq r$,

$$\sigma^j_{r+1} \circ \partial^i_r = \begin{cases} \partial^i_{r-1} \circ \sigma^{j-1}_r, & i < j, \\ \mathrm{Id}, & i = j, i = j+1, \\ \partial^{i-1}_{r-1} \circ \sigma^j_r, & i > j+1 \end{cases}$$

In the case that $\mathcal{C}$ is an additive category, define

$$d^r : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{U},r+1}, \quad d^r = \sum_{i=0}^{r+1} \partial^i_r.$$

Prove that $d^{r+1} \circ d^r$ equals 0.

(c)(Refinements and Cosimplicial Homotopies) For every refinement, $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$, for every integer $r \geq 0$, define

$$h_{A,\phi,r} : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{V},r}$$

to be the unique natural transformation such that for every $(V_0, \ldots, V_r) \in \mathfrak{V}^{r+1}$, the composition $\pi_{(V_0,\ldots,V_r)} \circ h_{A,q,r}$ equals the composition of projection

$$\pi_{(\phi(V_0),\ldots,\phi(V_r))} : h_{A,\mathfrak{U},r} \to h_{A(\phi(V_0)\cap\cdots\cap\phi(V_r))}$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota\phi(V_0) \cap \cdots \cap \iota\phi(V_r)) \to A(\iota(V_0) \cap \cdots \cap \iota(V_r)).$$

Prove that the natural transformations $(h_{A,\phi,r})_{r\geq 0}$ are compatible with the natural transformations ∂^i_r and σ^i_{r+1}. For every pair of refinements, $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ and $\psi : (V, \mathfrak{V}) \geq (W, \mathfrak{W})$, for the composition refinement $\phi \circ \psi : (U, \mathfrak{U}) \geq (W, \mathfrak{W})$, **prove** that $h_{A,\phi\circ\psi,r}$ equals $h_{A,\psi,r} \circ h_{A,\phi,r}$, and also **prove** that $h_{A,\mathrm{Id}_{\mathfrak{U}},r}$ equals $\mathrm{Id}_{h_{A,\mathfrak{U},r}}$. Thus $h_{A,\phi,r}$ is functorial in ϕ.

Let $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ and $\psi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$ be refinements. For every integer $r \geq 0$, for every integer $i = 0, \ldots, r$, define

$$g^i_{A,\phi,\psi,r+1} : h_{A,\mathfrak{U},r+1} \to h_{A,\mathfrak{V},r}$$

to be the unique natural transformation such that for every $(V_0,\dots,V_r)\in\mathfrak{V}^{r+1}$, $\pi_{(V_0,\dots,V_r)}\circ g^i_{A,\phi,\psi,r+1}$ equals the composition of the projection,

$$\pi_{\psi(V_0),\dots,\psi(V_i),\phi(V_i),\dots,\phi(V_r)}: h_{A,\mathfrak{U},r+1}\to h_{A(\iota(\psi(V_0),\dots,\psi(V_i),\phi(V_i),\dots,\phi(V_r)))},$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota\psi(V_0)\cap\dots\cap\iota\psi(V_i)\cap\iota\phi(V_i)\cap\dots\cap\iota\phi(V_r))\to A(\iota(V_0)\cap\dots\cap\iota(V_i)\cap\dots\cap\iota(V_r)).$$

Prove the following identities (cosimplicial homotopy identities),

$$g^0_{A,\phi,\psi,r+1}\circ\partial^0_{A,\mathfrak{U},r}=h_{A,\phi,r},\quad g^r_{A,\phi,\psi,r+1}\circ\partial^{r+1}_{A,\mathfrak{U},r}=h_{A,\psi,r},$$

$$g^j_{A,\phi,\psi,r+1}\circ\partial^i_{A,\mathfrak{U},r}=\begin{cases}\partial^i_{A,\mathfrak{V},r-1}\circ g^{j-1}_{A,\phi,\psi,r}, & 0\le i<j\le r,\\ g^{i-1}_{A,\phi,\psi,r+1}\circ\partial^i_{A,\mathfrak{U},r}, & 0<i=j\le r,\\ \partial^{i-1}_{A,\mathfrak{V},r-1}\circ g^j_{A,\phi,\psi,r}, & 1\le j+1<i\le r+1.\end{cases}$$

$$g^j_{A,\phi,\psi,r}\circ\sigma^i_{A,\mathfrak{U},r+1}=\begin{cases}\sigma^i_{A,\mathfrak{V},r}\circ g^{j+1}_{A,\phi,\psi,r+1}, & 0\le i\le j\le r-1,\\ \sigma^{i-1}_{A,\mathfrak{V},r}\circ g^j_{A,\phi,\psi,r+1}, & 0\le j<i\le r.\end{cases}$$

For the identity refinement $\mathrm{Id}_{\mathfrak{U}}:\mathfrak{U}\ge\mathfrak{U}$, **prove** that $g^j_{A,\mathrm{Id},\mathrm{Id},r+1}$ equals $\sigma^j_{A,\mathfrak{U},r+1}$. Also **prove** that for refinements $\chi:\mathfrak{V}\to\mathfrak{W}$ and $\xi:\mathfrak{T}\to\mathfrak{U}$, $g^j_{A,\phi\circ\chi,\psi\circ\chi,r+1}$ equals $h_{A,\chi,r}\circ g^j_{A,\phi,\psi,r+1}$ and $g^j_{A,\xi\circ\phi,\xi\circ\psi,r+1}$ equals $g^j_{A,\phi,\psi,r+1}\circ h_{A,\xi,r+1}$.

(d)(Functoriality in A) For every morphism of $\mathcal{C}$-presheaves, $\alpha:A\to A'$, define

$$h_{\alpha,\mathfrak{U},r}:h_{A,\mathfrak{U},r}\to h_{B,\mathfrak{U},r},$$

to be the unique natural transformation whose postcomposition with each projection $\pi_{B,(U_0,\dots,U_r)}$ equals the composition of $\pi_{A,(U_0,\dots,U_r)}$ with the natural transformation induced by the morphism

$$\alpha_{\iota(U_0,\dots,U_r)}:A(\iota(U_0,\dots,U_r))\to A'(\iota(U_0,\dots,U_r)).$$

Prove that these maps are compatible with the cosimplicial operations ∂^i_r and σ^i_{r+1}, as well as the operations $h_{A,\phi,r}$ associated to a refinement $\phi:\mathfrak{U}\ge\mathfrak{V}$, and the cosimplicial homotopies $g^i_{A,\phi,\psi,r+1}$ associated to a pair of refinements, $\phi,\psi:\mathfrak{U}\ge\mathfrak{V}$. **Prove** that this is functorial in α. Conclude that (up to serious set-theoretic issues), for every open cover $\mathfrak{U}$, morally these rules define a functor from the category of $\mathcal{C}$-presheaves to the "category" of cosimplicial objects in the category of contravariant functors from $\mathcal{C}$ to **Sets**. Stated differently, to every open cover $\mathfrak{U}$ there is an associated cosimplicial object in the category $\mathbf{Fun}(\mathcal{C}-\mathrm{Presh},\mathbf{Fun}(\mathcal{C},\mathbf{Sets}))$ of covariant functors from the category of $\mathcal{C}$-presheaves to the category of contravariant functors $\mathcal{C}\to\mathbf{Sets}$. This rule is covariant for refinement of open covers. Moreover, up to simplicial homotopy, it is independent of the choice of refinement.

(e)(Coadjunction of Sections) As a particular case, for the left adjoint ℓx of x, observe that there is a canonical refinement

$$\eta_{U,\mathfrak{U}} : \ell x \circ x(U,\mathfrak{U}) \geq (U,\mathfrak{U}), \text{ i.e., } (U,\{U\}) \geq (U,\mathfrak{U}).$$

Prove that $h_{A,\{U\},r}$ is the constant / diagonal cosimplicial object that for every r associates $h_{A(U)}$ and with ∂^i and σ^i equal to the identity morphism. Conclude that for every cover $(U,\mathfrak{U})$ in σ, there is a natural coaugmentation,

$$g^r_{A,\mathfrak{U}} : h_{A(U)} \to h_{A,\mathfrak{U},r},$$

that is functorial in A, functorial in $(U,\mathfrak{U})$ with respect to refinements, and that equalizes the simplicial homotopies associated to a pair of refinements in the sense that

$$g^j_{A,\phi,\psi,r+1} \circ g^{r+1}_{A,\mathfrak{U}} = g^r_{A,\mathfrak{V}} \circ h_{A^U_V}.$$

Define the functor

$$\text{const} : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C})$$

that associates to a functor $B : \sigma \to \mathcal{C}$ the functor $\text{const}_B : \sigma \to \mathbf{Fun}(\Delta,\mathcal{C})$ whose value on every $(U,\mathfrak{U})$ is the constant / diagonal cosimplicial object $r \mapsto B(U,\mathfrak{U})$ for every r with every ∂^i and σ^i defined to be the identity morphism. Conclude that the rule $U \mapsto (r \mapsto h_{A(U)})$ above is the Yoneda functor associated to $\text{const} \circ *_x(A)$.

(f)(Čech cosimplicial object) Assume now that $\mathcal{C}$ has all finite products. Thus, for every open covering $(U,\mathfrak{U})$ and for every integer $r \geq 0$, there exists an object

$$\check{C}^r(\mathfrak{U},A) = \prod_{(U_0,\dots,U_r)\in\mathfrak{U}} A(U_0 \cap \dots \cap U_r),$$

such that $h_{A,\mathfrak{U},r}$ equals $h_{\check{C}^r(\mathfrak{U},A)}$. Use the Yoneda Lemma to **prove** that there are associated morphisms in $\mathcal{C}$,

$$\partial^i_{A,\mathfrak{U},r} : \check{C}^r(\mathfrak{U},A) \to \check{C}^{r+1}(\mathfrak{U},A),$$
$$\sigma^i_{A,\mathfrak{U},r+1} : \check{C}^{r+1}_{\mathfrak{U}}(A) \to \check{C}^r_{\mathfrak{U}}(A),$$
$$\check{C}^r(\phi,A) : \check{C}^r(\mathfrak{U},A) \to \check{C}^r(\mathfrak{V},A),$$
$$\check{C}^{r+1,i}(\phi,\psi,A) : \check{C}^{r+1}(\mathfrak{U},A) \to \check{C}^r(\mathfrak{V},A),$$
$$\check{C}^r(\mathfrak{U},\alpha) : \check{C}^r(\mathfrak{U},A) \to \check{C}^r(\mathfrak{U},A'),$$

whose associated morphisms of Yoneda functors equal the morphisms defined above. Thus, in this case, $\check{C}^*(\mathfrak{U},A)$ is a cosimplicial object in $\mathcal{C}$. **Prove** that this defines a covariant functor

$$\check{C}(\mathfrak{U},-) : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\Delta,\mathcal{C}).$$

Incorporating the role of $\mathfrak{U}$, **prove** that this defines a functor

$$\check{C} : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C}).$$

Prove that this is, typically, *not* equivalent to the composite functor,

$$\text{const} \circ *_x : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma, \mathcal{C}).$$

However, **prove** that the coadjunction in the last part does give rise to a natural transformation,

$$g : \text{const} \circ *_x \Rightarrow \check{C}.$$

(g) Assume now that there exists a functor,

$$Z^0 : \mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors, i.e., assume that $\mathcal{C}$ is a Cartesian category. Use Problem 4(d) to conclude that there exists a functor,

$$Z^0 : \mathbf{Fun}(\Delta \times \sigma, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}),$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta \times \sigma, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors, $(\text{const}, Z^0, \eta, \theta)$ such that θ is a natural isomorphism. Moreover, for every $A^\bullet : \Delta \times \sigma \to \mathcal{C}$, for every object $(U, \mathfrak{U})$ of σ, **prove** that $\eta : Z^0(A^\bullet(\mathfrak{U}) \to A^0(\mathfrak{U}))$ is an equalizer of $\partial_0^0, \partial_0^1 : A^0(\mathfrak{U}) \to A^1(\mathfrak{U})$. Finally, the composition of natural transformations, $(Z^0 \circ g) \circ (\theta \circ *_x)$, is a natural transformation

$$Z^0(g) : *_x \Rightarrow Z^0 \circ \text{const} \circ *_x \Rightarrow Z^0 \circ \check{C}.$$

In particular, conclude that for a refinement $\phi : (U, \mathfrak{U}) \geq (V, \mathfrak{V})$, the induced morphism $Z^0(\check{C}^\bullet(\mathfrak{U}, A)) \to Z^0(\check{C}^\bullet(\mathfrak{V}, A))$ is independent of the choice of refinement.

(h) Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an object of σ. Let $\phi : (U, \mathfrak{U}) \geq (U, \{U\})$ be a refinement, i.e., $* = \phi(U)$ is an element of $\mathfrak{U}$ such that $\iota(*)$ equals U. Thus, $(U, \mathfrak{U})$ admits both the identity refinement of $(U, \mathfrak{U})$ and also the composite of ϕ with the canonical refinment from (e), $\eta_{U,\mathfrak{U}} circ \phi$. Using (c), **prove** that the identity on $\check{C}^\bullet(\mathfrak{U}, -)$ is homotopy equivalent to $\check{C}(\eta_{U,\mathfrak{U}}, -) \circ \check{C}(\phi, -)$. On the other hand, the refinement $\phi \circ \eta_{U,\mathfrak{U}}$ of $(U, \{U\})$ is the identity refinement. Thus the composite $\check{C}(\phi, -) \circ \check{C}(\eta_{U,\mathfrak{U}}, -)$ equals the identity on $\check{C}^\bullet(\{U\}, -)$. **Prove** that $\check{C}^\bullet(\mathfrak{U}, A)$ is homotopy equivalent to the constant simplicial object $\text{const}_{A(U)}$, and these homotopy equivalences are natural in A and open coverings $(U, \mathfrak{U})$ that refine to $(U, \{U\})$.

Sheaves Exercise. Let (X, τ_X) be a topological space. Let $\mathcal{C}$ be a category. A $\mathcal{C}$-*sheaf* on (X, τ_X) is a $\mathcal{C}$-presheaf A such that for every open subset U in τ_X, for every open covering $\iota : \mathfrak{U} \to \tau_U$ of U, the associated sequence of Yoneda functors,

$$h_{A(U)} \xrightarrow{g^0_{A,\mathfrak{U}}} h_{A,\mathfrak{U},0} \rightrightarrows h_{A,\mathfrak{U},1},$$

is exact, where the two arrows are $\partial^0_{A,\mathfrak{U},0}$ and $\partial^1_{A,\mathfrak{U},0}$. Stated more concretely, for every object S of $\mathcal{C}$, for every collection $(s_{U_0} : S \to A(\iota(U_0)))_{U_0 \in \mathfrak{U}}$ of $\mathcal{C}$-morphisms such that for every $(U_0, U_1) \in \mathfrak{U}^2$, the following two compositions are equal,

$$S \xrightarrow{s_{U_0}} A(\iota(U_0)) \xrightarrow{A^{\iota(U_0)}_{\iota(U_0)\cap\iota(U_1)}} A(\iota(U_0) \cap \iota(U_1)), \quad S \xrightarrow{s_{U_1}} A(\iota(U_1)) \xrightarrow{A^{\iota(U_1)}_{\iota(U_0)\cap\iota(U_1)}} A(\iota(U_0) \cap \iota(U_1)),$$

there exists a unique morphism $s_U : S \to A(U)$ such that for every $U_0 \in \mathfrak{U}$, s_{U_0} equals $A^U_{\iota(U_0)} \circ s_U$.

(a)(Sheaf Axiom via Čech Objects) For simplicity, assume that $\mathcal{C}$ is a Cartesian category that has all small products. In particular, assume that the functors $\check{C}$ and Z^0 of the previous exercise are defined. **Prove** that a $\mathcal{C}$-presheaf on (X, τ_X) is a sheaf if and only if the morphism

$$Z^0(g) : \star_x(A) \to Z^0(\check{C}(A))$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$ is an isomorphism.

(b)(Associated Sheaf / Sheafification Functor) Now assume that $\mathcal{C}$ has all small colimits. In particular, assume that there exists a functor

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C}),$$

such that $(L_x, \star_x)$ extends to an adjoint pair of functors. Using Exercise 0(g), **prove** that for every open U in τ_X and for every functor,

$$B : \sigma \to \mathcal{C},$$

$L_x(B)(U)$ is the colimit of the restriction of B to the fiber category $\sigma_{x,U}$. In particular, since open coverings $(U, \iota : \mathfrak{U} \to U)$ such that ι is a monomorphism are cofinal in the category $\sigma_{x,U}$, it suffices to compute the colimit over such open coverings. For every functor,

$$A : \tau_X \to \mathcal{C},$$

prove that $L_x \circ \star_x(A) \to A$ is a natural isomorphism. Denote by $\mathrm{Sh} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C})$ the composite functor,

$$L_x \circ Z^0 \circ \check{C} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_x, \mathcal{C}).$$

Prove that there exists a unique natural transformation,

$$\mathrm{sh} : \mathrm{Id}_{\mathbf{Fun}(\tau_x, \mathcal{C})} \Rightarrow \mathrm{Sh},$$

whose composition with the natural isomorphism above equals $L_x(Z^0(g))$. For every sheaf A, **prove** that

$$\mathrm{sh} : A \to \mathrm{Sh}(A)$$

is an isomorphism.

(c)(The Associated Sheaf is a Sheaf) Let $(U, \iota : \mathfrak{U} \to \tau_U)$ an object of σ, and let,

$$(\iota(U_0), \kappa_{U_0} : \mathfrak{V}_{U_0} \to \tau_{\iota(U_0)}),$$

be a collection of open coverings of each $\iota(U_0)$. For every pair $(U_0, U_1) \in \mathfrak{U}^2$, let

$$(\iota(U_0, U_1), \kappa_{U_0,U_1} : \mathfrak{V}_{U_0,U_1} \to \tau_{\iota(U_0,U_1)}),$$

be an open covering together with refinements

$$\phi_0^0 : (\iota(U_0), \mathfrak{V}_{U_0}) \geq (\iota(U_0, U_1), \mathfrak{V}_{U_0,U_1}), \quad \phi_0^1 : (\iota(U_1), \mathfrak{V}_{U_1}) \geq (\iota(U_0, U_1), \mathfrak{V}_{U_0,U_1}).$$

Define

$$\mathfrak{V} := (\sqcup_{U_0 \in \mathfrak{U}} \mathfrak{V}_{U_0}) \sqcup \left(\sqcup_{(U_0,U_1)\in\mathfrak{U}^2} \mathfrak{V}_{U_0,U_1}\right),$$

define

$$\kappa : \mathfrak{V} \to \tau_U,$$

to be the unique set map whose restriction to every $\mathfrak{V}_{U_0}$ equals κ_{U_0} and whose restriction to every $\mathfrak{V}_{U_0,U_1}$ equals κ_{U_0,U_1}. For every $U_0 \in \mathfrak{U}$, define

$$\phi_{U_0} : (U, \kappa : \mathfrak{V} \to \tau_U) \geq (\iota(U_0), \kappa_{U_0} : \mathfrak{V}_{U_0} \to \tau_{\iota(U_0)}),$$

to be the obvious refinement. For every $U_0 \in \mathfrak{U}$, define $Z(U_0, A) = Z^0(\check{C}^\bullet(\mathfrak{V}_{U_0}, A))$. For every $(U_0, U_1) \in \mathfrak{U}^2$, define $Z^0(U_0, U_1, A) = Z^0(\check{C}^\bullet(\mathfrak{V}_{U_0,U_1}, A))$. Define

$$Z^0(\mathfrak{U}, A) := \prod_{U_0 \in \mathfrak{U}} Z^0(U_0, A),$$

$$Z^1(\mathfrak{U}, A) := \prod_{(U_0,U_1)\in\mathfrak{U}^2} Z^0(U_0, U_1, A),$$

$$\partial_0^0 : Z^0(\mathfrak{U}, A) \to Z^1(\mathfrak{U}, A), \ \partial_0^i(z_{U_0}) = (A^{U_i}_{U_0\cap U_1}(z_{U_i}))_{U_0,U_1}.$$

Prove that the restriction morphism,

$$Z^0(\phi^\bullet) : Z^0(\mathfrak{V}, A) \to Z^0(Z^\bullet(\mathfrak{U}, A)),$$

is a $\mathfrak{C}$-isomorphism. Conclude that $\mathrm{Sh}(A)$ is a sheaf. Denote by,

$$\Phi : \mathcal{C} - \mathrm{Sh}_{(X,\tau_X)} \to \mathcal{C} - \mathrm{Presh}_{(X,\tau_X)},$$

the full embedding of the category of sheaves in the category of presheaves. Thus, Sh is a functor,

$$\mathrm{Sh} : \mathcal{C} - \mathrm{Presh}_{(X,\tau_X)} \to \mathcal{C} - \mathrm{Sh}_{(X,\tau_X)},$$

and sh is a natural transformation $\mathrm{Id}_{\mathcal{C}-\mathrm{Presh}_X} \Rightarrow \Phi \circ \mathrm{Sh}$. Conclude that $(\mathrm{Sh}, \Phi, \mathrm{sh})$ extends to an adjoint pair of functors.

(d)(Pushforward and Inverse Image) For a continuous map $f : (X, \tau_X) \to (Y, \tau_Y)$, **prove** that the composite functor,

$$\mathcal{C} - \mathrm{Sh}_{(X,\tau_X)} \xrightarrow{\Phi} \mathcal{C} - \mathrm{Presh}_{(X,\tau_X)} \xrightarrow{f_*} \mathcal{C} - \mathrm{Presh}_{(Y,\tau_Y)},$$

factors uniquely through $\Phi : \mathcal{C}-\mathrm{Sh}_{(Y,\tau_Y)} \to \mathcal{C}-\mathrm{Presh}_{(Y,\tau_Y)}$, i.e., there is a functor

$$f_* : \mathcal{C}-\mathrm{Sh}_{(X,\tau_X)} \to \mathcal{C}-\mathrm{Sh}_{(Y,\tau_Y)},$$

such that $f_* \circ \Phi$ equals $\Phi \circ f_*$. On the other hand, **prove** by example that the composite

$$\mathcal{C}-\mathrm{Sh}_{(Y,\tau_Y)} \xrightarrow{\Phi} \mathcal{C}-\mathrm{Presh}_{(Y,\tau_Y)} \xrightarrow{f^{-1}} \mathcal{C}-\mathrm{Presh}_{(X,\tau_X)}$$

need not factor through Φ. Define

$$f^{-1} : \mathcal{C}-\mathrm{Sh}_{(Y,\tau_Y)} \to \mathcal{C}-\mathrm{Sh}_{(X,\tau_X)},$$

to be the composite of the previous functor with $\mathrm{Sh} : \mathcal{C}-\mathrm{Presh}_{(X,\tau_X)} \to \mathcal{C}-\mathrm{Sh}_{(X,\tau_X)}$. **Prove** that the functors (f^{-1}, f_*) extend to an adjoint pair of functors between $\mathcal{C}-\mathrm{Sh}_{(X,\tau_X)}$ and $\mathcal{C}-\mathrm{Sh}_{(Y,\tau_Y)}$.

Espace Étalé Exercise. Let (X,τ_X) be a topological space. A *space over* X is a continuous map of topological spaces, $f : (Y,\tau_Y) \to (X,\tau_X)$. For spaces over X, $f : (Y,\tau_Y) \to (X,\tau_X)$ and $g : (Z,\tau_Z) \to (X,\tau_X)$, a *morphism of spaces over* X from f to g is a continuous map $u : (Y,\tau_Y) \to (Z,\tau_X)$ such that $g \circ u$ equals f.

(a)(The Category of Spaces over X) For every space over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, prove that $\mathrm{Id}_Y : (Y,\tau_Y) \to (Y,\tau_Y)$ is a morphism from f to f. For spaces over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, $g : (Z,\tau_Z) \to (X,\tau_X)$ and $h : (W,\tau_W) \to (X,\tau_X)$, for every morphism from f to g, $u : (Y,\tau_Y) \to (Z,\tau_Z)$, and for every morphism from g to h, $v : (Z,\tau_Z) \to (W,\tau_W)$, prove that the composition $v \circ u : (Y,\tau_Y) \to (W,\tau_W)$ is a morphism from f to h. Conclude that these notions form a category, denoted $\mathbf{Top}_{(X,\tau_X)}$.

(b)(The Sheaf of Sections) For every space over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, for every open U of τ_X, define $\mathrm{Sec}_f(U)$ to be the set of continuous functions $s : (U,\tau_U) \to (Y,\tau_Y)$ such that $f \circ s$ is the inclusion morphism $(U,\tau_Y) \to (X,\tau_X)$. For every inclusion of τ_X-open subsets, $U \supseteq V$, for every s in $\mathrm{Sec}_f(U)$, define $s|_V$ to be the restriction of s to the open subset V. **Prove** that $s|_V$ is an element of $\mathrm{Sec}_f(V)$. **Prove** that these rules define a functor

$$\mathrm{Sec}_f : \tau_X \to \mathbf{Sets}.$$

Prove that this functor is a sheaf of sets on (X,τ_X).

(c)(The Sections Functor) For spaces over X, $f(Y,\tau_Y) \to (X,\tau_X)$ and $g : (Z,\tau_Z) \to (X,\tau_X)$, for every morphism from f to g, $u : (Y,\tau_Y) \to (Z,\tau_Z)$, for every τ_X-open set U, for every s in $\mathrm{Sec}_f(U)$, **prove** that $u \circ s$ is an element of $\mathrm{Sec}_g(U)$. For every inclusion of τ_X-open sets, $U \supseteq V$, **prove** that $u \circ (s|_V)$ equals $(u \circ s)|_V$. Conclude that these rules define a morphism of sheaves of sets,

$$\mathrm{Sec}_u : \mathrm{Sec}_f \to \mathrm{Sec}_g.$$

Prove that $\mathrm{Sec}_{\mathrm{Id}_Y}$ is the identity morphism of Sec_f. For spaces over X, $f : (Y,\tau_Y) \to (X,\tau_X)$, $g : (Z,\tau_Z) \to (X,\tau_X)$ and $h : (W,\tau_W) \to (X,\tau_X)$, for every morphism from f to g, $u : (Y,\tau_Y) \to (Z,\tau_Z)$, and for every morphism from g to h, $v : (Z, \tau_Z) \to (W, \tau_W)$, **prove** that $\mathrm{Sec}_{v \circ u}$ equals $\mathrm{Sec}_v \circ \mathrm{Sec}_u$. Conclude that these rules define a functor,

$$\mathrm{Sec} : \mathbf{Top}_{(X,\tau_X)} \to \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)}.$$

(d)(The Éspace Étalé) For every presheaf of sets over X, $\mathcal{F}$, define $\mathrm{Esp}_{\mathcal{F}}$ to be the set of pairs (x, ϕ_x) of an element x of X and an element ϕ_x of the stalk $\mathcal{F}_x = \mathrm{colim}_{x \in U} \mathcal{F}(U)$; such an element is called a *germ* of $\mathcal{F}$ at x. Denote by

$$\pi_{\mathcal{F}} : \mathrm{Esp}_{\mathcal{F}} \to X,$$

the set map sending (x, ϕ_x) to x. For every open subset U of X and for every element ϕ of $\mathcal{F}(U)$, define $B(U, \phi) \subset \mathrm{Esp}_{\mathcal{F}}$ to be the image of the morphism,

$$\widetilde{\phi} : U \to \mathrm{Esp}_{\mathcal{F}},\ x \mapsto \phi_x.$$

Let (U, ψ) and (V, χ) be two such pairs. Let (x, ϕ_x) be an element of both $B(U, \psi)$ and $B(V, \chi)$. **Prove** that there exists an open subset W of $U \cap V$ containing x such that $\psi|_W$ equals $\chi|_W$. Denote this common restriction by $\phi \in \mathcal{F}(W)$. Conclude that (x, ϕ_x) is contained in $B(W, \phi)$, and this is contained in $B(U, \psi) \cap B(V, \chi)$. Conclude that the collection of all subset $B(U, \phi)$ of $\mathrm{Esp}_{\mathcal{F}}$ is a topological basis. Denote by $\tau_{\mathcal{F}}$ the associated topology on $\mathrm{Esp}_{\mathcal{F}}$. **Prove** that $\tau_{\mathcal{F}}$ is the finest topology on $\mathrm{Esp}_{\mathcal{F}}$ such that for every τ_X-open set U and for every $\phi \in \mathcal{F}(U)$, the set map $\widetilde{\phi}$ is a continuous map $(U, \tau_U) \to (\mathrm{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}})$. In particular, since every composition $\pi_{\mathcal{F}} \circ \widetilde{\phi}$ is the continuous inclusion of (U, τ_U) in (X, τ_X), conclude that every $\widetilde{\phi}$ is continuous for the topology $\pi_{\mathcal{F}}^{-1}(\tau_X)$ on $\mathrm{Esp}_{\mathcal{F}}$. Since $\tau_{\mathcal{F}}$ refines this topology, **prove** that

$$\pi_{\mathcal{F}} : (\mathrm{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}}) \to (X, \tau_X)$$

is a continuous map, i.e., $\pi_{\mathcal{F}}$ is a space over X.

(e)(The Éspace Functor) For every morphism of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$, for every (x, ϕ_x) in $\mathrm{Esp}_{\mathcal{F}}$, define $\mathrm{Esp}_\alpha(x, \phi_x)$ to be $(x, \alpha_x(\phi_x))$, where $\alpha_x : \mathcal{F}_x \to \mathcal{G}_x$ is the induced morphism of stalks. For every τ_X-open set U and every $\phi \in \mathcal{F}(U)$, **prove** tht the composition $\mathrm{Esp}_\alpha \circ \widetilde{\phi}$ equals $\widetilde{\alpha_U(\phi)}$ as set maps $U \to \mathrm{Esp}_{\mathcal{G}}$. By construction, $\widetilde{\alpha_U(\phi)}$ is continuous for the topology $\tau_{\mathcal{G}}$. Conclude that $\widetilde{\phi}$ is continuous for the topology $(\mathrm{Esp}_\alpha)^{-1}(\tau_{\mathcal{G}})$ on $\mathrm{Esp}_{\mathcal{F}}$. Conclude that $\tau_{\mathcal{F}}$ refines this topology, and thus Esp_α is a continuous function,

$$\mathrm{Esp}_\alpha : (\mathrm{Esp}_{\mathcal{F}}, \tau_{\mathcal{F}}) \to (\mathrm{Esp}_{\mathcal{G}}, \tau_{\mathcal{G}}).$$

Prove that $\mathrm{Esp}_{\mathrm{Id}_{\mathcal{F}}}$ equals the identity map on $\mathrm{Esp}_{\mathcal{F}}$. For morphisms of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$ and $\beta : \mathcal{G} \to \mathcal{H}$, **prove** that $\mathrm{Esp}_{\beta \circ \alpha}$ equals $\mathrm{Esp}_\beta \circ \mathrm{Esp}_\alpha$. Conclude that these rules define a functor,

$$\mathrm{Esp} : \mathbf{Sets} - \mathrm{Presh}_{(X,\tau_X)} \to \mathbf{Top}_{(X,\tau_X)}.$$

(f)(The Adjointness Natural Transformations) For every presheaf of sets over X, $\mathcal{F}$, for every τ_X-open set U, for every $\phi \in \mathcal{F}(U)$, **prove** that $\widetilde{\phi}$ is an element of $\mathrm{Sec}_{\pi_{\mathcal{F}}}(U)$. For every τ_X-open subset $U \supseteq V$, **prove** that $\widetilde{\phi}|_V$ equals $\widetilde{\phi|_V}$. Conclude that $\phi \mapsto \widetilde{\phi}$ is a morphism of presheaves of sets over X,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F}).$$

For every morphism of presheaves of sets over X, $\alpha : \mathcal{F} \to \mathcal{G}$, for every τ_X-open set U, for every $\phi \in \mathcal{F}(U)$, **prove** that $\mathrm{Esp}_\alpha \circ \theta_{\mathcal{F},U}(\phi)$ equals $\widetilde{\alpha_U(\phi)}$, and this in turn equals $\theta_{\mathcal{G},U} \circ \alpha_U(\phi)$. Conclude that $\mathrm{Sec} \circ \mathrm{Esp}(\alpha) \circ \theta_{\mathcal{F}}$ equals $\theta_{\mathcal{G}} \circ \alpha$. Therefore θ is a natural transformation of functors,

$$\theta : \mathrm{Id}_{\mathbf{Sets}-\mathrm{Presh}_{(X,\tau_X)}} \Rightarrow \mathrm{Sec} \circ \mathrm{Esp}.$$

(g)(Alternative Description of Sheafification) Since $\mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F})$ is a sheaf, **prove** that there exists a unique morphism

$$\widetilde{\theta}_{\mathcal{F}} : \mathrm{Sh}(\mathcal{F}) \to \mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F})$$

factoring $\theta_{\mathcal{F}}$. For every element $t \in \mathrm{Sec} \circ \mathrm{Esp}(\mathcal{F})(U)$, a *$t$-pair* is a pair (U_0, s_0) of a τ_X-open subset $U \supseteq U_0$ and an element $s_0 \in \mathcal{F}(U_0)$ such that $t^{-1}(B(U_0, s_0))$ equals U_0. Define $\mathfrak{U}$ to be the set of t-pairs, and define $\iota : \mathfrak{U} \to \tau_U$ to be the set map $(U_0, s_0) \mapsto U_0$. **Prove** that $(U, \iota : \mathfrak{U} \to \tau_U)$ is an open covering. For every pair of t-pairs, (U_0, s_0) and (U_1, s_1), for every $x \in U_0 \cap U_1$, prove that there exists a τ_X-open subset $U_{0,1} \subset U_0 \cap U_1$ containing x such that $s_0|_{U_{0,1}}$ equals $s_1|_{U_{0,1}}$. **Prove** that this data gives rise to a section $s \in \mathrm{Sh}(\mathcal{F})(U)$ such that $\widetilde{\theta}_{\mathcal{F}}(s)$ equals t. Conclude that $\widetilde{\theta}$ is an epimorphism. On the other hand, for every $r, s \in \mathcal{F}(U)$, if $\theta_{\mathcal{F},x}(r_x)$ equals $\theta_{\mathcal{F},x}(s_x)$, **prove** that $\widetilde{r}(x)$ equals $\widetilde{s}(x)$, i.e., r_x equals s_x. Conclude that every morphism $\widetilde{\theta}_x$ is a monomorphism, and hence $\widetilde{\theta}$ is a monomorphism of sheaves. Thus, finally **prove** that $\widetilde{\theta}_{\mathcal{F}}$ is an isomorphism of sheaves. Conclude that $\widetilde{\theta}$ is a natural isomorphism of functors,

$$\widetilde{\theta} : \mathrm{Sh} \Rightarrow \mathrm{Sec} \circ \mathrm{Esp}.$$

(h) For every space over X, $f : (Y, \tau_Y) \to (X, \tau_X)$, for every τ_X-open U, for every $s \in \mathrm{Sec}_f(U)$, and for every $x \in U$, define a set map,

$$\eta_{f,U,x} : \mathrm{Sec}_f(U) \to Y, \ s \mapsto s(x).$$

Prove that for every τ_X-open subset $U \supseteq V$ that contains x, $\eta_{f,V,x}(s|_V)$ equals $\eta_{f,U,x}(s)$. Conclude that the morphisms $\eta_{f,U,x}$ factor through set maps,

$$\eta_{f,x} : (\mathrm{Sec}_f)_x \to Y, \ s_x \mapsto s(x).$$

Define a set map,

$$\eta_f : \mathrm{Esp}_{\mathrm{Sec}_f} \to Y, \ (x, s_x) \mapsto \eta_{f,x}(s_x).$$

Prove that $\eta_f \circ \widetilde{s}$ equals s as set maps $U \to Y$. Since s is continuous for τ_Y, conclude that $\widetilde{s}$ is continuous for the inverse image topology $(\eta_f)^{-1}(\tau_Y)$ on $\mathrm{Esp}_{\mathrm{Sec}_f}$. Conclude that τ_{Sec_f} refines this topology, and thus η_f is a continuous map,

$$\eta_f : (\mathrm{Esp}_{\mathrm{Sec}_f}, \tau_{\mathrm{Sec}_f}) \to (Y, \tau_Y).$$

Also **prove** that $f \circ \eta_f$ equals π_{Sec_f}. Conclude that η_f is a morphism of spaces over X. Finally, for spaces over X, $f : (Y, \tau_Y) \to (X, \tau_X)$ and $g : (Z, \tau_Z) \to (X, \tau_X)$, and for every morphism from f to g, $u : (Y, \tau_Y) \to (Z, \tau_Z)$, **prove** that $u \circ \eta_f$ equals $\eta_g \circ \mathrm{Esp} \circ \mathrm{Sec}(u)$. Conclude that $f \mapsto \eta_f$ defines a natural transformation of functors,

$$\eta : \mathrm{Esp} \circ \mathrm{Sec} \Rightarrow \mathrm{Id}_{\mathbf{Top}_{(X,\tau_X)}}.$$

(i)(The Adjoint Pair) **Prove** that $(\mathrm{Esp}, \mathrm{Sec}, \theta, \eta)$ is an adjoint pair of functors.

Alternative Description of Inverse Image Exercise. Let $f : (Y, \tau_Y) \to (X, \tau_X)$ be a continuous function of topological spaces. Since the category of topological spaces is a Cartesian category (by Problem 2(e) on Problem Set 8), for every space over X, $g : (Z, \tau_Z) \to (X, \tau_X)$, there is a fiber product diagram in **Top**,

$$\begin{array}{ccc} (Z,\tau_Z) \times_{(X,\tau_X)} (Y,\tau_Y) & \xrightarrow{g^*f} & (Z,\tau_Z) \\ {\scriptstyle f^*g}\downarrow & & \downarrow{\scriptstyle g} \\ (Y,\tau_Y) & \xrightarrow[f]{} & (X,\tau_X) \end{array}.$$

Denote the fiber product by $f^*(Z, \tau_Z)$.

(a) For spaces over X, $g : (Z, \tau_Z) \to (X, \tau_X)$ and $h : (W, \tau_W) \to (X, \tau_X)$, for every morphism of spaces over X, $u : (Z, \tau_Z) \to (W, \tau_W)$, **prove** that there is a unique morphism of topological spaces,

$$f^*u : f^*(Z, \tau_Z) \to f^*(W, \tau_W),$$

such that $f^*h \circ f^*u$ equals f^*g and $h^*f \circ f^*u$ equals $u \circ g^*f$. **Prove** that $f^*\mathrm{Id}_Z$ is the identity morphism of $f^*(Z, \tau_Z)$. For spaces over X, $g : (Z, \tau_Z) \to (X, \tau_X)$, $h : (W, \tau_W) \to (X, \tau_X)$ and $i : (M, \tau_M) \to (X, \tau_X)$, for every morphism from g to h, $u : (Z, \tau_Z) \to (W, \tau_W)$, and for every morphism from h to i, $v : (W, \tau_W) \to (M, \tau_M)$, **prove** that $f^*(v \circ u)$ equals $f^*v \circ f^*u$. Conclude that these rules define a functor,

$$f^*_{\mathrm{Sp}} : \mathbf{Top}_{(X,\tau_X)} \to \mathbf{Top}_{(Y,\tau_Y)}.$$

Prove that this functor is contravariant in f. In particular, there is a composite functor,

$$f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)} : \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)} \to \mathbf{Top}_{(Y,\tau_Y)}.$$

(b) Consider the composite functor,

$$f_* \circ \mathrm{Sec}_{(Y,\tau_Y)} : \mathbf{Top}_{(Y,\tau_Y)} \to \mathbf{Sets} - \mathrm{Sh}_{(Y,\tau_Y)} \to \mathbf{Sets} - \mathrm{Sh}_{(X,\tau_X)}.$$

Prove directly (without using the inverse image functor on sheaves) that $(f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)}, f_* \circ \mathrm{Sec}_{(Y,\tau_Y)})$ extends to an adjoint pair of functors. Use this to conclude that the composite $\mathrm{Sec}_{(Y,\tau_Y)} \circ f^*_{\mathrm{Sp}} \circ \mathrm{Esp}_{(X,\tau_X)}$ is naturally isomorphic to the inverse image functor on sheaves of sets.

19 The Adjoint Pair of Discontinuous Sections (Godement Resolution)

Flasque Sheaves Exercise. Let (X, τ_X) be a topological space, and let $\mathcal{C}$ be a category. A $\mathcal{C}$-presheaf F on (X, τ_X) is *flasque* (or *flabby*) if for every inclusion of τ_X-open sets, $U \supseteq V$, the restriction morphism $A_V^U : A(U) \to A(V)$ is an epimorphism.

(a)(Pushforward Preserves Flasque Sheaves) For every continuous function $f : (X, \tau_X) \to (Y, \tau_Y)$, for every flasque $\mathcal{C}$-presheaf F on (X, τ_X), **prove** that f_*F is a flasque $\mathcal{C}$-presheaf on (Y, τ_Y).

(b)(Restriction to Opens Preserves Flasque Sheaves) For every τ_X-open subset U, for the continuous inclusion $i : (U, \tau_U) \to (X, \tau_X)$, for every flasque $\mathcal{C}$-presheaf F on (X, τ_X), **prove** that $i^{-1}F$ is a flasque $\mathcal{C}$-presheaf. Also, for every $\mathcal{C}$-sheaf F on (X, τ_X), **prove** that the presheaf inverse image $i^{-1}F$ is already a sheaf, so that the sheaf inverse image agrees with the presheaf inverse image.

(c)(H^1-Acyclicity of Flasque Sheaves) Let $\mathcal{A}$ be an Abelian category realized as a full subcategory of the category of left R-modules (via the embedding theorem). Let

$$0 \longrightarrow A' \xrightarrow{q} A \xrightarrow{p} A'' \longrightarrow 0$$

be a short exact sequence of $\mathcal{A}$-sheaves on (X, τ_X). Let U be a τ_X-open set. Let $t : A''(U) \to T$ be a morphism in $\mathcal{A}$ such that $t \circ p(U)$ is the zero morphism. Assume that A' is flasque. **Prove** that t is the zero morphism as follows. Let $a'' \in A''(U)$ be any element. Let $\mathcal{S}$ be the set of pairs (V, a) of a τ_X-open subset $V \subseteq U$ and an element $a \in A(V)$ such that $p(V)(a)$ equals $a''|_V$. For elements (V, a) and $(\widetilde{V}, \widetilde{a})$ of $\mathcal{S}$, define $(V, a) \leq (\widetilde{V}, \widetilde{a})$ if $V \subseteq V'$ and $\widetilde{a}|_V$ equals a. **Prove** that this defines a partial order on $\mathcal{S}$. Use the sheaf axiom for A to **prove** that every totally ordered subset of $\mathcal{S}$ has a least upper bound in $\mathcal{S}$. Use Zorn's Lemma to conclude that there exists a maximal element (V, a) in $\mathcal{S}$. For every x in U, since p is an epimorphism of sheaves, **prove** that there exists (W, b) in $\mathcal{S}$ such that $x \in W$. Conclude that on $V \cap W$, $a|_{V \cap W} - b|_{V \cap W}$ is in the kernel of $p(V \cap W)$. Since the sequence above is exact, **prove** that there exists unique $a' \in A'(V \cap W)$ such that $q(V \cap W)(a')$ equals $a|_{V \cap W} - b|_{V \cap W}$. Since A' is flasque, **prove** that there exists $a'_W \in A'(W)$ such that $a'_W|_{V \cap W}$ equals a'. Define $a_W = b + q(W)(a'_W)$. **Prove** that (W, a_W) is in $\mathcal{S}$ and $a|_{V \cap W}$ equals $a_W|_{V \cap W}$. Use the sheaf axiom for A once more to **prove** that there exists unique $(V \cap W, a_{V \cap W})$ in $\mathcal{S}$ with $a_{V \cap W}|_V$ equals a and $a_{V \cap W}|_W$ equals a_W. Since (V, a) is maximal, conclude that $W \subset V$, and thus x is in V. Conclude that V equals U. Thus, a'' equals $p(U)(a)$. Conclude that $t(a'')$ equals 0, and thus t is the zero morphism. (For a real challenge, modify this argument to avoid any use of the embedding theorem.)

(d)(H^r-Acyclicity of Flasque Sheaves) Let $C^\bullet = (C^q, d_C^q)_{q \geq 0}$ be a complex of $\mathcal{A}$-sheaves on (X, τ_X). Assume that every C^q is flasque. Let $r \geq 0$ be an integer, and assume that the cohomology sheaves $h^q(C^\bullet)$ are zero for $q = 0, \ldots, r$. Use (c) and induction on r to prove that for the associated complex in $\mathcal{C}$,

$$C^\bullet(U) = (C^q(U), d_C^q(U))_{q \geq 0}$$

also $h^q(C^\bullet(U))$ is zero for $q = 0, \ldots, r$.

Enough Injective $\Lambda - \Pi$-modules Exercise. Let (X, τ_X) be a topological space. Let Λ and Π be presheaves of associative, unital rings on (X, τ_X). The most common case is to take both Λ and Π to be the constant presheaf with values $\mathbb{Z}$. Assume, for simplicity, that $\Lambda(\varnothing)$ and $\Pi(\varnothing)$ are the zero ring. A *presheaf of* $\Lambda - \Pi$*-bimodules* on (X, τ_X) is a presheaf M of Abelian groups on (X, τ_X) together with a structure of $\Lambda(U) - \Pi(U)$-bimodule on every Abelian group $M(U)$ such that for every open subset $U \supseteq V$, relative to the restriction homomorphisms of associative, unital rings,

$$\Lambda^U_V : \Lambda(U) \to \Lambda(V), \quad \Pi^U_V : \Pi(U) \to \Pi(V),$$

every restriction homomorphism of Abelian groups,

$$M^U_V : M(U) \to M(V),$$

is a homomorphism of $\Lambda(U) - \Pi(U)$-bimodules. For presheaves of $\Lambda - \Pi$-bimodules on (X, τ_X), M and N, a *morphism of presheaves of* $\Lambda - Pi$*-bimodules* is a morphism of presheaves of Abelian groups $\alpha : M \to N$ such that for every open U, the Abelian group homomorphism,

$$\alpha(U) : M(U) \to N(U),$$

is a homomorphism of $\Lambda(U) - \Pi(U)$-bimodules.

(a)(The Category of Presheaves of $\Lambda - \Pi$-Bimodules) **Prove** that these notions form a category $\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}$. Prove that this is an Abelian category that satisfies Grothendieck's axioms (AB1), (AB2), (AB3), (AB3*), (AB4) and (AB5).

(b)(Discontinuous $\Lambda - \Pi$-Bimodules) A *discontinuous* $\Lambda - \Pi$-bimodule is a specification K for every nonempty τ_X-open U of a $\Lambda(U) - \Pi(U)$-bimodule $K(U)$, but without any specification of restriction morphisms. For discontinuous $\Lambda - \Pi$-bimodules K and L, a *morphism* of discontinuous $\Lambda - \Pi$-bimodules $\alpha : K \to L$ is a specification for every nonempty τ_X-open U of a homomorphism $\alpha(U) : K(U) \to L(U)$ of $\Lambda(U) - \Pi(U)$-bimodules. **Prove** that with these notions, there is a category $\Lambda - \Pi - \text{Disc}_{(X,\tau_X)}$ of discontinuous $\Lambda - \Pi$-bimodules. Prove that this is an Abelian category that satisfies Grothendieck's axioms (AB1), (AB2), (AB3), (AB3*), (AB4), (AB4*) and (AB5).

(c)(The Presheaf Associated to a Discontinuous $\Lambda - \Pi$-Bimodule) For every discontinuous $\Lambda - \Pi$-bimodule K, for every nonempty τ_X-open subset U, define

$$\widetilde{K}(U) = \prod_{W \subseteq U} K(W)$$

as a $\Lambda(U) - \Pi(U)$-bimodule, where the product is over nonempty open subsets $W \subseteq U$ (in particular also $W = U$ is allowed), together with its natural projections $\pi^U_W : \widetilde{K}(U) \to K(W)$. Also define $\widetilde{K}(\varnothing)$ to be a zero object. For every inclusion of τ_X-open subsets $U \supseteq V$, define

$$\widetilde{K}^U_V : \prod_{W \subseteq U} K(W) \to \prod_{W \subseteq V} K(W),$$

to be the unique morphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every $W \subset V$, $\pi_W^V \circ \widetilde{K}_V^U$ equals π_W^U. **Prove** that $\widetilde{K}$ is a presheaf of $\Lambda-\Pi$-bimodules. For discontinuous $\Lambda-\Pi$-bimodules K and L, for every morphism of discontinuous $\Lambda-\Pi$-bimodules, $\alpha : K \to L$, for every τ_X-open set U, define

$$\widetilde{\alpha}(U) : \prod_{W \subseteq U} K(W) \to \prod_{W \subseteq U} L(W)$$

to be the unique morphism of $\Lambda(U)-\Pi(U)$-bimodules such that for every $W \subseteq U$, $\pi_{L,W}^U \circ \widetilde{\alpha}(U)$ equals $\pi_{K,W}^U$. **Prove** that $\widetilde{\alpha}$ is a morphism of presheaves of $\Lambda-\Pi$-bimodules. **Prove** that these notions define a functor,

$$\widetilde{\ast} : \Lambda-\Pi-\mathrm{Disc}_{(X,\tau_X)} \to \Lambda-\Pi-\mathrm{Presh}_{(X,\tau_X)}.$$

Prove that this is an exact functor that preserves arbitrary limits and finite colimits.

(d)(The Čech Object of a Discontinuous $\Lambda-\Pi$-Bimodule is Acyclic) For every open covering $(U, \iota : \mathfrak{U} \to \tau_U)$, define

$$\tau_{\mathfrak{U}} = \bigcup_{U_0 \in \mathfrak{U}} \tau_{\iota(U_0)} = \{W \in \tau_U | \exists U_0 \in \mathfrak{U}, W \subset \iota(U_0)\}.$$

For every discontinuous $\Lambda-\Pi$-bimodule K, define

$$\widetilde{K}(\mathfrak{U}) := \prod_{W \in \tau_{\mathfrak{U}}} K(W)$$

together with its projections $\pi_W : \widetilde{K}(\mathfrak{U}) \to K(W)$. In particular, define

$$\pi_{\mathfrak{U}}^U : \widetilde{K}(\mathfrak{U}) \to \widetilde{K}(\mathfrak{U})$$

to be the unique $\Lambda(U)-\Pi(U)$-morphism such that for every $W \in \tau_{\mathfrak{U}}$, $\pi_W \circ \pi_{\mathfrak{U}}^U$ equals π_W.

For every nonempty $W \in \tau_{\mathfrak{U}}$, define

$$\mathfrak{U}^W := \{U_0 \in \mathfrak{U} | W \subset \iota(U_0)\}.$$

Prove that

$$\check{C}^r(\mathfrak{U}, \widetilde{K}) = \prod_{(U_0,\dots,U_r)\in\mathfrak{U}^{r+1}} \prod_{W \subseteq \iota(U_0,\dots,U_r)} K(W)$$

together with its projection $\pi_{(U_0,\dots,U_r;W)} : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to K(W)$ for every nonempty $W \subset \iota(U_0,\dots,U_r)$; if $\iota(U_0,\dots,U_r)$ is empty, the corresponding factor is a zero object. For every integer $r \geq 0$, for every $i = 0,\dots,r+1$, **prove** that the morphism

$$\partial_r^i : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to \check{C}^{r+1}(\mathfrak{U}, \widetilde{K}),$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0,\dots,U_r,U_{r+1}) \in (\mathfrak{U}^W)^{r+2}$, $\pi_{U_0,\dots,U_r,U_{r+1};W} \circ \partial_r^i$ equals $\pi_{U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1};W}$. For every integer $r \geq 0$ and for every $i = 0,\dots,r$, **prove** that the morphism

$$\sigma_{r+1}^i : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K}) \to \check{C}^r(\mathfrak{U}, \widetilde{K}),$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r;W} \circ \sigma^i_{r+1}$ equals equals $\pi_{U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_{r+1};W}$. For every integer $r \geq 0$, prove that the morphism

$$g^r_{\widetilde{K},\mathfrak{U}} : \widetilde{K}(\mathfrak{U}) \to \check{C}^r(\mathfrak{U}, \widetilde{K})$$

is the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every nonempty $W \in \tau_{\mathfrak{U}}$ and for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r;W} \circ g^r$ equals π_W.

For every nonempty $W \in \tau_{\mathfrak{U}}$, for every $r \geq 0$, define

$$\check{C}^r(\mathfrak{U}, \widetilde{K})^W := \prod_{(U_0,\dots,U_r)\in(\mathfrak{U}^W)^{r+1}} K(W),$$

with its projections

$$\pi_{U_0,\dots,U_r|W} : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to K(W).$$

Define

$$\pi^r_{-;W} : \check{C}^r(\mathfrak{U}, \widetilde{K}) \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

to be the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ \pi^r_{-;W}$ equals $\pi_{U_0,\dots,U_r;W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r+1$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$\partial^i_r : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W,$$

such that $\partial^i_r \circ \pi^r_{-;W}$ equals $\pi^{r+1}_{-;W} \circ \partial^i_r$, and **prove** that for every $(U_0, \dots, U_r, U_{r+1}) \in (\mathfrak{U}^W)^{r+2}$, $\pi_{U_0,\dots,U_r,U_{r+1}|W} \circ \partial^i_r$ equals $\pi_{U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1}|W}$. For every integer $r \geq 0$ and for every $i = 0, \dots, r$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$\sigma^i_{r+1} : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W,$$

such that $\sigma^i_{r+1} \circ \pi^{r+1}_{-;W}$ equals $\pi^r_{-;W} \circ \sigma^i_{r+1}$, and **prove** that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ \sigma^i_{r+1}$ equals equals $\pi_{U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_{r+1}|W}$. For every integer $r \geq 0$, **prove** that there exists a unique $\Lambda(U) - \Pi(U)$-morphism

$$g^r : K(W) \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

such that $\pi^r_{-;W} \circ g^r$ equals $g^r \circ \pi_W$, and **prove** that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ g^r$ equals $\mathrm{Id}_{K(W)}$. Conclude that

$$\pi^\bullet_{-;W} : \check{C}^\bullet(\mathfrak{U}, \widetilde{K}) \to \check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$$

is a morphism of cosimplicial $\Lambda(U) - \Pi(U)$-bimodules that is compatible with the coaugmentations $g^\bullet$. **Prove** that these morphisms realize $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ in the category $S^\bullet\Lambda(U) - \Pi(U) - \text{Bimod}$ as a product,

$$\check{C}^\bullet(\mathfrak{U}, \widetilde{K}) = \prod_{W\in\tau_{\mathfrak{U}}} \check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W.$$

Using the Axiom of Choice, prove that there exists a set map

$$\phi : \tau_{\mathfrak{U}} \smallsetminus \{\emptyset\} \to \mathfrak{U}$$

such that for every nonempty $W \in \tau_{\mathfrak{U}}$, $\phi(W)$ is an element in $\mathfrak{U}^W$. For every integer $r \geq 0$, define

$$\check{C}^r(\phi, \widetilde{K})^W : \check{C}^r(\mathfrak{U}, \widetilde{K})^W \to K(W)$$

to be $\pi_{\phi(W),\dots,\phi(W)|W}$. **Prove** that for every integer $r \geq 0$ and for every $i = 0, \dots, r+1$, $\check{C}^{r+1}(\phi, \widetilde{K})^W \circ \partial_r^i$ equals $\check{C}^r(\phi, \widetilde{K})^W$. **Prove** that for every integer $r \geq 0$ and for every $i = 0, \dots, r$, $\check{C}^r(\phi, \widetilde{K})^W \circ \sigma_{r+1}^i$ equals $\check{C}^{r+1}(\phi, \widetilde{K})^W$. Conclude that

$$\check{C}^\bullet(\phi, \widetilde{K})^W \to \text{const}_{K(W)}$$

is a morphism of cosimplicial $\Lambda(U) - \Pi(U)$-bimodules. **Prove** that $\check{C}^\bullet(\phi, \widetilde{K})^W \circ g^\bullet$ equals the identity morphism of $\text{const}_{K(W)}$. For every nonempty $W \in \tau_{\mathfrak{U}}$, for every integer $r \geq 0$, for every integer $i = 0, \dots, r$, define

$$g_{\phi,r+1}^i : \check{C}^{r+1}(\mathfrak{U}, \widetilde{K})^W \to \check{C}^r(\mathfrak{U}, \widetilde{K})^W$$

to be the unique $\Lambda(U)$–$\Pi(U)$-morphism such that for every $(U_0, \dots, U_r) \in (\mathfrak{U}^W)^{r+1}$, $\pi_{U_0,\dots,U_r|W} \circ g_{\phi,r+1}^i$ equals $\pi_{U_0,\dots,U_i,\phi(W),\dots,\phi(W)|W}$. **Prove** the following identities (cosimplicial homotopy identities),

$$g_{\phi,r+1}^0 \circ \partial_r^0 = g^r \circ \check{C}^r(\phi, \widetilde{K})^W, \quad g_{\phi,r+1}^r \circ \partial_r^{r+1} = \text{Id}_{\check{C}^r(\mathfrak{U},\widetilde{K})^W},$$

$$g_{\phi,r+1}^j \circ \partial_r^i = \begin{cases} \partial_{r-1}^i \circ g_{\phi,r}^{j-1}, & 0 \leq i < j \leq r, \\ g_{\phi,r+1}^{i-1} \circ \partial_r^i, & 0 < i = j \leq r, \\ \partial_{r-1}^{i-1} \circ g_{\phi,r}^j, & 1 \leq j+1 < i \leq r+1. \end{cases}$$

$$g_{\phi,r}^j \circ \sigma_{r+1}^i = \begin{cases} \sigma_r^i \circ g_{\phi,r+1}^{j+1}, & 0 \leq i \leq j \leq r-1, \\ \sigma_r^{i-1} \circ g_{\phi,r+1}^j, & 0 \leq j < i \leq r. \end{cases}$$

Conclude that $g^\bullet$ and $\check{C}^\bullet(\phi, \widetilde{K})^W$ are homotopy equivalences between $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$ and $\text{const}_{K(W)}$. Conclude that $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ is homotopy equivalent to $\text{const}_{\widetilde{K}(\mathfrak{U})}$. In particular, **prove** that the associated cochain complex of $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})^W$ is acyclic with $\check{H}^0(\mathfrak{U}, \widetilde{K})^W$ equal to $K(W)$. Similarly, **prove** that the associated cochain complex of $\check{C}^\bullet(\mathfrak{U}, \widetilde{K})$ is acyclic with $\check{H}^0(\mathfrak{U}, \widetilde{K})$ equal to $K(\mathfrak{U})$.

(e)(The Forgetful Functor to Discontinuous $\Lambda - \Pi$-Bimodules; Preservation of Injectives) For every presheaf M of $\Lambda - \Pi$-bimodules on (X, τ_X), define $\Phi(M)$ to be the discontinuous $\Lambda - \Pi$-bimodule $U \mapsto M(U)$. For presheaves of $\Lambda - \Pi$-bimodules, M and N, for every morphism of presheaves of $\Lambda - \Pi$-bimodules, $\alpha : M \to N$, define $\Phi(\alpha) : \Phi(M) \to \Phi(N)$ to be the assignment $U \mapsto \alpha(U)$. **Prove** that these rules define a functor

$$\Phi : \Lambda - \Pi - \text{Presh}_{(X,\tau_X)} \to \Lambda - \Pi - \text{Disc}_{(X,\tau_X)}.$$

Prove that this is a faithful exact functor that preserves arbitrary limits and finite colimits. For every presheaf M of $\Lambda - \Pi$-bimodules, for every τ_X-open U, define

$$\theta_{M,U} : M(U) \to \prod_{W \subseteq U} M(W)$$

to be the unique homomorphism of $\Lambda(U) - \Pi(U)$-bimodules such that for every τ_X-open subset $W \subset U$, $\pi^U_W \circ \theta_{M,U}$ equals M^U_W. **Prove** that $U \mapsto \theta_{M,U}$ is a morphism of presheaves of $\Lambda - \Pi$-bimodules,

$$\theta_M : M \to \widetilde{\Phi(M)}.$$

For every morphism of presheaves of $\Lambda - \Pi$-bimodules, $\alpha : M \to N$, for every τ_X-open set U, **prove** that $\widetilde{\Phi(\alpha)} \circ \theta_M$ equals $\theta_N \circ \alpha$. Conclude that θ is a natural transformation of functors,

$$\theta : \text{Id}_{\Lambda - \Pi - \text{Presh}_{(X,\tau_X)}} \Rightarrow \widetilde{\ast} \circ \Phi.$$

For every discontinuous $\Lambda - \Pi$-bimodule K, for every τ_X-open U, define

$$\eta_{K,U} : \prod_{W \subseteq U} K(W) \to K(U)$$

to be π^U_W. **Prove** that $U \mapsto \eta_{K,U}$ is a morphism of discontinuous $\Lambda - \Pi$-bimodules. For every pair of discontinuous $\Lambda - \Pi$-bimodules, K and L, for every morphism of discontinuous $\Lambda - \Pi$-bimodules, $\beta : K \to L$, **prove** that $\eta_L \circ \Phi(\widetilde{\beta})$ equals $\beta \circ \eta_L$. Conclude that η is a natural transformation of functors,

$$\eta : \Phi \circ \widetilde{\ast} \Rightarrow \text{Id}_{\Lambda - \Pi - \text{Disc}_{(X,\tau_X)}}.$$

Prove that $(\Phi, \widetilde{\ast}, \theta, \eta)$ is an adjoint pair of functors. Since Φ preserves monomorphisms, use Problem 3(d), Problem Set 5 to **prove** that $\widetilde{\ast}$ sends injective objects to injective objects. Since the forgetful morphism from sheaves to presheaves preserves monomorphisms, **prove** that the sheafification functor Sh sends injective objects to injective objects. Conclude that $\text{Sh} \circ \widetilde{\ast}$ sends injective objects to injective objects.

(f)(Enough Injectives) Recall from Problems 3 and 4 of Problem Set 5 that for every τ_X-open set U, there are enough injective $\Lambda(U) - \Pi(U)$-bimodules. Using the Axiom of Choice, conclude that $\Lambda - \Pi - \text{Disc}_{(X,\tau_X)}$ has enough injective objects. In particular, for every presheaf M of $\Lambda - \Pi$-bimodules, for every open set U, let there be given a monomorphism of $\Lambda(U) - \Pi(U)$-bimodules,

$$\epsilon_U : M(U) \to I(U),$$

with $I(U)$ an injective $\Lambda(U) - \Pi(U)$-bimodule. Conclude that $\widetilde{I}$ is an injective presheaf of $\Lambda - \Pi$-bimodules, and the composition

$$M \xrightarrow{\theta_M} \widetilde{\Phi(M)} \xrightarrow{\widetilde{\epsilon}} \widetilde{I}$$

is a monomorphism of presheaves of $\Lambda - \Pi$-bimodules. If M is a sheaf, conclude that $\text{Sh}(\widetilde{I})$ is an injective sheaf of $\Lambda - \Pi$-bimodules. Also, use (d) to prove that the composition

$$M \xrightarrow{\theta_M} \widetilde{\Phi(M)} \xrightarrow{\widetilde{\epsilon}} \widetilde{I} \xrightarrow{sh} \text{Sh}(\widetilde{I})$$

is a monomorphism of sheaves of $\Lambda-\Pi$-bimodules. (**Hint:** Since $\sigma_{x,U}$ is a filtering small category, use Problem 0 to reduce to the statement that for every open covering $(U,\mathfrak{U})$, the morphism $M(U)\to\widetilde{M}(\mathfrak{U})$ is a monomorphism. Realize this a part of the Sheaf Axiom for M.) Conclude that both the category $\Lambda-\Pi-\text{Presh}_{(X,\tau_X)}$ and $\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}$ have enough injective objects. In particular, for an additive, left-exact functor F, resp. G, on the category of presheaves of $\Lambda-\Pi$-bimodules, resp. the category of sheaves of $\Lambda-\Pi$-bimodules, there are right derived functors $((R^nF)_n,(\delta^n)_n)$, resp. $((R^nG)_n,(\delta^n)_n)$. Finally, since $\widetilde{*}$ is exact and sends injective objects to injective objects, use the Grothendieck Spectral Sequence (or universality of the cohomological δ-functor) to **prove** that $(R^nF)\circ\widetilde{*}$ is $R^n(F\circ\widetilde{*})$.

(g)(Enough Flasque Sheaves; Injectives are Flasque) Let K be a discontinuous $\Lambda-\Pi$-bimodule on X. For every τ_X-open set U, **prove** that $\widetilde{K}(U)\to\text{Sh}(\widetilde{K})(U)$ is the colimit over all open coverings $\mathfrak{U}\subset\tau_U$ (ordered by refinement as usual) of the morphism

$$\pi^U_{\mathfrak{U}}:\widetilde{K}(U)\to\widetilde{K}(\mathfrak{U}).$$

In particular, since every morphism $\widetilde{K}(U)\to\widetilde{K}(\mathfrak{U})$ is surjective (by the Axiom of Choice), conclude that also

$$\text{sh}(U):\widetilde{K}(U)\to\text{Sh}(\widetilde{K})(U)$$

is surjective. Use this to **prove** that $\text{Sh}(\widetilde{K})$ is a flasque sheaf.

For every injective $\Lambda-\Pi$-sheaf I, for the monomorphism $\theta_I:I\to\text{Sh}(\widetilde{\Phi(I)})$, there exists a retraction $\rho:\text{Sh}(\widetilde{\Phi(I)})\to I$. Also $\text{Sh}(\widetilde{\Phi(I)})$ is flasque. Use this to **prove** that also I is flasque.

(h)(Sheaf Cohomology; Flasque Sheaves are Acyclic) For every τ_X-open set U, prove that the functor

$$\Gamma(U,-):\Lambda-\Pi-\text{Presh}_{(X,\tau_X)}\to\Lambda(U)-\Pi(U)-\text{Bimod},\quad M\mapsto M(U)$$

is an exact functor. Also prove that the functor

$$\Gamma(U,-):\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}\to\Lambda(U)-\Pi(U)-\text{Bimod}$$

is an additive, left-exact functor. Use (g) to conclude that every sheaf M of $\Lambda-\Pi$-modules admits a resolution, $\epsilon:M\to I^\bullet$ by injective sheaves of $\Lambda-\Pi$-modules that are also flasque. Conclude that $\Gamma(U,-)$ extends to a universal cohomological δ-functor formed by the right derived functors, $((H^n(U,-))_n,(\delta^n)_n)$. Finally, assume that M is flasque. Use Problem 4(d) to **prove** that $I^\bullet(U)$ is an acyclic complex of $\Lambda(U)-\Pi(U)$-bimodules. Conclude that for every flasque sheaf M of $\Lambda-\Pi$-bimodules, for every $n\geq 0$, $H^n(U,M)$ is zero, i.e., flasque sheaves of $\Lambda-\Pi$-bimodules are acyclic for the right derived functors of $\Gamma(U,-)$.

(i)(Computation of Sheaf Cohomology via Flasque Resolutions; Canonical Resolutions; Independence of $\Lambda-\Pi$) Use (h) and the hypercohomlogy spectral sequence to **prove** that for every sheaf M of $\Lambda-\Pi$-bimodules, for every acyclic resolution $\epsilon_M:M\to M^\bullet$ of M by sheaves of $\Lambda-\Pi$-bimodules that are flasque, for every integer $n\geq 0$, there is a canonical isomorphism of $H^n(U,M)$

with $h^n(M^\bullet(U))$. In particular, the functor $\top = \text{Sh}\circ\widetilde{\star}\circ\Phi$, the natural transformation $\theta:\text{Id}\Rightarrow\top$, and the natural transformation

$$\text{Sh}\circ\widetilde{\star}\circ\eta\circ\Phi : \top\top\Rightarrow\top,$$

form a *triple* on the category $\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}$. There is an associated cosimplicial functor,

$$L_\top : \Lambda-\Pi-\text{Sh}_{(X,\tau_X)} \to S^\bullet\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}$$

and a functorial coaugmentation,

$$\theta_M : \text{const}^\bullet_M \to L^\bullet_\top(M).$$

The associated (unnormalized) cochain complex of this cosimplicial object is an acyclic resolution of M by flasque sheaves of $\Lambda-\Pi$-bimodules, and it is *canonical*, depending on no choices of injective resolutions.

Finally, let $\widehat{\Lambda}\to\Lambda$ and $\widehat{\Pi}\to\Pi$ be morphisms of presheaves of associative, unital rings. This induces a functor,

$$\Lambda-\Pi-\text{Sh}_{(X,\tau_X)} \to \widehat{\Lambda}-\widehat{\Pi}-\text{Sh}_{(X,\tau_X)}.$$

For every sheaf M of $\Lambda-\Pi$-bimodules, and for every acyclic resolution $\epsilon : M\to M^\bullet$ of M by flasque sheaves of $\Lambda-\Pi$-bimodules, this is also an acyclic, flasque resolution of M with the associated structure of sheaves of $\widehat{\Lambda}-\widehat{\Pi}$-bimodules. For the natural map of cohomological δ-functors from the derived functors of $\Gamma(U,-)$ on $\Lambda-\Pi-\text{Sh}_{(X,\tau_X)}$ to the derived functors of $\Gamma(U,-)$ on $\widehat{\Lambda}-\widehat{\Pi}-\text{Sh}_{(X,\tau_X)}$, **prove** that this natural map is a natural isomorphism of cohomological δ-functors. This justifies the notation $H^n(U,-)$ that makes no reference to the underlying presheaves Λ and Π, and yet is naturally a functor to $\Lambda(U)-\Pi(U)-\text{Bimod}$ whenever M is a sheaf of $\Lambda-\Pi$-bimodules.

Problem 6.(Flasque Sheaves are Čech-Acyclic) Let (X,τ_X) be a topological space. Let M be a presheaf of $\Lambda-\Pi$-bimodules on (X,τ_X). Let U be a τ_X-open set. Let $(U,\iota:\mathfrak{U}\to\tau_U)$ be an open covering. For every τ_X-open subset V, define $(V,\iota_V:\mathfrak{U}\to\tau_V)$ to be the open covering $\iota_V(U_0) = V\cap\iota(U_0)$. For simplicity, denote this by $(V,\mathfrak{U}_V)$. For every integer $r\geq 0$, define $\underline{\check{C}}^r(\mathfrak{U},M)(V)$ to be the $\Lambda(V)-\Pi(V)$-bimodule $\check{C}^r(\mathfrak{U}_V,M)$. Moreover, define

$$\partial^i_{V,r} : \underline{\check{C}}^r(\mathfrak{U},M)(V)\to\underline{\check{C}}^{r+1}(\mathfrak{U},M)(V),\quad \sigma^i_{V,r+1} : \underline{\check{C}}^{r+1}(\mathfrak{U},M)(V)\to\underline{\check{C}}^{r}(\mathfrak{U},M)(V),$$

to be the face and degeneracy maps on $\check{C}^\bullet(\mathfrak{U}_V,M)$. Finally, let $\eta^r_V : M(V)\to\underline{\check{C}}^r(\mathfrak{U},M)(V)$ be the coadjunction of sections from Problem 5(e), Problem Set 8. For every inclusion of τ_X-open subsets $W\cap V\cap U$, the identity map $\text{Id}_{\mathfrak{U}}$ is a refinement of open coverings,

$$\phi^V_W : (V,\iota_V:\mathfrak{U}\to\tau_V)\to(W,\iota_W:\mathfrak{U}\to\tau_W).$$

By Problem 5(f) from Problem Set 8, $\check{C}^r(\phi^V_W,M)$ is an associated morphism of $\Lambda(V)-\Pi(V)$-bimodules, denoted

$$\underline{\check{C}}^r(\mathfrak{U},M)^V_W : \underline{\check{C}}^r(\mathfrak{U},M)(V)\to\underline{\check{C}}^r(\mathfrak{U},M)(W).$$

(a)(The Presheaf of Čech Objects) **Prove** that the rules $V \mapsto \underline{\check{C}}^r(\mathfrak{U}, M)(V)$ and $\underline{\check{C}}^r(\mathfrak{U}, M)^V_W$ define a presheaf $\underline{\check{C}}^r(\mathfrak{U}, M)$ of $\Pi - \Lambda$-bimodules on U. Moreover, **prove** that the rules $V \mapsto \partial^i_{V,r}$, resp. $V \mapsto \sigma^i_{V,r+1}$, $V \mapsto \eta^r_V$, define morphisms of presheaves of $\Lambda - \Pi$-bimodules,

$$\partial^i_r : \underline{\check{C}}^r(\mathfrak{U}, M) \to \underline{\check{C}}^{r+1}(\mathfrak{U}, M), \quad \sigma^i_{r+1} : \underline{\check{C}}^{r+1}(\mathfrak{U}, M) \to \underline{\check{C}}^r(\mathfrak{U}, M), \quad \eta^r : M|_U \to \underline{\check{C}}^r(\mathfrak{U}, M).$$

Use Problem 5(f) from Problem Set 8 again to prove that these morphisms define a functor,

$$\underline{\check{C}}^\bullet : \sigma \times \Lambda - \Pi - \text{Presh}_{(X,\tau_X)} \to S^\bullet\Lambda - \Pi - \text{Presh}_{(U,\tau_U)},$$

compatible with cosimplicial homotopies for pairs of refinements and together with a natural transformation of cosimplicial objects,

$$\eta^\bullet : \text{const}^\bullet_{M|_U} \to \underline{\check{C}}^\bullet(\mathfrak{U}, M).$$

(b)(The Čech Resolution Preserves Sheaves and Flasques) For every $(U_0, \dots, U_r)$ in $\mathfrak{U}^{r+1}$, denote by $i_{U_0,\dots,U_r} : (\iota(U_0, \dots, U_r), \tau_{\iota(U_0,\dots,U_r)}) \to (U, \tau_U)$ the continuous inclusion map. **Prove** that $\underline{\check{C}}^r(\mathfrak{U}, M)$ is isomorphic as a presheaf of $\Lambda - \Pi$-bimodules to

$$\prod_{(U_0,\dots,U_r)} (\iota_{U_0,\dots,U_r})_* \iota^{-1}_{U_0,\dots,U_r} M.$$

Use Problem 4(a) and (b) to **prove** that $\underline{\check{C}}^r(\mathfrak{U}, M)$ is a sheaf whenever M is a sheaf, and it is flasque whenever M is flasque.

(c)(Localy Acyclicity of the Čech Resolution) Assume now that M is a sheaf. For every τ_X-open subset $V \subset U$ such that there exists $* \in \mathfrak{U}$ with $V \subset \iota(*)$, conclude that $(V, \mathfrak{U}_V)$ refines to $(V, \{V\})$. Using Problem 5(h), Problem Set 8, **prove** that

$$\eta^\bullet_V : \text{const}^\bullet_{M(V)} \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)(V)$$

is a homotopy equivalence. Conclude that for the cochain differential associated to this cosimplicial object,

$$d^r = \sum_{i=0}^{r} (-1)^i \partial^i_r,$$

the coaugmentation

$$\eta_V : M(V) \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)(V)$$

is an acyclic resolution. Conclude that the coaugmentation of complexes of sheaves of $\Pi - \Lambda$-bimodules,

$$\eta : M|_U \to \underline{\check{C}}^\bullet(\mathfrak{U}, M)$$

is an acyclic resolution.

Now assume that M is flasque. **Prove** that η is a flasque resolution of the flasque sheaf $M|_U$. Using Problem 5(i), **prove** that the cohomology of the complex of $\Lambda(U)-\Pi(U)$-bimodules,

$$\check{H}^n(\mathfrak{U}, M) := h^n(\check{C}^\bullet(\mathfrak{U}, M), d^\bullet)$$

equals $H^\bullet(U, M)$. Using Problem 5(h), **prove** that $H^0(U, M)$ equals $M(U)$ and $H^n(U, M)$ is zero for every integer $n > 0$. Conclude that for every flasque sheaf M of $\Lambda-\Pi$-bimodules, for every open covering $(U, \mathfrak{U})$, $M(U) \to \check{H}^0(\mathfrak{U}, M)$ is an isomorphism and $\check{H}^n(\mathfrak{U}, M)$ is zero for every integer $n > 0$.

Čech Cohomology as a Derived Functor Exercise. Let (X, τ_X) be a topological space. Let U be a τ_X-open set. Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an open covering. For every presheaf A of $\Lambda-\Pi$-bimodules, denote by $\check{C}^\bullet(\mathfrak{U}, A)$ the object in $\mathbf{Ch}^{\geq 0}(\Lambda-\Pi-\text{Bimod})$ associated to the cosimplicial object.

(a)(Exactness of the Functor of Čech Complexes; The δ-Functor of Čech Cohomologies) Use Problem 5 of Problem Set 8 to **prove** that this is an additive functor

$$\check{C}^\bullet(\mathfrak{U}, -) : \Lambda-\Pi-\text{Presh}_{(X,\tau_X)} \to \text{Ch}^{\geq 0}(\Lambda-\Pi-\text{Bimod}).$$

Prove that for every short exact sequence of presheaves of $\Lambda-\Pi$-bimodules,

$$0 \longrightarrow A' \xrightarrow{q} A \xrightarrow{p} A'' \longrightarrow 0,$$

the associated sequence of cochain complexes,

$$0 \longrightarrow \check{C}^\bullet(\mathfrak{U}, A') \xrightarrow{\check{C}^\bullet(\mathfrak{U},q)} \check{C}^\bullet(\mathfrak{U}, A) \xrightarrow{\check{C}^\bullet(\mathfrak{U},p)} \check{C}^\bullet(\mathfrak{U}, A'') \longrightarrow 0,$$

is a short exact sequence. Use this to prove that the Čech cohomology functor $\check{H}^0(\mathfrak{U}, A) = h^0(\check{C}^\bullet(\mathfrak{U}, A))$ is an additive, left-exact functor, and the sequence of Čech cohomologies,

$$\check{H}^r(\mathfrak{U}, A) = h^r(\check{C}^\bullet(\mathfrak{U}, A)),$$

extend to a cohomological δ-functor from $\Lambda-\Pi-\text{Presh}_{(X,\tau_X)}$ to $\Lambda(U)-\Pi(U)-\text{Bimod}$.

(b)(Effaceability of Čech Cohomology) For every presheaf A of $\Lambda-\Pi$-bimodules, use Problem 5(e) and 5(f) to **prove** that $\theta_A : A \to \widetilde{\Phi(A)}$ is a natural monomorphism of presheaves of $\Lambda-\Pi$-bimodules. Use Problem 5(d) to prove that for every $r \geq 0$, $\check{H}^r(\mathfrak{U}, \widetilde{\Phi(A)})$ is zero. Conclude that $\check{H}^r(\mathfrak{U}, -)$ is effaceable. **Prove** that the cohomological δ-functor $((\check{H}^r(\mathfrak{U}, A))_r, (\delta^r)_r)$ is universal. Conclude that the natural transformation of cohomological δ-functors from the right derived functor of $\check{H}^0(\mathfrak{U}, -)$ to the Čech cohomology δ-functor is a natural isomorphism of cohomological δ-functors.

(c)(Hypotheses of the Grothendieck Spectral Sequence) Denote by

$$\Psi : \Lambda-\Pi-\text{Sh}_{(X,\tau_X)} \to \Lambda-\Pi-\text{Presh}_{(X,\tau_X)},$$

the additive, fully faithful embedding (since we are already using Φ for the forgetful morphism to discontinuous $\Lambda-\Pi$-bimodules). Recall from Problem 6(c) on Problem Set 8 that this extends to an adjoint pair of functors (Sh, Φ). Recall the construction of Sh as a filtering colimit of Čech cohomologies $\check{H}^0(\mathfrak{U}, -)$. Since $\check{H}^0(\mathfrak{U}, -)$ is left-exact, and since $\Lambda - \Pi - \mathrm{Presh}_{(X,\tau_X)}$ satisfies Grothendieck's condition (AB5), **prove** that Sh is left-exact. Use Problem 3(d), Problem Set 5 to **prove** that Ψ sends injective objects to injective objects. Use Problem 5(g) to **prove** that every injective sheaf I of $\Lambda - \Pi$-bimodules is flasque. Use Problem 6(c) to **prove** that $\Psi(I)$ is acyclic for $\check{H}^\bullet(\mathfrak{U}, -)$. **Prove** that the pair of functors Ψ and $\check{H}^0(\mathfrak{U}, -)$ satisfy the hypotheses for the Grothendieck Spectral Sequence. Conclude that there is a convergent, first quadrant cohomological spectral sequence,

$$^{I}E_2^{p,q} = \check{H}^p(\mathfrak{U}, R^q\Psi(A)) \Rightarrow H^{p+q}(U, A).$$

(d)(The Derived Functors of Ψ are the Presheaves of Sheaf Cohomologies) For every sheaf A of $\Lambda - \Pi$-bimodules, for every integer $r \geq 0$, for every τ_X-open set U, denote $\mathcal{H}^r(A)(U)$ the additive functor $H^r(U, A)$. In particular, $\mathcal{H}^0(A)(U)$ is canonically isomorphic to $A(U)$. Thus, for all τ_X-open sets, $V \subset U$, there is a natural transformation

$$*|_V^U : \mathcal{H}^0(-)(U) \to \mathcal{H}^0(-)(V).$$

Use universality to **prove** that this uniquely extends to a morphism of cohomological δ-functors,

$$*|_V^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(V))_r, (\delta^r)_r).$$

Prove that for all τ_X-open sets, $W \subset V \subset U$, both the composite morphism of cohomological δ-functors,

$$*|_W^V \circ *|_V^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(V))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(W))_r, (\delta^r)_r),$$

and the morphism of cohomological δ-functors,

$$*|_W^U : ((\mathcal{H}^r(-)(U))_r, (\delta^r)_r) \to ((\mathcal{H}^r(-)(W))_r, (\delta^r)_r),$$

extend the functor $*|_W^V \circ *|_V^U = *|_W^U$ from $\mathcal{H}^0(-)(U)$ to $\mathcal{H}^0(-)(W)$. Use the uniqueness in the universality to conclude that these two morphisms of cohomological δ-functors are equal. **Prove** that $((\mathcal{H}^r(-))_r, (\delta^r)_r)$ is a cohomological δ-functor from $\Lambda - \Pi - \mathrm{Sh}_{(X,\tau_X)}$ to $\Lambda - \Pi - \mathrm{Presh}_{(X,\tau_X)}$. Use Problem 5(h) to **prove** that every flasque sheaf is acyclic for this cohomological δ-functor. Combined with Problem 5(i), **prove** that the higher functors are effaceable, and thus this cohomological δ-functor is universal. Conclude that this the canonical morphism of cohomological δ-functors from the right derived functors of Ψ to this cohomological δ-functor is a natural isomorphism of cohomological δ-functors. In particular, combined with the last part, this gives a convergent, first quadrant spectral sequence,

$$^{I}E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(U, A).$$

This is the *Čech-to-Sheaf Cohomology Spectral Sequence*. In particular, conclude the existence of monomorphic abutment maps,

$$\check{H}^r(\mathfrak{U}, A) \to H^r(U, A).$$

as well as abutment maps,

$$H^r(U,A) \to H^0(\mathfrak{U}, \mathcal{H}^r(A)).$$

(e)(The Colimit of Čech Cohomology with Respect to Refinement) Since Čech complexes are compatible with refinement, and the refinement maps are well-defined up to cosimplicial homotopy, the induced refinement maps on Čech cohomology are independent of the choice of refinement. Use this to define a directed system of Čech cohomologies. Denote the colimit of this direct system as follows,

$$\check{H}^\bullet(U,-) = \operatorname*{colim}_{\mathfrak{U}\in\sigma_{x,U}} \check{H}^\bullet(\mathfrak{U},-).$$

Prove that this extends uniquely to a cohomological δ-functor such that for every open covering $(U,\mathfrak{U})$, the induced sequence of natural transformations,

$$*|_U^{\mathfrak{U}} : ((\check{H}^r(\mathfrak{U},-))_r, (\delta^r)_r) \to ((\check{H}^r(U,-))_r, (\delta^r)_r),$$

is a natural transformation of cohomological δ-functors. Repeat the steps above to deduce the existence of a unique convergent, first quadrant spectral sequence,

$$^I E_2^{p,q} = \check{H}^p(U, \mathcal{H}^q(A)) \Rightarrow H^{p+q}(U,A),$$

such that for every open covering $(U,\mathfrak{U})$, the natural maps

$$*|_U^{\mathfrak{U}} : \check{H}^p(\mathfrak{U}, \mathcal{H}^q(A)) \to \check{H}^p(U, \mathcal{H}^q(A))$$

extend uniquely to a morphism of spectral sequences. In particular, conclude the existence of monomorphic abutment maps

$$\check{H}^r(U,A) \to H^r(U,A)$$

as well as abutment maps

$$H^r(U,A) \to \check{H}^0(U, \mathcal{H}^r(A)).$$

Use the first abutment maps to define subpresheaves $\check{\mathcal{H}}^r(A)$ of $\mathcal{H}^r(A)$ by $V \mapsto \check{H}^r(V,A)$.

(f)(Reduction of the Spectral Sequence; $\check{H}^1(U,A)$ equals $H^1(U,A)$) For every $r > 0$, **prove** that the associated sheaf of $\mathcal{H}^r(A)$ is a zero sheaf. (**Hint**. Prove the stalks are zero by using commutation of sheaf cohomology with filtered colimits combined with exactness of the stalks functor.) Conclude that $\check{H}^0(U, \mathcal{H}^r(A))$ is zero. In particular, conclude that the natural abutment map,

$$\check{H}^1(U,A) \to H^1(U,A)$$

is an isomorphism. Thus, also $\check{\mathcal{H}}^1(A) \to \mathcal{H}^1(A)$ is an isomorphism. Use this to produce a "long exact sequence of low degree terms" of the spectral sequence,

$$0 \to \check{H}^2(U,A) \to H^2(U,A) \to \check{H}^1(U, \check{\mathcal{H}}^1(A)) \xrightarrow{\delta} \check{H}^3(U,A).$$

(g)(Sheaves that Are Čech-Acyclic for "Enough" Covers are Acyclic for Sheaf Cohomology) Let $\mathcal{B} \subset \tau_X$ be a basis that is stable for finite intersection. For every open U in $\mathcal{B}$, let Cov_U be a collection of open coverings of U by sets in $\mathcal{B}$ such that Cov_U is cofinal with respect to refinement in $\sigma_{x,U}$. Let A be such that for every U in $\mathcal{B}$, for every $(U, \mathfrak{U})$ in Cov_U, for every $r > 0$, $\check{H}^r(\mathfrak{U}, A)$ is zero. **Prove** that $\mathcal{H}^r(U, A)$ is zero. Use the spectral sequence to inductively **prove** that for every $r > 0$, $\mathcal{H}^r(A)(U)$ is zero, $H^r(U, A)$ is zero and $\mathcal{H}^r(A)(U)$ is zero. Conclude that for every open covering $(X, \iota : \mathfrak{V} \to \mathcal{B})$, the Čech-to-Sheaf Cohomology Spectral Sequence relative to $\mathfrak{V}$ degenerates to isomorphisms

$$\check{H}^r(\mathfrak{V}, A) \to H^r(X, A).$$

If you are an algebraic geometer, let $(X, \mathcal{O}_X)$ be a separated scheme, let $\Lambda = \Pi = \mathcal{O}_X$, let $\mathcal{B}$ be the basis of open affine subsets, let Cov_U be the collection of basic open affine coverings, and let A be a quasi-coherent sheaf. Read the proof that for every basic open affine covering $(U, \mathfrak{U})$ of an affine scheme, for every quasi-coherent sheaf A, $\check{H}^r(\mathfrak{U}, A)$ is zero for all $r \geq 0$ (this is essentially exactness of the Koszul cochain complex for a regular sequence, combined with commutation with colimits). Use this to conclude that quasi-coherent sheaves are acyclic for sheaf cohomology on any affine scheme. Conclude that, on a separated scheme, for every quasi-coherent sheaf, sheaf cohomology is computed as Čech cohomology of any open affine covering.

A Propositional calculus

The language of category theory uses the language of classes. The most common formulation of class theory in pure mathematics is a second-order theory built on top of the first-order theory of predicate calculus and Zermelo – Fraenkel set theory.

Every formal language has an *alphabet* of symbols, A. The *Kleene star* A^* of the alphabet A is the set of all *strings* of elements of A, i.e., the elements of A^* are those ordered pairs whose first entry is a nonnegative (true) integer n, the *length* of the string, and whose second entry is itself an ordered n-tuple of elements of A (sometimes called a *literal*). Thus, for every nonempty set A, the first projection is a surjective function from A^* to the set of all nonnegative integers whose fiber over each nonnegative integer n is (naturally bijective to) the set A^n of all ordered n-tuples of elements of A.

Every formal language also has a specified subset of A^* whose elements are called *well-formed formulas*. In formalizing mathematics, a formal language is usually defined to be an ordered pair of an alphabet A and of this subset of A^*. Most often this subset is specified by a subset of *atomic strings* and a collection of *production rules* for producing new well-formed formulas from existing well-formed formulas. Then the well-formed formulas are all strings obtained by iteratively applying the production rules to the atomic strings (this is formalized mathematically using the Chomsky hierarchy, automata, the Chomsky-Schützenberger theorem, etc.).

In the formal language for propositional logic, the alphabet includes one symbol for the propositional variable as well as the following symbols for the usual logical connectives (we use the pipe to separate items in a list).

$$p \mid \Leftarrow \mid \Rightarrow \mid \neg \mid \wedge \mid \vee \mid \Leftrightarrow \mid (\mid)$$

When immediately preceded and succeeded by symbols other than p, the string p is p_1, and, for every (true) positive integer n, the concatenated string $p_n * p$ is p_{n+1}. Thus, the alphabet represents denumerably many

A string in propositional logic is a well-formed formula (called a *proposition*) if and only if it can be obtained, starting from propositional variables p_n for all (true) positive integers n, by iterated application of the following production rules. For all well-formed formulas f and g, also the following are well-formed formulas:

$$(f) \mid (f) \Rightarrow (g) \mid (f) \Leftarrow (g) \mid (f) \Leftrightarrow (g) \mid \neg(f) \mid (f) \wedge (g) \mid (f) \vee (g)$$

The alphabet of predicate calculus also includes a symbol t for a term variable. As with the propositional variables, when preceded and succeded by symbols other than t, the string t is t_1, and, for every (true) positive integer n, the concatenated string $t_n * t$ is t_{n+1}. The alphabet also includes a symbol – the comma "," – for separating term variables in a list. [HERE]

that are certain finite strings of symbols from the alphabet. In predicate calculus, there are symbols that allow to produce *variables*. Every well-formed formula, or *predicate*, in predicate calculus has a specified set of *free variables* whose number is a nonnegative integer called the *arity*. Every predicat also has a specified set of *bound variables*, each bound by precisely one quantifier ($\forall$ or $\exists$) and distinct from all free variables of the predicate. There is also a deductive system of axioms and inference rules that iteratively produce all *theorems* of Zermelo – Fraenkel set theory. Often one thinks of theorems as well-formed formulas for a second formal language structure on the same alphabet.

The alphabet of predicate calculus includes the usual logical connectives

Many strings of elements from the alphabet are not well-formed formulas. The well-formed formulas are those that are obtained by iteratively applying the *production rules* of the formal language to the list of *atomic strings*. The alphabet of *zeroth-order propositional calculus* includes a (countably enumerated) list of propositional variables, i.e., p_1, p_2, etc. It is common to have only one propositional symbol in the alphabet, say p, which when repeated consecutively gives all other propositional variables, e.g., repeated once p is p_1, repeated twice pp is p_2, repeated three times ppp is p_3, etc. The production rules of propositional calculus are as follows: every propositional variable is a well-formed formula, and for all well-formed formulas f and g

With the usual (i.e., intended) meaning of the logical connectives, the well-formed formulas are precisely those strings of elements from the alphabet that have an unambiguous (Boolean) value of true, $\top$, or false, $\bot$, for each *model*, i.e., for each assignment to each propositional variable of a (Boolean) value of $\top$ or $\bot$.

This formal language becomes a *Hilbert system* by introducing a second list of production rules – called *axioms* (if they have arity 0) and *inference rules* (if they have arity > 0) – that are *sound* for the (intended) models: i.e., for every model, every inference rule preserves the set of those well-formed formulas that take the value $\top$ on that model (when we give each logical connective its intended meaning). One common Hilbert system, the Łukasieweicz system, is obtained by first adopting *modus ponens*, i.e., the production rule that associates to each pair of well-formed formulas of the form f and $(f) \Rightarrow (g)$ (that both take the value $\top$ for the model) the well-formed formula g (this also takes the value $\top$ for the model, for the intended interpretation of $\Rightarrow$, so that modus ponens is sound). In symbols, this inference rule is as follows.

Modus Ponens for f and g. $f, (f) \Rightarrow (g) \vdash g$

We also have three additional axiom schemata for the Łukasieweicz system (where f, g and h are substituted with all triples of well-formed formulas, whether or not those formulas take the value $\top$ for a particular model).

Ł1 for f and g. $\vdash (f) \Rightarrow ((g) \Rightarrow (f))$

Ł2 for f, g and h. $\vdash ((f) \Rightarrow ((g) \Rightarrow (h))) \Rightarrow (((f) \Rightarrow (g)) \Rightarrow ((f) \Rightarrow (h)))$

Ł3 for f and g. $\vdash ((\neg(f)) \Rightarrow (\neg(g))) \Rightarrow ((g) \Rightarrow (f))$

Since we are also using other logical connectives than just $\Rightarrow$ and $\neg$, we add as axioms the definitions of those logical connectives in terms of $\Rightarrow$ and $\neg$.

Conjunction. $\vdash \neg((f) \Rightarrow (\neg(g))) \Rightarrow ((f) \wedge (g))$

$\vdash ((f) \wedge (g)) \Rightarrow \neg((f) \Rightarrow (\neg(g)))$

Disjunction. $\vdash ((\neg(f)) \Rightarrow (g)) \Rightarrow ((f) \vee (g))$

$\vdash ((f) \vee (g)) \Rightarrow ((\neg(f)) \Rightarrow (g))$

Reverse Implication. $\vdash ((f) \Rightarrow (g)) \Rightarrow ((g) \Leftarrow (f))$,

$\vdash ((g) \Leftarrow (f)) \Rightarrow ((f) \Rightarrow (g))$

Logical Equivalence. $\vdash (((f) \Rightarrow (g)) \wedge ((g) \Rightarrow (f))) \Rightarrow ((f) \Leftrightarrow (g))$

$\vdash ((f) \Leftrightarrow (g)) \Rightarrow (((f) \Rightarrow (g)) \wedge ((g) \Rightarrow (f)))$

A *theorem* of this Hilbert system is a well-formed formula obtained by iteratively applying modus ponens beginning with the axioms above. For some theorems, the iterative procedure is fairly short. For instance, for every well-formed formula f, substituting f for g in Axiom Ł1 gives the following.

Ł1 for f and f. $\vdash (f) \Rightarrow ((f) \Rightarrow (f))$

Next, substituting $(f) \Rightarrow (f)$ for g in Axiom Ł1 gives the following.

Ł1 for f and $(f) \Rightarrow (f)$. $\vdash (f) \Rightarrow (((f) \Rightarrow (f)) \Rightarrow (f))$

Substitute $(f) \Rightarrow (f)$ for g and substitute f for h in Axiom Ł2 to get the following.

Ł2 for f, $(f) \Rightarrow (f)$ and f. $\vdash ((f) \Rightarrow (((f) \Rightarrow (f)) \Rightarrow (f))) \Rightarrow (((f) \Rightarrow ((f) \Rightarrow (f))) \Rightarrow ((f) \Rightarrow (f)))$

Apply Modus Ponens to the previous two well-formed formulas gives the following.

Modus Ponens for Ł$1_{f,(f)\Rightarrow(f)}$ and Ł$2_{f,(f)\Rightarrow(f),f}$. $\vdash ((f) \Rightarrow ((f) \Rightarrow (f))) \Rightarrow ((f) \Rightarrow (f))$

Finally, apply Modus Ponens once more to the first well-formed formula and to this last well-formed formula gives the following.

Modus Ponens. $\vdash (f) \Rightarrow (f)$

Thus, using the syntactic procedure explained above, we have deduced the statement $(f) \Rightarrow (f)$ for every well-formed formula f. It is standard to define $\top$ to be this well-formed formula, $(f) \Rightarrow (f)$. Then define $\bot$ to be the well-formed formula $\neg(\top)$.

Of course other theorems have more content and longer proofs. For instance, basic theorems about the nonnegative integers (via some formalization of Peano arithmetic) and finite sets give the *Deduction Theorem*: for every finite collection $A = \{A_1, \dots, A_n\}$ of well-formed formulas, if there is a finite sequence of applications of the inference rules to the well-formed formulas in A and to the Łukasieweicz axioms that leads to a proof of the well-formed formula B, then we also have a finite sequence of applications of the inference rules to the Łukasieweicz axioms that leads to a proof of the well-formed formula

$$(A_1 \wedge (A_2 \wedge (\dots (A_{n-1} \wedge A_n) \dots))) \Rightarrow B.$$

Conversely, Modus Ponens applied to A and this well-formed formula gives B, so that the Deduction Theorem becomes an "if and only if" statement.

The *theory* of the Łukasieweicz deductive system is the set of all theorems, considered as a (nonempty, proper) subset of the set of all well-formed formulas. By Post's completeness theorem, a well-formed formula is a theorem if and only if, for every model, the well-formed formula takes the value $\top$. (For beginning mathematics students, a well-formed formula is a theorem if and only if it always gives the value $\top$ for every row in the *truth table*).

B Predicate calculus

First-order predicate calculus extends the propositional variables to predicate variables of arbitrary *arity* (a nonnegative integer), where the propositional variables are predicate variables of arity 0. A predicate variable of arity n immediately followed by an ordered n-tuple of term variables enclosed by parentheses is an *atomic predicate*. The variables occurring in the ordered n-tuple are the *free variables* of the predicate. The number of distinct free variables is the arity of the predicate. For a propositional variable p_n of arity 0, by convention we write p_n as the predicate rather than $p_n()$. We also have one special predicate of arity 2, equality $=$. This is always written in infix notation, i.e., we always write $t = s$ rather than $= (t, s)$. Every atomic predicate is a well-formed formula of predicate calculus.

We also add the universal quantifier, $\forall$, and the existential quantifier, $\exists$, with their usual syntax: each of these quantifiers is immediately followed by a term variable which is then followed by a well-formed formula (of arbitrary arity) that may, or may not, include the term variable among its list of free variables, i.e., $\exists s\ f$ and $\forall s\ f$, but must never include the term variable among the bound variables of f. The bound variables of this new predicate is the set of all bound variables of f together with s. The free variables of the new predicate is the set of all free variables of f different from s.

One way to prevent the bound term variable s from being among the bound variables of f is to use *variable substitution*. For each term variable t, for each term variable u, for each predicate f such that either t does not occur in f or such that u is not a bound variable of f, there is a predicate $f[u/t]$, by replacing each instance of t in f with u. The free variables of $f[u/t]$ are obtained from the free variables of f by replacing t by u (if t occurs in the list of free variables of f), and the bound variables of $f[u/t]$ are obtained from bound variables of f by replacing t by u (if t occurs in the list of bound variables of f).

Similarly, when combining well-formed formulas using the logical connectives, we substitute term variables bounded by quantifiers so that each bound variable of one constituent well-formed formula in the connective is not a variable occurring in the other constituent well-formed formula. Then the free variables of the connective are those in the union of the sets of free variables of the constituents, and likewise for the set of bound variables of the constituents (which are now disjoint sets). The predicates are all of the well-formed formulas produced by these production rules. Each predicate has an arity n that is a nonnegative integer together with a set of n free variables, and a set of bound variables. Using variable substitution, we insure that no free variable is also a bound variable, and every bound variable is bound precisely once in the predicate.

For all predicates f and g, we add the following axioms.

Universal Generalization. $\vdash f[t/s] \Rightarrow (\forall u\ f[u/s])$

Here s is any term variable that is not a bound variable of f, and t and u are term variables that first appear at the application of universal generalization (i.e., for a new term variable t not mentioned in any earlier step of a derivation, if we can derive $f[t/s]$, then we deduce that $f[u/s]$ holds for all u).

Universal Instantiation. $\vdash (\forall u\ f[u/s]) \Rightarrow f[t/s]$

Here s and t are term variables that are not bound variables of f, and u is a term variable that does not occur in f.

Existential Generalization. $\vdash p[t/s] \Rightarrow (\exists u\ f[u/s])$

Here s and t are term variables that are not bound variables of f, and u is a term variable that first appears at the application of existential generalization.

Existential Instantiation. $\vdash (\exists u\ f[u/s]) \Rightarrow f[t/s]$

Here s is a term variable that is not a bound variable of f, and u and t are term variables that first appear at the application of existential instantiation.

For every predicate f, and for all term variables s, t and u, we add the following axioms for the arity-2 predicate of equality.

Substitution. $\vdash (t = u) \Rightarrow (f[t/s] \Leftrightarrow f[u/s])$

Reflexivity. $\vdash t = t$

Transitivity. $\vdash ((t = u) \wedge (u = v)) \Rightarrow (t = v)$

Symmetry. $\vdash (t = u) \Rightarrow (u = t)$

Here s is a free variable of f, and t and u are not bound variables of f.

For predicate calculus, a *model* consists of a nonempty set, called a *universe*, together with an assignment to each predicate variable of arity n of a true / false valued function (Boolean function) on the n-fold self-product of the universe. For this model, we use the usual (i.e., intended) meaning for the equality symbol, the logical connectives, and the quantifiers. For the well-formed formulas of arity 0, there is a well-defined true / false value of the well-formed formula for each model. For well-formed formulas of higher arity, they are considered true if and only if they are true once all free variables are universally quantified (this corresponds to the inference rule of universal generalization). The axioms and inference rules of first-order predicate calculus are sound for these models with the intended interpretation of the logical symbols (and assuming consistency of some version of Peano arithmetic). By Gödel's Completeness Theorem, a well-formed formula of arity 0 is a theorem produced by the axioms and inference rules of predicate calculus if and only if the Boolean value of the well-formed formula is true for every model (Gödel's Completeness Theorem assumes consistency of an appropriate fragment of Zermelo-Fraenkel set theory).

C Zermelo-Fraenkel axioms

The only additional symbol in Zermelo-Fraenkel set theory is an arity-2 predicate written in infix notation, $x \in y$, read "x is an element of y" or "y contains x as an element." Adding this predicate, the production rules produce the well-formed formulas of set theory, i.e., the Zermelo – Fraenkel predicates. To the axioms and inference rules of predicate calculus, we also add the following axioms of Zermelo – Fraenkel set theory.

Axiom C.1 (Axiom of Extensionality)**.** For every set a and for every set b, the set a equals the set b if and only if, for every set x, the set x is an element of a if and only if x is an element of b.

$$\forall a\ \forall b\ (\forall x\ ((x \in a) \Leftrightarrow (x \in b))) \Leftrightarrow (a = b)$$

Axiom C.2 (Axiom of Regularity)**.** For every set a such that there exists a set x that is an element of a, there exists an element y of a such that every element of y is not an element of a.

$$\forall a\ ((\exists x\ (x \in a)) \Rightarrow (\exists y\ (y \in a) \wedge (\forall z\ (z \in y) \Rightarrow \neg(z \in a))))$$

Together with the other axioms, the axiom of regularity implies a strong form of foundation: there does not exist a sequence of sets $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that for every element n of $\mathbb{Z}_{\geq 0}$ the set a_{n+1} is an element of the set a_n (every formalization of this requires first formalizing natural numbers).

The next axiom is sometimes also called the "Axiom of Separation." It is an axiom schema: there is one axiom for each predicate $f(s,t)$ in the first-order language of set theory together with an ordered pair (s,t) of (all of) the free variables of the predicate.

Axiom C.3 (Axiom Schema of Specification)**.** For every set b, for every set c, there exists a set a whose elements are precisely those elements x of b such that the predicate $f(c,x)$ is true.

$$\forall b\ \forall c\ \exists a(\forall x\ ((x \in a) \Leftrightarrow ((x \in b) \wedge p(c,x)))$$

In particular, assuming that the universe of sets has at least one member (which we do assume), for the predicate $p(s,t)$ of s equals s and t does **not** equal t, for each set $a = \varnothing$ produced by the axiom (for any set b and for any set c), for every set x, the set x is **not** an element of $\varnothing$. The Axiom of Extensionality guarantees that this **empty set** is unique. So (together with the tacit axiom that the universe of sets has at least one member), the Axiom Schema of Specification gives the existence of an empty set.

Please note, we certainly do need to guard the quantifier of x in the Axiom Schema of Specification, restricting x to an element of the specified set b, to avoid asserting that there exists a "set whose elements are all sets that do not include themselves as an element" (which leads to Russell's Paradox about whether the set is an element of itself). Also note, we do not claim that we can recover the predicate $p(s,t)$ from the subset of b. For one thing, different predicates can be logically equivalent, so the best we could hope for is to recover the truth-valued function whose domain equals b determined by the predicate. A subset a of b is equivalent to such a truth-valued function, and every such subset arises from substitution of a for s in the specific predicate p: $t \in s$. So this axiom schema is producing "every" subset that it should. Even though predicates are specified via a finite string of symbols from an (at most) countable alphabet, this certainly does not imply that we have (at most) countably many distinct subsets of b (in a given model of Zermelo – Fraenkel set theory), since the subset c can range freely. As Cantor proved, for every set b, there does not exist a surjective function from b to the set of all subsets of b.

Axiom C.4 (Axiom of Pairing)**.** For every set a, for every set b, there exists a set $\{a,b\}$ whose elements are precisely a and b.

$$\forall a\ \forall b\ \exists c\ \forall x\ ((x \in c) \Leftrightarrow ((x = a) \vee (x = b)))$$

Please note, for every set a and for every set b, the set $\{a,b\}$ equals the singleton set $\{a\}$ if (and only if) b equals a. Thus, this axiom also gives the existence of the Kuratowski ordered pair,

$(a,b) := \{\{a\},\{a,b\}\}$, by applying the axiom to the singleton set $\{a\}$ and the doubleton set $\{a,b\}$. By the Axiom of Extensionality, for every set a, for every set b, for every set a', for every set b', the ordered pair (a,b) equals the ordered pair (a',b') if and only if both a equals a' and b equals b'.

Axiom C.5 (Axiom of Union)**.** For every set a, there exists a set b such that, for every set x, the set x is an element of b if and only if there exists an element y of a such that x is an element of y.

$$\forall a\ \exists b\ \forall x\ ((x \in b) \Leftrightarrow (\exists y\ ((x \in y) \wedge (y \in a))))$$

By the Axiom of Extensionality, the *union set* produced by this axiom is unique. In particular, for every set a, for every set b, the Axiom of Union applied to the set $\{a,b\}$ guarantees the existence of a set, denoted $a \cup b$, such that, for every set x, the set x is an element of $a \cup b$ if (and only if) either x is an element of a or x is an element of b (or both).

Similar to the Axiom Schema of Specification, the next axiom schema has one axiom for each predicate $f(x,b,y)$ in the first-order language of set theory together with an ordered triple (x,b,y) of (all of) the free variables of the predicate.

Axiom C.6 (Axiom Schema of Replacement)**.** For every set b and for every set d such that, for every element x of d there exists a unique set y satisfying $f(x,b,y)$, then there exists a set c whose elements are precisely those sets y such that there exists an element x of d such that $f(x,b,y)$ holds.

$$\forall b\ \forall d\ ((\forall x\ ((x \in d) \Rightarrow (\exists y\ p(x,b,y)) \wedge (\forall z\ \forall w((p(x,b,z) \wedge p(x,b,w)) \Rightarrow (y = z)))) \Rightarrow$$
$$(\exists c\ \forall y'\ ((y' \in c) \Leftrightarrow (\exists x\ (x \in d) \wedge p(x,b,y')))))$$

Consider the predicate f with an ordered triple of free variables (x,b,y): the set y equals (x,b), i.e., y equals $\{\{x\},\{x,b\}\}$. By the Axiom of Pairing, for every set a, for every set b, and for every element x of a, there exists a unique set y satisfying the predicate $p(x,b,y)$. Thus, the Axiom Schema of Replacement guarantees the existence of a set, denoted $a \times \{b\}$, such that for every set y, the set y is an element of $a \times \{b\}$ if (and only if) there exists an element x of a such that y equals (x,b). Moreover, by the Axiom of Extensionality, this set $a \times \{b\}$ is unique.

Next, consider the predicate f' with an ordered triple of free variables (x',b',y'): y' equals $b' \times \{x'\}$. By the previous paragraph, for every set a', for every set a, and for every element x of a', there exists a unique set $a \times \{x\}$ satisfying the predicate $f'(x',a,y')$. Thus, the Axiom Schema of Replacement and the Axiom of Union guarantees the existence of a set, denoted $a \times a'$, such that for every set x'', the set x'' is an element of $a \times a'$ if (and only if) there exists an element x of a and there exists an element x' of a' such that x'' equals (a,a'). Therefore, for every set a and for every set a', the Axiom Schema of Replacement (together with the earlier axioms) guarantees the existence of a Cartesian product set $a \times a'$. By the Axiom of Extensionality, the Cartesian product set $a \times a'$ is unique.

This, finally, leads to the essential meaning of the Axiom Schema of Replacement. For every ordered triple $(b, d, g(x, z, y))$ of a set b, of a domain set d, and of a "function" predicate $g(x, z, y)$ for (b, d), i.e., such that for every element x of d there exists a unique set y such that $g(x, b, y)$ holds, there exists an *image set* c for $(b, d, g(x, z, y))$, and also there exists a Cartesian product set $d \times c$. Finally, by the Axiom Schema of Specification, the predicate $g(x, z, y)$ and the set b (substituted for z) determines a subset $\text{graph}(g(x, b, y))$ of $d \times c$ that equals the graph of a unique set function from d onto c. Therefore, for every domain set d, for every "parameter" set b, and for every predicate $g(x, b, y)$ that determines a function in the "traditional" sense on the domain set d, there exists a unique image set $c = \text{cod}_{d,b,g}$ and a unique surjective set function $\text{func}_{d,b,g}$ from d to $\text{cod}_{d,b,g}$ such that for every element x of d, for every set y, the predicate $g(x, b, y)$ holds if and only if both y is an element of $\text{cod}_{d,b,g}$ and y equals the value of $\text{func}_{d,b,g}$ on x. Thus, to every function in the "traditional" sense on the domain set d, there exists a function in the set-theoretical sense of a subset of a Cartesian product $d \times c$ satisfying the "vertical line test."

As with the Axiom Schema of Specification, the Axiom Schema of Replacement is producing all the set-theoretical functions from d to c, since we can let $g(x, b, y)$ be the predicate that x is an element of d, that y is an element of c, that (x, y) is an element of b, and that b is a subset of $c \times d$ such that for every element x of d, there exists a unique element y of c for which (x, y) is an element of b (i.e., b is a subset of $c \times d$ that satisfies the "vertical line test").

Axiom C.7 (Axiom of Infinity)**.** There exists a set $\mathbb{Z}_{\geq 0}$ such that (i) the empty set, $\varnothing$, is one element of $\mathbb{Z}_{\geq 0}$, such that (ii) for every element $n \in \mathbb{Z}_{\geq 0}$ the set $n \cup \{n\}$ is an element in $\mathbb{Z}_{\geq 0}$, and such that (iii) the set $\mathbb{Z}_{\geq 0}$ is a subset of every set that satisfies both (i) and (ii).

$$\exists z \; ((\varnothing \in z) \wedge (\forall n \; ((n \in z) \Rightarrow (n \cup \{n\} \in z)))) \wedge$$
$$(\forall z' \; (((\varnothing \in z') \wedge (\forall n' \; ((n' \in z') \Rightarrow (n' \cup \{n'\} \in z')))) \Rightarrow (\forall n'' \; (n'' \in z) \Rightarrow (n'' \in z'))))$$

Consider the predicate $g(x, b, y)$ with three free variables: b equals b and y equals $x \cup \{x\}$. This is a predicate as in the Axiom Schema of Replacement, i.e., it can be used to define a set function, succ (for "successor"), in the "traditional" sense for each specification of domain set. Since the empty set contains no element $\{n\}$, the empty set can never be an element of the image set of such a function. The empty set *can* be an element of the domain set, i.e., $\{\varnothing\}$ can be a subset of the domain that is disjoint from the image set. The Axiom of Infinity guarantees the existence of a domain set for this function such that the domain set equals the disjoint union of the image set and the singleton set $\{\varnothing\}$.

For each such domain set, the intersection of all subsets of the domain set satisfying these conditions is a unique subset, by the Axiom Schema of Specification and the Axiom of Extensionality. So, up to replacing any domain set as above by this unique subset, there exists a unique domain set $\mathbb{Z}_{\geq 0}$ for succ that equals the disjoint union of the image set and the singleton set $\{\varnothing\}$, and such that every domain set satisfying these conditions contains $\mathbb{Z}_{\geq 0}$ as a subset. For every model of Zermelo – Fraenkel set theory, the set $\mathbb{Z}_{\geq 0}$, the element $\varnothing$ of $\mathbb{Z}_{\geq 0}$ (interpreted as the element 0), and the associated set function succ from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$ is a model of the (second order) axiom schema of Peano arithmetic. So the Axiom of Infinity inteprets Peano arithmetic within Zermelo-Fraenkel set theory. This addresses the difficulty, mentioned earlier, that many of the metamathematical notions about this axiomatization of set theory implicitly use some formalization of the natural numbers.

Axiom C.8 (Axiom of Power Sets)**.** For every set b, there exists a set, denoted $\mathcal{P}(b)$, such that for every set a, the set a is an element of $\mathcal{P}(b)$ if and only if the set a is a subset of b, i.e., if and only if, for every set x, if x is an element of a then a is an element of b.

$$\forall b\ \exists b'\ \forall a\ ((a \in b') \Leftrightarrow (\forall x\ ((x \in a) \Rightarrow (x \in b))))$$

Really the axiom of power sets is only the first in a continuing list of axioms (e.g., "large cardinal" axioms) considered by set theorists that allow more and more of the operations on sets that are relevant in both mathematics and metamathematics.

The following axiom, the Axiom of Choice, is **not** part of the Zermelo – Fraenkel axiom system, but it is accepted by most current mathematicians. Assuming the consistency of the Zermelo – Fraenkel axiom system, Cohen and Gödel proved the independence of the Axiom of Choice: the Zermelo – Fraenkel axiom system remains consistent if we add the Axiom of Choice, and the Zermelo – Fraenkel axiom system remains consistent if we add the negation of the Axiom of Choice (obviously it is not consistent if we add both simultaneously).

Axiom C.9 (Axiom of Choice)**.** For every set a, for every set b, for every set c, if c is a subset of $a \times b$ such that for every element y of b there exists an element (x, y) of c, then there exists a subset d of c such that for every element y of b there exists a unique element (x, y) of d.

$$\forall a\ \forall b\ \forall c\ ((\forall y\ ((y \in b) \Rightarrow (\exists x\ ((x, y) \in c)))) \Rightarrow$$
$$(\exists d\ (\forall w\ ((w \in b) \Rightarrow ((\exists z\ ((z, w) \in d)) \wedge (\forall v\ \forall u\ ((((v, w) \in d) \wedge ((u, w) \in d)) \Rightarrow (v = u)))))))$$

As discussed in all books on set theory, in the presence of the Zermelo – Fraenkel axioms, the Axiom of Choice is equivalent to the Well-Order Principle (every set has a well-order), it is equivalent to Zorn's lemma, etc.

D Classes

The definition of category uses the notion of a class. Classes can be axiomatized as a first-order theory, as done by von Neumann – Bernays – Gödel or by Morse – Kelley. The approach here is a second-order theory using the metalanguage of (first-order) Zermelo – Fraenkel set theory. This can be formalized, for instance, by using a Gödel numbering of the well-formed formulas of (first-order) Zermelo – Fraenkel set theory, but we prefer the verbose alternative of writing out the predicates of Zermelo – Fraenkel set theory. The classes produced in this way are the *parametrically definable* classes. For every model of Zermelo – Fraenkel set theory, the parameterically definable classes in that model form a model of class theory (the model most often intended in analysis, algebra, geometry, and topology). In particular, the (Kuratowski) ordered pair $(a,b) := \{\{a\},\{a,b\}\}$ converts predicates of higher arity into predicates of lower arity, i.e., every predicate $p(t_1,t_2,\dots,t_{n-1},t_n)$ of arity $n \geq 1$ (with n a "true" natural number) in the first-order language of Zermelo – Fraenkel set theory converts to the following predicate $\widetilde{p}(t)$ of arity 1 with unique free variable t,

$$\exists t_1\ \exists t_2\ \dots \exists t_{n-1}\ \exists t_n\ ((t_1,(t_2,\dots,(t_{n-1},t_n)\dots)) = t) \wedge p(t_1,t_2,\dots,t_{n-1},t_n).$$

Definition D.1 (Parametrically definable classes)**.** For every ordered pair $((p(s,t),a),(p'(s',t'),a'))$ of (first-order, Zermelo – Fraenkel) predicates p, respectively p', with a specified ordered pair (s,t), resp. (s',t'), of (all of) its free variables and of a set a, resp. a', the ordered pair $(p(s,t),a)$ is **Lindenbaum-Tarski equivalent** to $(p'(s',t'),a')$ if (and only if)

$$\forall b\ \left(p'(a',b) \Leftrightarrow p(a,b)\right).$$

Because logical equivalence is reflexive, transitive and symmetric, also Lindenbaum-Tarski equivalence is reflexive, transitive and symmetric. A parametrically definable **class** is a Lindenbaum-Tarski equivalence class $[p(s,t),a]$ (i.e., we are extending the usual equality predicate $a = a'$ to a predicate $[p(s,t),a] = [p'(s',t'),a']$ via Lindenbaum-Tarski equivalence). For every class $[p(s,t),a]$, a set b is a **member** of $[p(s,t),a]$ if (and only if) $p(a,b)$ holds (i.e., we are extending the set membership predicate $b \in a$ to a predicate of membership of b in the class $[p(s,t),a]$ as above). For every class $\mathbf{C}$, a class $\mathbf{B}$ is a **subclass** of $\mathbf{C}$ if (and only if) every member of $\mathbf{B}$ is a member of $\mathbf{C}$ (i.e., we are extending the subset predicate $b \subseteq c$ to a subclass predicate).

With this definition, we have a variant of extensionality for classes.

Lemma D.2 (Extensionality)**.** *For every class $\mathbf{B}$, for every class $\mathbf{B}'$, the class $\mathbf{B}$ equals the class $\mathbf{B}'$ if and only if, for every set x, the set x is a member of $\mathbf{B}$ if and only if x is a member of $\mathbf{B}'$.*

Proof. This is just a restatement of Lindenbaum-Tarski equivalence. □

By construction we also have the axiom of class formation.

Lemma D.3 (Class Formation)**.** *For every (first-order, Zermelo – Fraenkel) predicate $p(s,t)$ with an ordered pair (s,t) of (all of) its free variables, for every set a, there exists a unique class $\mathbf{C}$ such that, for every set b, the set b is a member of $\mathbf{C}$ if and only if $p(a,b)$ holds.*

Proof. The class $\mathbf{C} := [p(s,t),a]$ is one such class. By the previous lemma, this is unique. □

In particular, we have a universal class.

Lemma D.4. *There exists a unique class $\mathbf{V}$ such that every set is a member of $\mathbf{V}$.*

Proof. Let $p(s,t)$ be tautological, e.g., $(s = s) \wedge (t = t)$. Then for every set a, say $a = \varnothing$, every set is a member of the class $\mathbf{V} := [(s = s) \wedge (t = t), a]$. By Lemma D.2, this class is unique. □

Also, we have a class for each set (including for the empty set). In most axiomatizations of class theory, each set is identified with its associated class (but we prefer not to do this).

Lemma D.5. *For every set a, there exists a unique class whose members are the elements of a. In particular, for a equal to the empty set, the associated class has no members. Two sets are equal if and only if their associated classes are equal.*

Proof. The members of the class $[t \in s, a]$ are precisely the sets the elements of a. By Lemma D.2, this class is unique. By the Axiom of Extensionality, two sets are equal if and only if their associated classes are equal. □

We also have a variant for classes of the axiom of foundation.

Lemma D.6 (Foundation)**.** *For every class $\mathbf{C}$, there does not exist a sequence $(a_n)_{n \in \mathbb{Z}_{\geq 0}}$ of members of $\mathbf{C}$ such that, for every element n of $\mathbb{Z}_{\geq 0}$, the set a_{n+1} is an element of the set a_n. In particular, for every class $\mathbf{C}$ that has at least one member, there exists a member a of $\mathbf{C}$ such that for every element of a, that element is not a member of $\mathbf{C}$.*

Proof. By foundation for Zermelo – Fraenkel set theory, there does not exist any sequence $(a_n)_{n \in \mathbb{Z}_{\geq 0}}$ of sets such that, for every element n of $\mathbb{Z}_{\geq 0}$, the set a_{n+1} is an element of the set a_n. Thus, there exists no such sequence satisfying the additional condition that every set a_n is a member of $\mathbf{C}$.

For every class $\mathbf{C}$ that has a member, there exists a set a_0 that is a member of $\mathbf{C}$. If there exists an element a_1 of a_0 that is also a member of $\mathbf{C}$, then this gives a finite sequence (a_0, a_1) of members of $\mathbf{C}$ such that a_1 is an element of a_0. If there exists an element a_2 of a_1 that is also a member of $\mathbf{C}$, then this gives a finite sequence (a_0, a_1, a_2) of member of $\mathbf{C}$ such that a_1 is an element of a_0 and a_2 is an element of a_1. Continuing inductively, either there exists a sequence $(a_0, a_1, \ldots, a_n)$ of members of $\mathbf{C}$ such that a_1 is an element of a_0, etc., a_n is an element of a_{n-1} and every element of a_n is not a member of $\mathbf{C}$, or there exists a sequence $(a_n)_{n \in \mathbb{Z}_{\geq 0}}$ of members of $\mathbf{C}$ such that, for every element n of $\mathbb{Z}_{\geq 0}$, the member a_{n+1} is an element of a_n. This second case is forbidden by the previous paragraph. Thus, there exists a member a_n of $\mathbf{C}$ such that every element of a_n is not a member of $\mathbf{C}$. □

The axioms in the previous section define Zermelo – Fraenkel set theory, i.e., ZF, but do not include the Axiom of Choice that gives ZFC set theory. The lemmas above verify the axioms of NBG, von Neumman – Bernays – Gödel class theory, for the model of parameterically definable classes in each model of ZF set theory, except for the Axiom of Limitation of Size, which is essentially a global analogue of the Axiom of Choice.

Of course there are many additional results about classes. Many of these are the analogues for classes of well-known results for sets.

Lemma D.7. *For every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, the class* $\mathbf{B}$ *equals the class* $\mathbf{B}'$ *if and only if both* $\mathbf{B}$ *is a subclass of* $\mathbf{B}'$ *and* $\mathbf{B}'$ *is a subclass of* $\mathbf{B}$*.*

Proof. Of course if $\mathbf{B}$ equals $\mathbf{B}'$, then every member of $\mathbf{B}$ is a member of $\mathbf{B}'$, i.e., $\mathbf{B}$ is a subclass of $\mathbf{B}'$, and every member of $\mathbf{B}'$ is a member of $\mathbf{B}$, i.e., $\mathbf{B}'$ is a subclass of $\mathbf{B}$.

Conversely, if both $\mathbf{B}$ is a subclass of $\mathbf{B}'$ and $\mathbf{B}'$ is a subclass of $\mathbf{B}$, then for every set x that is a member of $\mathbf{B}$, also x is a member of $\mathbf{B}'$, and for every set x that is a member of $\mathbf{B}'$, also x is a member of $\mathbf{B}$. By Lemma D.2, the class $\mathbf{B}$ equals the class $\mathbf{B}'$. □

Definition D.8. The class that has every set as a member is the **von Neumann class**, sometimes called the **von Neumann universe** or the **universal class**, denoted $\mathbf{V}$ or $\text{ob}_{\mathbf{Set}}$. For every set a, the class that has as members precisely the elements of a is the **class** of the set a, denoted $\mathbf{Cl}_a$.

Lemma D.9. *The von Neumann class* $\mathbf{V}$ *is the unique class such that, for every class* $\mathbf{B}$*, the class* $\mathbf{B}$ *is a subclass of* $\mathbf{V}$*. For every set* a*, the class* $\mathbf{Cl}_a$ *is the unique class such that, for every class* $\mathbf{B}$*, the class* $\mathbf{Cl}_a$ *is a subclass of* $\mathbf{B}$ *if and only if* x *is a member of* $\mathbf{B}$ *for every element* x *of* a*.*

Proof. By definition of $\mathbf{V}$, every set is a member of $\mathbf{V}$. Thus, every class is a subclass of $\mathbf{V}$. For every class $\mathbf{B}$, if also $\mathbf{V}$ is a subclass of $\mathbf{B}$, then $\mathbf{B}$ equals $\mathbf{V}$ by Lemma D.7. Thus, if every class is a subclass of $\mathbf{B}$, so that $\mathbf{V}$ is a subclass of $\mathbf{B}$ in particular, then $\mathbf{B}$ equals $\mathbf{V}$. Therefore $\mathbf{V}$ is the unique class such that every class is a subclass of $\mathbf{V}$.

For every set a, for every class $\mathbf{B}$, by the definition of subclass, the class $\mathbf{Cl}_a$ is a subclass of $\mathbf{B}$ if and only if, for every set x that is a member of $\mathbf{Cl}_a$, also x is a member of $\mathbf{B}$. By the definition of $\mathbf{Cl}_a$, this holds if and only if, for every set x that is an element of a, also x is a member of $\mathbf{B}$. □

Lemma D.10. *For every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, there exists a unique class* $\mathbf{B}\wedge\mathbf{B}'$ *whose members are those sets that are simultaneously members of* $\mathbf{B}$ *and members of* $\mathbf{B}'$*. The subclasses of* $\mathbf{B}\wedge\mathbf{B}'$ *are precisely the classes that are simultaneously subclasses of both* $\mathbf{B}$ *and* $\mathbf{B}'$*. For every ordered pair* (b,b') *of sets, the class* $\mathbf{Cl}_b\wedge\mathbf{Cl}_{b'}$ *equals* $\mathbf{Cl}_{b\cap b'}$*. Finally, for every class* $\mathbf{B}$ *there exists a class* $\cap\mathbf{B}$ *whose members are all sets* x *such that for every member* b *of* $\mathbf{B}$*, the set* x *is an element of* b*. In particular, for every set* c*, the class* $\cap\mathbf{Cl}_c$ *equals* $\mathbf{Cl}_{\cap c}$*.*

Proof. For every class $\mathbf{B}=[p(s,t),a]$, for every class $\mathbf{B}'=[p'(s',t'),a']$ the class $[p''(s'',t''),(a,a')]$ for the following predicate has as members precisely those sets that are simultaneously members of $\mathbf{B}$ and members of $\mathbf{B}'$.

$$\exists s\ \exists s'\ (p(s,t'')\wedge p'(s',t''))\wedge(s''=(s,s')).$$

By Lemma D.2, the class $\mathbf{B}\wedge\mathbf{B}'=[p''(s'',t''),(a,a')]$ is the unique class whose members are precisely those sets that are simultaneously members of $\mathbf{B}$ and members of $\mathbf{B}'$.

By definition, a class $\mathbf{C}$ is a subclass of $\mathbf{B}\wedge\mathbf{B}'$ if and only if, for every member x of $\mathbf{C}$, also x is a member of $\mathbf{B}\wedge\mathbf{B}'$. By the definition of $\mathbf{B}\wedge\mathbf{B}'$, for every set x, a set x is a member of $\mathbf{B}\wedge\mathbf{B}'$ if and only if both x is a member of $\mathbf{B}$ and x is a member of $\mathbf{B}'$. Thus, $\mathbf{C}$ is a subclass of $\mathbf{B}\wedge\mathbf{B}'$ if and only if, for every member x of $\mathbf{C}$, both x is a member of $\mathbf{B}$ and x is a member of $\mathbf{B}'$. By the definition of subclass, $\mathbf{C}$ is a subclass of $\mathbf{B} \wedge \mathbf{B}'$ if and only if both $\mathbf{C}$ is a subclass of $\mathbf{B}$ and $\mathbf{C}$ is a subclass of $\mathbf{B}'$.

For every ordered pair (b, b') of sets, by the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b$ if and only if x is an element of b, and the set x is a member of $\mathbf{Cl}_{b'}$ if and only if x is an element of b'. Thus, for every set x, the set x is a member of $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ if and only if both x is an element of b and x is an element of b'. By the definition of intersection, for every set x, the set x is both an element of b and an element of b' if and only if x is an element of $b \cap b'$. Thus, again using the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ if and only x is a member of $\mathbf{Cl}_{b\cap b'}$. By Lemma D.2, the class $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ equals $\mathbf{Cl}_{b\cap b'}$.

Finally, for every class $\mathbf{B} = [p(s,t), a]$, for the class $\cap\mathbf{B} := [\forall t\ (p(s,t) \Rightarrow (t' \in t)), a]$ with the ordered pair of free variables (s, t'), for every set x, the set x is a member of $\cap\mathbf{B}$ if and only if, for every member b of $\mathbf{B}$, the set x is an element of b. By Lemma D.2, the class $\cap\mathbf{B}$ is the unique class such that, for every set x, the set x is a member of $\cap\mathbf{B}$ if and only if, for every member b of $\mathbf{B}$, the set x is an element of b. In particular, for every set c, the class $\cap\mathbf{Cl}_c$ equals $\mathbf{Cl}_{\cap c}$. □

Lemma D.11. *For every class $\mathbf{B}$, for every class $\mathbf{B}'$, there exists a unique class $\mathbf{B} \vee \mathbf{B}'$ whose members are those sets that are either members of $\mathbf{B}$ or members of $\mathbf{B}'$ (or both). The classes that have $\mathbf{B} \vee \mathbf{B}'$ as a subclass are precisely the classes that both have $\mathbf{B}$ as a subclass and have $\mathbf{B}'$ as a subclass. For every ordered pair (b, b') of sets, the class $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ equals $\mathbf{Cl}_{b\cup b'}$. Finally, for every class $\mathbf{B}$ there exists a class $\cup\mathbf{B}$ whose members are all sets x such that there exists a member b of $\mathbf{B}$ with x an element of b. In particular, for every set c, the class $\cup\mathbf{Cl}_c$ equals $\mathbf{Cl}_{\cup c}$.*

Proof. For every class $\mathbf{B} = [p(s,t), a]$, for every class $\mathbf{B}' = [p'(s',t'), a']$, the class $[p''(s'',t''), a'']$ with $a'' = (a, a'')$ and with the following predicate has as members precisely those sets that are either members of $\mathbf{B}$ or members of $\mathbf{B}'$.

$$\exists s\ \exists s'\ (p(s,t'') \vee p'(s',t'')) \wedge (s'' = (s,s')).$$

By Lemma D.2, the class $\mathbf{B} \vee \mathbf{B}' = [p''(s'',t''), (a,a')]$ is the unique class whose members are precisely those sets that are either members of $\mathbf{B}$ or members of $\mathbf{B}'$.

By definition, a class $\mathbf{C}$ has $\mathbf{B} \vee \mathbf{B}'$ as a subclass if and only if, for every member x of $\mathbf{B} \vee \mathbf{B}'$, also x is a member of $\mathbf{C}$. By the definition of $\mathbf{B} \vee \mathbf{B}'$, for every set x, a set x is a member of $\mathbf{B} \vee \mathbf{B}'$ if and only if either x is a member of $\mathbf{B}$ or x is a member of $\mathbf{B}'$. Thus, $\mathbf{B} \wedge \mathbf{B}'$ is a subclass of $\mathbf{C}$ if and only if, both every member x of $\mathbf{B}$ is a member of $\mathbf{C}$ and every member x of $\mathbf{B}'$ is a member of $\mathbf{C}$. By the definition of subclass, $\mathbf{B} \wedge \mathbf{B}'$ is a subclass of $\mathbf{C}$ if and only if both $\mathbf{B}$ is a subclass of $\mathbf{C}$ and $\mathbf{B}'$ is a subclass of $\mathbf{C}$.

For every ordered pair (b, b') of sets, by the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b$ if and only if x is an element of b, and the set x is a member of $\mathbf{Cl}_{b'}$ if and only if x is an element of b'. Thus, for every set x, the set x is a member of $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ if and only if either x is an element of b or x is an element of b'. By the definition of union, for every set x, the set x is either an element of b or an element of b' if and only if x is an element of $b \cup b'$. Thus, again using the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ if and only x is a member of $\mathbf{Cl}_{b\cup b'}$. By Lemma D.2, the class $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ equals $\mathbf{Cl}_{b\cup b'}$.

Finally, for every class $\mathbf{B} = [p(s,t), a]$, for the class $\cup\mathbf{B} := [\exists t\ (p(s,t) \Rightarrow (t' \in t)), a]$ with the ordered pair of free variables (s, t'), for every set x, the set x is a member of $\cup\mathbf{B}$ if and only if there exists a member b of $\mathbf{B}$ that has x as an element. By Lemma D.2, the class $\cup\mathbf{B}$ is the unique class such that, for every set x, the set x is a member of $\cup\mathbf{B}$ if and only if there exists a member b of $\mathbf{B}$ that has x as an element. In particular, for every set c, the class $\cup\mathbf{Cl}_c$ equals $\mathbf{Cl}_{\cup c}$. ☐

Lemma D.12. *For every class $\mathbf{B}$, there exists a unique class $\neg\mathbf{B}$ whose members are those sets that are not members of $\mathbf{B}$. A class is a subclasses of $\neg\mathbf{B}$ if and only if every member of the class is not a member of $\mathbf{B}$. The class $\neg(\neg\mathbf{B})$ equals $\mathbf{B}$. For every class $\mathbf{B}'$, both $\neg(\mathbf{B} \wedge \mathbf{B}')$ equals $(\neg\mathbf{B}) \vee (\neg\mathbf{B}')$ and $\neg(\mathbf{B} \vee \mathbf{B}')$ equals $(\neg\mathbf{B}) \wedge (\neg\mathbf{B}')$. Also $\neg(\cap\mathbf{B})$ equals $\cup(\neg\mathbf{B})$, and $\neg(\cup\mathbf{B})$ equals $\cap(\neg\mathbf{B})$. For every set b, for every set b', the class $\mathbf{Cl}_b \wedge (\neg\mathbf{Cl}_{b'})$ equals $\mathbf{Cl}_{b\setminus b'}$.*

Proof. For every class $\mathbf{B} = [p(s,t), a]$, the members of $[\neg p(s,t), a]$ are precisely the sets that are not members of $\mathbf{B}$, and this class is unique by Lemma D.2.

By definition, a class $\mathbf{C}$ is a subclass of $\neg\mathbf{B}$ if and only if, for every member x of $\mathbf{C}$, also x is a member of $\neg\mathbf{B}$. By definition, for every set x, the set x is a member of $\neg\mathbf{B}$ if and only if x is not a member of $\mathbf{B}$. Therefore, $\mathbf{C}$ is a subclass of $\neg\mathbf{B}$ if and only if every member x of $\mathbf{C}$ is not a member of $\mathbf{B}$. In particular, a class $\mathbf{C}$ is a subclass of $\neg(\neg\mathbf{B})$ if and only if every member x of $\mathbf{C}$ is not a member of $\neg\mathbf{B}$, i.e., if and only if every member x of $\mathbf{C}$ is a member of $\mathbf{C}$. By Lemma D.7, the class $\neg(\neg\mathbf{B})$ equals $\mathbf{B}$.

For every class $\mathbf{B} = [p(s,t), a]$ and for every class $\mathbf{B}' = [p'(s',t'), a']$, since $\neg(p(a,t'') \wedge p'(a',t''))$ is logically equivalent to $(\neg p(a,t'')) \vee (\neg p'(a',t''))$, also the class $\neg(\mathbf{B} \wedge \mathbf{B}')$ equals $(\neg\mathbf{B}) \vee (\neg\mathbf{B}')$. Similarly, the class $\neg(\mathbf{B} \vee \mathbf{B}')$ equals the class $(\neg\mathbf{B}) \wedge (\neg\mathbf{B}')$.

Since the following two predicates are logically equivalent,

$$\neg\left(\exists t\ (x \in t) \wedge p(a,t)\right),$$

$$\forall t\ (x \in t) \Rightarrow \neg p(a,t),$$

the class $\neg(\cup\mathbf{B})$ equals $\cap(\neg\mathbf{B})$. Similarly, the class $\neg(\cap\mathbf{B})$ equals $\cup(\neg\mathbf{B})$.

Finally, for every set b, for every set b', for every set x, the set x is a member of $\mathbf{Cl}_b \wedge (\neg\mathbf{Cl}_{b'})$ if and only if both x is an element of b and x is not an element of b', i.e., if and only if x is an element of $b \setminus b'$. Thus, the class $\mathbf{Cl}_b \wedge (\neg\mathbf{Cl}_{b'})$ equals $\mathbf{Cl}_{b\setminus b'}$ by Lemma D.2. ☐

Lemma D.13. *For every class $\mathbf{B}$, for every class $\mathbf{B}'$, there exists a unique class $\mathbf{B} \times \mathbf{B}'$ whose members are ordered pairs (b, b') of a member b of $\mathbf{B}$ and a member b' of $\mathbf{B}'$. A class is a subclass of $\mathbf{B} \times \mathbf{B}'$ if and only if every member of the class is of the form (b, b') for a member b of $\mathbf{B}$ and a member b' of $\mathbf{B}'$. For every set c, for every set c', the class $\mathbf{Cl}_c \times \mathbf{Cl}_{c'}$ equals $\mathbf{Cl}_{c\times c'}$.*

Proof. For every class $\mathbf{B} = [p(s,t),a]$, for every class $\mathbf{B}' = [p'(s',t'),a']$, the class $[p''(s'',t''),a'']$ with $a'' = (a,a'')$ and with the following predicate has as members precisely those sets (b,b') such that b is a member of $\mathbf{B}$ and such that b' is a member of $\mathbf{B}'$.

$$\exists s\ \exists s'\ \exists t\ \exists t'\ (p(s,t) \vee p'(s',t')) \wedge (s'' = (s,s')) \wedge (t'' = (t,t')).$$

By Lemma D.2, the class $\mathbf{B}\times\mathbf{B}' = [p''(s'',t''),(a,a')]$ is the unique class whose members are precisely those sets (b,b') such that b is a member of $\mathbf{B}$ and such that b' is a member of $\mathbf{B}'$.

By definition, a class $\mathbf{C}$ is a subclass of $\mathbf{B}\times\mathbf{B}'$ if and only if, for every member b'' of $\mathbf{C}$ is also a member of $\mathbf{B}\times\mathbf{B}'$. By the definition of $\mathbf{B}\times\mathbf{B}'$, a set b'' is a member of $\mathbf{B}\times\mathbf{B}'$ if and only if b'' equals (b,b') for a member b' of $\mathbf{B}$ and for a member b' of $\mathbf{B}'$. Thus, $\mathbf{C}$ is a subclass of $\mathbf{B}\times\mathbf{B}'$ if and only if every member of $\mathbf{C}$ equals (b,b') for a member b of $\mathbf{B}$ and for a member b' of $\mathbf{B}'$.

For every set c, for every set c', by the definition of $\mathbf{Cl}$, a set is an element of $\mathbf{Cl}_c\times\mathbf{Cl}_{c'}$ if and only if the set equals (b,b') for an element b of c and for an element b' of c', i.e., if and only if the set is an element of $c\times c'$. Therefore, by Lemma D.2, the class $\mathbf{Cl}_c\times\mathbf{Cl}_{c'}$ equals $\mathbf{Cl}_{c\times c'}$. ☐

Lemma D.14. *For every class $\mathbf{R}$, there exists a unique subclass rel$(\mathbf{R})$ of $\mathbf{R}$ whose members are those members of $\mathbf{R}$ of the form (b,c) for a set b and for a set c. In particular, for every set r, the class rel$(\mathbf{Cl}_r)$ is the class of the unique maximal subset rel(r) of r such that rel(r) is a binary relation.*

Proof. For every class $\mathbf{R} = [p(s,t),a]$, the class of the following predicate $\text{rel}(p)(s,t)$, the members of the class $[\text{rel}(p)(s,t),a]$ are those members of $\mathbf{R}$ of the form (b,c) for a set b and for a set c.

$$\exists u\ \exists v\ (t = (u,v)) \wedge p(s,(u,v)).$$

By Lemma D.2, this subclass of $\mathbf{R}$ is unique.

For every set r, by the Axiom Schema of Specification, there exists a unique subset $\text{rel}(r)$ of r consisting of those elements of r of the form (b,c) for some set b and for some set c. By the Axiom Schema of Replacement, there exists a unique set $\text{active}(\text{rel}(r))$ and a unique set $\text{image}(\text{rel}(r))$ such that $\text{rel}(r)$ is a subset of $\text{active}(\text{rel}(r))\times\text{image}(\text{rel}(r))$ and each of the two projection functions are surjective. ☐

Definition D.15. For every class $\mathbf{R}$, the class $\mathbf{R}$ is a **class relation** if (and only if) the subclass $\text{rel}(\mathbf{R})$ equals $\mathbf{R}$, i.e., if (and only if) every member of $\mathbf{R}$ is of the form (b,c) for a set b and for a set c.

Lemma D.16. *For every class $\mathbf{R}$, for every subclass of $\mathbf{R}$, the subclass is a class relation if and only if it is a subclass of rel$(\mathbf{R})$. For every class relation $\mathbf{R}$, there exists a unique class relation $\mathbf{R}^{opp}$ whose members are those sets of the form (c,b) such that (b,c) is a member of $\mathbf{R}$. Also there exists a unique class image$(\mathbf{R})$ whose members are all sets c such that (b,c) is a member of $\mathbf{R}$ for some set b. Similarly, there exists a unique class active$(\mathbf{R})$ = image$((\mathbf{R})^{opp})$ whose members are all sets b such that (b,c) is a member of $\mathbf{R}$ for some set c. More generally, for every class relation*

$\mathbf{R}$ and for every class $\mathbf{B}$, there exists a unique class $\mathbf{R}[\mathbf{B}]$ whose members are those sets c such that there exists a member b of $\mathbf{B}$ for which (b,c) is a member of $\mathbf{R}$. Similarly, for every class $\mathbf{R}$ and for every class $\mathbf{C}$, there exists a unique class $\mathbf{R}^{opp}[\mathbf{C}]$ whose members are those sets b such that there exists a member c of $\mathbf{C}$ for which (b,c) is a member of $\mathbf{R}$.

Proof. For every class $\mathbf{R} = [p(s,t),a]$, the class of the following predicate $\text{rel}(p)(s,t)$, the members of the class $[\text{rel}(p)(s,t),a]$ are those members of $\mathbf{R}$ of the form (b,c) for a set b and for a set c.

$$\exists u \ \exists v \ (t = (u,v)) \wedge p(s,(u,v)).$$

By Lemma D.2, this subclass of $\mathbf{R}$ is unique.

Similarly, for the following predicate $\text{rel}(p)^{\text{opp}}(s,t)$, the members of the class $[\text{rel}(p)^{\text{opp}}(s,t),a]$ are those sets of the form (c,b) such that (b,c) is a member of $\mathbf{R}$.

$$\exists u \ \exists v \ (t = (v,u)) \wedge p(s,(u,v)).$$

By Lemma D.2, this class is unique.

For the following predicate $\text{image}(\text{rel}(p))(s,t)$, the members of the class $[\text{image}(\text{rel}(p))(s,t),a]$ are those sets c such that (b,c) is a member of $\mathbf{R}$ for some set b.

$$\exists u \ p(s,(u,t)).$$

Similarly, for the following predicate $\text{active}(\text{rel}(p))(s,t)$, the members of the class $[\text{active}(\text{rel}(p))(s,t),a]$ are those sets b such that (b,c) is a member of $\mathbf{R}$ for some set c.

$$\exists v \ p(s,(t,v)).$$

By Lemma D.2, this class is unique.

For every class $\mathbf{R}$, for every class $\mathbf{B}$, the members of the class $\text{image}(\text{rel}(\mathbf{R})) \wedge \mathbf{B}$ are those sets c such that there exists a member b of $\mathbf{B}$ for which (b,c) is a member of $\mathbf{R}$. By Lemma D.2, this class $\text{rel}(\mathbf{R})[\mathbf{B}]$ is unique.

Similarly, for every class $\mathbf{R}$, for every class $\mathbf{C}$, the members of the class $\mathbf{C} \wedge \text{active}(\text{rel}(\mathbf{R}))$ are those sets b such that there exists a member c of $\mathbf{C}$ for which (b,c) is a member of $\mathbf{R}$. By Lemma D.2, this class $[\mathbf{C}]\text{rel}(\mathbf{R})$ is unique. □

Lemma D.17. *For every class $\mathbf{R}$, for every set b, there exists a unique class $\mathbf{R}_b$ whose members are those sets c such that (b,c) is a member of $\mathbf{R}$.*

Proof. For every class $\mathbf{R} = [p(s,t),a]$, for every set b, for following predicate $\text{rel}(p)(s,t)$, the members of the class $[\text{rel}(p)(s,t),(a,b)]$ are those sets c such that (b,c) is a member of $\mathbf{R}$.

$$\exists u \ \exists v \ (s = (u,v)) \wedge p(u,(v,t)).$$

By Lemma D.2, this subclass of $\mathbf{R}$ is unique. □

Definition D.18. For every class $\mathbf{B}$ and for every class $\mathbf{C}$, a subclass $\mathbf{R}$ of $\mathbf{B} \times \mathbf{C}$ is a **relation** from $\mathbf{B}$ to $\mathbf{C}$. In particular, for every class $\mathbf{B}$, a **$\mathbf{B}$-class** is a relation from $\mathbf{B}$ to the von Neumann class $\mathbf{V}$. For every $\mathbf{B}$-class $\mathbf{R}$, for every member b of $\mathbf{B}$, the **fiber class** $\mathbf{R}_b$ of $\mathbf{R}$ over b is the class whose members are all sets c such that (b,c) is a member of $\mathbf{R}$.

Lemma D.19. *For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{R}$, for every $\mathbf{B}$-class $\mathbf{R}'$, the $\mathbf{B}$-class $\mathbf{R}$ equals $\mathbf{R}'$ if and only if, for every member b of $\mathbf{B}$, the fiber class $\mathbf{R}_b$ equals $\mathbf{R}'_b$.*

Proof. By Lemma D.2, the class $\mathbf{R}$ equals $\mathbf{R}'$ if and only if, for every set x, the set x is a member of $\mathbf{R}$ if and only if x is a member of $\mathbf{R}'$. Since $\mathbf{R}$ is a $\mathbf{B}$-class, every member x of $\mathbf{R}$ is of the form (b,c) for a unique member b of $\mathbf{B}$ and for a unique set c. Since $\mathbf{R}'$ is a $\mathbf{B}$-class, every member x' of $\mathbf{R}'$ is of the form (b',c') for a unique member b' of $\mathbf{B}$ and for a unique set c. By the defining property of Kuratowski ordered pairs, the Kuratowski ordered pair (b,c) equals (b',c') if and only if both b equals b' and c equals c'.

Thus, the following two conditions are equivalent: (i) for every set x, the set x is a member of $\mathbf{R}$ if and only if x is a member of $\mathbf{R}'$; (ii) for every member b of $\mathbf{B}$, for every set c, the Kuratowski ordered pair (b,c) is a member of $\mathbf{R}$ if and only if (b,c) is a member of $\mathbf{R}'$. Therefore $\mathbf{R}$ equals $\mathbf{R}'$ if and only if, for every member b of $\mathbf{B}$, the class $\mathbf{R}_b$ equals $\mathbf{R}'_b$. ☐

Lemma D.20. *For every class $\mathbf{B}$, for every class $\mathbf{B}'$, there exists a unique $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{B},\mathbf{B}')$ whose 0-fiber equals $\mathbf{B}$ and whose 1-fiber equals $\mathbf{B}'$, where 0 is $\varnothing$ and 1 is $\{\varnothing\}$. For every $\mathbf{Cl}_{\{0,1\}}$-class $\mathbf{R}$, for every class $\mathbf{B}$, for every class $\mathbf{B}'$, the $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{B},\mathbf{B}')$ equals $\mathbf{R}$ if and only if both $\mathbf{B}$ equals the fiber class $\mathbf{R}_0$ and $\mathbf{B}'$ equals the fiber class $\mathbf{R}_1$. In particular, for every class $\mathbf{C}$, for every class $\mathbf{C}'$, the $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{C},\mathbf{C}')$ equals $(\mathbf{B},\mathbf{B}')$ if and only if both $\mathbf{B}$ equals $\mathbf{C}$ and $\mathbf{B}'$ equals $\mathbf{C}'$.*

Proof. For every class $\mathbf{B} = [p(s,t),a]$ and for every class $\mathbf{B}' = [p'(s',t'),a']$, for the following predicate $p''(s'',t'')$, the members of the class $[p''(s'',t''),(a,a')]$ are those sets of the form $(0,b)$ for a member b of $\mathbf{B}$ and those sets of the form $(1,c)$ for a member c of $\mathbf{C}$.

$$\exists s\ \exists s'\ \exists u\ \exists v\ (s'' = (s,s')) \wedge (t'' = (u,v)) \wedge (((u=0) \wedge p(s,v)) \vee ((u=1) \wedge p'(s',v))).$$

By Lemma D.2, this subclass of $\mathbf{R}$ is unique.

For every $\mathbf{Cl}_{\{0,1\}}$-class $\mathbf{R}$, by the previous lemma, the $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{B},\mathbf{B}')$ equals $\mathbf{R}$ if and only if both the 0-fiber class $\mathbf{B}$ equals $\mathbf{R}_0$ and $\mathbf{B}'$ equals $\mathbf{R}_1$. In particular, for every class $\mathbf{C}$, for every class $\mathbf{C}'$, the $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{B},\mathbf{B}')$ equals $(\mathbf{C},\mathbf{C}')$ if and only if both the 0-fiber $\mathbf{B}$ equals $\mathbf{C}$ and the 1-fiber $\mathbf{B}'$ equals $\mathbf{C}'$. ☐

E Morphisms and spans between classes

For defining categories, a bit more useful than class morphisms or relations is the notion of spans.

Definition E.1. For every class $\mathbf{B}$, for every class $\mathbf{C}$, a **$(\mathbf{B},\mathbf{C})$-span** $\mathbf{M}$ is a $\mathbf{B}\times\mathbf{C}$-class. For every member b of $\mathbf{B}$, for every member c of $\mathbf{C}$, the **fiber class** $\mathbf{M}_c^b$ of $\mathbf{M}$ over (b,c) is the fiber class $\mathbf{M}_{(b,c)}$. A $\mathbf{B}$-class is a **$\mathbf{B}$-set** if (and only if) every fiber class is a class of a set. Similarly, a $(\mathbf{B},\mathbf{C})$-span is a **$(\mathbf{B},\mathbf{C})$-set** if (and only if) it is a $\mathbf{B}\times\mathbf{C}$-set. Finally, for every class $\mathbf{O}$, an **$\mathbf{O}$-Hom span** is an $(\mathbf{O},\mathbf{O})$-set $\mathbf{M}$, i.e., for every ordered pair (b,c) of members of $\mathbf{O}$, the fiber class $\mathbf{M}_c^b$ is the class of a set.

Example E.2. For every class $\mathbf{B}$, the **identity relation** $\mathrm{Id}_{\mathbf{B}}$ from $\mathbf{B}$ to itself is the class whose members are all ordered pairs (b,b) such that b is a member of $\mathbf{B}$. In particular, for every set a, $\mathrm{Id}_{\mathbf{Cl}_a}$ equals $\mathbf{Cl}_{\mathrm{Id}_a}$ for the usual identity set relation Id_a whose elements are all ordered pairs (b,b) such that b is an element of a. For every class $\mathbf{O}$, the **identity $\mathbf{O}$-Hom span $\mathbf{Id}_{\mathbf{O}}$** is the class whose members are all ordered pairs $((b,b),\mathrm{Id}_b)$ such that b is a member of $\mathbf{O}$. In particular, the identity $\mathbf{Cl}_a$-Hom span is the $\mathbf{Cl}_{\mathrm{Id}_a}$-class whose fiber class $(\mathbf{Id}_{\mathbf{Cl}_a})_c^b$ has a unique member Id_b if c equals b is an element of a and otherwise has no member.

Definition E.3. For every class $\mathbf{B}$, a $\mathbf{B}$-class $\mathbf{F}$ is a **class morphism** from $\mathbf{B}$ if (and only if), for every member b of $\mathbf{B}$, the fiber class $\mathbf{F}_b$ is the class of a singleton set, i.e., there exists a unique set c such that (b,c) is a member of $\mathbf{F}$. For every class $\mathbf{B}$, for every class $\mathbf{C}$, a **class morphism** from $\mathbf{B}$ to $\mathbf{C}$ is a relation from $\mathbf{B}$ to $\mathbf{C}$ that is also a class morphism from $\mathbf{B}$. The class morphism is a **class isomorphism** if also, for every member c of $\mathbf{C}$, there exists a unique member b of $\mathbf{B}$ such that (b,c) is a member of the class morphism.

Example E.4. For every class $\mathbf{B}$, the identity $\mathrm{Id}_{\mathbf{B}}$ is a class isomorphism from $\mathbf{B}$ to itself.

Example E.5. For every class $\mathbf{B}$, for every morphism of classes $\mathbf{F}$ from $\mathbf{B}$, there is a $\mathbf{B}$-class $\mathrm{cl}_{\mathbf{B},\mathbf{F}}$ whose members are all ordered pairs (b,c) of a member b of $\mathbf{B}$ and of an element c of the set $\mathbf{F}(b)$.

Exercise E.6. For every class $\mathbf{B}$, for every morphism of classes $\mathbf{F}$ from $\mathbf{B}$, check that $\mathrm{cl}_{\mathbf{B},\mathbf{F}}$ is a $\mathbf{B}$-set. Conversely, for every $\mathbf{B}$-set $\mathbf{D}$, check that there is a unique morphism of classes $\mathrm{fun}_{\mathbf{B},\mathbf{D}}$ from $\mathbf{B}$ associating to every member b of $\mathbf{B}$ the unique set whose associated class is the fiber class $\mathbf{D}_b$. Check that these two operations determine an equivalence between $\mathbf{B}$-sets and morphisms of classes from $\mathbf{B}$.

Definition E.7. For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{Q}$, for every $\mathbf{B}$-class $\mathbf{R}$, a **$\mathbf{B}$-class morphism** from $\mathbf{Q}$ to $\mathbf{R}$ is a class morphism $\mathbf{F}$ from $\mathbf{Q}$ to $\mathbf{R}$ such that, for every member b of $\mathbf{B}$, for every member c of $\mathbf{Q}_b$, there exists a unique member d of $\mathbf{R}_b$ such that $((b,c),(b,d))$ is a member of $\mathbf{F}$. In this case, the **fiber class morphism** $\mathbf{F}_b$ from $\mathbf{Q}_b$ to $\mathbf{R}_b$ associated to $\mathbf{F}$ is the class morphism whose members are all ordered pairs (c,d) such that $((b,c),(b,d))$ is a member of $\mathbf{F}$. A $\mathbf{B}$-class morphism $\mathbf{F}$ from $\mathbf{Q}$ to $\mathbf{R}$ is a **$\mathbf{B}$-class isomorphism** if and only if $\mathbf{F}$ is a class isomorphism from $\mathbf{R}$ to $\mathbf{Q}$.

In particular, for every class $\mathbf{B}$, for every class $\mathbf{C}$, for every $(\mathbf{B},\mathbf{C})$-span $\mathbf{M}$, for every $(\mathbf{B},\mathbf{C})$-span $\mathbf{N}$, a **$(\mathbf{B},\mathbf{C})$-span morphism** from $\mathbf{M}$ to $\mathbf{N}$ is a $\mathbf{B}\times\mathbf{C}$-class morphism from $\mathbf{M}$ to $\mathbf{N}$. This is a **$(\mathbf{B},\mathbf{C})$-span isomorphism $\mathbf{F}$** from $\mathbf{M}$ to $\mathbf{N}$ if (and only if) it is a $\mathbf{B}\times\mathbf{C}$-class isomorphism from $\mathbf{M}$ to $\mathbf{N}$.

Example E.8. For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{Q}$, the identity class isomorphism $\mathrm{Id}_{\mathbf{Q}}$ is a $\mathbf{B}$-class isomorphism from $\mathbf{Q}$ to itself such that $(\mathrm{Id}_{\mathbf{Q}})_b$ equals $\mathrm{Id}_{\mathbf{Q}_b}$ for every member b of $\mathbf{B}$. Similarly, for every class $\mathbf{B}$, for every class $\mathbf{C}$, for every $(\mathbf{B},\mathbf{C})$-span $\mathbf{M}$, the identity class isomorphism $\mathrm{Id}_{\mathbf{M}}$ is a $(\mathbf{B},\mathbf{C})$-span isomorphism such that $(\mathrm{Id}_{\mathbf{M}})^b_c$ equals $\mathrm{Id}_{\mathbf{M}^b_c}$ for every member (b,c) of $\mathbf{B}\times\mathbf{C}$. In particular, for every class $\mathbf{O}$, the identity $\mathrm{Id}_{\mathbf{Id_O}}$ class morphism from $\mathbf{Id_O}$ to itself is an isomorphism of $\mathbf{O}$-Hom spans.

The notion of composition of functions and relations between sets extends to composition of morphisms and relations between classes, as well as composition of spans.

Definition E.9. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every relation $\mathbf{Q}$ from $\mathbf{B}$ to $\mathbf{C}$, for every relation $\mathbf{R}$ from $\mathbf{C}$ to $\mathbf{D}$, a class the **composition** $\mathbf{R}\circ\mathbf{Q}$ of $\mathbf{R}$ and $\mathbf{Q}$ is the class whose members are all ordered pairs (b,d) such that there exists a member c of $\mathbf{C}$ with both (b,c) a member of $\mathbf{Q}$ and (c,d) a member of $\mathbf{R}$.

Definition E.10. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, for every span $\mathbf{N}$ from $\mathbf{C}$ to $\mathbf{D}$, the **span composition** $\mathbf{N}\circ\mathbf{M}$ of $\mathbf{N}$ and $\mathbf{M}$ is the span from $\mathbf{B}$ to $\mathbf{D}$ such that for every member (b,d) of $\mathbf{B}\times\mathbf{D}$, the members of the fiber class $(\mathbf{N}\circ\mathbf{M})^b_d$ are all ordered pairs $(c,(n,m))$ of a member c of $\mathbf{C}$ and members n and m of the respective fiber categories $\mathbf{N}^c_d$ and $\mathbf{M}^b_c$.

Example E.11. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, there is an isomorphism of $(\mathbf{B},\mathbf{C})$-spans $\mathsf{r}_{\mathbf{M}}$ from $\mathbf{M}\circ\mathbf{Id_B}$ to $\mathbf{M}$, respectively $\mathsf{l}_{\mathbf{M}}$ from $\mathbf{Id_C}\circ\mathbf{M}$ to $\mathbf{M}$, sending every member $((b,c),(b,(m,\mathrm{Id}_b)))$ of $\mathbf{M}\circ\mathbf{Id_B}$ to $((b,c),m)$, respectively sending every member $((b,c),(c,(\mathrm{Id}_c,m)))$ of $\mathbf{Id_C}\circ\mathbf{M}$ to $((b,c),m)$. The isomorphism $\mathsf{r}_{\mathbf{M}}$, respectively $\mathsf{l}_{\mathbf{M}}$, is the **right unitor** of $\mathbf{M}$, resp. the **left unitor** of $\mathbf{M}$.

Example E.12. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every class $\mathbf{E}$, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, for every span $\mathbf{N}$ from $\mathbf{C}$ to $\mathbf{D}$, and for every span $\mathbf{P}$ from $\mathbf{D}$ to $\mathbf{E}$, there is an isomorphism of $(\mathbf{B},\mathbf{E})$-spans $\mathsf{a}_{\mathbf{P},\mathbf{N},\mathbf{M}}$ from $(\mathbf{P}\circ\mathbf{N})\circ\mathbf{M}$ to $\mathbf{P}\circ(\mathbf{N}\circ\mathbf{M})$ sending every member $((b,e),(c,((d,(p,n)),m)))$ of $(\mathbf{P}\circ\mathbf{N})\circ\mathbf{M}$ to the member $((b,e),(d,(p,(c,(n,m)))))$ of $\mathbf{P}\circ(\mathbf{Q}\circ\mathbf{R})$. In other words, for every member (b,e) of $\mathbf{B}\times\mathbf{E}$, the induced isomorphism of fiber classes from $((\mathbf{P}\circ\mathbf{N})\circ\mathbf{M})^b_e$ to $(\mathbf{P}\circ(\mathbf{N}\circ\mathbf{M}))^b_e$ sends $(c,((d,(p,n)),m))$ to $(d,(p,(c,(n,m))))$, i.e., it transposes c and d while leaving p, n and m in the same order. The isomorphism $\mathsf{a}_{\mathbf{P},\mathbf{N},\mathbf{M}}$ is the **associator** of $\mathbf{P}$, $\mathbf{N}$ and $\mathbf{M}$.

Example E.13. For the von Neumann class $\mathbf{V}$ of all sets, consider the span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to $\mathbf{V}$ such that for every set b and for every set c, the members of the fiber class over (b,c) are all subsets of $b\times c$ that are (graphs of) functions from b to c. In other words, for every member (b,c) of $\mathbf{V}\times\mathbf{V}$, the fiber class is the class of the set $\mathrm{Fun}(b,c)$ of all functions from b to c. The span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself, together with the *usual composition law*, is the category $\mathbf{Set}$ of all sets.

Proposition E.14. *Composition of relations between classes is strictly associative, and the identity relations are strict left-right identities for this composition. Composition of spans is associative up* *to the specified associator* a*, and the identity spans are left-right identities for this composition up to the left and right unitors* l *and* r*. The associator and unitors satisfy the triangle (coherence) identity and the pentagon (coherence) identity of monoidal categories.*

There is a notion of morphisms of spans. Together with the composition, associator and unitors, spans satisfy the axioms of (a version of) *double category*. Of course spans are classes that may not be sets, so extreme care is necessary in forming any kind of category of spans.

Exercise E.15. Read about double categories. Formulate and verify the axioms of a double category that are satisfied by the operations above for spans.

Spans admit a more general notion of morphisms that is useful in formulating natural transformations.

Definition E.16. For every ordered triple $(\mathbf{B},\mathbf{C},\mathbf{M})$ of classes $\mathbf{B}$ and $\mathbf{C}$ and a span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, for every ordered triple $(\mathbf{B}',\mathbf{C}',\mathbf{M}')$ of classes $\mathbf{B}'$ and $\mathbf{C}'$ and a span $\mathbf{M}'$ from $\mathbf{B}'$ to $\mathbf{C}'$, a **span cell** from $(\mathbf{B},\mathbf{C},\mathbf{M})$ to $(\mathbf{B}',\mathbf{C}',\mathbf{M}')$ is a class $\mathbf{F} = ((\mathsf{s}(\mathbf{F}),\mathsf{t}(\mathbf{F})),\mathbf{F}_{\text{mor}})$ of a morphism of classes $\mathsf{s}(\mathbf{F})$ from $\mathbf{B}$ to $\mathbf{B}'$, of a morphism of classes $\mathsf{t}(\mathbf{F})$ from $\mathbf{C}$ to $\mathbf{C}'$, and of a morphism of classes $\mathbf{F}_{\text{mor}}$ from $\mathbf{M}$ to $\mathbf{M}'$ such that for every member $((b,c),m)$ of $\mathbf{M}$, for the unique member $((b',c'),m')$ of $\mathbf{M}'$ such that $(((b,c),m),((b',c'),m'))$ is a member of $\mathbf{F}$, also (b,b') is a member of $s(\mathbf{F})$ and (c,c') is a member of $t(\mathbf{F})$.

Example E.17. For every ordered triple $(\mathbf{B},\mathbf{C},\mathbf{M})$ of class $\mathbf{B}$ and $\mathbf{C}$ and a span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, the **identity span cell** is $(\text{Id}_{\mathbf{B}},\text{Id}_{\mathbf{C}},\text{Id}_{\mathbf{M}})$.

Exercise E.18. Check that the identity span cell is a span cell.

Example E.19. For every ordered triple $(\mathbf{B},\mathbf{C},\mathbf{M})$ of a span $\mathbf{M}$ from a class $\mathbf{B}$ to a class $\mathbf{C}$, for every ordered triple $(\mathbf{B}',\mathbf{C}',\mathbf{M}')$ of a span $\mathbf{M}'$ from a class $\mathbf{B}'$ to a class $\mathbf{C}'$, for every ordered triple $(\mathbf{B}'',\mathbf{C}'',\mathbf{M}'')$ of a span $\mathbf{M}''$ from a class $\mathbf{B}''$ to a class $\mathbf{C}''$, for every span cell $\mathbf{F} = (s(\mathbf{F}),t(\mathbf{F}),\mathbf{F}_{\text{mor}})$ from $(\mathbf{B},\mathbf{C},\mathbf{M})$ to $(\mathbf{B}',\mathbf{C}',\mathbf{M}')$, and for every span cell $\mathbf{F}' = (s(\mathbf{F}'),s(\mathbf{F}'),\mathbf{F}'_{\text{mor}})$ from $(\mathbf{B}',\mathbf{C}',\mathbf{M}')$ to $(\mathbf{B}'',\mathbf{C}'',\mathbf{M}'')$, the **composition span cell** is $(s(\mathbf{F}')\circ s(\mathbf{F}),t(\mathbf{F}')\circ t(\mathbf{F}),\mathbf{F}'_{\text{mor}}\circ\mathbf{F}_{\text{mor}})$ from $(\mathbf{B},\mathbf{C},\mathbf{M})$ to $(\mathbf{B}'',\mathbf{C}'',\mathbf{M}'')$.

Exercise E.20. Check that the composition span cell is a span cell.

Exercise E.21. Check that composition of span cells is strictly associative. Also check that identity span cells are strict left-right identities for composition of span cells.

One advantage of relations, and more generally of spans, over morphisms is that they have *opposites*.

Definition E.22. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every relation $\mathbf{R}$ from $\mathbf{B}$ to $\mathbf{C}$, the **opposite relation** $\mathbf{R}^{\text{opp}}$ from $\mathbf{C}$ to $\mathbf{B}$ is the unique subclass of $\mathbf{C}\times\mathbf{B}$ whose members are all ordered pairs (c,b) such that (b,c) is a member of $\mathbf{R}$.

More generally, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, the **opposite span** $\mathbf{M}^{\text{opp}}$ from $\mathbf{C}$ to $\mathbf{B}$ is the $\mathbf{C}\times\mathbf{B}$-class such that for every member b of $\mathbf{B}$ and for every member c of $\mathbf{C}$, the fiber class $(\mathbf{M}^{\text{opp}})^c_b$ equals the fiber class $\mathbf{M}^b_c$.

Exercise E.23. Formulate the notion of the opposite of a span cell. Check that the opposite span of a span composite is naturally span isomorphic to the span composite of the span opposites of the factors (in the opposite order). Read about *dagger categories*. Formulate and check the axioms of a dagger category that hold for spans.

F Definition of categories

A category is a span from a class to itself whose fiber classes are required to be (classes of) sets, and equipped with a morphism of spans to itself from the span composite of the span with itself that is associative and unital.

Definition F.1. For every class $\mathbf{O}$, a **$\mathbf{O}$-Hom span** is a $(\mathbf{O},\mathbf{O})$-set $\mathbf{M}$, i.e., a class in which every member is of the form $((a,b),f)$ for members a and b of $\mathbf{O}$ and a set f, and each fiber class $\mathbf{M}_b^a$ of all sets f such that $((a,b),f)$ is a member of $\mathbf{M}$ is the class of a set, the **Hom set** of $\mathbf{M}$ over (a,b). For every clas $\mathbf{O}$, for every $\mathbf{O}$-Hom span $\mathbf{M}'$, for every $\mathbf{O}$-Hom span $\mathbf{M}$, a **morphism** of $\mathbf{O}$-Hom spans from $\mathbf{M}'$ to $\mathbf{M}$ is a morphism of $(\mathbf{O},\mathbf{O})$-classes from $\mathbf{M}'$ to $\mathbf{M}$.

Breaking with our earlier convention, we sometimes denote the Hom set by $\mathbf{M}_b^a$. More often it is denoted $\text{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$, or just $\text{Hom}(a,b)$ when $\mathbf{O}$ and $\mathbf{M}$ are understood, i.e., the members of $\mathbf{M}$ are sets $((a,b),f)$ for members a and b of $\mathbf{O}$ and elements f of $\text{Hom}(a,b)$.

Example F.2. For every class $\mathbf{O}$, the empty $(\mathbf{O},\mathbf{O})$-span $\mathbf{M}$ with no members is an $\mathbf{O}$-Hom span, the **initial $\mathbf{O}$-Hom span**. For every class $\mathbf{O}$, the identity $\text{Id}_{\mathbf{O}\times\mathbf{O}}$ of $\mathbf{O}\times\mathbf{O}$, considered as a span from $\mathbf{O}$ to itself, is an $(\mathbf{O},\mathbf{O})$-span, the **final $\mathbf{O}$-Hom span**. Finally, the **identity Hom span** $\mathbf{Id}_{\mathbf{O}}$ is the class whose members are all ordered pairs $((b,b),\text{Id}_b)$ for b a member of $\mathbf{O}$. This is also called the **discrete $\mathbf{O}$-Hom span**.

Example F.3. For every set H, let $\mathbf{O}_H$ be a class with a unique member (say $\varnothing$, for definiteness), and let $\mathbf{M}_H$ be the unique $\mathbf{O}_H$-Hom span whose unique Hom set is H.

Example F.4. Recall the earlier example, where $\mathbf{O}$ is the von Neumann class $\mathbf{V}$ of all sets, the span $\text{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself is the class of all triples $((a,b),f)$ of a set a, of a set b, and of a function f from a to b. Thus, each Hom set $\text{Hom}_{\mathbf{V},\text{mor}(\mathbf{Set})}(a,b)$ is the set $\text{Fun}(a,b)$ of all functions from a to b.

Example F.5. For another example, again let $\mathbf{O}$ be the von Neumann class $\mathbf{V}$ of all sets, but now let the span $\text{mor}(\mathbf{Rel})$ from $\mathbf{V}$ to itself be the class of all triples $((a,b),R)$ of a set a, of a set b, and of a relation R from a to b, i.e., R is an (arbitrary) subset of $a\times b$. Thus, each Hom set $\text{Hom}_{\mathbf{V},\text{mor}(\mathbf{Rel})}(a,b)$ is the power set $\mathcal{P}(a\times b)$ of $a\times b$.

Example F.6. For every class $\mathbf{O}$, for every $\mathbf{O}$-Hom span $\mathbf{M}$, for every $\mathbf{O}$-Hom span $\mathbf{M}'$, for every $\mathbf{O}$-Hom span $\mathbf{M}''$, for every morphism $\mathbf{F}'$ of $\mathbf{O}$-Hom spans from $\mathbf{M}'$ to $\mathbf{M}$, and for every morphism $\mathbf{F}''$ of $\mathbf{O}$-Hom spans from $\mathbf{M}''$ to $\mathbf{M}$, the fiber product $\mathbf{M}'\times_{\mathbf{F}',\mathbf{M},\mathbf{F}''}\mathbf{M}''$, or just $\mathbf{M}'\times_{\mathbf{M}}\mathbf{M}''$ when confusion is unlikely, is also an $\mathbf{O}$-Hom span whose fiber class for each ordered pair (a,b) of members of $\mathbf{O}$ equals the fiber product set $(\mathbf{M}')^a_b \times_{\mathbf{M}^a_b} (\mathbf{M}'')^a_b$. In particular, $\mathbf{M}\times_{\mathbf{O}\times\mathbf{O}}\mathbf{M}$ is the $\mathbf{O}$-Hom span whose fiber class is just the product set $\mathbf{M}^a_b \times \mathbf{M}^a_b$ for every ordered pair (a,b) of members of $\mathbf{O}$.

Of course, for a Hom span $(\mathbf{O},\mathbf{M})$, the composite span $\mathbf{M}\circ\mathbf{M}$ from $\mathbf{O}$ to $\mathbf{O}$ is typically **not** a Hom span: for all members a and c of $\mathbf{O}$, the members of $(\mathbf{M}\circ\mathbf{M})^a_c$ are all ordered triples $(b,(g,f))$ of a member b of $\mathbf{O}$, of an element f of the set $\mathrm{Hom}(a,b)$ and of an element g of $\mathrm{Hom}(b,c)$. Since b varies over members of a class (that is typically not a set), the class $(\mathbf{M}\circ\mathbf{M})^a_c$ is typically not a set. This is a Hom span if and only if $\mathbf{O}$ is the class of a set.

Definition F.7. A Hom span $(\mathbf{O},\mathbf{M})$ is **small** if (and only if) the class $\mathbf{O}$ is the class of a set.

Example F.8. In the example **Set**, for every set a, for every set c, the fiber class $(\mathrm{mor}(\mathbf{Set})\circ\mathrm{mor}(\mathbf{Set}))^a_c$ is the class of all triples $(b,(g,f))$ of a set b, of a function f from a to b, and of a function g from b to c. This is not the class of a set, since the class of all sets b (i.e., the von Neumann class) is not the class of a set.

Example F.9. On the other hand, for every small Hom span $(\mathbf{O},\mathbf{M})$, for every nonnegative integer n, the n-fold composite of the $\mathbf{O}$-Hom span is again an $\mathbf{O}$-Hom span. Taking the union over all positive integers n gives a new $\mathbf{O}$-Hom span $(\mathbf{O},\mathbf{M}^*)$ where the fiber class over (a,b) is the set of **strings**, i.e., ordered pairs $(n,(a=a_0\xrightarrow{f_1}a_1,a_1\xrightarrow{f_2}a_2,\dots,a_{n-1}\xrightarrow{f_n}a_n=b))$ of a positive integer n and an ordered n-tuple of "composable" members of $\mathbf{M}$. We "complete" this by also adding a member $(0,(a=a_0,a_0=a))$ of $\mathbf{M}^*$ mapping to (a,a) in $\mathbf{O}\times\mathbf{O}$ for every member a of $\mathbf{O}$.

Definition F.10. For every class $\mathbf{O}$, for every Hom span $\mathbf{M}$ from $\mathbf{O}$ to itself, a $(\mathbf{O},\mathbf{M})$-**composition law** is a span morphism $\circ$ from the composition $(\mathbf{O},\mathbf{O})$-span $\mathbf{M}\circ\mathbf{M}$ to $\mathbf{M}$, i.e., a morphism of $\mathbf{O}\times\mathbf{O}$-classes such that, for all members a and c of $\mathbf{O}$, the induced fiber morphism from $(\mathbf{M}\circ\mathbf{M})^a_c$ to $\mathbf{M}^a_c$ sends each member $(b,(g,f))$ of $(\mathbf{M}\circ\mathbf{M})^a_c$ to a member $g\circ f$ of $\mathbf{M}^a_c$.

A composition law is **associative** if (and only if), for all members a, b, c and d of $\mathbf{O}$, for every element (h,g,f) of $\mathrm{Hom}(d,e)\times\mathrm{Hom}(c,d),\mathrm{Hom}(b,c)$, the composition $(h\circ g)\circ f$ equals $h\circ(g\circ f)$ as elements of $\mathrm{Hom}(a,e)$.

An associative composition law is **unital** if (and only if), for every member a of $\mathbf{O}$, there exists an element $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ of $\mathrm{Hom}(a,a)$ such that, for every member b of $\mathbf{O}$, both the left composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ from $\mathrm{Hom}(b,a)$ to itself is the identity, and the right composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ from $\mathrm{Hom}(a,b)$ to itself is the identity.

A **category** is a class $\mathbf{O}$, called the **class of objects**, an $\mathbf{O}$-Hom span $\mathbf{M}$, called the **class of morphisms**, the specification of the **source** and **target** morphisms from $\mathbf{M}$ to $\mathbf{O}$ sending every member $((a,b),f)$ of $\mathbf{M}$ to the member a of $\mathbf{O}$, respectively to the member b of $\mathbf{O}$, and a $(\mathbf{O},\mathbf{M})$-**composition law** $\circ$ that is both associative and unital. An **isomorphism** in a category is a morphism $((a,b),f)$ such that there exists a morphism $((b,a),g)$ with both $g\circ f$ equal to Id_a and $f\circ g$ equal to Id_b; in this case we denote g by f^{-1}.

For a category $\mathbf{C}$, the class $\mathbf{O}$ is often denoted $\text{ob}(\mathbf{C})$ and its members are called **C-objects** or **objects** of $\mathbf{C}$. The class $\mathbf{M}$ is often denoted $\text{mor}(\mathbf{C})$, each set $\text{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$ is denoted $\mathbf{C}_b^a$ or $\text{Hom}_{\mathbf{C}}(a,b)$ and its elements are called **C-morphisms** from a to b. The composition law is denoted $\circ^{\mathbf{C}}$, or just $\circ$ when confusion is unlikely. For every object a of $\mathbf{C}$, the left-right identity morphism from a to itself is usually denoted $\text{Id}_a^{\mathbf{C}}$ or Id_a when confusion is unlikely (this set may or may not equal the identity function from the set a to itself, so please use caution). A category is **small** if (and only if) the class of objects is (the class of) a set.

G Examples of categories

There are many elementary examples of categories, and there are many different properties that a category can possess.

Example G.1. For every class $\mathbf{O}$ that has at least one member, the empty $\mathbf{O}$-Hom span is not the underlying Hom span of any category structure, since there are not identity morphisms (there are not any morphisms at all). On the other hand, the final $\mathbf{O}$-Hom span, where every Hom set is a singleton set, has a unique composition law (since singleton sets are final objects in the category of sets), and this composition law is associative and unital. This is the **final category structure** on $\mathbf{O}$. Similarly, the discrete $\mathbf{O}$-Hom span (whose only morphisms are identity morphisms) also has a unique composition law, and this is associative. It is unital by construction. This is the **discrete category structure** on $\mathbf{O}$.

Definition G.2. For every small class $\mathbf{O}$, for every $\mathbf{O}$-Hom span $\mathbf{M}$, the **free category** on $(\mathbf{O},\mathbf{M})$ is the $\mathbf{O}$-Hom span $\mathbf{M}^*$ of composable strings of morphisms from $\mathbf{M}$ with composition law given by concatenation, i.e., for every morphism $f = (m,(a_0 \xrightarrow{f_1} a_1,\dots,a_{m-1} \xrightarrow{f_m} a_m))$ and for every morphism $g = (n,(b_0 \xrightarrow{g_1} b_1,\dots,b_{n-1} \xrightarrow{g_n} b_n))$ such that a_m equals b_0, the composition is

$$g \circ f := (n+m,(a_0 \xrightarrow{f_1} a_1,\dots,a_{m-1} \xrightarrow{f_m} a_m, b_0 \xrightarrow{g_1} b_1,\dots,b_{n-1} \xrightarrow{g_n} b_n)).$$

Of course each element $(0,(a_0,a_0))$ composes as a left-right identity.

Exercise G.3. Check that this is a small category.

Example G.4. The category **Set** of sets has object class $\text{obj}(\mathbf{Set})$ equal to the von Neumann class / universal class $\mathbf{V}$ of all sets, has morphism class $\text{mor}(\mathbf{Set})$ as above with fiber class $\mathbf{Set}_b^a$ equal to the (class of the) set $\text{Fun}(a,b)$ of all functions f from a to b, and has the usual composition of functions. Composition of functions is associative. The identity functions are the identity morphisms of this category.

Example G.5. The category **Rel** of relations again has object class equal to $\mathbf{V}$, but has morphism class as in the second example above with fiber class $\mathbf{Rel}_b^a$ equal to the (class of the) power set $\mathcal{P}(a \times b)$ of $a \times b$. Composition is composition of relations (as defined in the previous section). The identity functions (or their graphs) are the identity morphisms of this category.

Example G.6. For every small category $\mathbf{B}$, for every small category $\mathbf{C}$, each object of the category $\mathbf{Span}_{\mathbf{C}}^{\mathbf{B}}$ is a (set whose associated class is a) $\mathbf{B}\times\mathbf{C}$-set, i.e., a span from $\mathbf{B}$ to $\mathbf{C}$ whose fiber classes are all (classes of) sets. The morphisms between two such spans are span cells such that the class morphism from $\mathbf{B}$ to itself is the identity and the class morphism from $\mathbf{C}$ to itself is the identity. Composition is composition of span cells.

Definition G.7. A category is a **monoid** if (and only if) the object class is the class of a singleton set. A category is **thin** if (and only if) every nonempty Hom set is a singleton set. A category is a **groupoid** if (and only if) every morphism is an isomorphism. A thin groupoid is a **setoid**. A category is **skeletal** if (and only if) all isomorphic objects are equal. A skeletal setoid is a **discrete category**.

There are many ways to produce new categories from given categories.

Definition G.8. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, the **opposite category** is the category $(\mathbf{O}, \mathbf{M}^{\text{opp}}, \circ^{\text{opp}})$, where $\mathbf{M}^{\text{opp}}$ is the opposite span of $\mathbf{M}$, and where, for every member $((a,b),f)$ of $\mathbf{M}$ and for every member $((b,c),g)$ of $\mathbf{M}$, the opposite composition is defined by

$$((b,a),f)\circ^{\text{opp}}((c,b),g) = ((c,a), g\circ f).$$

Definition G.9. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every subclass $\mathbf{O}'$ of $\mathbf{O}$, the **full subcategory** of $\mathbf{C}$ with objects class $\mathbf{O}'$ is the category $(\mathbf{O}', \mathbf{M}|_{\mathbf{O}'}, \circ')$ where, for all members a and b of $\mathbf{O}'$, the class $(\mathbf{M}|_{\mathbf{O}'})^a_b$ equals $\mathbf{M}^a_b$, and where $\circ'$ is the restriction of $\circ$. More generally, a (not necessarily full) **subcategory** of $\mathcal{C}$ consists of a subclass $\mathbf{O}'$ of $\mathbf{O}$ and a subclass of $\mathbf{M}'$ of $\mathbf{M}|_{\mathbf{O}'}$ that contains all identity morphisms of objects of $\mathbf{O}'$ and that is stable for composition, thus defining a restriction composition on the subcategory.

Example G.10. The category **Set** of sets is a non-full subcategory of the category **Rel** of all relations. The category of all finite sets is a full subcategory of the category of all sets.

Example G.11. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, the discrete category on $\mathbf{O}$ is (uniquely) a (typically not full) subcategory of $\mathbf{C}$.

Example G.12. For every set H together with a binary operation $\bullet$ from $H\times H$ to H that is associative and unital, there exists a monoid category $B(H,\bullet)$ whose unique object is, say, the set H itself (perhaps considered as a right act over itself), and whose unique Hom set is H with $\bullet$ giving the binary operation. For every category, for every object of that category, the restriction of composition to the Hom set of that object is a monoid as above. In particular, every category with a unique object is strongly equivalent to the category of the monoid of the unique Hom set with its composition operation.

Example G.13. In particular, for every set S, for a category $\mathbf{O}$ with a unique object $*$, for the $\mathbf{O}$-Hom span $\mathbf{M}_S$ whose unique Hom set is S, for the free category $(\mathbf{O}, \mathbf{M}_S^*, \circ)$, the associated monoid is the **free monoid** on the set S. The unique Hom set S^* is also called the free monoid on S, and it is also the **Kleene star** of S, i.e.,

$$S^* = (\{0\}\times\{\text{Id}_*\})\sqcup(\{1\}\times S)\sqcup(\{2\}\times(S\times S))\sqcup(\{3\}\times(S\times S\times S))\sqcup\dots.$$

Example G.14. For every monoid $(H, \bullet)$, the opposite category of $B(H, \bullet)$ is (canonically equivalent to) the category of the **opposite monoid** $(H, \bullet^{\text{opp}})$ where $a \bullet^{\text{opp}} b$ is defined to equal $b \bullet a$ for all elements a and b of H.

Example G.15. For every monoid $(H, \bullet)$, the monoid is a group if and only if every element of H is invertible. In this case, the category of the group is a skeletal groupoid. In this case, the nerve of the small category $B(H, \bullet)$ is the **classifying simplicial set** of the (discrete) group $(H, \bullet)$. The geometric realization of this simplicial set is the **classifying space** of $(H, \bullet)$. For every groupoid, for every object $*$ of that groupoid, the restriction of composition to the Hom set from $*$ to itself is a group $(H, \bullet)$, and the full subcategory whose unique object is $*$ is strongly equivalent to $B(H, \bullet)$. In particular, every groupoid with a unique object is strongly equivalent to $B(H, \bullet)$ for the unique Hom set $(H, \bullet)$ with its composition operation.

Example G.16. We could "deskeletonize" the previous example by considering the category whose objects are all right acts over the monoid $(H, \bullet)$ that are principal homogeneous spaces, and whose morphisms are all morphisms of right $(H, \bullet)$-acts.

Example G.17. The category of right principal homogenous spaces for $(H, \bullet)$, as above, is a full subcategory of the category of all right $(H, \bullet)$-acts. Another full subcategory is the category of all right $(H, \bullet)$-acts that are trivial in the sense that every element of $(H, \bullet)$ acts identically on the set. This full subcategory is strongly equivalent to the category **Set** of all sets. Of course if $(H, \bullet)$ is itself a singleton monoid, then this full subcategory equals the entire category of right $(H, \bullet)$-acts, so that this category is strongly equivalent to **Set**.

Example G.18. The category **Monoid** has as objects all ordered pairs $(H, \bullet)$ of a set H together with a unital, associative binary operation $\bullet$ from H to itself, and has morphisms from $(H, \bullet)$ to $(H', \bullet')$ being all functions f from H to H' that preserve the identity and preserve the binary operation (i.e., usual morphisms of monoids): $f(e_H)$ equals $e_{H'}$ and $f(h \bullet k)$ equals $f(h) \bullet' f(k)$ for all elements h, k of H. The category **Monoid** has a full subcategory **Grp** whose objects are groups. The category **Grp** has a full subcategory **Ab** whose objects are Abelian groups. The category **Ab** has a full subcategory $\mathbb{Q} - mbfMod$ whose objects are Abelian groups such that multiplication by n is a bijection of the group to itself for every nonzero integer n, i.e., the Abelian group is a $\mathbb{Q}$-vector space. The category $\mathbb{Q} - \mathbf{Mod}$ has a full subcategory whose objects are finite-dimensional $\mathbb{Q}$-vector spaces, etc.

Example G.19. A hybrid of the previous two examples is the category whose objects are all ordered pairs $((H, \bullet), (S, \rho))$ of a monoid $(H, \bullet)$ together with a right H-act $\rho : S \times H \to S$. The morphisms from $((H, \bullet), (S, \rho))$ to $((H', \bullet'), (S', \rho'))$ are all ordered pairs (f, g) of a morphism f of monoids from $(H, \bullet)$ to $(H', \bullet')$ together with a function g from S to S' such that, for every element h of H and every element s of S, the image $g(\rho(s, h))$ equals $\rho'(g(s), f(h))$, i.e., g is a morphism of right H-acts for the induced right H-act on S' obtained from ρ' and f. This hybrid category is an example of the "Grothendieck construction" for fibered categories (one of the basic notions in extending from schemes to stacks).

For Abelian monoids, respectively for Abelian groups, there is an enrichment of the Hom sets to Abelian monoids, resp. to Abelian groups.

Definition G.20. For every Abelian monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, the **addition law** on $\mathrm{Hom}_{\mathbf{Monoid}}((H', \bullet'), (H, \bullet))$ is the binary operation that associates to every pair (f, g) of monoid homomorphisms from $(H', \bullet')$ to $(H, \bullet)$ the monoid homomorphism $f \bullet g$ that sends every element h' of H' to $f(h') \bullet g(h')$.

Example G.21. Check that the set function $f \bullet g$ is a monoid homomorphism. Check that the addition law is both associative and commutative, and it has a left-right identity consisting of the constant set function from H' with image the singleton of the monoid identity in H. Thus $\mathrm{Hom}_{\mathbf{Monoid}}((H', \bullet'), (H, \bullet))$ with this addition law is itself an Abelian monoid. If $(H, \bullet)$ is an Abelian group, check that also $\mathrm{Hom}_{\mathbf{Monoid}}((H', \bullet'), (H, \bullet))$ is an Abelian group.

Example G.22. For the full subcategory **Ab** of all Abelian groups, check that the addition laws makes composition into a biadditive map of Abelian groups

$$\mathrm{Hom}_{\mathbf{Ab}}((H', \bullet'), (H, \bullet)) \times \mathrm{Hom}_{\mathbf{Ab}}((H'', \bullet''), (H', \bullet')) \to \mathrm{Hom}_{\mathbf{Ab}}((H'', \bullet''), (H, \bullet)).$$

In particular, check that the addition law together with composition makes $\mathrm{Hom}_{\mathbf{Ab}}((H, \bullet), (H, \bullet))$ into an *associative, unital ring*, i.e., composition is a monoid structure that distributes with respect to addition both on the left and right.

This allows a concise definition of associative, unital rings. Moreover, for each associative, unital rings, there are (Abelian) categories of modules over that ring.

Definition G.23. An **associative, unital ring** $(R, +, \cdot)$ is an Abelian group $(R, +)$ together with an (injective) homomorphism of Abelian groups

$$L_\bullet : (R, +) \to \mathrm{Hom}_{\mathbf{Ab}}((R, +), (R, +)), \quad r \mapsto (L_r : (R, +) \to (R, +))$$

whose image is a submonoid under composition that is right unital, i.e., there exists a (unique) element 1 in R with $L_1 = \mathrm{Id}_R$ (so 1 is a left multiplicative identity) and also with $L_r(1) = r$ for every element r of R (so 1 is also a right right multiplicative identity), and, for every (r, s) in $R \times R$, there exists a (unique) element $r \cdot s$ of R such that $L_r \circ L_s$ equals $L_{r \cdot s}$ (notice that $L_r(s) = L_r(L_s(1)) = L_{r \cdot s}(1) = r \cdot s$). For every associative, unital ring $(R, +, \cdot)$, for every associative, unital ring $(R', +', \cdot')$, a **morphism** of associative, unital rings from $(R, +, \cdot)$ to $(R', +', \cdot')$ is a set function f from R to R' that is simultaneously a homomorphism from the Abelian group $(R, +)$ to $(R', +')$ and a homomorphism from the monoid $(R, \cdot)$ to the monoid $(R', \cdot')$.

Exercise G.24. Check that, for every associative, unital ring $(R, +, \cdot)$, the identity function Id_R is a morphism of associative, unital rings from $(R, +, \cdot)$ to itself. Also check that the composition function of morphisms of associative, unital rings is again a morphism of associative, unital rings.

Definition G.25. The **category of associative, unital rings**, denoted **Ring**, has as objects all associative, unital rings, has as morphisms the morphisms of associative, unital rings, and has the composition from the previous exercise.

Definition G.26. For every associative, unital ring $(R,+,\cdot)$, the **opposite product** $\cdot^{\text{opp}}$ is the binary operation on R defined by $s\cdot^{\text{opp}} r = r\cdot s$ for every element (r,s) of $R\times R$.

Exercise G.27. Check that for every associative, unital ring $(R,+,\cdot)$, also $(R,+,\cdot^{\text{opp}})$ is an associative, unital ring.

Definition G.28. A **commutative, associative, unital ring** is an associative, unital ring $(R,+,\cdot)$ such that $\cdot^{\text{opp}}$ equals $\cdot$, i.e., $r\cdot s$ equals $s\cdot r$ for every element (r,s) of $R\times R$. For every commutative, associative, unital ring $(R,+,\cdot)$, for every commutative, associative, unital ring $(R',+',\cdot')$, a **morphism** of commutative, associative, unital rings from $(R,+,\cdot)$ to $(R',+',\cdot')$ is a morphism of associative, unital rings from $(R,+,\cdot)$ to $(R',+',\cdot')$. The category **CRing** is the full subcategory of **Ring** whose objects are all commutative, associative, unital rings.

Definition G.29. For every associative, unital ring $(R,+,\cdot)$, for every Abelian group $(M,+)$, a **left** R**-module** structure on $(M,+)$ is a morphism of associative, unital rings λ from $(R,+,\cdot)$ to the associative, unital ring $\text{Hom}_{\mathbf{Ab}}((M,+),(M,+))$, i.e., for every element r of R, the function λ_r from M to itself is a group homomorphism, λ_1 equals Id_M, and, for every element (r,s) of $R\times R$, the image λ_{r+s} equals $\lambda_r+\lambda_s$ and $\lambda_{r\cdot s}$ equals $\lambda_r\circ\lambda_s$. Stated differently, this is a biadditive map $*$ from $R\times M$ to M that is also a monoid homomorphism for $\cdot$ on R and for composition of Abelian group homomorphisms of M, i.e., $1*m$ equals m and $(r\cdot s)*m$ equals $r*(s*m)$ for every element m of M and for every element (r,s) of $R\times R$. For every left R-module $(M,+,\lambda)$, for every left R-module $(M',+',\lambda')$, a **morphism** of left R-modules from $(M,+,\lambda)$ to $(M',+',\lambda')$ is a set function f from M to M' that is a homomorphism of Abelian groups from $(M,+)$ to $(M',+')$ and that commutes with λ and λ', i.e., $f\circ\lambda_r$ equals $\lambda'_r\circ f$ for every element r of R.

Exercise G.30. Check that the identity function from M to itself is a morphism of left R-modules from $(M,+,\lambda)$ to itself. Check that the composition function of two morphisms of left R-modules is again a morphism of left R-modules.

Definition G.31. For every associative, unital ring $(R,+,\cdot)$, the **category of left** R**-modules**, denoted $R-\mathbf{Mod}$, has objects that are all left R-modules, has morphisms that are all morphisms of left R-modules, and has composition as defined above.

Exercise G.32. For every left R-module $(M,+,\lambda)$, for every Abelian group $(M',+')$, define a left R-module structure on $\text{Hom}_{\mathbf{Ab}}((M',+'),(M,+))$ by $\lambda^{M'}_{M,r}(f)(m') := \lambda_r(f(m'))$ for every element r of R, for every element m' of M', and for every Abelian group homomorphism f from $(M',+')$ to $(M,+)$. Check that this is a structure of left R-module. For every ordered pair of left R-modules $(M,+,\lambda)$ and $(M',+',\lambda')$, for every Abelian group homomorphism f from $(M',+')$ to $(M,+)$, check that f is a morphism of left R-modules from $(M',+',\lambda')$ to $(M,+,\lambda)$ if and only if $\lambda^{M'}_{M,r}(f)$ equals $f\circ\lambda'_r$ for every element r of R.

Definition G.33. For every associative, unital ring $(R, +, \cdot)$, for every Abelian group $(M, +)$, a **right R-module** structure on $(M, +)$ is a morphism of associative, unital rings ρ from $(R, +, \cdot^{\text{opp}})$ to the associative, unital ring $\text{Hom}_{\mathbf{Ab}}((M, +), (M, +))$. Stated differently, this is a biadditive map $*$ from $M \times R$ to M that is also a monoid homomorphism for $\cdot$ on R and for composition of Abelian group homomorphisms of M, i.e., $m \cdot 1$ equals m and $m * (r \cdot s)$ equals $(m * r) * s$ for every element m of M and for every element (r, s) of $R \times R$. For every right R-module $(M, +, \rho)$, for every right R-module $(M', +', \rho')$, a **morphism** of right R-modules from $(M, +, \rho)$ to $(M', +', \rho')$ is a set function f from M to M' that is a homomorphism of Abelian groups from $(M, +)$ to $(M', +')$ and that commutes with ρ and ρ', i.e., $f \circ \rho_r$ equals $\rho'_r \circ f$ for every element r of R.

Exercise G.34. Check that the identity function from M to itself is a morphism of right R-modules from $(M, +, \rho)$ to itself. Check that the composition function of two morphisms of right R-modules is again a morphism of right R-modules.

Definition G.35. For every associative, unital ring $(R, +, \cdot)$, the **category of right R-modules**, denoted $\mathbf{Mod} - R$, has objects that are all right R-modules, has morphisms that are all morphisms of right R-modules, and has composition as defined above.

Exercise G.36. For every left R-module $(M, +, \lambda)$, for every Abelian group $(M', +')$, define a right R-module structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$ by $\rho^{M,r}_{M'}(f)(m) := f(\lambda_r(m))$ for every element r of R, for every element m of M, and for every Abelian group homomorphism f from $(M, +)$ to $(M', +')$. Check that this is a structure of right R-module. For every ordered pair of left R-modules $(M, +, \lambda)$ and $(M', +', \lambda')$, for every Abelian group homomorphism f from $(M, +)$ to $(M', +')$, check that f is a morphism of left R-modules from $(M, +, \lambda)$ to $(M', +', \lambda')$ if and only if $\rho^{M,r}_{M'}(f)$ equals $\lambda_r \circ f$ (which also equals $\lambda^{M}_{M',r}(f)$, by definition) for every element r of R.

Exercise G.37. Formulate and prove the analogous results for a right R-module structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$ associated to a right R-module structure on $(M', +')$ and for a left R-module structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$ associated to a right R-module structure on $(M, +)$.

Definition G.38. For every associative, unital ring $(R, +_R, \cdot_R)$, for every associative, unital ring $(S, +_S, \cdot_S)$, an $R - S$-**bimodule** is a quadruple $(M, +, \lambda, \rho)$ of an Abelian group $(M, +)$ with a left R-module structure λ and a right S-module structure ρ such that, for every element (r, r') of $R \times R'$, the Abelian group homomorphism $\lambda_r \circ \rho_{r'}$ equals $\rho_{r'} \circ \lambda_r$, i.e., the images of λ and ρ in $\text{Hom}_{\mathbf{Ab}}((M, +), (M, +))$ centralize one another. For every $R - S$-bimodule $(M, +, \lambda, \rho)$, for every $R - S$-bimodule $(M', +', \lambda', \rho')$, a **morphism** of $R - S$-bimodules from $(M, +, \lambda, \rho)$ to $(M', +', \lambda', \rho')$ is a set function f from M to M' that is simultaneously a morphism of left R-modules from $(M, +, \lambda)$ to $(M', +', \lambda')$ and a morphism of right S-modules from $(M, +, \rho)$ to $(M', +', \rho')$.

Exercise G.39. Check that the identity set function Id_M is a morphism of $R - S$-bimodules from $(M, +, \lambda, \rho)$ to itself. Also check that the composition function of morphisms of $R - S$-bimodules is again an $R - S$-bimodule.

Definition G.40. For every associative, unital ring $(R, +_R, \cdot_R)$, for every associative, unital ring $(S, +_S, \cdot_S)$, the **category of $R - S$-bimodules**, denoted $R - S - \mathbf{Mod}$, has objects that are all

$R-S$-bimodules, has morphisms that are all morphisms of $R-S$-bimodules, and has composition as defined above.

Exercise G.41. For every $R-S$-bimodule $(M,+,\lambda,\rho)$, for every Abelian group $(M',+')$, check that the operations $\lambda_{M,r}^{M'}$ and $\rho_{M,s}^{M'}$ make $\text{Hom}_{\mathbf{Ab}}((M',+'),(M,+))$ into an $R-S$-bimodule. Similarly, define an $S-R$-bimodule structure on $\text{Hom}_{\mathbf{Ab}}((M,+),(M',+'))$. For every Abelian group homomorphism f from an $R-S$-bimodule $(M,+,\lambda,\rho)$ to an $R-S$-bimodule $(M',+',\lambda',\rho')$, check that f is a morphism of $R-S$-bimodules if and only if both $\rho_{M'}^{M,r}(f)$ equals $\lambda_{M',r}^{M}(f)$ and $\rho_{M',s}^{M}(f)$ equals $\lambda_{M'}^{M,s}(f)$ for every element r of R and for every element s of S.

Of course there are also many notions of topological space and geometric object.

Definition G.42. For every set X, a **topology** (of open subsets of X) is a subset τ of the power set $\mathcal{P}(X)$ of X satisfying all of the following.

(i) Both $\varnothing$ and X are elements of τ.

(ii) For every ordered pair (U,V) of elements of τ, also $U\cap V$ is an element of τ.

(iii) For every subset I of τ, the union over all elements of I (considered as a subset of X) is an element of τ.

A **topological space** is an ordered pair (X,τ) of a set X and a topology τ on X.

For every ordered pair $((X,\tau),(X',\tau'))$ of topological spaces, a **continuous map** from (X,τ) to (X',τ') is a function f from X to X' such that for every element U' of τ', the preimage $f^{\text{pre}}(U')$ is an element of τ.

Exercise G.43. For every topological space (X,τ), check that Id_X is a continuous map from (X,τ) to itself. Check that the composition function of continuous maps is again a continuous map. Thus, topological spaces with continuous maps form a category, **Top**.

Definition G.44. For every set X, a **topological basis** (of open subsets of X) is a subset $\mathcal{B}$ of the power set $\mathcal{P}(X)$ of X satisfying all of the following.

(i) The set X is the union over all elements of $\mathcal{B}$.

(ii) For every ordered pair (U,V) of elements of $\mathcal{B}$, the set $U\cap V$ equals the union over all elements of $\mathcal{B}$ that are subset of $U\cap V$.

Occasionally, a function to $\mathcal{P}(X)$ whose image is a topological basis is also called a topological basis. The **topology generated** by a topological basis is the subset $\tau(\mathcal{B})$ of $\mathcal{P}(X)$ of all subsets U of X that equal the union over all elements of $\mathcal{B}$ that are a subset of U (thus, also $\varnothing$ is tautologically an element of $\mathcal{B}$).

Exercise G.45. Check that $\tau(\mathcal{B})$ is a topology for $\mathcal{X}$. For every topological space (X', τ') and for every function f from X' to X, check that f is a continuous map from (X', τ') to $(X, \tau(\mathcal{B})$ if and only if, for every element U of $\mathcal{B}$, the preimage subset $f^{\text{pre}}(U)$ is an element of τ'.

Associated to every category there is a maximal subcategory that is a groupoid.

Definition G.46. For every category $\mathbf{C}$, the **core** of $\mathbf{C}$ is the (usually non-full) groupoid subcategory with the same objects, but whose Hom set is the subset of invertible elements in the corresponding Hom set of $\mathbf{C}$.

Example G.47. For every monoid $(H, \bullet)$, the core of the category of $(H, \bullet)$ is $B(H, \bullet)^\times$, where $(H, \bullet)^\times$ is the submonoid (in fact, group) of $(H, \bullet)$ whose elements are all invertible elements of H. The core of **Set** is the the groupoid of sets whose morphisms are bijections of sets. This also equals the core of **Rel**. The core of a product of categories is (canonically equivalent to) the product of the cores of the categories.

Example G.48. In particular, in the core of the hybrid category, for every object $((H, \bullet), (H, r_H))$ where $(H, \bullet)$ is a group and where (H, r_H) is the right regular H-action on itself, the group of automorphisms of this object is the classical notion of *holomorph* of the group, i.e., the semidirect product of the group with its automorphism group.

Example G.49. For every set S together with a relation R from S to itself, consider the class of S as a class of objects, and consider R as a span from this class to itself. An associative, unital composition law extending this to a category is unique if it exists. In fact, this span extends to a category if and only if R is a **preorder**, i.e., if and only if R is both transitive and reflexive. In this case, the corresponding category is small and thin. Every small, thin category is strongly equivalent to the category of a preordered set. The core of the category of a preordered set (S, R) is (canonically equivalent to) the category of the associated **Bishop set**, i.e., the set S together with an equivalence relation $\sim_R$, where $a \sim_R b$ if and only if both (a, b) and (b, a) are elements of R. The category of a preordered set is skeletal if and only if $\sim_R$ is equality, i.e., if and only if the preorder is a partial order: a transitive, reflexive relation that is also asymmetric. Similarly, the category of a preordered set is a groupoid if and only if the relation R is already an equivalence relation, i.e., if and only if the transitive, reflexive relation is also symmetric. Every preordered set is the pullback of a partial order under a surjection whose associated equivalence relation is $\sim_R$ (and this surjection to a partially ordered set is unique up to unique isomorphism). If we accept the Axiom of Choice, there exists a subset of the original set that surjects isomorphically to the partially ordered set, and this defines a full subcategory of the category of the preordered set that is a skeleton.

Definition G.50. For every category $\mathbf{C}$, the objects of the **arrow category** $\text{Arr}(\mathbf{C})$ are objects of $\text{mor}(\mathbf{C})$, i.e., tuples $((s, t), f)$ of an ordered pair (s, t) of objects of $\mathbf{C}$ and a $\mathbf{C}$-morphism f from s to t. For every ordered pair of $\mathbf{C}$-morphisms, say $((s, t), f)$ and $((s', t'), f')$, the morphisms of $\text{Arr}(\mathbf{C})$ from $((s, t), f)$ to $((s', t'), f')$ are ordered pairs (σ, τ) of a $\mathbf{C}$-morphism σ from s to s' and a $\mathbf{C}$-morphism τ from t to t' such that $f' \circ \sigma$ equals $\tau \circ f$. Composition of morphisms is componentwise.

Example G.51. For every monoid $(H,\bullet)$, for the associated monoid category, the objects of the arrow category are elements h of H, and for every ordered pair (h,h') of elements of H, the morphisms from h to h' in the arrow category are ordered pairs (σ,τ) of elements of H such that $\tau\bullet h$ equals $h'\bullet\sigma$. In particular, if $(H,\bullet)$ is a group, this is the same as the set of pairs $(\sigma, h'\bullet\sigma\sigma h^{-1})$, which projects under pr_1 as a bijection to H (where composition corresponds to $\bullet$). Thus, for every group $(H,\bullet)$, the arrow category of $B(H,\bullet)$ is weakly equivalent to $B(H,\bullet)$.

Example G.52. For each partially ordered set (S,R), denote by $R^{(2)}$ the partial order on $S\times S$ whose elements are all elements $((s,t),(s',t'))$ of $(S\times S)\times(S\times S)$ such that both (s,s') and (t,t') are elements of R. Denote by $R^{(2)}|_R$ the restriction of this partial order to the subset R of $S\times S$. Then for the category of the partially ordered set (S,R), the arrow category is strongly equivalent to the category of the partially ordered set $(R,R^{(2)}|_R)$.

Definition G.53. For every category $\mathbf{C}$ and for every object b, the **under category** of $\mathbf{C}$ under the object b, denoted $\mathbf{C}_b$ or $b/\mathbf{C}$, is the subcategory of the arrow category whose objects are arrows $((s,t),f)$ such that t equals b, and whose morphisms from $((s,b),f)$ to $((s',b),f')$ are all ordered pairs (σ,Id_b) of a morphism σ from s to s' such that $f'\circ\sigma$ equals $\mathrm{Id}_b\circ f$, i.e., equals f.

Definition G.54. For every category $\mathbf{C}$ and for every object a, the **over category** of $\mathbf{C}$ over the object a, denoted $\mathbf{C}^a$ or $\mathbf{C}/a$, is the subcategory of the arrow category whose objects are arrows $((s,t),f)$ such that s equals a, and whose morphisms from $((a,t),f)$ to $((a,t'),f')$ are all ordered pairs (Id_a,τ) of a morphism τ from t to t' such that $\tau\circ f$ equals $f'\circ\mathrm{Id}_a$, i.e., equals f'.

Example G.55. For every monoid $(H,\bullet)$, for the associated monoid category, for the unique object, both for the under category and the over category, the objects are the elements h of H. For every ordered pair (h,h') of elements of H, the morphisms from h to h' in the under category are ordered pairs (σ,Id_H) of elements of H such that h equals $h'\bullet\sigma$, and the morphisms from h to h' in the over category are ordered pairs (Id_H,τ) such that $\tau\bullet h$ equals h'. In particular, if $(H,\bullet)$ is a group, both the over category and the under category are weakly equivalent to the discrete category with only one object and only one morphism (the identity morphism).

Example G.56. For each partially ordered set (S,R), for each element b of S, denote by S_b the **lower subset** of b in (S,R), i.e., the subset of all elements a of S such that (a,b) is an element of R. Denote by R_b the restriction of R to this subset. Then the under category of the category of (S,R) under the object b is strongly equivalent to the category of the partially ordered set (S_b,R_b). Similarly, the over category over an element a is strongly equivalent to the category of the partially ordered set (S^a,R^a), where S^a is the **upper subset** of a in (S,R), i.e., the subset of all elements b of S such that (a,b) is an element of R.

H Functors

The usual notion of morphisms between categories, called *functors*, are morphisms of spans that respect both composition and identities.

Definition H.1. For every ordered pair class $(\mathbf{O},\mathbf{M})$ of a class $\mathbf{O}$ and a $\mathbf{O}$-Hom span $\mathbf{M}$, for every ordered pair class $(\mathbf{O}',\mathbf{M}')$ of a class $\mathbf{O}'$ and a $\mathbf{O}'$-Hom span $\mathbf{M}'$, a **morphism** of Hom spans from $(\mathbf{O},\mathbf{M})$ to $(\mathbf{O}',\mathbf{M}')$ is an ordered pair class $\mathbf{F}=(\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{mor}})$ such that $(\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{mor}})$ is a span cell $(\mathbf{O},\mathbf{O},\mathbf{M})$ to $(\mathbf{O}',\mathbf{O}',\mathbf{M}')$, i.e., for every member (a,a') of $\mathbf{F}_{\text{obj}}$, for every member (b,b') of $\mathbf{F}_{\text{obj}}$, for every member $((a,b),f)$ of $\mathbf{M}$, the value of $((a,b),f)$ under the class morphism $\mathbf{F}_{\text{mor}}$ equals $((a',b'),f')$ for a unique member $((a',b'),f')$ of $\mathbf{M}'$.

Example H.2. For every ordered pair class $(\mathbf{O},\mathbf{M})$ with $\mathbf{M}$ a $\mathbf{O}$-Hom span, for every ordered pair class $(\mathbf{O}',\mathbf{M}')$ with $\mathbf{M}'$ a $\mathbf{O}'$-Hom span such that $\mathbf{O}'$ is a subclass of $\mathbf{O}$ and such that $\mathbf{M}'$ is a subclass of $\mathbf{M}$, the inclusion morphism from the class $\mathbf{O}'$ to $\mathbf{O}$ and the inclusion morphism from $\mathbf{M}'$ to $\mathbf{M}$ together define a morphism of Hom spans, $\text{incl}_{\mathbf{O},\mathbf{M}}^{\mathbf{O}',\mathbf{M}'}$ from $(\mathbf{O}',\mathbf{M}')$ to $(\mathbf{O},\mathbf{M})$, the **inclusion morphism**.

Exercise H.3. For every ordered pair class $(\mathbf{O},\mathbf{M})$ as above, check that the identity span cell from $(\mathbf{O},\mathbf{O},\mathbf{M})$ to itself is a morphism of Hom spans from $(\mathbf{O},\mathbf{M})$ to itself. Also, for every morphism of Hom spans $\mathbf{F}=(\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{mor}})$ from $(\mathbf{O},\mathbf{M})$ to $(\mathbf{O}',\mathbf{M}')$ and for every morphism of Hom spans $\mathbf{F}'=(\mathbf{F}'_{\text{obj}},\mathbf{F}'_{\text{obj}},\mathbf{F}_{\text{mor}})$ from $(\mathbf{O}',\mathbf{M}')$ to $(\mathbf{O}'',\mathbf{M}'')$, check that the composition $(\mathbf{F}'_{\text{obj}}\circ\mathbf{F}_{\text{obj}},\mathbf{F}'_{\text{mor}}\circ\mathbf{F}_{\text{mor}})$ is a morphism of Hom spans from $(\mathbf{O},\mathbf{M})$ to $(\mathbf{O}'',\mathbf{M}'')$. Use earlier exercises to deduce that composition of morphisms of Hom spans is associative, and the identity morphisms of Hom spans are left-right identities for this composition.

Definition H.4. For every category $\mathbf{C}=(\mathbf{O},\mathbf{M},\circ)$, for every category $\mathbf{C}'=(\mathbf{O}',\mathbf{M}',\circ')$, a (covariant) **functor** from $\mathbf{C}$ to $\mathbf{C}'$ is a morphism $\mathbf{F}$ of Hom spans from $(\mathbf{O},\mathbf{M})$ to $(\mathbf{O}',\mathbf{M}')$ that maps identities to identities and that is compatible with composition laws: for every object a of $\mathbf{C}$, the morphism $\mathbf{F}_{\text{mor}}$ maps $(a,a,\text{Id}_a^{\mathbf{C}})$ to $(a',a',\text{Id}_{a'}^{\mathbf{C}'})$, and for every ordered pair $((a,b),f)$, $((b,c),g)$ of members of $\mathbf{M}$ with images $((a',b'),f')$ and $((b',c'),g')$ under $\mathbf{F}_{\text{mor}}$, also $((a,c),g\circ f)$ has image $((a',c'),g'\circ' f')$.

Exercise H.5. For every category $\mathbf{C}$, check that the identity span cell of $\mathbf{C}$ is a functor from $\mathbf{C}$ to itself. This is the **identity functor**. Also, check that the composition of Hom spans of functors is again a functor. By the previous exercise, deduce that composition of functors is associative, and that identity functors are left-right identities for functor composition.

Example H.6. For every category $\mathbf{C}=(\mathbf{O},\mathbf{M},\circ)$ for every (not necessarily full) subcategory $\mathbf{C}'=(\mathbf{O}',\mathbf{M}',\circ)$ the inclusion morphism is a functor $\text{incl}_{\mathbf{C}}^{\mathbf{C}'}$ from $\mathbf{C}'$ to $\mathbf{C}$, the **inclusion functor**.

Example H.7. For every category $\mathbf{C}=(\mathbf{O},\mathbf{M},\circ)$, for every class morphism $\mathbf{F}_{\text{obj}}$ from a class $\mathbf{O}'$ to $\mathbf{O}$, this extends uniquely to a functor $(\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{mor}})$ from the discrete category structure on $\mathbf{O}'$ to $(\mathbf{O},\mathbf{M},\circ)$. Similarly for every class morphism $\mathbf{F}_{\text{obj}}$ from $\mathbf{O}$ to a class $\mathbf{O}'$, this extends uniquely to a functor $(\mathbf{F}_{\text{obj}},\mathbf{F}_{\text{mor}})$ from $(\mathbf{O},\mathbf{M},\circ)$ to the final category structure on $\mathbf{O}'$.

Definition H.8. For every category $\mathbf{C}=(\mathbf{O},\mathbf{M},\circ)$, a **C-Hom equivalence relation** is a subclass $\mathbf{R}$ of $\mathbf{M}\times_{\mathbf{O}\times\mathbf{O}}\mathbf{M}$, that, considered as an $\mathbf{O}$-Hom span, is stable for (component-wise) composition and whose fiber class, for every ordered pair (a,b) of members of $\mathbf{O}$, is an equivalence relation on $\mathbf{M}_b^a$.

Example H.9. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every $\mathbf{C}$-Hom equivalence relation $\mathbf{R}$, there exists a unique $\mathbf{O}$-Hom span $\mathbf{M}/\mathbf{R}$ and a unique composition law $\circ$ making $(\mathbf{O}, \mathbf{M}/\mathbf{R}, \circ)$ into a category such that both, for every ordered pair (a,b) of members of $\mathbf{O}$, the Hom set $(\mathbf{M}/\mathbf{R})^a_b$ is the set of $\mathbf{R}^a_b$-equivalence classes in $\mathbf{M}^a_b$, and the identity class morphism on $\mathbf{O}$ together with the quotient class morphism $\mathbf{M} \to \mathbf{M}/\mathbf{R}$ defines a full, strictly surjective functor from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O}, \mathbf{M}/\mathbf{R}, \circ)$.

Exercise H.10. Check that the composition law on $\mathbf{M}$ does factor through a composition law on $\mathbf{M}/\mathbf{R}$. Also check that this composition is associative and unital. Deduce that the class morphisms above define a full, strictly surjective functor.

Definition H.11. For every category $\mathbf{C}$, for every $\mathbf{C}$-Hom equivalence relation $\mathbf{R}$, the functor of the previous example is the **quotient functor** of $\mathbf{C}$ by the $\mathbf{C}$-Hom equivalence relation $\mathbf{R}$.

Example H.12. As examples of the previous construction, for every category $(\mathbf{O}, \mathbf{M}, \circ)$, for the equality class, the quotient functor is a strict equivalence of categories.

Example H.13. On the other hand, if $\mathbf{R}$ equals the entire $\mathbf{O}$-Hom span $\mathbf{M} \times_{\mathbf{O}\times\mathbf{O}} \mathbf{M}$, then each quotient Hom set $(\mathbf{M}/\mathbf{R})^a_b$ is either a singleton set if there exists an $\mathbf{M}$-morphism from a to b, or the empty set if $\mathbf{M}^a_b$ is empty. When $\mathbf{O}$ is the class of a set, this quotient category is equivalent to a preorder on that set.

Example H.14. For every monoid $(H, \bullet)$, for the associated monoid category, a Hom-equivalence relation is equivalent to an equivalence relation R on H such that, for every (h, h') in R and for every k in H, also $(k \bullet h, k \bullet h')$ and $(h \bullet k, h' \bullet k)$ are elements of R. In particular, if $(H, \bullet)$ is a group, Hom-equivalence relation are precisely the equivalence relations of (left or right) congruence modulo *normal* subgroups of $(H, \bullet)$, and the quotient functor corresponds to the quotient group homomorphism by the normal subgroup.

Definition H.15. For every functor $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O}', \mathbf{M}', \circ')$, the span cell of opposites spans is a functor of the opposite category, $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}^{\text{opp}}_{\text{mor}})$ from $(\mathbf{O}, \mathbf{M}^{\text{opp}}, \circ^{\mathbf{opp}})$ to $(\mathbf{O}', (\mathbf{M}')^{\text{opp}}, (\circ')^{\text{opp}})$. This is the **opposite functor**. The opposite functor of the opposite functor equals the original functor.

For every category $\mathbf{C}$, for every category $\mathbf{C}'$, a functor from $\mathbf{C}^{\text{opp}}$ to $\mathbf{C}'$ is then equivalent (up to taking opposites) to a functor from $\mathbf{C}$ to $(\mathbf{C}')^{\text{opp}}$, and these are both (somewhat confusingly) called **contravariant functors** from $\mathbf{C}$ to $\mathbf{C}'$.

Definition H.16. For every functor $(\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$ to $\mathbf{C}' = (\mathbf{O}', \mathbf{M}', \circ')$, the functor is **full**, respectively **faithful**, **fully faithful**, if for all members a and b of $\mathbf{O}$ with values $a' = \mathbf{F}_{\text{obj}}(a)$ and $b' = \mathbf{F}_{\text{obj}}(b)$, the function $\mathbf{F}^a_{\text{mor},b}$ from $\text{Hom}_{\mathbf{C}}(a,b)$ to $\text{Hom}_{\mathbf{C}'}(a',b')$ is surjective, resp. injective, bijective. A functor is **essentially surjective** if every object of $\mathbf{C}'$ is isomorphic to an object of the form $\mathbf{F}_{\text{obj}}(a)$ for some member a of $\mathbf{C}$. A faithful functor is **conservative** if (and only if) every morphism that is mapped to an isomorphism under the functor is already an isomorphism. A functor that is essentially surjective and fully faithful is a **weak equivalence** of categories.

Definition H.17. For every category, the **identity functor** from the category to itself maps every object to itself and maps every morphism to itself. For every functor $\mathbf{F} = (\mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{obj}}, \mathbf{F}_{\text{mor}})$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every functor $\mathbf{F}' = (\mathbf{F}'_{\text{obj}}, \mathbf{F}'_{\text{obj}}, \mathbf{F}'_{\text{mor}})$ from the category $\mathbf{C}'$ to a category $\mathbf{C}''$, the **composite functor** is the composite of span cells, $\mathbf{F}' \circ \mathbf{F} = (\mathbf{F}'_{\text{obj}} \circ \mathbf{F}_{\text{obj}}, \mathbf{F}'_{\text{obj}} \circ \mathbf{F}_{\text{obj}}, \mathbf{F}'_{\text{mor}} \circ \mathbf{F}_{\text{mor}})$ that sends every $\mathbf{C}$-object a to $\mathbf{F}'_{\text{obj}}(\mathbf{F}_{\text{obj}}(a))$ and that sends every $\mathbf{C}$-morphism $((a,b),f)$ in $\mathbf{C}^a_b$ to $\mathbf{F}'_{\text{mor}}(\mathbf{F}_{\text{mor}}((a,b),f))$.

Proposition H.18. *Composition of functors is associative, and it is unital for the identity functors. A composition of functors is faithful, respectively full, fully faithful, essentially surjective, if each of the component functors is of this type.*

Example H.19. The inclusion functor of a subcategory into a category is a faithful functor. The inclusion functor is full if and only if the subcategory is a full subcategory. An essentially surjective inclusion functor of a full skeletal subcategory in a category is a **skeleton** of the category. If we assume a strong version of the Axiom of Choice then every category has a skeleton.

Example H.20. A faithful functor from a category $\mathbf{C}$ to **Set** is a **concrete functor**, and this functor makes $\mathbf{C}$ into a **concrete category**. Most of the categories that arise in analysis, algebra, geometry, etc. are concrete, and typically the concrete functor is a "forgetful functor" that "forgets" some of the structure of the objects of $\mathbf{C}$. For example, the forgetful functor from **Monoid** to **Set** that forgets the binary operation is a faithful functor; in fact, it is conservative. Thus, we also get concrete (and conservative) functors by restricting to the full subcategories **Grp**, **Ab** and $\mathbb{Q}-\mathbf{Mod}$. Similarly, the forgetful functors on $R-\mathbf{Mod}$, on $\mathbf{Mod}-S$ and on $R-S-\mathbf{Mod}$ are concrete (and conservative) functors. The forgetful functor on **Ring** is a concrete (and conservative) functor, hence so is its restriction to the full subcategory **CRing**. Similarly, the forgetful functor from **Top** to **Set** is faithful, but it is not conservative (because there can be many different topologies on the same underlying set).

Exercise H.21. Of course the inclusion of **Set** as a (non-full) subcategory of **Rel** is faithful. Prove that the following defines a faithful functor $\mathcal{P}$ from **Rel** to **Set**: map every set a, considered as an object of **Rel**, to the power set $\mathcal{P}(a)$ of a, as an object of **Set**, and, for every ordered pair (a,b) of sets, map every element R of $\mathbf{Rel}^a_b = \mathcal{P}(a \times b)$ to the set function $\mathcal{P}_R$ from $\mathcal{P}(a)$ to $\mathcal{P}(b)$ by sending every subset a' of a to the subset $\mathcal{P}_R(a') = \text{pr}_{(a,b),2}(\text{pr}^{\text{pre}}_{(a,b),1}(a') \cap R)$ of b, where $\text{pr}_{(a,b),1}$, respectively $\text{pr}_{(a,b),2}$, is the usual projection function from the Cartesian product $a \times b$ to a, resp. to b. (Eventually we will see that this defines a right adjoint to the non-full inclusion of **Set** into **Rel**.)

Example H.22. For every monoid $(H,\bullet)$, for every monoid $(H',\bullet')$, for every monoid homomorphism f from $(H,\bullet)$ to $(H',\bullet')$, there is a unique functor from the category of $(H,\bullet)$ to the category of $(H',\bullet')$ that maps the unique object to the unique object, and that maps Hom sets via f. Every functor between these categories is of this form for a unique monoid homomorphism f. More generally, for every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every object a of $\mathbf{C}$ with image $a' = \mathbf{F}(a)$, the function $\mathbf{F}^a_a$ from $\mathbf{C}^a_a$ to $(\mathbf{C}')^{a'}_{a'}$ is a monoid homomorphism. Moreover, for every ordered pair (a,b) of objects of $\mathbf{C}$, for the set $\mathbf{C}^a_b$ with its natural left $\mathbf{C}^b_b$-act and its natural right $\mathbf{C}_a^a$-act, for the set $(\mathbf{C}')_{b'}^{a'}$ with the induced left $\mathbf{C}_b^b$-act and right $\mathbf{C}_a^a$-act arising from the monoid homomorphisms $\mathbf{F}_b^b$ and $\mathbf{F}_a^a$, the function $\mathbf{F}_b^a$ from $\mathbf{C}_b^a$ to $(\mathbf{C}')_{b'}^{a'}$ is compatible with the left and right acts.

Example H.23. Specializing the previous example to the case when $(H, \bullet)$ and $(H', \bullet')$ are groups, the functors from BH to BH' are equivalent to group homomorphisms from $(H, \bullet)$ to $(H', \bullet')$. More generally, every functor between groupoids induces group homomorphisms between automorphism groups of objects and the induced functioms between general Hom sets are compatible with both the left and right actions by these automorphism groups.

Example H.24. For every category $\mathbf{C}$, for every preordered set (S', R'), every functor from $\mathbf{C}$ to the category of (S', R') is equivalent to a morphism $\mathbf{F}_{\text{obj}}$ from $\text{obj}(\mathbf{C})$ to (the class of) S' that is *nondecreasing*, i.e., for every ordered pair (a, b) of objects of $\mathbf{C}$ such that $\mathbf{C}_b^a$ is nonempty, then $(f(a), f(b))$ is an element of R'.

Definition H.25. For every every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every object a' of the category $\mathbf{C}'$, the **constant functor** $\text{const}_{\mathbf{C}',a'}^{\mathbf{C}}$ from $\mathbf{C}$ to $\mathbf{C}'$ with value a' assigns the object a' to every object a of $\mathbf{C}$ and assigns the identity morphism $\text{Id}_{a'}^{\mathbf{C}'}$ to every $\mathbf{C}$-morphism. In other words, $\text{const}_{\mathbf{C}',a'}^{\mathbf{C}}$ is the composition of the unique functor from $\mathbf{C}$ to the trivial monoid $B\{e\}$ with the unique functor from $B\{e\}$ to $\mathbf{C}'$ sending the unique object of $B\{e\}$ to the object a' of $\mathbf{C}'$.

Example H.26. In particular, for the category **Set**, the functor $\mathbf{L} = \text{const}_{\mathbf{Set},\varnothing}^{\mathbf{Set}}$ from **Set** to itself has the special property that $\text{Hom}_{\mathbf{Set}}(\mathbf{L}(a), b)$ is always a singleton set.

Definition H.27. For every category $\mathbf{C}$, an object 0 of $\mathbf{C}$ is an **initial object** if (and only if), for every object a of $\mathbf{C}$, there exists a unique $\mathbf{C}$-morphism from 0 to a.

Example H.28. Similarly, for the category **Set**, for every singleton set, say $\mathbf{1} := \{\varnothing\}$, the functor $\mathbf{R} = \text{const}_{\mathbf{Set},\mathbf{1}}^{\mathbf{Set}}$ from **Set** to itself has the special property that $\text{Hom}_{\mathbf{Set}}(a, \mathbf{R}(b))$ is always a singleton set.

Definition H.29. For every category $\mathbf{C}$, an object 1 of $\mathbf{C}$ is a **final object** if (and only if), for every object a of $\mathbf{C}$, there exists a unique $\mathbf{C}$-morphism from a to 1. An object that is both initial and final is a **zero object**.

Exercise H.30. For every category $\mathbf{C}$ that has an initial object, prove that the initial object is unique up to unique isomorphism. Similarly, for every category $\mathbf{C}$ that has a final object, prove that the final object is unique up to unique isomorphism (you can use opposites to reduce to the previous assertion). Conclude that for every category $\mathbf{C}$ that has a zero object, the zero object is unique up to unique isomorphism.

Exercise H.31. Prove that **Set** has an initial object and a final object, but these are not isomorphic, hence **Set** does not have a zero object. Prove the same for **Top**, and the concrete forgetful functor maps the initial object of **Top**, respectively each final object of **Top**, to the initial object of **Set**, resp. to a final object of **Set**. On the other hand, prove that the empty set is the unique zero object of **Rel**. Similarly, prove that $\{e\}$ is the unique zero object in **Monoid**, in the full subcategory **Grp**, in the full subcategory **Ab**, etc.

Exercise H.32. Prove that the (standard) ring of integers $\mathbb{Z}$ is an initial object in the category **Ring** of associative, unital rings, and also in the full subcategory **CRing** of commutative, associative, unital rings. Prove that the zero ring is a final object in each of these categories.

I Natural transformations

The notion of functors admits its own notion of morphisms between functors, called *natural transformations*. This is very analogous to the operation on group homomorphisms of post-composition by a conjugation (inner) automorphism.

Definition I.1. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every covariant functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{C}'$, and for every covariant functor $\mathbf{G}$ from $\mathbf{C}$ to $\mathbf{C}'$, a **natural transformation** from $\mathbf{F}$ to $\mathbf{G}$ is a morphism of classes θ from $\text{ob}_{\mathbf{C}}$ associating to every object a of $\mathbf{C}$ an element θ_a of $\text{Hom}_{\mathbf{C}'}(\mathbf{F}(a), \mathbf{G}(a))$ such that, for every ordered pair (a, b) of objects of $\mathbf{C}$ and for every $\mathbf{C}$-morphism u from a to b, the $\mathbf{C}'$-composite $\theta_b \circ F_b^a(u)$ equals the $\mathbf{C}'$-composite $G_b^a(u) \circ \theta_a$. A natural transformation is a **natural equivalence** (or **natural isomorphism**) if (and only if) the morphism associated to each object is an isomorphism.

Example I.2. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, and for every covariant functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{C}'$, the **identity natural equivalence** from $\mathbf{F}$ to itself is the natural transformation that associates to every object a of $\mathbf{C}$ the identity morphism $\text{Id}^{\mathbf{C}'}_{\mathbf{F}(a)}$. This is denoted by $\text{Id}^{\mathbf{C}}_{\mathbf{C}',\mathbf{F}}$, or just $\text{Id}_{\mathbf{F}}$ when confusion is unlikely.

Exercise I.3. For the inclusion functor $\text{incl}^{\mathbf{Set}}_{\mathbf{Rel}}$ from **Set** to **Rel**, for the power set functor $\mathcal{P}$ from **Rel** to **Set**, check that the following defines a natural transformation θ from the identity functor $\text{Id}^{\mathbf{Set}}$ to the composite functor $\mathcal{P} \circ \text{incl}^{\mathbf{Set}}_{\mathbf{Rel}}$. For every set a, the set function θ_a from a to $\mathcal{P}(a)$ sends every element y of a to the singleton set $\{y\}$ considered as an element of $\mathcal{P}(a)$.

Exercise I.4. Continuing the previous exercise, check that the following defines a natural transformation η from $\text{incl}^{\mathbf{Set}}_{\mathbf{Rel}} \circ \mathcal{P}$ to the identity functor $\text{Id}^{\mathbf{Rel}}$. For every set a, the relation η_a from $\mathcal{P}(a)$ to a is the subset η_a of $\mathcal{P}(a) \times a$ of all ordered pairs (x, y) of a subset x of a and an element y of a, the ordered pair (x, y) is an element of η_a if and only if y is an element of x (i.e., η_a is the opposite relation of the relation ϵ_a of being an element of a set).

Definition I.5. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every ordered triple $(\mathbf{F}, \mathbf{G}, \mathbf{H})$ of covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, for every natural transformation η from $\mathbf{G}$ to $\mathbf{H}$, the (vertical) **composition natural transformation** $\eta \circ \theta$ from $\mathbf{F}$ to $\mathbf{H}$ is the natural transformation that associates to every object a of $\mathbf{C}$ the composite morphism $\eta_a \circ \theta_a$ from $\mathbf{F}(a)$ to $\mathbf{H}(a)$.

Exercise I.6. Check that the composition natural transformation is, indeed, a natural transformation. Also check that composition is (strictly) associative for natural transformations. Finally check that identity natural transformations are (strict) left-right identities for composition.

Example I.7. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every $\mathbf{C}'$-morphism f' from an object a' to an object b', there is an associated natural transformation $\text{const}^{\mathbf{C}}_{\mathbf{C}',f'}$ from the constant functor $\text{const}^{\mathbf{C}}_{\mathbf{C}',a'}$ to the constant functor $\text{const}^{\mathbf{C}}_{\mathbf{C}',b'}$ associating to every object a of $\mathbf{C}$ the morphism f'.

Exercise I.8. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every object a' of $\mathbf{C}'$, prove that $\text{const}^{\mathbf{C}}_{\mathbf{C}',\text{Id}_{a'}}$ is the identity natural transformation from $\text{const}^{\mathbf{C}}_{\mathbf{C}',a'}$ to itself. Also, for every ordered triple (a',b',c') of $\mathbf{C}'$-objects, for every $\mathbf{C}'$-morphism f' from a' to b' and for every $\mathbf{C}'$-morphism g' from b' to c', prove that $\text{const}^{\mathbf{C}}_{\mathbf{C}',g'\circ f'}$ equals the composition of natural transformations $\text{const}^{\mathbf{C}}_{\mathbf{C}',g'} \circ \text{const}^{\mathbf{C}}_{\mathbf{C}',f'}$.

Example I.9. For every monoid $(H,\bullet)$, for every monoid $(H',\bullet')$, for monoid homomorphisms f and g from $(H,\bullet)$ to $(H',\bullet')$, for the associated functors from the category of $(H,\bullet)$ to the category of $(H',\bullet')$, a natural transformation between these functors is an element h' of H' such that for every element h of H, the composite $h' \bullet' f(h)$ equals $g(h) \bullet' h'$. In particular, if g equals f, then the natural self-transformations of $g = f$ are equivalent to elements of H' that centralize the image of f. So the center of $(H',\bullet')$ is equivalent to the set of natural self-transformations of the identity functor of the category of $(H',\bullet')$.

Example I.10. Continuing the previous example, if the monoid $(H',\bullet')$ is a group (i.e., if every morphism is an isomorphism), then a natural transformation from f to g, monoid homomorphisms from $(H,\bullet)$ to $(H',\bullet')$, are equivalent to elements h' of H' such that g equals the composite $\text{inner}_{h'} \circ f$, where $\text{inner}_{h'}$ is the conjugation (inner) automorphism of $(H',\bullet')$ associated to h'.

Example I.11. Similarly, for every natural transformation θ between functors $\mathbf{F}$ and $\mathbf{G}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every object a of $\mathbf{C}$ that maps under both $\mathbf{F}$ and $\mathbf{G}$ to a common object a', the monoid homomorphisms $\mathbf{F}^a_a$ and $\mathbf{G}^a_a$ from $\mathbf{C}^a_a$ to $(\mathbf{C}')^{a'}_{a'}$ are *intertwined* by the element θ_a of $(\mathbf{C}')^{a'}_{a'}$ in the sense that $\theta_a \circ \mathbf{F}^a_a(u)$ equals $\mathbf{G}^a_a(u) \circ \theta_a$ for every element u of $\mathbf{C}^a_a$.

Example I.12. For every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, if there exists an initial object $0'$ of $\mathbf{C}'$, then there is a unique natural transformation from the constant functor $\text{const}^{\mathbf{C}}_{\mathbf{C}',0'}$ to $\mathbf{F}$ that associates to every object a of $\mathbf{C}$ the unique $\mathbf{C}'$-morphism from $0'$ to $\mathbf{F}(a)$.

Example I.13. For every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, if there exists a final object $1'$ of $\mathbf{C}'$, then there is a unique natural transformation from $\mathbf{F}$ to the constant functor $\text{const}^{\mathbf{C}}_{\mathbf{C}',1'}$ that associates to every object a of $\mathbf{C}$ the unique $\mathbf{C}'$-morphism from $\mathbf{F}(a)$ to $1'$.

Example I.14. For every category $\mathbf{C}$, for every preordered set (S',R'), for functors $\mathbf{F}$ and $\mathbf{G}$ from $\mathbf{C}$ to the category of (S,R), i.e., nondecreasing morphisms from $\text{obj}(\mathbf{C})$ to (the class of) S', there exists a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ if and only if $(\mathbf{F}(a),\mathbf{G}(a))$ is an element of R' for every object a of $\mathbf{C}$, and then the natural transformation is unique. Thus, there exists a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ if and only if, valuewise $\mathbf{F}_{\text{mor}}$ is "less than or equal to" $\mathbf{G}_{\text{mor}}$.

There is another notion of composition for functors.

Definition I.15. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every category $\mathbf{C}' = (\mathbf{O}', \mathbf{M}', \circ')$, for every category $\mathbf{C}'' = (\mathbf{O}'', \mathbf{M}'', \circ'')$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, for every ordered pair $(\mathbf{F}', \mathbf{G}')$ of covariant functors from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, for every natural transformation θ' from $\mathbf{F}'$ to $\mathbf{G}'$, the **horizontal composition natural transformation** of θ' and θ, sometimes called the **Godement product**, is the natural transformation $\theta' * \theta$ from $\mathbf{F}' \circ \mathbf{F}$ to $\mathbf{G}' \circ \mathbf{G}$ associating to every object a of $\mathbf{C}$ the $\mathbf{C}''$-morphism,

$$\theta'_{\mathbf{G}(a)} \circ'' (\mathbf{F}')^{\mathbf{F}(a)}_{\mathbf{G}(a)}(\theta_a) = (\theta' * \theta)_a = (\mathbf{G}')^{\mathbf{F}(a)}_{\mathbf{G}(a)}(\theta_a) \circ'' \theta'_{\mathbf{F}(a)}.$$

Exercise I.16. Check that the Godement product is a natural transformation from $\mathbf{F}' \circ \mathbf{F}$ to $\mathbf{G}' \circ \mathbf{G}$. Also check that the Godement product is associative in both θ and θ' separately for the (vertical) composition of natural transformations.

There are some important special cases of the Godement product.

Definition I.17. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every category $\mathbf{C}''$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, for every covariant functor $\mathbf{H}'$ from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **$\mathbf{H}'$-pushforward natural transformation** is $\mathbf{H}'_*\theta = \mathrm{Id}^{\mathbf{C}'}_{\mathbf{C}'',\mathbf{H}'} * \theta$, associating to every object a of $\mathbf{A}$ the $\mathbf{C}''$-morphism $(\mathbf{H}')^{\mathbf{F}(a)}_{\mathbf{G}(a)}(\theta_a)$.

Definition I.18. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every category $\mathbf{C}''$, for every covariant functor $\mathbf{E}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every ordered pair $(\mathbf{F}', \mathbf{G}')$ of covariant functors from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation θ from $\mathbf{F}'$ to $\mathbf{G}'$, the **$\mathbf{E}$-pullback natural transformation**, $\mathbf{E}^*\theta = \theta * \mathrm{Id}^{\mathcal{C}}_{\mathcal{C}',\mathbf{E}}$ associates to every object a of $\mathbf{C}$ the $\mathbf{C}''$-morphism $\theta_{\mathbf{E}(a)}$.

Exercise I.19. Check that the Godement product can be expanded in terms of pushforward, pullback and vertical composition as follows,

$$\mathbf{G}^*\eta \circ \mathbf{F}'_*\theta = \eta * \theta = \mathbf{G}'_*\theta \circ \mathbf{F}^*\eta.$$

J Products and Coproducts

Definition J.1. For every category $\mathbf{C}$, for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$, an **arrow over** (b_1, b_2) is an ordered pair (p_1, p_2) of a $\mathbf{C}$-morphism p_1 from an object a to b_1 and a $\mathbf{C}$-morphism p_2 from a to b_2. For every arrow $((p_1 : a \to b_1, p_2 : a \to b_2)$ over (b_1, b_2), for every arrow $(p'_1 : a' \to b_1, p'_2 : a' \to b_2))$ over (b_1, b_2), a **morphism** of arrows over (b_1, b_2) from (p_1, p_2) to (p'_1, p'_2) is a $\mathbf{C}$-morphism f from a to a' such that both $p'_1 \circ f$ equals p_1 and $p'_2 \circ f$ equals p_2. A **product** of (b_1, b_2) in $\mathbf{C}$ is an arrow over (b_1, b_2), say

$$(\mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),1} : b_1 \times b_2 \to b_1, \mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),2} : b_1 \times b_2 \to b_2),$$

such that for every arrow (p_1, p_2) over (b_1, b_2), there exists a unique morphism of arrows over (b_1, b_2) from (p_1, p_2) to $(\mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),1}, \mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),2})$. More generally, for every object c of $\mathbf{C}$, for every ordered pair $(g_1 : b_1 \to c, g_2 : b_2 \to c)$ of objects of the under category $\mathbf{C}_c$, a **fiber product** of (g_1, g_2) is a product of (g_1, g_2) in the under category $\mathbf{C}_c$, i.e., an arrow over (b_1, b_2),

$$(\mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),1} : b_1 \times_{g_1,c,g_2} b_2 \to b_1, \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),2} : b_1 \times_{g_1,c,g_2} b_2 \to b_2),$$

such that $g_1 \circ \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),1}$ equals $g_2 \circ \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),2}$, and such that, for every arrow over (b_1, b_2), say $(p_1 : a \to b_1, p_2 : a \to b_2)$ that satisfies $g_1 \circ p_1 = g_2 \circ p_2$, there exists a unique morphism of arrows over (b_1, b_2) from (p_1, p_2) to $(\mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),1}, \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),2})$.

Definition J.2. For every category $\mathbf{C}$, for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$, an **arrow under** (b_1, b_2) is an ordered pair (i_1, i_2) of a $\mathbf{C}$-morphism i_1 from b_1 to an object c and a $\mathbf{C}$-morphism p_2 from b_2 to c. For every arrow $((i_1 : b_1 \to c, i_2 : b_2 \to c)$ under (b_1, b_2), for every arrow $(i'_1 : b_1 \to c', i'_2 : b_2 \to c'))$ under (b_1, b_2), a **morphism** of arrows under (b_1, b_2) from (i_1, i_2) to (i'_1, i'_2) is a $\mathbf{C}$-morphism h from c to c' such that both $h \circ i_1$ equals i'_1 and $h \circ i_2$ equals i'_2. A **coproduct** of (b_1, b_2) in $\mathbf{C}$ is an arrow under (b_1, b_2), say

$$(\mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),1} : b_1 \to b_1 \sqcup b_2, \mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),2} : b_2 \to b_1 \sqcup b_2),$$

such that for every arrow (i'_1, i'_2) under (b_1, b_2), there exists a unique morphism of arrows under (b_1, b_2) from $(\mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),1}, \mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),2})$ to (i'_1, i'_2). More generally, for every object a of $\mathbf{C}$, for every ordered pair $(f_1 : a \to b_1 \to b, f_2 : a \to b_2)$ of objects of the over category $\mathbf{C}^a$, a **cofiber coproduct** of (f_1, f_2) is a coproduct of (f_1, f_2) in the over category $\mathbf{C}^a$, i.e., an arrow under (b_1, b_2),

$$(\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),1} : b_1 \to b_1 \sqcup^{f_1,a,f_2} b_2, \mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),2} : b_2 \to b_1 \sqcup^{f_1,a,f_2} b_2),$$

such that $\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),1} \circ f_1$ equals $\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),2} \circ f_2$, and such that for every arrow under (b_1, b_2), say $(i'_1 : b_1 \to c', i'_2 : b_2 \to c')$ that satisfies $f_1 \circ i'_1 = f_2 \circ i'_2$, there exists a unique morphism of arrows under (b_1, b_2) from $(\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),1}, \mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),2})$ to (i'_1, i'_2).

Lemma J.3. *When products exist, respectively when fiber products exist, when coproducts exist, when cofiber coproducts exist, they are unique up to unique isomorphism. Products and fiber products in the opposite category are coproducts and cofiber coproducts in the original category. Coproducts and cofiber coproducts in the opposite category are products and fiber products in the original category.*

Definition J.4. A category $\mathbf{C}$ has **all finite products**, respectively has **all finite coproducts**, if (and only if) for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$ there exists a product of (b_1, b_2) in $\mathbf{C}$, resp. there exists a coproduct of (b_1, b_2) in $\mathbf{C}$. A category has **all finite limits** if (and only if), for every object c of $\mathbf{C}$, the under category $\mathbf{C}_c$ has all finite products. A category has **all finite colimits** if (and only if), for every object a of $\mathbf{C}$, the over category $\mathbf{C}^a$ has all finite coproducts.

Example J.5. In the category **Set**, Cartesian products with the usual projection functions are products, and disjoint unions with the usual inclusion functions are coproducts. Thus, **Set** has all finite products, and it has all finite coproducts. Similarly, for fiber products, the equalizer subset in the Cartesian product of the pair of morphisms is a fiber product in the category of sets, and the coequalizer quotient set of the disjoint union for the pair of morphisms is a cofiber coproduct in the category of sets. Thus, **Set** has all finite limits, and it has all finite colimits. In the category **Rel**, again disjoint union with inclusion functions (considered as relations) are coproducts. The opposite relations of the inclusions functions make disjoint unions into products in the category of **Rel**. Thus, **Rel** has all finite products, and it has all finite coproducts. However, the category **Rel** does not have all fiber products, nor does it have all cofiber coproducts.

Definition J.6. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{C}'$, the functor **preserves finite products** if (and only if), for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$ and for every ordered pair $(\text{pr}^{\mathbf{C}}_{(b_1,b_2),1}, \text{pr}^{\mathbf{C}}_{(b_1,b_2),2})$ of $\mathbf{C}$-morphisms that is a product of (b_1, b_2), for the $\mathbf{C}'$-objects $b'_i = \mathbf{F}(b_i)$, and for the $\mathbf{C}'$-morphisms $\text{pr}^{\mathbf{C}'}_{(b'_1,b'_2),i} = \mathbf{F}(\text{pr}^{\mathbf{C}}_{(b_1,b_2),i})$, the ordered pair $(\text{pr}^{\mathbf{C}'}_{(b'_1,b'_2),1}, \text{pr}^{\mathbf{C}'}_{(b'_1,b'_2),2})$ is a product of (b'_1, b'_2) in $\mathbf{C}'$.

Similarly, the functor **preserves finite limits** if (and only if), for every object c of $\mathbf{C}$ with image object $c' = \mathbf{F}(c)$ of $\mathbf{C}'$, the associated functor $\mathbf{F}_c$ from the under category $\mathbf{C}_c$ to the under category $\mathbf{C}'_{c'}$ preserves finite products, i.e., $\mathbf{F}$ preserves (finite) fiber products.

Definition J.7. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{C}'$, the functor **preserves finite coproducts** if (and only if), for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$ and for every ordered pair $(\text{incl}^{\mathbf{C}}_{(b_1,b_2),1}, \text{incl}^{\mathbf{C}}_{(b_1,b_2),2})$ of $\mathbf{C}$-morphisms that is a coproduct of (b_1, b_2), for the $\mathbf{C}'$-objects $b'_i = \mathbf{F}(b_i)$, and for the $\mathbf{C}'$-morphisms $\text{incl}^{\mathbf{C}'}_{(b'_1,b'_2),i} = \mathbf{F}(\text{incl}^{\mathbf{C}}_{(b_1,b_2),i})$, the ordered pair $(\text{incl}^{\mathbf{C}'}_{(b'_1,b'_2),1}, \text{incl}^{\mathbf{C}'}_{(b'_1,b'_2),2})$ is a coproduct of (b'_1, b'_2) in $\mathbf{C}'$.

Similarly, the functor **preserves finite colimits** if (and only if), for every object a of $\mathbf{C}$ with image object $a' = \mathbf{F}(c)$ of $\mathbf{C}'$, the associated functor $\mathbf{F}_a$ from the over category $\mathbf{C}^a$ to the over category $(\mathbf{C}')^{a'}$ preserves finite coproducts, i.e., $\mathbf{F}$ preserves (finite) cofiber coproducts.

Exercise J.8. For every monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, define a binary operation on the Cartesian product $H \times H'$ by $(h_1, h'_1) * (h_2, h'_2) := (h_1 \bullet h_2, h'_1 \bullet' h'_2)$. Check that the projection function $\text{pr}_{(H,H'),1}$, respectively $\text{pr}_{(H,H'),2}$, is a monoid homomorphism from $(H \times H', *)$ to $(H, \bullet)$, resp. to $(H', \bullet')$. Check that this operation makes $(H \times H', *)$ into a product of $(H, \bullet)$ and $(H', \bullet')$ in the category of monoids. Conclude that **Monoid** has all finite products, and the forgetful concrete functor from **Monoid** to **Set** preserves finite products. Similarly, check that **Monoid** has all finite limits, and the forgetful concrete functor preserves finite limits.

Exercise J.9. Prove that a full subcategory of a category that has all finite products, respectively that has all finite coproducts, both has all finite products, resp. all finite coproducts, and the inclusion functor preserves all finite products, resp. all finite coproducts, if and only if every product in the ambient category, resp. every coproduct in the ambient category, of objects of the full subcategory is isomorphic to an object in the full subcategory. Formulate and prove the analogous result for finite limits, resp. for finite colimits.

Exercise J.10. Prove that the full subcategory **Grp** of **Monoid** has all finite limits and the inclusion functor preserves all finite limits. Similarly, prove that the full subcategory **Ab** of **Grp** has all finite limits and the inclusion functor preserves all finite limits. Similarly, prove that the full subcategory $\mathbb{Q}-$**Mod** of **Ab** has all finite limits and the inclusion functor preserves all finite limits. More generally, for all associative, unital rings R and S, for the forgetful functor to **Ab** from $R-$**Mod**, respectively from **Mod** $-S$, from $R-S-$**Mod**, mapping each module to its underlying additive group, prove that each of these categories has all finite limits and the forgetful functor preserves all finite limits.

Exercise J.11. For the forgetful functor from **Ring** to **Ab** that maps each associative, unital rings to its underling additive group, prove that **Ring** has all finite limits and the forgetful functor preserves all finite limits. Prove that the full subcategory **CRing** of **Ring** has all finite limits and the inclusion functor preserves all finite limits.

Exercise J.12. For every ordered pair $((X_1,\tau_1),(X_2,\tau_2))$ of topological spaces, prove that there exists a coarsest topology $\tau_1\otimes\tau_2$ on the product set $X_1\times X_2$ such that for both $i=1$ and $i=2$, the projection function $\text{pr}_{(X_1,X_2),i}$ is a continuous map from $(X_1\times X_2,\tau_1\otimes\tau_2)$ to (X_i,τ_i), namely the topology generated by the topological basis $\mathcal{B}$ of all subsets $\text{pr}^{\text{pre}}_{(X_1,X_2),1}(U_1)\cap\text{pr}^{\text{pre}}_{(X_1,X_2),2}(U_2)$ with U_1 an element of τ_1 and with U_2 an element of τ_2. This is the **product topology** on $X_1\times X_2$ of τ_1 and τ_2. Prove that the pair of continuous maps $((\text{pr}_{(X_1,X_2),1},\text{pr}_{(X_1,X_2),2})$ is a product of (X_1,τ_1) and (X_2,τ_2) in the category of topological spaces. Conclude that the category of topological spaces has all finite products.

Exercise J.13. For every topological space (X,τ), and for every subset X' of X, prove that the subset $\tau|_{X'}:=\{U\cap X'|U\in\tau\}$ of $\mathcal{P}(X')$ is the coarsest topology on X' such that the inclusion function $\text{incl}_X^{X'}$ is a continuous map from $(X',\tau|_{X'})$ to (X,τ). Show also that for every topological space (X'',τ''), for every continuous map f from (X'',τ'') to (X,τ), the image of f is contained in the subset X' if and only if there exists a continuous map f' from (X'',τ'') to $(X',\tau|_{X'})$ such that f equals $\text{incl}_X^{X'}\circ f'$, and then f' is unique. The topology $\tau|_{X'}$ is the **subspace topology**.

Exercise J.14. For every ordered triple of topological spaces, say (X_1,τ_1), (X_2,τ_2) and (X,τ), for every ordered pair of continuous maps g_1 from (X_1,τ_1) to (X,τ) and g_2 from (X_2,τ_2) to (X,τ), prove that the subspace topology on the subset $X_1\times_{g_1,X,g_2}X_2$ of $(X_1\times X_2,\tau_1\otimes\tau_2)$ gives a fiber product of g_1 and g_2 in the category of topological spaces. Conclude that the category of topological spaces has all finite limits, and the forgetful functor from **Top** to **Set** preserves all finite limits.

The description of coproduct in each of these concrete categories is different. The notion of left adjoint functors to each concrete forgetful functor gives a uniform construction of the coproducts.

K Product categories

Definition K.1. For every category $\mathbf{C}_1$ and for every category $\mathbf{C}_2$, the **product category** $\mathbf{C}_1\times\mathbf{C}_2$ of $\mathbf{C}_1$ and $\mathbf{C}_2$ is the category whose objects are ordered pairs (a_1,a_2) of a $\mathbf{C}_1$-object a_1 and a $\mathbf{C}_2$-object a_2. For every ordered pair $((a_1,a_2),(b_1,b_2))$ of objects a_1 and b_1 of $\mathbf{C}_1$ and objects a_2 and

b_2 of $\mathbf{C}_2$, the Hom set in $\mathbf{C}_1 \times \mathbf{C}_2$ is the product set

$$\mathrm{Hom}_{\mathbf{C}_1\times\mathbf{C}_2}((a_1,a_2),(b_1,b_2)) = \mathrm{Hom}_{\mathbf{C}_1}(a_1,b_1)\times\mathrm{Hom}_{\mathbf{C}_2}(a_2,b_2).$$

Finally, composition is defined componentwise: for every ordered pair $((g_1,g_2),(f_1,f_2))$ of $\mathbf{C}_1$-morphisms f_1 from a_1 to b_1 and g_1 from b_1 to c_1 and $\mathbf{C}_2$-morphisms f_2 from a_2 to b_2 and g_2 from b_2 to c_2, the composition $(g_1,g_2)\circ(f_1,f_2)$ is defined to equal $(g_1\circ_1 f_1, g_2\circ_2 f_2)$.

Example K.2. For every monoid $(H,\bullet)$, for every monoid $(H',\bullet')$, the product of the category of $(H,\bullet)$ and the category of $(H',\bullet')$ is (canonically equivalent to) the category of the direct product monoid $(H\times H',*)$ where $(a,a')*(b,b')$ equals $(a\bullet a', b\bullet b')$ for all elements a and b of H and for all elements a' and b' of H'. Note, this is (usually) quite different from the free product of the two monoids (which is the coproduct in the category of monoids), i.e., the quotient of the free monoid on the set $H\sqcup H'$ by the equivalence relation arising from the identities and group operations on H and on H'. The direct product is a further quotient by the equivalence relation identifying each product $(e,h')*(h,e')$ with the product $(h,e')*(e,h')$, for identity elements e and e' of H and H'.

Definition K.3. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, for the product category $\mathbf{C}_1\times\mathbf{C}_2$, the **first projection functor** $\mathrm{pr}^{\mathbf{C}_1,\mathbf{C}_2}_{\mathbf{C}_1,1}$ from $\mathbf{C}_1\times\mathbf{C}_2$ to $\mathbf{C}_1$ maps every object (a_1,a_2) of $\mathbf{C}_1\times\mathbf{C}_2$ to the object a_1 of $\mathbf{C}_1$ and maps (f_1,f_2) to f_1 for every ordered pair (f_1,f_2) of a $\mathbf{C}_1$-morphism f_1 from a_1 to b_1 and a $\mathbf{C}_2$-morphism f_2 from a_2 to b_2. This functor is denoted by pr_1 when confusion is unlikely.

Similarly, the **second projection functor** $\mathrm{pr}^{\mathbf{C}_1,\mathbf{C}_2}_{\mathbf{C}_2,2}$ from $\mathbf{C}_1\times\mathbf{C}_2$ to $\mathbf{C}_2$ maps every object (a_1,a_2) to a_2 and maps every $\mathbf{C}_1\times\mathbf{C}_2$-morphism (f_1,f_2) to f_2. This functor is denoted by pr_2 when confusion is unlikely.

Example K.4. For every monoid $(H_1,\bullet_1)$, for every monoid $(H_2,\bullet_2)$ the projection functors from the product category correspond to the projection monoid homomorphisms from the product monoid $(H_1\times H_2,\bullet_1\times\bullet_2)$ to the factors $(H_1,\bullet_1)$ and $(H_2,\bullet_2)$.

Proposition K.5. *For every category $\mathbf{B}$, for every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, for every functor $\mathbf{F}_1$ from $\mathbf{B}$ to $\mathbf{C}_1$, for every functor $\mathbf{F}_2$ from $\mathbf{B}$ to $\mathbf{C}_2$, there exists a unique functor $(\mathbf{F}_1,\mathbf{F}_2)$ from $\mathbf{B}$ to the product category $\mathbf{C}_1\times\mathbf{C}_2$ such that the composite functor $pr_1\circ(\mathbf{F}_1,\mathbf{F}_2)$ equals $\mathbf{F}_1$ and the composite functor $pr_2\circ(\mathbf{F}_1,\mathbf{F}_2)$ equals $\mathbf{F}_2$.*

Proposition K.6. *For every category $\mathbf{B}$, for every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, for every ordered pair $(\mathbf{F}_1,\mathbf{G}_1)$ of functors from $\mathbf{B}$ to $\mathbf{C}_1$, for every ordered pair $(\mathbf{F}_2,\mathbf{G}_2)$ of functors from $\mathbf{B}$ to $\mathbf{C}_2$, for every natural transformation θ_1 from $\mathbf{F}_1$ to $\mathbf{G}_1$, for every natural transformation θ_2 from $\mathbf{F}_2$ to $\mathbf{G}_2$, there exists a unique natural transformation (θ_1,θ_2) from $(\mathbf{F}_1,\mathbf{F}_2)$ to $(\mathbf{G}_1,\mathbf{G}_2)$ such that the pushforward of (θ_1,θ_2) by pr_1 equals θ_1 and the pushforward of (θ_1,θ_2) by pr_2 equals θ_2.*

Corollary K.7. *For every functor of categories, $\mathbf{F}_1$ from $\mathbf{C}_1$ to $\mathbf{D}_1$, for every functor of categories, $\mathbf{F}_2$ from $\mathbf{C}_2$ to $\mathbf{D}_2$, there is a unique functor $(\mathbf{F}_1\circ pr_1,\mathbf{F}_2\circ pr_2)$ from the product category $\mathbf{C}_1\times\mathbf{C}_2$ to the product category $\mathbf{D}_1\times\mathbf{D}_2$, such that the composite functor $pr_1\circ(\mathbf{F}_1\circ pr_1,\mathbf{F}_2\circ pr_2)$ equals $\mathbf{F}_1\circ pr_1$ and the composite functor $pr_2\circ(\mathbf{F}_1\circ pr_1,\mathbf{F}_2\circ pr_2)$ equals $\mathbf{F}_2\circ pr_2$.*

Corollary K.8. *For every ordered pair* $(\mathbf{F}_1, \mathbf{G}_1)$ *of functors from* $\mathbf{C}_1$ *to* $\mathbf{D}_1$*, for every ordered pair* $(\mathbf{F}_2, \mathbf{G}_2)$ *of functors from* $\mathbf{C}_2$ *to* $\mathbf{D}_2$*, for every natural transformation* θ_1 *from* $\mathbf{F}_1$ *to* $\mathbf{G}_1$*, for every natural transformation* θ_2 *from* $\mathbf{F}_2$ *to* $\mathbf{G}_2$*, there is a unique natural transformation* $(pr_1^*\theta_1, pr_2^*\theta_2)$ *from* $(\mathbf{F}_1 \circ pr_1, \mathbf{F}_2 \circ pr_2)$ *to* $(\mathbf{G}_1 \circ pr_1, \mathbf{G}_2 \circ pr_2)$ *whose pushforward by* pr_1 *equals the pullback* $pr_1^*\theta_1$ *and whose pushforward by* pr_2 *equals the pullback* $pr_2^*\theta_2$*.*

Definition K.9. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, for every category $\mathbf{D}$, a **bifunctor** (or **strict** 2**-functor**) $\mathbf{F}$ to $\mathbf{D}$ from $\mathbf{C}_1$ and $\mathbf{C}_2$ is an ordered triple class $(\mathbf{F}_{\text{obj,obj}}, (\mathbf{F}_{\text{mor,obj}}, \mathbf{F}_{\text{obj,mor}}))$ of a class morphism $\mathbf{F}_{\text{obj,obj}}$ from $\text{obj}(\mathbf{C}_1) \times \text{obj}(\mathbf{C}_2)$ to $\text{obj}(\mathbf{D})$, of a class morphism $\mathbf{F}_{\text{mor,obj}}$ from $\text{mor}(\mathbf{C}_1) \times \text{obj}(\mathbf{C}_2)$ to $\text{mor}(\mathbf{D})$, and of a class morphism $\mathbf{F}_{\text{obj,mor}}$ from $\text{obj}(\mathbf{C}_1) \times \text{mor}(\mathbf{C}_2)$ to $\text{mor}(\mathbf{D})$ such that, for every member (a_1, a_2) of $\text{obj}(\mathbf{C}_1) \times \text{obj}(\mathbf{C}_2)$, the ordered pair class $(\mathbf{F}_{\text{obj,obj}}(\bullet, a_2), \mathbf{F}_{\text{mor,obj}}(\bullet, a_2))$ is a functor from $\mathbf{C}_1$ to $\mathbf{D}$, the ordered pair class $(\mathbf{F}_{\text{obj,obj}}(a_1, \bullet), \mathbf{F}_{\text{obj,mor}}(a_1, \bullet))$ is a functor from $\mathbf{C}_2$ to $\mathbf{D}$, and we have

$$\mathbf{F}_{\text{obj,mor}}(b_1, f_1) \circ \mathbf{F}_{\text{mor,obj}}(f_1, a_2) = \mathbf{F}_{\text{mor,obj}}(f_1, b_2) \circ \mathbf{F}_{\text{obj,mor}}(a_1, f_2)$$

for every $\mathbf{C}_1$-morphism f_1 from a_1 to an object b_1 and for every $\mathbf{C}_2$-morphism f_2 from a_2 to an object b_2.

Example K.10. For every category $\mathbf{C}$, the **Hom bifunctor** $\text{Hom}_{\mathbf{C}}$, or just Hom when confusion is unlikely, is the bifunctor to **Set** from $\mathbf{C}^{\text{opp}}$ and $\mathbf{C}$ that maps every ordered pair (a, b) of objects of $\mathbf{C}$ to the set $\text{Hom}_{\mathbf{C}}(a, b)$, that maps every ordered pair $(u : a' \to a, b)$ of a $\mathbf{C}$-morphism u from a' to a and an object b of $\mathbf{C}$ to the set function $\text{Hom}_{\mathbf{C}}(u, b)$ from $\text{Hom}_{\mathbf{C}}(a, b)$ to $\text{Hom}_{\mathbf{C}}(a', b)$ of precomposition by u, and that maps every ordered pair $(a, v : b \to b')$ of an object a of $\mathbf{C}$ and a $\mathbf{C}$-morphism v from b to b' to the set function $\text{Hom}_{\mathbf{C}}(a, v)$ from $\text{Hom}_{\mathbf{C}}(a, b)$ to $\text{Hom}_{\mathbf{C}}(a, b')$ of postcomposition by v. This satisfies the bifunctor identities because of associativity of composition.

Exercise K.11. Check that this is a bifunctor.

Example K.12. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, the **braiding bifunctor** $B_{\mathbf{C}_1,\mathbf{C}_2}$ is the bifunctor to $\mathbf{C}_2 \times \mathbf{C}_1$ from $\mathbf{C}_1$ and $\mathbf{C}_2$ that maps every ordered pair (a_1, a_2) of an object a_1 of $\mathbf{C}_1$ and an object a_2 of $\mathbf{C}_2$ to the object (a_2, a_1) of $\mathbf{C}_2 \times \mathbf{C}_1$, that maps every ordered pair $(u_1 : a_1 \to a_1', a_2)$ of a $\mathbf{C}_1$-morphism u_1 from a_1 to a_1' and an object a_2 of $\mathbf{C}_2$ to the morphism (Id_{a_2}, u_1) from (a_2, a_1) to (a_2, a_1') in $\mathbf{C}_2 \times \mathbf{C}_1$, and that maps every ordered pair $(a_1, u_2 : a_2 \to a_2')$ of a $\mathbf{C}_2$-morphism u_2 from a_2 to a_2' and an object a_1 of $\mathbf{C}_1$ to the morphism (u_2, Id_{a_1}) from (a_2, a_1) to (a_2', a_1) in $\mathbf{C}_2 \times \mathbf{C}_1$.

Proposition K.13. *For every category* $\mathbf{C}_1$*, for every category* $\mathbf{C}_2$*, and for every category* $\mathbf{D}$*, every bifunctor to* $\mathbf{D}$ *from* $\mathbf{C}_1$ *and* $\mathbf{C}_2$ *extends uniquely to a functor from the product category* $\mathbf{C}_1 \times \mathbf{C}_2$ *to* $\mathbf{D}$*.*

Proposition K.14. *For every category* $\mathbf{C}_1$*, for every category* $\mathbf{C}_2$*, and for every category* $\mathbf{D}$*, for every ordered pair* $(\mathbf{F}, \mathbf{G})$ *of bifunctor to* $\mathbf{D}$ *from* $\mathbf{C}_1$ *and* $\mathbf{C}_2$*, for every class morphism* θ *from obj*$(\mathbf{C}_1) \times$ *obj*$(\mathbf{C}_2)$ *to mor*$(\mathbf{D})$*, this is a natural transformation from the functor of* $\mathbf{F}$ *to the* *functor of* $\mathbf{G}$ *if and only if, for every member* (a_1,a_2) *of* $obj(\mathbf{C}_1)\times obj(\mathbf{C}_2)$*, both* $\theta_{\bullet,a_2}$ *is a natural transformation from* $\mathbf{F}(\bullet,a_2)$ *to* $\mathbf{G}(\bullet,a_2)$ *and* $\theta_{a_1,\bullet}$ *is a natural transformation from* $\mathbf{F}(a_1,\bullet)$ *to* $\mathbf{G}(a_1,\bullet)$*.*

Example K.15. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, there is a **projection bifunctor** $\text{pr}^{\mathbf{C}_1,\mathbf{C}_2}_{\mathbf{C}_1,1}$, respectively $\text{pr}^{\mathbf{C}_1,\mathbf{C}_2}_{\mathbf{C}_2,2}$, from $\mathbf{C}_1$ and $\mathbf{C}_2$ to $\mathbf{C}_1$, resp. to $\mathbf{C}_2$, that sends every ordered pair (a_1,a_2) of an object a_1 of $\mathbf{C}_1$ and an object a_2 of $\mathbf{C}_2$ to the object a_1 of $\mathbf{C}_1$, resp. to the object a_2 of $\mathbf{C}_2$. For every $\mathbf{C}_1$-morphism f_1 from a_1 to b_1, the associated morphism from $\text{pr}_1(a_1,a_2)=a_1$ to $\text{pr}_1(b_1,a_2)=b_1$, resp. from $\text{pr}_2(a_1,a_2)=a_2$ to $\text{pr}_1(b_1,a_2)=a_2$, is f, resp. is Id_{a_2}. For every $\mathbf{C}_2$-morphism f_2 from a_s to b_2, the associated morphism from $\text{pr}_1(a_1,a_2)=a_1$ to $\text{pr}_1(a_1,b_2)=a_1$, resp. from $\text{pr}_2(a_1,a_2)=a_2$ to $\text{pr}_1(a_1,b_2)=b_2$, is Id_{a_1}, resp. is f_2.

Example K.16. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, the **associated bifunctor** $\mathbf{F}^{\text{opp}}\times\mathbf{F}$ of $\mathbf{F}$ from $\mathbf{C}^{\text{opp}}\times\mathbf{C}$ to $\mathbf{D}^{\text{opp}}\times\mathbf{D}$ is the unique functor such that both $\text{pr}_1\circ(\mathbf{F}^{\text{opp}}\times\mathbf{F})$ equals $\mathbf{F}^{\text{opp}}\circ\text{pr}_1$ and $\text{pr}_2\circ(\mathbf{F}^{\text{opp}}\times\mathbf{F})$ equals $\mathbf{F}\circ\text{pr}_2$. The **associated natural transformation of Hom bifunctors** $\mathbf{F}^{\bullet}_{\bullet}$ of $\mathbf{F}$ from $\text{Hom}_{\mathbf{C}}$ to $\text{Hom}_{\mathbf{D}}\circ(\mathbf{F}^{\text{opp}}\times\mathbf{F})$ maps every ordered pair (a,b) of objects of $\mathbf{C}$ to the set function $\mathbf{F}^a_b$ from $\text{Hom}_{\mathbf{C}}(a,b)$ to $\text{Hom}_{\mathbf{D}}(\mathbf{F}(a),\mathbf{F}(b))$.

Exercise K.17. Check that $\mathbf{F}^{\bullet}_{\bullet}$ is, indeed, a natural transformation of bifunctors. For every functor $\mathbf{G}$ from $\mathbf{D}$ to a category $\mathbf{E}$, check that $(\mathbf{G}^{\text{opp}}\times\mathbf{G})\circ(\mathbf{F}^{\text{opp}}\times\mathbf{F})$ equals $(\mathbf{G}\circ\mathbf{F})^{\text{opp}}\times(\mathbf{G}\circ\mathbf{F})$. Also check that the composition natural transformation $(\mathbf{F}^{\text{opp}}\times\mathbf{F})_*\mathbf{G}^{\bullet}_{\bullet}\circ\mathbf{F}^{\bullet}_{\bullet}$ equals $(\mathbf{G}\circ\mathbf{F})^{\bullet}_{\bullet}$.

Exercise K.18. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for functors $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ from $\mathbf{C}$ to $\mathbf{D}$, for every natural equivalence θ from $\mathbf{F}$ to $\widetilde{\mathbf{F}}$ with inverse natural equivalence $\widetilde{\theta}$, prove that $\widetilde{\theta}^{\text{opp}}\times\theta$ from $\mathbf{F}^{\text{opp}}\times\mathbf{F}$ to $\widetilde{\mathbf{F}}^{\text{opp}}\times\widetilde{\mathbf{F}}$ is a natural equivalence. Formulate and prove the compatibility of this natural transformation with the natural transformations $\mathbf{F}^{\bullet}_{\bullet}$ and $\widetilde{\mathbf{F}}^{\bullet}_{\bullet}$.

Definition K.19. For every category $\mathbf{C}$, a **product bifunctor** is an ordered triple $(-\times-,\text{pr}^{\mathbf{C}}_1,\text{pr}^{\mathbf{C}}_2)$ of a bifunctor $-\times-$ to $\mathbf{C}$ from $\mathbf{C}$ and $\mathbf{C}$, a natural transformation $\text{pr}^{\mathbf{C}}_1$ from the bifunctor $-\circ-$ to the bifunctor $\text{pr}^{\mathcal{C},\mathcal{C}}_{\mathbf{C},1}$, and a natural transformation $\text{pr}^{\mathbf{C}}_2$ from the bifunctor $-\circ-$ to the bifunctor $\text{pr}^{\mathcal{C},\mathcal{C}}_{\mathbf{C},2}$ such that, for every ordered pair (a_1,a_2) of objects of $\mathbf{C}$, the following ordered pair is a product of a_1 and a_2 in $\mathbf{C}$,

$$(\text{pr}^{\mathbf{C}}_{(a_1,a_2),1}:a_1\times a_2\to a_1,\text{pr}^{\mathbf{C}}_{(a_1,a_2),2}:a_1\times a_2\to a_2).$$

Exercise K.20. For every category $\mathbf{C}$, if a product bifunctor exists, prove that it is unique up to unique natural equivalence.

Exercise K.21. Let $\mathbf{C}$ be a small category such that for every ordered pair (a_1,a_2) of objects of $\mathbf{C}$, there exists a product. Using the Axiom of Choice, prove that there is a product bifunctor. Up to some much stronger Axiom of Choice, every category that admits finite products has a product bifunctor.

Example K.22. For every product bifunctor on a category $\mathbf{C}$, a product bifunctor on the arrow category $\text{Arr}(\mathbf{C})$ maps every ordered pair $(((s_1,t_1),f_1),((s_2,t_2),f_2))$ to $((s_1\times s_2,t_1\times t_2),f_1\times f_2)$, maps every morphism (σ_1, τ_1) from $((s_1, t_1), f_1)$ to $((s_1', t_1'), f_1')$ to the morphism $(\sigma_1 \times \mathrm{Id}_{s_2}, \tau_1 \times \mathrm{Id}_{t_2})$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_1' \times s_2, t_1' \times t_2), f_1' \times f_2)$, and maps every morphism (σ_2, τ_2) from $((s_2, t_2), f_2)$ to $((s_2', t_2'), f_2')$ to the morphism $(\mathrm{Id}_{s_1} \times \sigma_2, \mathrm{Id}_{t_1} \times \tau_2)$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_1 \times s_2', t_1 \times t_2'), f_1 \times f_2')$. The projection natural transformation $\mathrm{pr}_1^{\mathrm{Arr}(\mathbf{C})}$ maps every ordered pair $(((s_1, t_1), f_1), ((s_2, t_2), f_2))$ to the projection morphism $(\mathrm{pr}^{\mathbf{C}}_{(s_1,s_2),1}, \mathrm{pr}^{\mathbf{C}}_{(t_1,t_2),1})$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_1, t_1), f_1)$. The projection natural transformation $\mathrm{pr}_2^{\mathrm{Arr}(\mathbf{C})}$ maps every ordered pair $(((s_1, t_1), f_1), ((s_2, t_2), f_2))$ to the projection morphism $(\mathrm{pr}^{\mathbf{C}}_{(s_1,s_2),2}, \mathrm{pr}^{\mathbf{C}}_{(t_1,t_2),2})$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_2, t_2), f_2)$.

Example K.23. For the category **Set**, the bifunctor $- \times -$ associates to every ordered pair (a_1, a_2) of sets the Cartesian product set $a_1 \times a_2$, associates to every function f_1 from a set a_1 to a set a_1' the function $f_1 \times \mathrm{Id}_{a_2}$ from $a_1 \times a_2$ to $a_1' \times a_2$, and associates to every function f_2 from a_2 to a_2' the function $\mathrm{Id}_{a_1} \times f_2$ from $a_1 \times a_2$ to $a_1 \times a_2'$. The natural transformation $\mathrm{pr}_1^{\mathbf{Set}}$ associates to every ordered pair (a_1, a_2) of sets the first projection function $\mathrm{pr}^{\mathbf{Set}}_{(a_1,a_2),1}$ from $a_1 \times a_2$ to a_1. The natural transformation $\mathrm{pr}_2^{\mathbf{Set}}$ associates to every ordered pair (a_1, a_2) of sets the second projection function $\mathrm{pr}^{\mathbf{Set}}_{(a_1,a_2),1}$ from $a_1 \times a_2$ to a_2.

Exercise K.24. Check that this defines a product bifunctor on the category **Set**.

Exercise K.25. Each of the categories **Monoid**, **Grp**, **Ab**, **Ring**, **CRing**, $R-$**Mod**, **Mod**$-S$ and $R-S-$**Mod**, and **Top** has all finite products, and the (standard) concrete forgetful functor from each to **Set** preserves all finite products. Use this "lift" to each of these categories the product bifunctor for **Set**, thus proving that each of these categories has a product bifunctor.

Exercise K.26. Formulate and prove analogues of each of the general theorems about for a *co-product functor* (e.g., by applying the theorems above to the opposite category). However, the standard concrete forgetful functors in the previous exercise do not preserve all coproducts, except for the concrete functor on **Top**. Adjoint pairs give coproducts in the other cases.

Exercise K.27. For every monoid $(H, \bullet)$, there is a bifunctor $\sqcup$ to the category $H-$**Act** of left H-acts from the category $H-$**Act** and $H-$**Act** that sends every ordered pair $((S, \rho), (S', \rho'))$ of left H-acts to the left H-act $\rho \sqcup \rho'$ on the disjoint union set $S \sqcup S'$. Deduce that $H-$**Act** has all finite coproducts, and the concrete forgetful functor to **Set** preserves all finite coproducts. If $(H, \bullet)$ is a group, prove that the left H-actions that are indecomposable with respect to $\sqcup$ are precisely the left regular action of H on the right coset space H/K of a subgroup K of H.

Exercise K.28. For every monoid $(H, \bullet)$, use the same technique as earlier to construct a product bifunctor $\times$ to $H-$**Act** from the category $H-$**Act** and $H-$**Act** sending every ordered pair $((S, \rho), (S', \rho'))$ of left H-acts to the left H-act $\rho \times \rho'$ on the Cartesian product set $S \times S'$. Deduce that the category $H-$**Act** has all finite products, and the concrete forgetful functor to **Set** preserves finite products. In particular, if $(H, \bullet)$ is a group, then for $\sqcup$-indecomposible left H-actions H/K and H/K' for subgroups K and K' of H, the $\sqcup$-components of $(H/K) \times (H/K')$ are of the form H/K'' for K'' a subgroup of the form $(hKh^{-1}) \cap (h'K'(h')^{-1})$. Thus, the $\sqcup$-components are all isomorphic (so that $(H/K) \times (H/K')$ is "isotypic") if at least two of K, K' and $K \cap K'$ are normal.

Exercise K.29. For associative, unital rings $(R, +_R, \cdot_R)$, $(S, +_S, \cdot_S)$ and $(T, +_T, \cdot_T)$, there is a bifunctor $\otimes_S$ to the category $R - T - \mathbf{Mod}$ of $R - T$-bimodules from the category $R - S - \mathbf{Mod}$ and $S - T - \mathbf{Mod}$ that sends every ordered pair $((M, +, (\rho, \sigma)), (M', +', (\sigma', \tau'))$ of an $R - S$-bimodule and a $S - T$-bimodule to the associated tensor product $R - T$-bimodule $M \otimes_S M'$, where the set function from $M \times M'$ to the Abelian group $M \otimes_S M'$ is initial among all biadditive maps from $M \times M'$ to an Abelian group that are S-**balanced**: for every element (m, m') of $M \times M'$ and for every element s of S, both $(m \cdot s, m')$ and $(m, s \cdot m')$ have the same image. Formulate and prove existence of associator isomorphisms $(M \otimes_S M') \otimes_T M'' \cong M \otimes_S (M' \otimes_T M'')$ for every T-module M''. Formulate and prove existence of left / right unitor isomorphisms of $S \otimes_S M' \cong M'$ and $M \otimes_S S \cong M$. Formulate and prove the triangle (coherence) identity and the pentagon (coherence) identity for the unitors and associators.

Example K.30. For every Abelian monoid $(H, \bullet)$, there is a bifunctor $\text{sum}_{H,\bullet}$ to $B(H, \bullet)$ from $B(H, \bullet)$ and $B(H, \bullet)$ that maps the unique object $(*, *)$ to the unique object $*$, and, for every element h of H, maps both $(h, *)$ and $(*, h)$ to h. The bifunctor axiom is precisely the Abelian hypothesis on the monoid.

L Comma categories

M Adjoint pairs

Definition M.1. For every category $\mathbf{C}$, for every category $\mathbf{D}$, an **adjoint pair** of covariant functors between $\mathbf{C}$ and $\mathbf{D}$ is $((\mathbf{L}, \mathbf{R}), (\theta, \eta))$ consisting of an ordered pair of covariant functors,

$$\mathbf{L} : \mathbf{C} \to \mathbf{D},$$

$$\mathbf{R} : \mathbf{D} \to \mathbf{C},$$

and an ordered pair of natural transformations of covariant functors,

$$\theta : \text{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L}, \ \theta(a) : a \to \mathbf{R}(\mathbf{L}(a)),$$

$$\eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \text{Id}_{\mathbf{D}}, \ \eta(b) : \mathbf{L}(\mathbf{R}(b)) \to b,$$

such that the following composition of natural transformations equals $\text{Id}_{\mathbf{R}}$, respectively equals $\text{Id}_{\mathbf{L}}$,

$$(*_{\mathbf{R}}) : \mathbf{R} \overset{\theta \circ \mathbf{R}}{\Rightarrow} \mathbf{R} \circ \mathbf{L} \circ \mathbf{R} \overset{\mathbf{R} \circ \eta}{\Rightarrow} \mathbf{R},$$

$$(*_{\mathbf{L}}) : \mathbf{L} \overset{\mathbf{L} \circ \theta}{\Rightarrow} \mathbf{L} \circ \mathbf{R} \circ \mathbf{L} \overset{\eta \circ \mathbf{L}}{\Rightarrow} \mathbf{L}.$$

For every object a of $\mathbf{C}$ and for every object b of $\mathbf{D}$, define set maps,

$$H^{\mathbf{L}}_{\mathbf{R}}(a, b) : \text{Hom}_{\mathbf{D}}(\mathbf{L}(a), b) \to \text{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)),$$

$$(\mathbf{L}(a) \xrightarrow{\phi} b) \mapsto \left(a \xrightarrow{\theta(a)} \mathbf{R}(\mathbf{L}(a)) \xrightarrow{\mathbf{R}(\phi)} \mathbf{R}(b)\right),$$

and

$$H_{\mathbf{L}}^{\mathbf{R}}(a,b) : \mathrm{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)) \to \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(a), b),$$
$$(a \xrightarrow{\psi} \mathbf{R}(b)) \mapsto \left(\mathbf{L}(a) \xrightarrow{\mathbf{L}(\psi)} \mathbf{L}(\mathbf{R}(b)) \xrightarrow{\eta(b)} b\right).$$

Exercise M.2. For $\mathbf{L}$, $\mathbf{R}$, θ and η as above, prove that the conditions $(*_{\mathbf{R}})$ and $(*_{\mathbf{L}})$ hold if and only if, for every object a of $\mathbf{C}$ and for every object b of $\mathbf{D}$, the morphisms $H_{\mathbf{R}}^{\mathbf{L}}(a,b)$ and $H_{\mathbf{L}}^{\mathbf{R}}(a,b)$ are inverse bijections.

Exercise M.3. Prove that both $H_{\mathbf{R}}^{\mathbf{L}}(a,b)$ and $H_{\mathbf{L}}^{\mathbf{R}}(a,b)$ are binatural in a and b.

Exercise M.4. For functors $\mathbf{L}$ and $\mathbf{R}$, and for binatural inverse bijections $H_{\mathbf{R}}^{\mathbf{L}}(a,b)$ and $H_{\mathbf{L}}^{\mathbf{R}}(a,b)$ between the bifunctors

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(a), b), \mathrm{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)) : \mathbf{C} \times \mathbf{D} \to \mathbf{Set},$$

prove that there exist unique θ and η extending $\mathbf{L}$ and $\mathbf{R}$ to an adjoint pair such that $H_{\mathbf{R}}^{\mathbf{L}}$ and $H_{\mathbf{L}}^{\mathbf{R}}$ agree with the binatural inverse bijections defined above.

Exercise M.5. Let $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ be an adjoint pair as above. For every covariant functor,

$$\widetilde{\mathbf{R}} : \mathbf{D} \to \mathbf{C},$$

for every natural transformation η from $\mathbf{L} \circ \widetilde{\mathbf{R}}$ to $\mathrm{Id}_{\mathbf{D}}$, prove that $\widetilde{\eta}' := R_* \widetilde{\eta} \circ \widetilde{\mathbf{R}}^* \theta$ is the unique natural transformation from $\widetilde{\mathbf{R}}$ to $\mathbf{R}$ such that $\widetilde{\eta}$ equals $\eta \circ L_* \widetilde{\eta}'$. Conversely, for every natural transformation $\widetilde{\eta}'$ from $\widetilde{\mathbf{R}}$ to $\mathbf{R}$, prove that $\widetilde{\eta} := \eta \circ L_* \widetilde{\eta}'$ is the unique natural transformation from $\mathbf{L} \circ \widetilde{\mathbf{R}}$ such that $\widetilde{\eta}'$ equals $R_* \widetilde{\eta} \circ \widetilde{\mathbf{R}}^* \theta$. Formulate and prove the analogous correspondence between natural transformations $\widetilde{\theta}$ from $\mathrm{Id}_{\mathbf{C}}$ to $\mathbf{R} \circ \widetilde{\mathbf{L}}$ and natural transformations $\widetilde{\theta}'$ from $\mathbf{L}$ to a functor $\widetilde{\mathbf{L}}$.

Exercise M.6. Let $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ be an adjoint pair as above. Let a covariant functor

$$\widetilde{\mathbf{R}} : \mathbf{D} \to \mathbf{C},$$

and natural transformations,

$$\widetilde{\theta} : \mathrm{Id}_{\mathbf{C}} \Rightarrow \widetilde{\mathbf{R}} \circ \mathbf{L}, \widetilde{\eta} : \mathbf{L} \circ \widetilde{\mathbf{R}} \Rightarrow \mathrm{Id}_{\mathbf{D}},$$

be natural transformations such that $(\mathbf{L}, \widetilde{\mathbf{R}}, \widetilde{\theta}, \widetilde{\eta})$ is also an adjoint pair. For every object b of $\mathbf{D}$, define $\iota(b)$ in $\mathrm{Hom}_{\mathbf{D}}(\mathbf{R}(b), \widetilde{\mathbf{R}}(b))$ to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathbf{D}}(b,b) \xrightarrow{\mathrm{Hom}_{\mathbf{D}}(\theta(b),b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(\mathbf{R}(b)), b) \xrightarrow{H_{\mathbf{L}}^{\widetilde{\mathbf{R}}}(\mathbf{R}(b),b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{R}(b), \widetilde{\mathbf{R}}(b)).$$

Similarly, define $\kappa(b)$ in $\text{Hom}_{\mathbf{D}}(\widetilde{\mathbf{R}}(b), \mathbf{R}(b))$, to be the image of Id_b under the composition,

$$\text{Hom}_{\mathbf{D}}(b,b) \xrightarrow{\text{Hom}_{\mathbf{D}}(\widetilde{\theta}(b),b)} \text{Hom}_{\mathbf{D}}(\mathbf{L}(\widetilde{\mathbf{R}}(b)),b) \xrightarrow{H^{\mathbf{R}}_{\mathbf{L}}(\widetilde{\mathbf{R}}(b),b)} \text{Hom}_{\mathbf{D}}(\widetilde{\mathbf{R}}(b), \mathbf{R}(b)).$$

Prove that ι and κ are the unique natural transformations of functors,

$$\iota : \mathbf{R} \Rightarrow \widetilde{\mathbf{R}}, \ \kappa : \widetilde{\mathbf{R}} \Rightarrow \mathbf{R},$$

such that $\widetilde{\theta}$ equals $(\iota \circ \mathbf{L}) \circ \theta$, θ equals $(\kappa \circ \mathbf{L}) \circ \widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta \circ (\mathbf{L} \circ \iota)$, and η equals $\widetilde{\eta} \circ (\mathbf{L} \circ \kappa)$. Moreover, prove that ι and κ are inverse natural equivalences. In this sense, every extension of a functor $\mathbf{L}$ to an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ is unique up to unique natural isomorphisms (ι, κ). Formulate and prove the symmetric statement for all extensions of a functor $\mathbf{R}$ to an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ (you could use opposite categories to simplify this).

Exercise M.7. For every adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$, prove that also $(\mathbf{R}^{\text{opp}}, \mathbf{L}^{\text{opp}}, \eta^{\text{opp}}, \theta^{\text{opp}})$ is an adjoint pair.

Exercise M.8. Formulate the corresponding notions of adjoint pairs when $\mathbf{L}$ and $\mathbf{R}$ are contravariant functors (just replace one of the categories by its opposite category).

Exercise M.9. For every ordered triple of categories, $(\mathbf{C}, \mathbf{D}, \mathcal{E})$ for all covariant functors,

$$\mathbf{L}' : \mathbf{C} \to \mathbf{D}$$

$$\mathbf{R}' : \mathbf{D} \to \mathbf{C},$$

for all natural transformations that form an adjoint pair,

$$\theta' : \text{Id}_{\mathbf{C}} \Rightarrow \mathbf{R}'\mathbf{L}',$$

$$\eta' : \mathbf{L}'\mathbf{R}' \Rightarrow \text{Id}_{\mathbf{D}},$$

for all covariant functors,

$$\mathbf{L}'' : \mathbf{D} \to \mathcal{E},$$

$$\mathbf{R}'' : \mathcal{E} \to \mathbf{D},$$

and for all natural transformations that form an adjoint pair,

$$\theta'' : \text{Id}_{\mathbf{D}} \Rightarrow \mathbf{R}''\mathbf{L}'',$$

$$\eta'' : \mathbf{L}''\mathbf{R}'' \Rightarrow \text{Id}_{\mathcal{E}},$$

define covariant functors

$$\mathbf{L} : \mathbf{C} \to \mathcal{E}, \ \mathbf{R} : \mathcal{E} \to \mathbf{C}$$

by $\mathbf{L} = \mathbf{L}'' \circ \mathbf{L}'$, $\mathbf{R} = \mathbf{R}' \circ \mathbf{R}''$, define the natural transformation,

$$\theta : \text{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L},$$

to be the composition of natural transformations,

$$\mathrm{Id}_{\mathbf{C}} \overset{\theta'}{\Rightarrow} \mathbf{R}' \circ \mathbf{L}' \overset{\mathbf{R}'\circ\theta''\circ\mathbf{L}'}{\Rightarrow} \mathbf{R}' \circ \mathbf{R}'' \circ \mathbf{L}'' \circ \mathbf{L}',$$

and define the natural transformation,

$$\eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathcal{E}},$$

to be the composition of natural transformations,

$$\mathbf{L}'' \circ \mathbf{L}' \circ \mathbf{R}' \circ \mathbf{R}'' \overset{\mathbf{L}''\circ\eta'\circ\mathbf{R}''}{\Rightarrow} \mathbf{L}'' \circ \mathbf{R}'' \overset{\eta''}{\Rightarrow} \mathrm{Id}_{\mathcal{E}}.$$

Prove that $\mathbf{L}$, $\mathbf{R}$, θ and η form an adjoint pair of functors. This is the **composition** of $(\mathbf{L}', \mathbf{R}', \theta', \eta')$ and $(\mathbf{L}'', \mathbf{R}'', \theta'', \eta'')$.

Exercise M.10. If $\mathbf{C}$ equals $\mathbf{D}$, if $\mathbf{L}'$ and $\mathbf{R}'$ are the identity functors, and if θ' and η' are the identity natural transformations, prove that $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ equals $(\mathbf{L}'', \mathbf{R}'', \theta'', \eta'')$. Similarly, if $\mathbf{D}$ equals $\mathcal{E}$, if $\mathbf{L}''$ and $\mathbf{R}''$ are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ equals $(\mathbf{L}', \mathbf{R}', \theta', \eta')$. Finally, prove that composition of three adjoint pairs is associative.

Example M.11. Let $\mathbf{C}$ be a category that has a final object f, and let $\mathbf{D}$ be a category that has an initial object e. Let $\mathbf{L}$ be $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},e}$, and let $\mathbf{R}$ be $\mathrm{const}^{\mathbf{D}}_{\mathbf{C},f}$. Thus, $\mathbf{R} \circ \mathbf{L}$ equals $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$, and $\mathbf{L} \circ \mathbf{R}$ equals $\mathrm{const}^{\mathbf{D}}_{\mathbf{D},e}$. Since f is a final object of $\mathbf{C}$, there is a unique natural transformation from every endofunctor of $\mathbf{C}$ to $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$. In particular, there exists a unique natural transformation θ from the identity functor to $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$. Since e is an initial object of $\mathbf{C}$, there is a unique natural transformation from $\mathrm{const}^{\mathbf{D}}_{\mathbf{D},e}$ to every endofunctor of $\mathbf{D}$. In particular, there exists a unique natural transformation η from $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$ to the identity functor. Together, these define an adjoint pair giving binatural bijections for every object a of $\mathbf{C}$ and every object b of $\mathbf{D}$,

$$\mathrm{Hom}_{\mathbf{D}}(\mathrm{const}^{\mathbf{C}}_{\mathbf{D},e}(a), b) \cong \mathrm{Hom}_{\mathbf{C}}(a, \mathrm{const}^{\mathbf{D}}_{\mathbf{C},f}(b)).$$

Example M.12. Let $(S, \leq)$ and $(S', \leq')$ be partially ordered sets. Let $\mathbf{L}$ be a nondecreasing function from $(S, \leq)$ to $(S', \leq')$ considered as a functor between the associated categories. Let $\mathbf{R}$ be a nondecreasing function from $(S', \leq')$ to $(S, \leq)$ considered as a functor between the associated categories. There exist natural transformations completing this to an adjoint pair if and only if, for every element a of S, for every element a' of S', we have $L(a) \leq' a'$ if and only $a \leq R(a')$. In this case, the natural transformations extending to an adjoint pair are unique.

Definition M.13. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every adjoint pair

$$(\mathbf{L} : \mathbf{C} \to \mathbf{D}, \mathbf{R} : \mathbf{D} \to \mathbf{C}, \theta : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L}, \eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathbf{D}}),$$

the adjoint pair is a **strict equivalence** from $\mathbf{C}$ to $\mathbf{D}$ if (and only if) both θ is a natural equivalence and η is a natural equivalence.

Exercise M.14. Prove that identity adjoint pairs are strict equivalences. Prove that the composition adjoint pair of strict equivalences is a strict equivalence. For every strict equivalence from **C** to **D** as above, prove that also $(\mathbf{R}, \mathbf{L}, \eta^{-1}, \theta^{-1})$ is a strict equivalence from **D** to **C** that is a left-right inverse of the original strict equivalence.

Exercise M.15. Prove that each of the functors in a strict equivalence is a weak equivalence. Prove that every composition of weak equivalences is a weak equivalence.

Exercise M.16. Let **C** and **D** be strictly small categories. Prove that for every weak equivalence L from **C** to **D** there exists a strict equivalence (L, R, θ, η) from **C** to **D**, and this strict equivalence is unique up to isomorphism (which is not necessarily unique). Thus, using a strong variant of the Axiom of Choice, every weak equivalence should arise (non-uniquely) from a strict equivalence.

N More about categories

The category of presheaves on a topological space (containing the category of sheaves as a full subcategory) is an example of a functor category. Functor categories also give the cleanest formulation of the Yoneda lemma and of limits / colimits.

N.1 Functor categories

Functors give a formalism for working with labelled collections of objects in some fixed category **D**, where the labels or indices are themselves objects of some small category **C** (such as a partially ordered set). The indexed collections then form objects of a new category, called a functor category.

Definition N.1. For every small category **C**, for every category **D**, the class $\text{obj}(\mathbf{D}^{\mathbf{C}})$, sometimes also denoted $\text{obj}([\mathbf{C}, \mathbf{D}])$ or $\text{obj}(\text{Fun}(\mathbf{C}, \mathbf{D}))$, is the unique class whose members are precisely the sets whose classes give functors from **C** to **D**.

For every ordered pair $(\mathbf{F}, \mathbf{G})$ of functors from **C** to **D**, again because **C** is small, every natural transformation from **F** to **G** is the class of a set, and the class of all sets whose classes are natural transformations from **F** to **G** is itself a set.

Definition N.2. For every small category **C**, for every category **D**, the class $\text{mor}(\mathbf{D}^{\mathbf{C}})$, sometimes also denoted $\text{mor}([\mathbf{C}, \mathbf{D}])$ or $\text{mor}(\text{Fun}(\mathbf{C}, \mathbf{D})$, is the span from $\text{obj}(\mathbf{D}^{\mathbf{C}})$ whose fiber class over each ordered pair $(\mathbf{F}, \mathbf{G})$ of sets whose classes are functors from **C** to **D** is the class whose members are precisely the sets whose classes give natural transformation from **F** to **G**.

Together this defines a category.

Definition N.3. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **functor category** $\mathbf{D}^{\mathbf{C}}$ from $\mathbf{C}$ to $\mathbf{D}$, also denoted $\text{Fun}(\mathbf{C}, \mathbf{D})$ or $[\mathbf{C}, \mathbf{D}]$, is the category with objects class $\text{obj}(\mathbf{D}^{\mathbf{C}})$ and with morphisms class $\text{mor}(\mathbf{D}^{\mathbf{C}})$. So the objects of the class are equivalent to functors from $\mathbf{C}$ to $\mathbf{D}$, and the morphisms of the class are equivalent to natural transformations. The composition law of this category is composition of natural transformations.

Please note, the way we formalize (parametrically definable) classes there is a distinction between sets and the associated classes. Thus the objects of the functor category are sets whose classes are functors from $\mathbf{C}$ to $\mathbf{D}$, and the morphisms are sets whose classes are natural transformations between such functors. Nonetheless, we shall treat this category as if the objects are functors and as if the morphisms are natural transformations.

Definition N.4. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, the **H-composition functor** $\mathbf{H}^{\mathbf{C}}$ from $\mathbf{D}^{\mathbf{C}}$ to $(\mathbf{D}')^{\mathbf{C}}$ maps every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$ to the composite functor $\mathbf{H} \circ \mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}'$, and maps every natural transformation θ from a functor $\mathbf{F}$ to a functor $\mathbf{G}$ to the $\mathbf{H}$-pushforward natural transformation $\mathbf{H}_*\theta$.

Exercise N.5. Prove that the $\mathbf{H}$-composition functor is a functor. Prove that the $\text{Id}_{\mathbf{D}}$-composition functor is the identity functor from $\mathbf{D}^{\mathbf{C}}$ to itself. Prove that for every ordered pair $(\mathbf{I}, \mathbf{H})$ of a functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$ and a functor $\mathbf{I}$ from $\mathbf{D}'$ to $\mathbf{D}''$, the $\mathbf{I} \circ \mathbf{H}$-composition functor $(\mathbf{I} \circ \mathbf{H})^{\mathbf{C}}$ equals the composition of functors $\mathbf{I}^{\mathbf{C}} \circ \mathbf{H}^{\mathbf{C}}$.

Definition N.6. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every functor $\mathbf{J}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every category $\mathbf{D}$, the **J-precomposition functor** $\mathbf{D}^{\mathbf{J}}$ from $\mathbf{D}^{\mathbf{C}'}$ to $\mathbf{D}^{\mathbf{C}}$ maps every functor $\mathbf{F}'$ from $\mathbf{C}'$ to $\mathbf{D}$ to the composite functor $\mathbf{F}' \circ \mathbf{J}$ from $\mathbf{C}$ to $\mathbf{D}$, and maps every natural transformation θ' from a functor $\mathbf{F}'$ to a functor $\mathbf{G}'$ to the $\mathbf{J}$-pullback natural transformation $\mathbf{J}_*\theta'$.

Exercise N.7. Prove that the $\mathbf{J}$-precomposition functor is a functor. Prove that the $\text{Id}_{\mathbf{C}}$-precomposition functor is the identity functor from $\mathbf{D}^{\mathbf{C}}$ to itself. Prove that for every ordered pair $(\mathbf{K}, \mathbf{J})$ of a functor $\mathbf{J}$ from $\mathbf{C}$ to $\mathbf{C}'$ and a functor $\mathbf{K}$ from $\mathbf{C}'$ to $\mathbf{C}''$, the $\mathbf{K} \circ \mathbf{I}$-precomposition functor $\mathbf{D}^{\mathbf{K} \circ \mathbf{J}}$ equals the composition of functors $\mathbf{D}^{\mathbf{J}} \circ \mathbf{D}^{\mathbf{K}}$.

Exercise N.8. Prove that for every functor $\mathbf{J}$ from a small category $\mathbf{C}$ to a small category $\mathbf{C}'$ and for every functor $\mathbf{H}$ from a category $\mathbf{D}$ to a category $\mathbf{D}'$, the composite functor $(\mathbf{D}')^{\mathbf{J}} \circ \mathbf{H}^{\mathbf{C}'}$ equals the composite functor $\mathbf{H}^{\mathbf{C}} \circ \mathbf{D}^{\mathbf{J}}$.

Definition N.9. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, the **θ-composition natural transformation** $\theta^{\mathbf{C}}$ from the functor $\mathbf{H}_1^{\mathbf{C}}$ to the functor $\mathbf{H}_2^{\mathbf{C}}$ maps every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$ to the $\mathbf{F}$-pullback natural transformation $\mathbf{F}^*\theta$ from $\mathbf{H}_1 \circ \mathbf{F}$ to $\mathbf{H}_2 \circ \mathbf{F}$.

Exercise N.10. Prove that $\theta^{\mathbf{C}}$ is a natural transformation. For every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, for the identity natural transformation $\text{Id}_{\mathbf{H}}$ from $\mathbf{H}$ to itself, prove that $(\text{Id}_{\mathbf{H}})^{\mathbf{C}}$ is the identity natural transformation from $\mathbf{H}^{\mathbf{C}}$ to itself. For every ordered pair (θ_2, θ_1) of a natural transformation θ_1 of functors from $\mathbf{H}_1$ to $\mathbf{H}_2$ and of a natural transformation θ_2 of functors from $\mathbf{H}_2$ to $\mathbf{H}_3$, prove that $(\theta_2 \circ \theta_1)^{\mathbf{C}}$ equals the composite natural transformation $\theta_2^{\mathbf{C}} \circ \theta_1^{\mathbf{C}}$.

Exercise N.11. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every category $\mathbf{D}''$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, for every ordered pair $(\mathbf{H}_1', \mathbf{H}_2')$ of functors from $\mathbf{D}'$ to $\mathbf{D}''$, for every natural transformation θ' from $\mathbf{H}_1'$ to $\mathbf{H}_2'$, prove that for the Godement product $\theta' * \theta$ natural transformation from $\mathbf{H}_1' \circ \mathbf{H}_1$ to $\mathbf{H}_2' \circ \mathbf{H}_2$, also $(\theta' * \theta)^{\mathbf{C}}$ equals the Godement product $(\theta')^{\mathbf{C}} * \theta^{\mathbf{C}}$. Deduce special cases of compatibility of $(-)^{\mathbf{C}}$ with pushforward and pullback by functors of natural transformations.

Definition N.12. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every ordered pair $(\mathbf{I}_1, \mathbf{I}_2)$ of functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation η from $\mathbf{I}_1$ to $\mathbf{I}_2$, for every category $\mathbf{D}$, the **η-precomposition natural transformation $\mathbf{D}^{\eta}$** from the functor $\mathbf{D}^{\mathbf{I}_1}$ to the functor $\mathbf{D}^{\mathbf{I}_2}$ maps every functor $\mathbf{F}'$ from $\mathbf{C}'$ to $\mathbf{D}$ to the $\mathbf{F}'$-pushforward natural transformation $(\mathbf{F}')_*\eta$ from $\mathbf{F}' \circ \mathbf{I}_1$ to $\mathbf{F}' \circ \mathbf{I}_2$.

Exercise N.13. Prove that $\mathbf{D}^{\eta}$ is a natural transformation. For every functor $\mathbf{I}$ from $\mathbf{C}$ to $\mathbf{C}'$, for the identity natural transformation $\mathrm{Id}_{\mathbf{I}}$ from $\mathbf{I}$ to itself, prove that $\mathbf{D}^{\mathrm{Id}_{\mathbf{I}}}$ is the identity natural transformation from $\mathbf{D}^{\mathbf{I}}$ to itself. For every ordered pair (η_2, η_1) of a natural transformation η_1 of functors from $\mathbf{I}_1$ to $\mathbf{I}_2$ and of a natural transformation η_2 of functors from $\mathbf{I}_2$ to $\mathbf{I}_3$, prove that $\mathbf{D}^{\eta_2 \circ \eta_1}$ equals the composite natural transformation $\mathbf{D}^{\eta_2} \circ \mathbf{D}^{\eta_1}$. Also prove that $\mathbf{D}^{\bullet}$ is compatible with Godement products.

Exercise N.14. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every small category $\mathbf{C}''$, for every ordered pair $(\mathbf{I}_1, \mathbf{I}_2)$ of functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation η from $\mathbf{I}_1$ to $\mathbf{I}_2$, for every ordered pair $(\mathbf{I}_1', \mathbf{I}_2')$ of functors from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation η' from $\mathbf{I}_1'$ to $\mathbf{I}_2'$, for every category $\mathbf{D}$, prove that for the Godement product $\eta' * \eta$ natural transformation from $\mathbf{I}_1' \circ \mathbf{I}_1$ to $\mathbf{I}_2' \circ \mathbf{I}_2$, also $\mathbf{D}^{\eta' * \eta}$ equals the Godement product $\mathbf{D}^{\eta} * \mathbf{D}^{\eta'}$ (in the opposite order). Deduce special cases of compatibility of $\mathbf{D}^{\bullet}$ with pushforward and pullback by functors of natural transformations.

Exercise N.15. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every ordered pair $(\mathbf{I}_1, \mathbf{I}_2)$ of functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation η from $\mathbf{I}_1$ to $\mathbf{I}_2$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, prove that the Godement product $\theta^{\mathbf{C}} * \mathbf{D}^{\eta}$ equals the Godement product $(\mathbf{D}')^{\eta} * \theta^{\mathbf{C}'}$. Deduce special cases for pushforward and pullback by functors of natural transformations.

N.2 Constant functors

Definition N.16. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **constant functor** $\mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}}$ from $\mathbf{D}$ to $\mathbf{D}^{\mathbf{C}}$ maps every object a of $\mathbf{D}$ to the object $\mathrm{const}_{\mathbf{D},a}^{\mathbf{C}}$ of $\mathbf{D}^{\mathbf{C}}$, and maps every $\mathbf{D}$-morphism f from a to b to the natural transformation $\mathrm{const}_{\mathbf{D},f}^{\mathbf{C}}$ from $\mathrm{const}_{\mathbf{D},a}^{\mathbf{C}}$ to $\mathrm{const}_{\mathbf{D},b}^{\mathbf{C}}$.

Exercise N.17. Prove that this is a functor.

Exercise N.18. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, prove that the composite functor $\mathbf{H}^{\mathbf{C}} \circ \mathrm{const}^{\mathbf{C}}_{\mathbf{D},\bullet}$ equals the composite functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}',\bullet} \circ \mathbf{H}$ as functors from $\mathbf{D}$ to $(\mathbf{D}')^{\mathbf{C}}$.

Exercise N.19. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every functor $\mathbf{I}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every category $\mathbf{D}$, prove that the composite functor $\mathbf{D}^{\mathbf{I}} \circ \mathrm{const}^{\mathbf{C}'}_{\mathbf{D},\bullet}$ equals $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},\bullet}$.

Exercise N.20. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every functor $\mathbf{I}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, use the compatibilities above to deduce the compatibilities between the functors $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},\bullet}$, $\mathrm{const}^{\mathbf{C}'}_{\mathbf{D},\bullet}$, $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}',\bullet}$, $\mathrm{const}^{\mathbf{C}'}_{\mathbf{D}',\bullet}$, $\mathbf{H}^{\mathbf{C}}$, $\mathbf{H}^{\mathbf{C}'}$, $\mathbf{D}^{\mathbf{I}}$ and $(\mathbf{D}')^{\mathbf{I}}$, e.g., the composite functor $\mathbf{H}^{\mathbf{C}} \circ \mathbf{D}^{\mathbf{I}} \circ \mathrm{const}^{\mathbf{C}'}_{\mathbf{D},\bullet}$ equals the composite functor $(\mathbf{D}')^{\mathbf{I}} \circ \mathrm{const}^{\mathbf{C}'}_{\mathbf{D}',\bullet} \circ \mathbf{H}$ as functors from $\mathbf{D}$ to $(\mathbf{D}')^{\mathbf{C}}$.

Exercise N.21. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, prove that the pullback natural transformation $(\mathrm{const}^{\mathbf{C}}_{\mathbf{D},\bullet})^* \theta^{\mathbf{C}}$ equals the pushforward natural transformation $(\mathrm{const}^{\mathbf{C}}_{\mathbf{D}',\bullet})_* \theta$ as natural transformation between functors from $\mathbf{D}$ to $(\mathbf{D}')^{\mathbf{C}}$.

N.3 Category of small categories

Definition N.22. The class of **small categories** is the class $\mathrm{obj}(\mathbf{Cat})$ whose members are sets whose associated class is a small category. The class of **functors of small categories** is the span $\mathrm{mor}(\mathbf{Cat})$ from $\mathrm{obj}(\mathbf{Cat})$ to itself whose fiber class over each pair $(\mathbf{C}, \mathbf{D})$ has for members those sets whose associated class is a functor from $\mathbf{C}$ to $\mathbf{D}$. Composition of functors defines a composition law that completes these classes to a category $\mathbf{Cat}$, the **category of small categories**.

Technically we distinguish each set from its associated class, and thus the objects of $\mathbf{Cat}$ are sets whose associated class is a small category, rather than the small category itself (since we do not allow classes to be members of other classes). Similarly, the morphisms of $\mathbf{Cat}$ are sets whose associated class is a functor between small categories, rather than the functor itself.

The standard usage is different: most authors identify each set with the associated class (this is built in to the axioms of von Neumann – Bernays – Gödel class theory). At any rate, even though it is technically incorrect, we will refer to small categories as objects of $\mathbf{Cat}$, and we will refer to functors between small categories as morphisms of $\mathbf{Cat}$.

Exercise N.23. Read about (strict) 2-categories. Formulate and prove the assertion that the natural transformations between functors make $\mathbf{Cat}$ into a 2-category.

Definition N.24. The **opposite functor** from $\mathbf{Cat}$ to $\mathbf{Cat}$ is the functor that maps every small category $\mathbf{C}$ to its opposite category $\mathbf{C}^{\mathrm{opp}}$, that maps every functor $\mathbf{F}$ from a small category $\mathbf{C}$ to a small category $\mathbf{D}$ to the functor $\mathbf{F}^{\mathrm{opp}}$ from $\mathbf{C}^{\mathrm{opp}}$ to $\mathbf{D}^{\mathrm{opp}}$. The 2-**cell dual** of the 2-category $\mathbf{Cat}$ is the 2-category $\mathbf{Cat}^{\mathrm{co}}$ with the same objects as $\mathbf{Cat}$, respectively the same and 1-morphisms as $\mathbf{Cat}$, namely small categories, resp. functors between small categories, yet with opposite 2-morphism sets. The **opposite 2-functor** is the strict 2-functor from $\mathbf{Cat}^{\mathrm{co}}$ to $\mathbf{Cat}$ extending the opposite functor by mapping every natural transformation θ from a functor $\mathbf{F}$ to a functor $\mathbf{G}$ (both from a small category $\mathbf{C}$ to a small category $\mathbf{D}$) to the natural transformation θ^{opp} from $\mathbf{G}^{\mathrm{opp}}$ to $\mathbf{F}^{\mathrm{opp}}$.

Exercise N.25. Prove that this is a strict 2-functor from $\mathbf{Cat}^{\mathrm{co}}$ to $\mathbf{Cat}$.

N.4 Evaluation bifunctor

Definition N.26. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **evaluation bifunctor** $\mathrm{Hom}_{\mathbf{D}}^{\mathbf{C}}$, or just Hom when confusion is unlikely, is the bifunctor to $\mathbf{D}$ from $\mathbf{D}^{\mathbf{C}}$ and $\mathbf{C}$ that maps every ordered pair $(\mathbf{F}, a)$ of an object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ and an object a of $\mathbf{C}$ to the object $\mathbf{F}_{\mathrm{obj}}(a)$ of $\mathbf{D}$, that maps every ordered pair (θ, a) of a $\mathbf{D}^{\mathbf{C}}$-morphism θ from $\mathbf{F}$ to $\mathbf{G}$ and of an object a of $\mathbf{C}$ to the $\mathbf{D}$-morphism θ_a from $\mathbf{F}_{\mathrm{obj}}(a)$ to $\mathbf{G}_{\mathrm{obj}}(a)$, and that maps every ordered pair $(\mathbf{F}, u)$ of an object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ and of a $\mathbf{C}$-morphism u from a to b to the $\mathbf{D}$-morphism $\mathbf{F}_{\mathrm{mor}}(u)$ from $\mathbf{F}_{\mathrm{obj}}(a)$ to $\mathbf{F}_{\mathrm{obj}}(b)$.

Exercise N.27. Prove that this is a bifunctor.

Definition N.28. For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every bifunctor $\mathbf{F}$ to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, the **classifying functor** $S_{\mathbf{D}}^{\mathbf{B},\mathbf{C}}\mathbf{F}$, or just $S\mathbf{F}$ when confusion is unlikely, from $\mathbf{B}$ to $\mathbf{D}^{\mathbf{C}}$ maps every object b of $\mathbf{B}$ to the functor $\mathbf{F}(b, \bullet)$ from $\mathbf{C}$ to $\mathbf{D}$ and maps every $\mathbf{B}$-morphism u from b to b' to the natural transformation $\mathbf{F}(u, \bullet)$ from $\mathbf{F}(b, \bullet)$ to $\mathbf{F}(b', \bullet)$.

Exercise N.29. Prove that $S\mathbf{F}$ is a functor.

Proposition N.30. *For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every bifunctor $\mathbf{F}$ to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, the functor $S\mathbf{F}$ from $\mathbf{B}$ to $\mathbf{D}^{\mathbf{C}}$ is the unique functor such that the pullback of the bifunctor Hom by the functor $S\mathbf{F} \times Id_{\mathbf{C}}$ equals F.*

Exercise N.31. Formulate and prove functoriality of the construction $S_{\mathbf{D}}^{\mathbf{B},\mathbf{C}}$ in $\mathbf{B}$, in $\mathbf{C}$, and in $\mathbf{D}$.

Definition N.32. For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of bifunctors to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **classifying natural transformation** $S_{\mathbf{D}}^{\mathbf{B},\mathbf{C}}\theta$, or just $S\theta$ when confusion is unlikely, from $S\mathbf{F}$ to $S\mathbf{G}$ maps every object b of $\mathbf{B}$ to the natural transformation $\theta_{b,\bullet}$ from $\mathbf{F}(b, \bullet)$ to $\mathbf{G}(b, \bullet)$.

Exercise N.33. Prove that $S\theta$ is a natural transformation.

Proposition N.34. *For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of bifunctors to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, for every natural transformation from $S\mathbf{F}$ to $S\mathbf{G}$ there exists a unique natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$ such that the natural transformation equals $S\theta$.*

Exercise N.35. Use this universal property (or any other argument) to formulate and prove compatibility of the operations S with Godement products of natural transformations. Specialize this to formulate and prove compatibility of S with pushforwards and pullbacks by functors of natural transformations.

N.5 Limits and colimits

Definition N.36. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every object U of $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, the **U-section object** of $\mathbf{F}$ over U is the object $\Gamma_{\mathbf{D}}^{\mathbf{C}}(U, \mathbf{F}) := \mathbf{F}(U)$ of $\mathbf{D}$, denoted also $\Gamma(U, \mathbf{F})$ when confusion is unlikely. For every ordered pair $(\mathbf{F}, \mathbf{G})$ of functors from $\mathbf{C}$ to $\mathbf{D}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **U-section morphism** of $\mathbf{F}$ over U is the $\mathbf{D}$-morphism $\Gamma(U, \theta) := \theta_U$ from $\mathbf{F}(U)$ to $\mathbf{G}(U)$.

Exercise N.37. Prove that these rules preserve identities and composition.

Definition N.38. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every object U of $\mathbf{C}$, the **U-sections functor** from $\mathbf{D}^{\mathbf{C}}$ to $\mathbf{D}$ maps every object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ to $\Gamma(U, \mathbf{F}) := \mathbf{F}(U)$ and maps every $\mathbf{D}^{\mathbf{C}}$-morphism θ from $\mathbf{F}$ to $\mathbf{G}$ to the $\mathbf{D}$-morphism $\Gamma(U, \theta) := \theta_U$.

For every ordered pair (V, U) of objects of $\mathbf{C}$, for every $\mathbf{C}$-morphism r from V to U, the **r-sections natural transformation** from $\Gamma(V, \bullet)$ to $\Gamma(U, \bullet)$ maps every object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ to the $\mathbf{D}$-morphism $\Gamma(r, \mathbf{F}) := \mathbf{F}_U^V(r)$ from $\mathbf{F}(V)$ to $\mathbf{F}(U)$.

Exercise N.39. Prove that $\Gamma(r, \bullet)$ is a natural transformation. For every object U of $\mathbf{C}$, prove that $\Gamma(\mathrm{Id}_U, \bullet)$ is the identity natural transformation from $\Gamma(U, \bullet)$ to itself. For every triple (W, V, U) of objects of $\mathbf{C}$, for every $\mathbf{C}$-morphism r from W to V, for every $\mathbf{C}$-morphism s from V to U, prove that $\Gamma(s \circ r, \bullet)$ equals the composition of natural transformations $\Gamma(s, \bullet) \circ \Gamma(r, \bullet)$.

Exercise N.40. For every small category $\mathbf{C}$, for every small category $\mathbf{D}$, prove that $\mathbf{D}^{\mathbf{C}}$ is a small category.

Definition N.41. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **sections bifunctor** is the functor $\Gamma_{\mathbf{D}}^{\mathbf{C}}(-, \bullet)$, or just $\Gamma(-, \bullet)$ when confusion is unlikely, from the product category $\mathbf{C} \times \mathbf{D}^{\mathbf{C}}$ to the category $\mathbf{D}$ that sends every object $(U, \mathbf{F})$ of $\mathbf{C} \times \mathbf{D}^{\mathbf{C}}$ to $\Gamma(U, \mathbf{F}) := \mathbf{F}(U)$, that sends every $\mathbf{C}$-morphism r from a to b to the $\mathbf{D}$-morphism $\Gamma(r, \mathbf{F})$, and that sends every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$ to the $\mathbf{D}$-morphism $\Gamma(U, \theta) := \theta_U$.

Exercise N.42. Prove that the sections bifunctor is a bifunctor.

Exercise N.43. Formulate and prove the statement that formation of $\mathbf{D}^{\mathbf{C}}$ is covariant in the category $\mathbf{D}$ and is contravariant in the small category $\mathbf{C}$. In particular, for every small category $\mathbf{C}$, prove that the covariant Yoneda functor of $\mathbf{C}$ in $\mathbf{Cat}$ enriches to a functor from $\mathbf{Cat}$ to itself. Similarly, for every small category $\mathbf{D}$, prove that the contravariant Yoneda functor of $\mathbf{D}$ in $\mathbf{Cat}$ enriches to a functor from $\mathbf{Cat}^{\mathrm{opp}}$ to $\mathbf{Cat}$.

N.6 Yoneda embedding

Definition N.44. For every category $\mathbf{C}$, for every object a of $\mathbf{C}$, the set-valued **covariant Yoneda functor** of a from $\mathbf{C}$ maps every $\mathbf{C}$-object b to the set $\mathbf{C}_b^a = \mathrm{Hom}_{\mathbf{C}}(a, b)$. This is also denoted $h_{\mathbf{C}}^a(b)$, or just $h^a(b)$ when confusion is unlikely. Also, for every $\mathbf{C}$-morphism v from b to b', the functor maps u to left-composition with v from $\mathbf{C}^a_b$ to $\mathbf{C}^a_{b'}$. This is denoted $h^a(v)$. Similarly, for every set S, the set-valued functor $S \times h^a$ maps every $\mathbf{C}$-object b to $S \times h^a(b)$ and maps every $\mathbf{C}$-morphism v to $\mathrm{Id}^{\mathbf{Set}}_S \times h^a(v)$ from $S \times h^a(b)$ to $S \times h^a(b')$.

Similarly, for every object b of $\mathbf{C}$, the set-valued **contravariant Yoneda functor** of b is the covariant functor from $\mathbf{C}^{\mathrm{opp}}$ that maps every $\mathbf{C}$-object a to the set $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a,b)$. This is also denoted $h_{\mathbf{C},b}(a)$, or just $h_b(a)$ when confusion is unlikely. Also, for every $\mathbf{C}$-morphism u from a to a', the functor map u to the right-composition with u from $\mathbf{C}^{a'}_b$ to $\mathbf{C}^a_b$ (note this is contravariant). This is denoted $h_b(u)$. Similarly, for every set S, the set-valued functor $S \times h_b$ maps every $\mathbf{C}$-object a to $S \times h_b(a)$ and maps every $\mathbf{C}$-morphism u to $\mathrm{Id}^{\mathbf{Set}}_S \times h_b(u)$ from $S \times h_b(a')$ to $S \times h_b(a)$.

Exercise N.45. Check that each of these does preserve identities and composition, so that it is a functor.

Example N.46. Let $(S, \leq)$ be a partially ordered set. For every element a of S, for every element b of S, the Yoneda functor $h^a(b)$ is a singleton set if and only if $a \leq b$, and otherwise it is empty, i.e., the image in **Set** is either an initial object or a final object. If we define the **support** of such a function to be the subset of S where the image is not the empty set, then the support of h^a is the subset $S_{\geq a}$ of all elements b with $a \leq b$. Similarly, the support of h_b is the subset $S_{\leq b}$ of all elements of b with $a \leq b$.

Example N.47. For every monoid $(H, \bullet)$, for the unique object (which, recall, is chosen to be H itself considered as a set), the Yoneda functor h^H associates to the unique object (i.e., H) the set H, and associates to each element a of H, considered as a morphism from the unique object to itself, the associated bijection of H of left-multiplication by a, i.e., h^H is the left regular representation of $(H, \bullet)$. Similarly h_H is the right regular representation of $(H, \bullet)$.

Definition N.48. For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism u from a to a', the **Yoneda natural transformation of covariant functors** from $h^{a'}$ to h^a associates to every object b the set function of right-composition with u from $h^{a'}(b) = \mathbf{C}^{a'}_b$ to $h^a(b) = \mathbf{C}^a_b$. This natural transformation is denoted by h^u. Similarly, for every set S, the natural transformation $\mathrm{Id}^{\mathbf{Set}}_S \times h^u$ maps every set $S \times h^{a'}(b)$ to $S \times h^a(b)$ by $\mathrm{Id}^{\mathbf{Set}}_S \times h^u(b)$.

For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism v from b to b', the **Yoneda natural transformation of contravariant functors** from h_b to $h_{b'}$ associates to every object a of the set function of left-composition with v from $h_b(a)$ to $h_{b'}(a)$. This natural transformation is denoted by h_v. Similarly, for every set S, the natural transformation $\mathrm{Id}^{\mathbf{Set}}_S \times h_v$ maps every set $S \times h_b(a)$ to $S \times h_{b'}(a)$ by $\mathrm{Id}^{\mathbf{Set}}_S \times h_v(a)$.

Exercise N.49. Check that each of these is a natural transformation of set-valued functors from $\mathbf{C}$.

Exercise N.50. For every $\mathbf{C}$-morphism u from a to a', for every $\mathbf{C}$-morphism u' from a' to a'', check that $h^u \circ h^{u'}$ equals $h^{u' \circ u}$; thus, also, $(\mathrm{Id}^{\mathbf{Set}}_S \times h^u) \circ (\mathrm{Id}^{\mathbf{Set}}_s \times h^{u'})$ equals $\mathrm{Id}^{\mathbf{Set}}_S \times h^{u' \circ u}$. Conclude *contravariance* of the assignment to every $\mathbf{C}$-object a of the covariant Yoneda functor h^a and to every $\mathbf{C}$-object u of the Yoneda natural transformation h^u.

Exercise N.51. For every $\mathbf{C}$-morphism v from b to b', for every $\mathbf{C}$-morphism v' from b' to b'', check that $h_{v'} \circ h_v$ equals $h_{v' \circ v}$; thus, also, $(\mathrm{Id}_S^{\mathbf{Set}} \times h_{v'}) \circ (\mathrm{Id}_S^{\mathbf{Set}} \times h_v)$ equals $\mathrm{Id}_S^{\mathbf{Set}} \times h_{v' \circ v}$. Conclude *covariance* of the assignment to every $\mathbf{C}$-object a of the contravariant Yoneda functor h_a and to every $\mathbf{C}$-object v of the Yoneda natural transformation h_v.

Exercise N.52. For every set-valued functor $\mathbf{F}$ from $\mathbf{C}$, respectively from $\mathbf{C}^{\mathrm{opp}}$, for every set S, for the set-valued functor $S \times \mathbf{F}$ from $\mathbf{C}$, resp. from $\mathbf{C}^{\mathrm{opp}}$, check covariance in S.

Definition N.53. For every category $\mathbf{B}$, for every set-valued covariant functor $\mathbf{F}$ from $\mathbf{B}^{\mathrm{opp}}$, for every $\mathbf{C}$-object b, for every element γ of the set $\mathbf{F}(b)$, the **Yoneda evaluation natural transformation** from h_b to $\mathbf{F}$ associates to every $\mathbf{C}$-object a the set-function from $h_b(a) = \mathrm{Hom}_{\mathbf{C}}(a,b)$ to $\mathbf{F}(a)$ sending each element w of $\mathrm{Hom}_{\mathbf{C}}(a,b)$ to the image of γ under the set function $\mathbf{F}(w)$ from $\mathbf{F}(b)$ to $\mathbf{F}(a)$. This natural transformation is denoted by $\eta_b^{\gamma,\bullet}(\mathbf{F})$, so that w maps to $\eta_b^{\gamma,\bullet}(\mathbf{F})(w)$. Similarly, $\eta_b(\mathbf{F})$ is the natural transformation from $\mathbf{F}(b) \times h_b$ to $\mathbf{F}$ that associates to every $\mathbf{C}$-object a the set-function from $\mathbf{F}(b) \times h_b(a)$ to $\mathbf{F}(a)$ sending every element (γ, w) to $\eta_b^{\gamma,\bullet}(\mathbf{F})(w)$.

For every category $\mathbf{B}$, for every set-valued covariant functor $\mathbf{F}$ from $\mathbf{B}$, for every $\mathbf{C}$-object a, for every element δ of the set $\mathbf{F}(a)$, the **Yoneda evaluation natural transformation** from h^a to $\mathbf{F}$ associates to every $\mathbf{C}$-object b the set-function from $h^a(b) = \mathrm{Hom}_{\mathbf{C}}(a,b)$ to $\mathbf{F}(b)$ sending each element w of $\mathrm{Hom}_{\mathbf{C}}(a,b)$ to the image of δ under the set function $\mathbf{F}(w)$ from $\mathbf{F}(a)$ to $\mathbf{F}(b)$. This natural transformation is denoted by $\eta_{\delta,\bullet}^{b}(\mathbf{F})$, so that w maps to $\eta_{\gamma,\bullet}^{a}(\mathbf{F})(w)$. Similarly, $\eta^a(\mathbf{F})$ is the natural transformation from $\mathbf{F}(a) \times h^a$ to $\mathbf{F}$ that associates to every $\mathbf{C}$-object b the set-function from $\mathbf{F}(a) \times h^a(b)$ to $\mathbf{F}(b)$ sending every element (δ, w) to $\eta_{\gamma,\bullet}^{a}(\mathbf{F})(w)$.

Exercise N.54. Check that $\eta_b(\mathbf{F})$ and $\eta^a(\mathbf{F})$ are natural transformations.

Exercise N.55. For every natural transformation α from $\mathbf{F}$ to $\mathbf{G}$ of set-valued covariant functors from $\mathbf{C}$, check that $\alpha \circ \eta_b(\mathbf{F})$ equals the composition of $\eta_b(\mathbf{G})$ with the natural transformation of functors $\alpha(b) \times \mathrm{Id}_{h_b}$ from $\mathbf{F}(b) \times h_b$ to $\mathbf{G}(b) \times h_b$ induced by the set function $\alpha(b)$ from $\mathbf{F}(b)$ to $\mathbf{G}(b)$. Thus, $\eta_b(\mathbf{F})$ is "covariant" in $\mathbf{F}$.

Lemma N.56 (Yoneda Lemma)**.** *For every category $\mathbf{C}$, for every covariant set-valued functor $\mathbf{F}$ from $\mathbf{C}^{opp}$, for every $\mathbf{C}$-object b, every natural transformation Γ from h_b to $\mathbf{F}$ is of the form $\eta_b^{\gamma,\bullet}(\mathbf{F})$ for a unique element γ of $\mathbf{F}(b)$, namely the image under Γ of the element $Id_b^{\mathbf{C}}$ of $h_b(b) = Hom_{\mathbf{C}}(b,b)$.*

Exercise N.57. Formulate and prove the analogous result for covariant set-valued functors from $\mathbf{C}$ and the Yoneda functors h^a.

Definition N.58. For every set S, the **identity section** is the set function from S to $S \times h_b(b) = S \times \mathrm{Hom}_{\mathbf{C}}(b,b)$ that pairs each element of S with $\mathrm{Id}_b^{\mathbf{C}}$.

Exercise N.59. Check that the identity section is covariant in S.

Definition N.60. For every small category $\mathbf{C}$, for every $\mathbf{C}$-object b, the set-valued **left Yoneda functor** $\mathbf{L}_b$ from the functor category $\mathbf{Set}^{(\mathbf{C}^{\mathrm{opp}})}$ associates to every set-valued covariant functor

$\mathbf{F}$ from $\mathbf{C}^{\text{opp}}$ the set $\mathbf{F}(b)$ and associates to every natural transformation α from $\mathbf{F}$ to $\mathbf{G}$ the set function $\alpha(b)$ from $\mathbf{F}(b)$ to $\mathbf{G}(b)$.

Similarly, the **right Yoneda functor** $\mathbf{R}_b$ from **Set** to $\mathbf{Set}^{\mathbf{C}^{\text{opp}}}$ associates to every set S the covariant set-valued functor $S \times h_b$ from $\mathbf{C}^{\text{opp}}$, and associates to every set function f from S to S' the natural transformation $f \times \text{Id}_{h_b}$ from $S \times h_b$ to $S' \times h_b$.

Exercise N.61. Check that each of these is a functor. Check that the identity section is a natural transformation from the identity functor of **Set** to the composite functor $\mathbf{R}_b \circ \mathbf{L}_b$.

Lemma N.62 (Yoneda Lemma II)**.** *For every small category* $\mathbf{C}$*, for every* $\mathbf{C}$*-object* b*, the left Yoneda functor and the right Yoneda functor extend to an adjoint pair of functors using the natural transformation* η_b *above and the identity section natural transformation.*

Exercise N.63. For every small category $\mathbf{C}$, conclude that the Yoneda functor from $\mathbf{C}$ to $\mathbf{Set}^{(\mathbf{C}^{\text{opp}}}$ sending every $\mathbf{C}$-object b to h_b is a fully faithful embedding of categories.

In the sense of the previous lemma, the Yoneda functors give examples of adjoint pairs. Conversely, extension of a functor to an adjoint pair is an example of a *representability problem*.

Definition N.64. For every category $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}^{\text{opp}}$ to **Set**, a **representation** of $\mathbf{F}$ is an ordered pair (a, x) of an object a of $\mathbf{C}$ and an element x of the set $\mathbf{F}(a)$ such that the induced natural transformation $h_a \Rightarrow \mathbf{F}$ is a natural equivalence. A functor from $\mathbf{C}^{\text{opp}}$ is **representable** if (and only if) there exists a representation.

Exercise N.65. Formulate the opposite notion of representable for functors from $\mathbf{C}$ to **Set**.

Exercise N.66. For every category $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}^{\text{opp}}$ to **Set**, for every representation (a, x) of $\mathbf{F}$, for every representation (a', x') of $\mathbf{F}$, prove that there exists a unique $\mathbf{C}$-isomorphism f from a to a' that pulls x' back to x. Conclude that a representation of a representable functor is unique up to unique isomorphism. Formulate and prove the opposite result for covariant functors from $\mathbf{C}$ to **Set**.

Exercise N.67. For every category $\mathbf{C}$, for every small category $\mathbf{D}$, for every covariant functor $\mathbf{L}$ from $\mathbf{C}$ to $\mathbf{D}$ such that the set-valued functor $\text{Hom}_{\mathbf{D}}(\mathbf{L}(\bullet), b)$ on $\mathbf{C}^{\text{opp}}$ is representable for every object b of $\mathbf{D}$, prove that there exists an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ (which is unique up to unique natural equivalences by an earlier exercise). Thus, show that extension of a functor to an adjoint pair is a special case of representability of functors.

Exercise N.68. Prove the variant of the previous result for opposite categories: for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every covariant functor $\mathbf{R}$ from $\mathbf{D}$ to $\mathbf{C}$, if the set-valued functor $\text{Hom}_{\mathbf{C}}(a, \mathbf{R}(\bullet))$ on $\mathbf{D}$ is representable for every object $\mathbf{a}$ of $\mathbf{C}$, prove there exists an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$.

N.7 Functor categories

Notation N.69. For every small category τ, for every category $\mathcal{C}$, for every object a of $\mathcal{C}$, denote by

$$\underline{a}_\tau : \tau \to \mathcal{C}$$

the constant functor $\mathrm{const}^{\tau}_{\mathcal{C},a}$ that sends every object to a and that sends every morphism to Id_a. For every morphism in $\mathcal{C}$, $p : a \to b$, denote by

$$\underline{p}_\tau : \underline{a}_\tau \Rightarrow \underline{b}_\tau$$

the natural transformation that assigns to every object U of τ the morphism $p : a \to b$. Finally, for every object U of τ, denote

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U), \quad \Gamma(U, \theta) = \theta(U),$$

and for every morphism $r : U \to V$ of τ, denote

$$\Gamma(r, \mathcal{F}) = \mathcal{F}(r).$$

Functor Categories and Section Functors.

Recall that associated to the small category τ and the category $\mathcal{C}$ there is the functor category $\mathcal{C}^\tau$ $\mathrm{Fun}(\tau, \mathcal{C})$ whose objects are functors and whose morphisms are natural transformations

Exercise N.70. For every small category τ, for every category $\mathcal{C}$, prove that the functor $\mathrm{const}^{\tau}_{\mathcal{C},\bullet}$ from $\mathcal{C}$ to $\mathcal{C}^\tau$ preserves isomorphisms.

Adjointness of Constant / Diagonal Functors and the Global Sections Functor.

Exercise N.71. For every small category τ, for every category $\mathcal{C}$, if $\mathcal{C}$ has an initial object X, prove that $(\underline{\star}_\tau, \Gamma(X, -))$ extends to an adjoint pair of functors.

N.8 Limits and colimits

Definition N.72. For every small category τ, for every category $\mathcal{C}$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, a **limit** of the τ-family $\mathcal{F}$ is a natural transformation $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ that is final among all such natural transformations, i.e., for every natural transformation $\theta : \underline{b}_\tau \Rightarrow \mathcal{F}$, there exists a unique morphism $t : b \to a$ in $\mathcal{C}$ such that θ equals $\eta \circ \underline{t}_\tau$.

Exercise N.73. For every small category τ, for every category $\mathcal{C}$, for all τ-families $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}$, for every morphism ϕ of τ-families from $\mathcal{F}$ to $\mathcal{G}$, for all limits $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ and $\theta : \underline{b}_\tau \Rightarrow \mathcal{G}$, prove that there exists a unique morphism $f : a \to b$ such that $\theta \circ \underline{p}_\tau$ equals $\phi \circ \eta$. In particular, prove that if a limit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. Thus, for every object a of $\mathcal{C}$, the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a limit of $\underline{a}_\tau$.

Adjointness of Constant / Diagonal Functors and Limits.

Definition N.74. A category $\mathcal{C}$ is **complete** if (and only if), for every small category τ, every τ-family has a limit (which is then unique up to unique isomorphism by the previous exercise).

For every complete category $\mathcal{C}$, some version of the Axiom of Choice (e.g., Hilbert's epsilon operator) produces a rule Γ_τ that assigns to every τ-family $\mathcal{F}$ an object $\Gamma_\tau(\mathcal{F})$ and a natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \to \mathcal{F}$ that is a limit. (In many concrete categories, there is an explicit "construction" of such a rule.)

Exercise N.75. For every small category τ, for every complete category $\mathcal{C}$, and for every rule Γ_τ as above, prove that there is an extension to a functor,

$$\Gamma_\tau : \mathrm{Fun}(\tau, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation of functors

$$\eta : \underline{\star}_\tau \circ \Gamma_\tau \Rightarrow \mathrm{Id}_{\mathrm{Fun}(\tau,\mathcal{C})}.$$

Moreover, the rule sending every object a of $\mathcal{C}$ to the identity natural transformation θ_a is a natural transformation $\theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Gamma_\tau \circ \underline{\star}_\tau$. The quadruple $(\underline{\star}_\tau, \Gamma, \theta, \eta)$ is an adjoint pair of functors. In particular, the limit functor Γ_τ preserves monomorphisms and sends injective objects of $\mathrm{Fun}(\tau, \mathcal{C})$ to injective objects of $\mathcal{C}$.

Adjointness of Colimits and Constant / Diagonal Functors.

Exercise N.76. For every small category τ, for every category $\mathcal{C}$, if $\mathcal{C}$ has a final object O, prove that $(\Gamma(O, -), \underline{\star}_\tau)$ extends to an adjoint pair of functors.

Definition N.77. For every small category τ, for every category $\mathcal{C}$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, a **colimit** of the τ-family $\mathcal{F}$ is a natural transformation $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ that is final among all such natural transformations, i.e., for every natural transformation $\eta : \mathcal{F} \Rightarrow \underline{b}_\tau$, there exists a unique morphism $h : a \to b$ in $\mathcal{C}$ such that $\underline{h}_\tau \circ \theta$ equals η.

Exercise N.78. For every small category τ, for every category $\mathcal{C}$, for all τ-families $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}$, for every morphism ϕ of τ-families from $\mathcal{F}$ to $\mathcal{G}$, for all colimits $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ and $\eta : \mathcal{G} \Rightarrow \underline{b}_\tau$, prove that there exists a unique morphism $f : a \to b$ such that $\underline{f}_\tau \circ \theta$ equals $\eta \circ \phi$. In particular, prove that if a colimit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. Thus, for every object a of $\mathcal{C}$, the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a colimit of $\underline{a}_\tau$. Finally, repeat the previous results with colimits in place of limits. Deduce that colimits (if they exist) preserve epimorphisms and projective objects. (You can use opposite categories to reduce most of this to the case of limits.)

Functoriality in the Source.

Definition N.79. For every complete category $\mathcal{C}$, for every functor x from a small category σ to a small category τ, for every τ-family $\mathcal{F}$, the x-**pullback** $\mathcal{F}_x$ of $\mathcal{F}$ is the composite functor $\mathcal{F} \circ x$, which is a σ-family. For every morphism of τ-families, say ϕ from $\mathcal{F}$ to $\mathcal{G}$, the x-**pullback** ϕ_x from the σ-family $\mathcal{F}_x$ to $\mathcal{G}_x$ is $\phi \circ x$, which is a morphism of σ-families.

Exercise N.80. For every complete category $\mathcal{C}$, for every functor x from a small category σ to a small category τ, prove that x-pullback defines a functor

$$\star_x : \text{Fun}(\tau, \mathcal{C}) \to \text{Fun}(\sigma, \mathcal{C}).$$

For the identity functor $\text{Id}_\tau : \tau \to \tau$, prove that Id_τ-pullback is the identity functor from $\text{Fun}(\tau, \mathcal{C})$ to itself. For every functor y from a small category ρ to σ, prove that $x \circ y$-pullback equals the composite $\star_y \circ \star_x$. In this sense, deduce that pullback is contravariant in x.

Definition N.81. For every complete category $\mathcal{C}$, for every small category σ, for every small category τ, for all functors x and x' from σ to τ, and for every natural transformation n from x to x', the **associated morphism** of σ-families is the natural transformation $\mathcal{F}_n$ from $\mathcal{F}_x$ to $\mathcal{F}_{x'}$ that sends every σ-object V to the morphism $\mathcal{F}(n(V))$ from $\mathcal{F}(x(V))$ to $\mathcal{F}(x'(V))$.

Exercise N.82. Prove that $\mathcal{F}_n$ is a morphism of σ-families. Also, for every morphism of τ-families, ϕ from $\mathcal{F}$ to $\mathcal{G}$, prove that $\phi_{x'} \circ \mathcal{F}_n$ equals $\mathcal{G}_n \circ \phi_x$. Thus, the operation $\star_n$ is a natural transformation from the functor $\star_x$ to $\star_{x'}$. For the identity natural transformation Id_x from x to itself, also $\star_{\text{Id}_x}$ is the identity natural transformation of $\star_x$. Finally, for every functor x'' from σ to τ, and for every natural transformation m from x' to x'', the morphism of σ-families $\mathcal{F}_{m \circ n}$ equals $\mathcal{F}_m \circ \mathcal{F}_n$. In this sense, the operation $\star_x$ is also compatible with natural transformations. In particular, if (x, y, θ, η) is an adjoint pair of functors, then also $(\star_y, \star_x, \star_\theta, \star_\eta)$ is an adjoint pair of functors.

Fiber Categories The following notion of *fiber category* is a special case of the notion of *2-fiber product* of functors of categories. Let $x : \sigma \to \tau$ be a functor; this is also called a *category over* τ. For every object U of τ, a *$\sigma_{x,U}$-object* is a pair $(V, r : x(V) \to U)$ of an object V of σ and a τ-isomorphism $r : x(V) \to U$. For two objects $\sigma_{x,U}$-objects (V, r) and (V', r') of $\sigma_{x,U}$, a *$\sigma_{x,U}$-morphism* from (V, r) to (V', r') is a morphism of σ, $s : V \to V'$, such that $r' \circ x(s)$ equals r. **Prove** that Id_V is a $\sigma_{x,U}$-morphism from (V, r) to itself; more generally, the $\sigma_{x,U}$-morphisms from (V, r) to (V, r) are precisely the σ-morphisms $s : V \to V$ such that $x(s)$ equals $\text{Id}_{x(V)}$. For every pair of $\sigma_{x,U}$-morphisms, $s : (V, r) \to (V', r')$ and $s' : (V', r') \to (V'', r'')$, **prove** that $s' \circ s$ is a $\sigma_{x,U}$-morphism from (V, r) to (V'', r''). Conclude that these rules form a category, denoted $\sigma_{x,U}$. **Prove** that the rule $(V, r) \mapsto V$ and $s \mapsto s$ defines a faithful functor,

$$\Phi_{x,U} : \sigma_{x,U} \to \sigma,$$

and $r : x(V) \to U$ defines a natural isomorphism $\theta_{x,U} : x \circ \Phi_{x,U} \Rightarrow \underline{U}_{\sigma_{x,U}}$. Finally, for every category σ', for every functor $\Phi' : \sigma' \to \sigma$, and for every natural isomorphism $\theta' : x \circ \Phi' \Rightarrow \underline{U}_{\sigma'}$, **prove** that there exists a unique functor $F : \sigma' \to \sigma_{x,U}$ such that Φ' equals $\Phi_{x,U} \circ F$ and θ' equals $\theta_{x,U} \circ F$. In this sense, $(\Phi_{x,U}, \theta_{x,U})$ is final among pairs (Φ', θ') as above.

For every pair of functors $x, x_1 : \sigma \to \tau$, and for every natural *isomorphism* $n : x \Rightarrow x_1$, for every $\sigma_{x_1,U}$-object $(V, r_1 : x_1(V) \to U)$, **prove** that $(V, r_1 \circ n_V : x(V) \to U)$ is an object of $\sigma_{x,U}$. For every morphism in $\sigma_{x_1,U}$, $s : (V, r_1) \to (V', r_1')$, **prove** that s is also a morphism $(V, r_1 \circ n_V) \to (V', r_1' \circ n_{V'})$. Conclude that these rules define a functor,

$$\sigma_{n,U} : \sigma_{x_1,U} \to \sigma_{x,U}.$$

Prove that this functor is a *strict equivalence* of categories: it is a bijection on Hom sets (as for all equivalences), but it is also a bijection on objects (rather than merely being essentially surjective). **Prove** that $\sigma_{n,U}$ is functorial in n, i.e., for a second natural isomorphism $m : x_1 \Rightarrow x_2$, prove that $\sigma_{m\circ n,U}$ equals $\sigma_{n,U} \circ \sigma_{m,U}$.

For every pair of functors, $x : \sigma \to \tau$ and $y : \rho \to \tau$, and for every functor $z : \sigma \to \rho$ such that x equals $y \circ z$ equals x, for every $\sigma_{x,U}$-object (V, r), **prove** that $(z(V), r)$ is a $\rho_{y,U}$-object. For every $\sigma_{x,U}$-morphism $s : (V, r) \to (V', r')$, **prove** that $z(s)$ is a $\rho_{y,U}$-morphism $(z(V), r) \to (z(V'), r')$. **Prove** that $z(\mathrm{Id}_V)$ equals $\mathrm{Id}_{z(V)}$, and **prove** that z preserves composition. Conclude that these rules define a functor,

$$z_U : \sigma_{x,U} \to \rho_{y,U}.$$

Prove that this is functorial in z: $(\mathrm{Id}_\sigma)_U$ equals $\mathrm{Id}_{\sigma_{x,U}}$, and for a third functor $w : \pi \to \tau$ and functor $z' : \rho \to \pi$ such that y equals $w \circ z'$, then $(z' \circ z)_U$ equals $z'_U \circ z_U$. For an object (W, r_W) of $\rho_{y,U}$, for each object $((V, r_V), q : Z(V) \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, define the *associated* object of $\sigma_{z,W}$ to be (V, q). For an object $((V', r_{V'}), q' : Z(V') \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, for every morphism $s : (V, r_V) \to (V', r_{V'})$ such that q equals $q' \circ z(s)$, define the *associated* morphism of $\sigma_{z,W}$ to be s. **Prove** that this defines a functor

$$\widetilde{z}_{U,(W,r_W)} : (\sigma_{x,U})_{z_U,(W,r_W)} \to \sigma_{z,W}.$$

Prove that this functor is a strict equivalence of categories. **Prove** that this equivalence is functorial in z. Finally, for two functors $z, z_1 : \sigma \to \rho$ such that x equals both $y \circ z$ and $y \circ z_1$, and for a natural transformation $m : z \Rightarrow z_1$, for every object $(V, r : x(V) \to U)$ of $\sigma_{x,U}$, **prove** that m_V is a morphism in $\rho_{y,U}$ from $(z(V), r)$ to $(z_1(V), r)$. Moreover, for every morphism in $\sigma_{x,U}$, $s : (V, r) \to (V', r')$, **prove** that $m_{V'} \circ z(s)$ equals $z_1(s) \circ m_V$. Conclude that this rule is a natural transformation $m_U : z_U \Rightarrow (z_1)_U$. **Prove** that this is functorial in m. If m is a natural isomorphism, **prove** that also m_U is a natural isomorphism, and the strict equivalence $(m_U)_{(W,r_W)}$ is compatible with the strict equivalence m_W. Finally, **prove** that $m \mapsto m_U$ is compatible with precomposition and postcomposition of m with functors of categories over τ.

(vii)(Colimits and Limits along an Essentially Surjective Functor) Let $x : \sigma \to \tau$ be a functor of small categories. **Prove** that every fiber category $\sigma_{x,U}$ is small. Next, assume that x is *essentially surjective*, i.e., for every object U of τ, there exists a $\sigma_{x,U}$-object (V, r). Let $y : \tau \to \sigma$ be a functor, and let $\alpha : \mathrm{Id}_\sigma \Rightarrow y \circ x$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\alpha_V) : x(V) \to x(y(x(V)))$ is an isomorphism and $(y(x(V)), x(\alpha_V)^{-1})$ is a final object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and α extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has a final object.) For every adjoint pair (x, y, α, β), also $(\star_y, \star_x, \star_\alpha, \star_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and α, yet let L_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $L_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\theta_{\mathcal{F}} : \mathcal{F} \to \star_x \circ L_x(\mathcal{F}),$$

of objects in $\mathbf{Fun}(\sigma,\mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\theta_{\mathcal{F},x,U} : \mathcal{F}\circ\Phi_{x,U} \Rightarrow L_x(\mathcal{F})\circ \underline{U}_{\sigma_{x,U}},$$

of objects in $\mathbf{Fun}(\sigma_{x,U},\mathcal{C})$. Assume that each $(L_x(\mathcal{F})(U),\theta_{\mathcal{F},x,U})$ is a colimit of $\mathcal{F}\circ\Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$L_x : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}),$$

and a natural transformation

$$\theta_x : \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})} \Rightarrow \star_x \circ L_x.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau,\mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G}\circ x\circ\Phi_{x,U} \to \mathcal{G}\circ \underline{U}_{\sigma_{x,U}},$$

factors uniquely through a $\mathcal{C}$-morphism $L_x(\mathcal{G}\circ x)(U)\to\mathcal{G}(U)$. **Prove** that this defines a morphism $\eta_{\mathcal{G}} : L_x(\mathcal{G}\circ x)\to\mathcal{G}$ in $\mathbf{Fun}(\tau,\mathcal{C})$. **Prove** that is a natural transformation,

$$\eta : L_x\circ\star_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

Prove that $(L_x,\star_x,\theta,\eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F}\circ\Phi_{x,U}$ admits a colimit, then there exists a Γ^x and θ as above.)

Next, as above, let $x:\sigma\to\tau$ be a functor of small catgories that is essentially surjective. Let $y:\tau\to sigma$ be a functor, and let $\beta : y\circ x\Rightarrow \mathrm{Id}_\sigma$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x,y,α,β) if and only if for every object V of σ, the morphism $x(\beta_v) : x(y(x(V)))\to x(V)$ is an isomorphism and $(y(x(V)),x(\beta_v))$ is an initial object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and β extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has an initial object.) For every adjoint pair $(y,x,\alpha,beta)$ also $(\star_x,\star_y,\star_\alpha,\star_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and β, yet let R_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma,\mathcal{C})$ an object $R_x(\mathcal{F})$ of $\mathbf{Fun}(\tau,\mathcal{C})$ and a natural transformation,

$$\eta_{\mathcal{F}} : \star_x\circ R_x(\mathcal{F})\to\mathcal{F},$$

of objects in $\mathbf{Fun}(\sigma,\mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\eta_{\mathcal{F},x,U} : R_x(\mathcal{F})\circ\underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{F}\circ\Phi_{x,U},$$

of objects in $\mathbf{Fun}(\sigma_{x,U},\mathcal{C})$. Assume that each $(R_x(\mathcal{F})(U),\eta_{\mathcal{F},x,U})$ is a limit of $\mathcal{F}\circ\Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$R_x : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}),$$

and a natural transformation,

$$\eta : \star_x\circ R_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})}.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau,\mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G} \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{G} \circ x \circ \Phi_{x,U},$$

factors uniquely through a $\mathcal{G}(U) \to \mathcal{C}$-morphism $R_x(\mathcal{G} \circ x)(U)$. **Prove** that this defines a morphism $\theta_{\mathcal{G}} : \mathcal{G} \to R_x(\mathcal{G} \circ x)$ in $\mathbf{Fun}(\tau,\mathcal{C})$. **Prove** that this is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow R_x \circ \star_x.$$

Prove that $(\star_x, R_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a R_x and η as above.)

(viii)(Adjoints Relative to a Full, Upper Subcategory) In a complementary direction to the previous case, let $x : \sigma \to \tau$ be an embedding of a full subcategory (thus, x is essentially surjective if and only if x is an equivalence of categories). In this case, the functor

$$\star_x : \mathbf{Fun}(\tau,\mathcal{C}) \to \mathbf{Fun}(\sigma,\mathcal{C})$$

is called *restriction*. Assume further that σ is *upper* (a la the theory of partially ordered sets) in the sense that every morphism of τ whose source is an object of σ also has target an object of σ. Assume that $\mathcal{C}$ has an initial object, $\odot$. Let $\mathcal{G}$ be a σ-family of objects of $\mathcal{C}$. Also, let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of σ-families. For every object U of τ, if U is an object of σ, then define ${}_x\mathcal{G}(U)$ to be $\mathcal{G}(U)$, and define ${}_x\phi(U)$ to be $\phi(U)$. For every object U of τ that is not an object of σ, define ${}_x\mathcal{G}(U)$ to be $\odot$, and define ${}_x\phi(U)$ to be $\mathrm{Id}_{\odot}$. For every morphism $r : U \to V$, if U is an object of σ, then r is a morphism of σ. In this case, define ${}_x\mathcal{G}(r)$ to be $\mathcal{G}(r)$. On the other hand, if U is not an object of σ, then $\mathcal{G}(U)$ is the initial object $\odot$. In this case, define ${}_x\mathcal{G}(r)$ to be the unique morphism ${}_x\mathcal{G}(U) \to {}_x\mathcal{G}(V)$. **Prove** that ${}_x\mathcal{G}$ is a τ-family of objects, i.e., the definitions above are compatible with composition of morphisms in τ and with identity morphisms. Also **prove** that ${}_x\phi$ is a morphism of τ-families. **Prove** that ${}_x\mathrm{Id}_{\mathcal{G}}$ equals $\mathrm{Id}_{{}_x\mathcal{G}}$. Also, for a second morphism of σ-families, $\psi : \mathcal{H} \to \mathcal{I}$, **prove** that ${}_x(\psi \circ \phi)$ equals ${}_x\psi \circ {}_x\phi$. Conclude that these rules form a functor,

$${}_x\star : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}).$$

Prove that $({}_x\star, \star_x)$ extends to an adjoint pair of functors. In particular, conclude that $\star_x$ preserves epimorphisms and ${}_x\star$ preserves monomorphisms.

Next assume that $\mathcal{C}$ is an Abelian category that satisfies (AB3). For every τ-family $\mathcal{F}$, for every object U of τ, define $\theta_{\mathcal{F}}(U) : \mathcal{F}(U) \to {}^x\mathcal{F}(U)$ to be the cokernel of $\mathcal{F}(U)$ by the direct sum of the images of

$$\mathcal{F}(s) : \mathcal{F}(T) \to \mathcal{F}(U),$$

for all morphisms $s : T \to U$ with V *not* in σ (possibly empty, in which case $\theta_{\mathcal{F}}(U)$ is the identity on $\mathcal{F}(U)$). In particular, if U is not in σ, then ${}^x\mathcal{F}(U)$ is zero. For every morphism $r : U \to V$ in τ, **prove** that the composition $\theta_{\mathcal{F}}(V) \circ \mathcal{F}(r)$ equals ${}^x\mathcal{F}(r) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$${}^x\mathcal{F}(r) : {}^x\mathcal{F}(U) \to {}^x\mathcal{F}(V).$$

Prove that ${}^x\mathcal{F}(\mathrm{Id}_U)$ is the identity morphism of ${}^x\mathcal{F}(U)$. **Prove** that $r \mapsto {}^x\mathcal{F}(r)$ is compatible with composition in τ. Conclude that ${}^x\mathcal{F}$ is a τ-family, and $\theta_{\mathcal{F}}$ is a morphism of τ-families. For every morphism $\phi : \mathcal{F} \to \mathcal{E}$ of τ-families, for every object U of τ, **prove** that $\theta_{\mathcal{E}}(U) \circ \phi(U)$ equals ${}^x\phi(U) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$${}^x\phi(U) : {}^x\mathcal{F}(U) \to {}^x\mathcal{E}(U).$$

Prove that the rule $U \mapsto {}^x\phi(U)$ is a morphism of τ-families. **Prove** that ${}^x\mathrm{Id}_{\mathcal{F}}$ is the identity on ${}^x\mathcal{F}$. Also **prove** that $\phi \mapsto {}^x\phi$ is compatible with composition. Conclude that these rules define a functor

$${}^x* : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}).$$

Prove that the rule $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is a natural transformation $\mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow {}^x*$. **Prove** that the natural morphism of τ-families,

$${}^x\mathcal{F} \to {}_x(({}^x\mathcal{F})_x),$$

is an isomorphism. Conclude that there exists a unique functor,

$$*^x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}),$$

and a natural isomorphism ${}^x* \Rightarrow {}_x(*^x)$. **Prove** that $(*^x, {}_x*, \theta)$ extends to an adjoint pair of functors. In particular, conclude that ${}_x*$ preserves epimorphisms and $*^x$ preserves monomorphisms.

Finally, drop the assumption that $\mathcal{C}$ has an initial object, but assume that σ is upper, assume that σ has an initial object, W_σ, and assume that there is a functor

$$y : \tau \to \sigma$$

and a natural transformation $\theta : \mathrm{Id}_\tau \Rightarrow x \circ y$, such that for every object U of τ, the unique morphism $W_\sigma \to y(U)$ and the morphism $\theta_U : U \to y(U)$ make $y(U)$ into a coproduct of W_σ and U in τ. For simplicity, for every object U of σ, assume that $\theta_U : U \to y(U)$ is the identity Id_U (rather than merely being an isomorphism), and for every morphism $r : U \to V$ in σ, assume that $y(r)$ equals r. Thus, for every object V of σ, the identity morphism $y(V) \to V$ defines a natural transformation $\eta : y \circ x \Rightarrow \mathrm{Id}_\sigma$. **Prove** that (y, x, θ, η) is an adjoint pair of functors. Conclude that $(*_x, *_y, *_\theta, *_\eta)$ is an adjoint pair of functors. In particular, conclude that $*_x$ preserves monomorphisms and $*_y$ preserves epimorphisms.

(ix)(Compatibility of Limits and Colimits with Functors) Denote by 0 the "singleton category" 0 with a single object and a single morphism. **Prove** that $\Gamma(0, -)$ is an equivalence of categories. For an arbitrary category τ, for the unique natural transformation $\hat{\tau} : \tau \to 0$, **prove** that $*_{\hat{\tau}}$ equals the composite $\underline{*}_\tau \circ \Gamma(0, -)$ so that $\underline{*}_\tau$ is an example of this construction. In particular, for every functor $x : \sigma \to \tau$, **prove** that $(\underline{a}_\tau)_x$ equals $\underline{a}_\sigma$. If $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ is a limit of a τ-family $\mathcal{F}$, and if $\theta : \underline{b}_\sigma \Rightarrow \mathcal{F}_x$ is a limit of the associated σ-family $\mathcal{F}_x$, then **prove** that there is a unique morphism $h : a \to b$ in $\mathcal{C}$ such that η_x equals $\theta \circ \underline{p}_\sigma$. If there are right adjoints Γ_τ of $\underline{*}_\tau$ and Γ_σ of $\underline{*}_\sigma$, conclude that there exists a unique natural transformation

$$\Gamma_x : \Gamma_\tau \Rightarrow \Gamma_\sigma \circ *_x$$

so that $\eta_{\mathcal{F}_x} \circ \underline{\Gamma_x(\mathcal{F})}_\sigma$ equals $(\eta_{\mathcal{F}})_x$. **Repeat** this construction for colimits.

(x)(Limits / Colimits of a Concrete Category) Let σ be a small category in which the only morphisms are identity morphisms: identify σ with the underlying set of objects. Let $\mathcal{C}$ be the category **Sets**. For every σ-family $\mathcal{F}$, **prove** that the rule

$$\Gamma_\sigma(\mathcal{F}) \coloneqq \prod_{U \in \Sigma} \Gamma(U, \mathcal{F})$$

together with the morphism

$$\eta_{\mathcal{F}} : \underline{\Gamma_\sigma(\mathcal{F})}_\sigma \Rightarrow \mathcal{F},$$

$$\eta_{\mathcal{F}}(V) = \mathrm{pr}_V : \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}),$$

is a limit of $\mathcal{F}$. Next, for every small category τ, define σ to be the category with the same objects as τ, but with the only morphisms being identity morphisms. Define $x : \sigma \to \tau$ to be the unique functor that sends every object to itself. Define $\Gamma_\tau(\mathcal{F})$ to be the subobject of $\Gamma_\sigma(\mathcal{F}_x)$ of data $(f_U)_{U \in \Sigma}$ such that for every morphism $r : U \to V$, $\mathcal{F}(r)$ maps f_U to f_V. **Prove** that with this definition, there exists a unique natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \Rightarrow \mathcal{F}$ such that the natural transformation $\underline{\Gamma_\tau(\mathcal{F})}_\sigma \Rightarrow \underline{\Gamma_\sigma(\mathcal{F}_x)} \Rightarrow \mathcal{F}_x$ equals $(\eta_{\mathcal{F}})_x$. **Prove** that $\eta_{\mathcal{F}}$ is a limit of $\mathcal{F}$. Conclude that **Sets** has all small limits. Similarly, for associative, unital rings R and S, **prove** that the forgetful functor

$$\Phi : R - S - \mathrm{mod} \to \mathbf{Sets}$$

sends products to products. Let $\mathcal{F}$ be a τ-family of $R-S$-modules. **Prove** that the defining relations for $\Gamma_\tau(\Phi \circ \mathcal{F})$ as a subset of $\Gamma_\sigma(\Phi \circ \mathcal{F})$ are the simultaneous kernels of $R-S$-module homomorphisms. Conclude that there is a natural $R-S$-module structure on $\Gamma_\tau(\Phi \circ \mathcal{F})$, and use this to **prove** that $R-S$-mod has all limits.

(xi)(Functoriality in the Target) For every functor of categories,

$$H : \mathcal{C} \to \mathcal{D},$$

for every τ-family $\mathcal{F}$ in $\mathcal{C}$, **prove** that $H \circ \mathcal{F}$ is a τ-family in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$, **prove** that $H \circ \phi$ is a morphism of τ-families in $\mathcal{D}$. **Prove** that this defines a functor

$$H_\tau : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{D}).$$

For the identity functor $\mathrm{Id}_{\mathcal{C}}$, **prove** that $(\mathrm{Id}_{\mathcal{C}})_\tau$ is the identity functor. For $I : \mathcal{D} \to \mathcal{E}$ a functor of categories, **prove** that $(I \circ H)_\tau$ is the composite $I_\tau \circ H_\tau$. In this sense, deduce that H_τ is functorial in H.

For two functors, $H, I : \mathcal{C} \to \mathcal{D}$, and for a natural transformation $N : H \Rightarrow I$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, define $N_\tau(\mathcal{F})$ to be

$$N \circ \mathcal{F} : H \circ \mathcal{F} \Rightarrow I \circ \mathcal{F}.$$

Prove that $N_\tau(\mathcal{F})$ is a morphism of τ-families in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $N_\tau(\mathcal{G}) \circ H_\tau(\phi)$ equals $I_\tau(\phi) \circ N_\tau(\mathcal{F})$. In this sense, conclude that N_τ is a natural transformation $H_\tau \Rightarrow I_\tau$. For the identity natural transformation $\mathrm{Id}_H : H \Rightarrow H$, **prove** that $(\mathrm{Id}_H)_\tau$ is the identity natural transformation of H_τ. For a second natural transformation $M : I \Rightarrow J$, **prove** that $(M \circ N)_\tau$ equals $M_\tau \circ N_\tau$. In this sense, deduce that $(-)_\tau$ is also compatible with natural transformations.

(xii)(Reductions of Limits to Finite Systems for Concrete Categories) A category is *cofiltering* if for every pair of objects U and V there exists a pair of morphisms, $r : W \to U$ and $s : W \to V$, and for every pair of morphisms, $r, s : V \to U$, there exists a morphism $t : W \to V$ such that $r \circ t$ equals $s \circ t$ (both of these are automatic if the category has an initial object X). Assume that the category $\mathcal{C}$ has limits for all categories τ with finitely many objects, and also for all small cofiltering categories. For an arbitrary small category τ, define $\widehat{\tau}$ to be the small category whose objects are finite full subcategories σ of τ, and whose morphisms are inclusions of subcategories, $\rho \subset \sigma$, of τ. **Prove** that $\widehat{\tau}$ is cofiltering. Let $\mathcal{F}$ be a τ-family in $\mathcal{C}$. For every finite full subcategory $\sigma \subset \tau$, denote by $\mathcal{F}_\sigma$ the restriction as in (f) above. By hypothesis, there is a limit $\eta_\sigma : \underline{\widehat{\mathcal{F}}(\sigma)}_\sigma \Rightarrow \mathcal{F}_\sigma$. Moreover, by (g), for every inclusion of full subcategories $\rho \subset \sigma$, there is a natural morphism in $\mathcal{C}$, $\widehat{\mathcal{F}}(\rho) \to \widehat{\mathcal{F}}(\sigma)$, and this is functorial. Conclude that $\widehat{\mathcal{F}}$ is a $\widehat{\tau}$-family in $\mathcal{C}$. Since $\widehat{\tau}$ is filtering, there is a limit

$$\eta_{\widehat{\mathcal{F}}} : \underline{a}_{\widehat{\tau}} \Rightarrow \widehat{\mathcal{F}}.$$

Prove that this defines a limit $\eta_{\mathcal{F}} \underline{a}_\tau \Rightarrow \mathcal{F}$.

Finally, use this to **prove** that limits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

(xiii)(bis, Colimits) Repeat the steps above for colimits in place of limits. Use this to **prove** that colimits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where R and S are associative, unital rings).

Practice with Limits and Colimits Exercise. In each of the following cases, say whether the given category (a) has an initial object, (b) has a final object, (c) has a zero object, (d) has finite products, (e) has finite coproducts, (f) has arbitrary products, (g) has arbitrary coproducts, (h) has arbitrary limits (sometimes called *inverse limits*), (i) has arbitrary colimits (sometimes called *direct limits*), (j) coproducts / filtering colimits preserve monomorphisms, (k) products / cofiltering limits preserve epimorphisms.

(i) The category **Sets** whose objects are sets, whose morphisms are set maps, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(ii) The opposite category $\mathbf{Sets}^{\mathrm{opp}}$.

(iii) For a given set S, the category whose objects are elements of the set, and where the only morphisms are the identity morphisms from an element to that same element. What if the set is the empty set? What if the set is a singleton set?

(iv) For a partially ordered set $(S, \leq)$, the category whose objects are elements of S, and where the Hom set between two elements x, y of S is a singleton set if $x \leq y$ and empty otherwise. What if the partially ordered set $(S, \leq)$ is a **lattice**, i.e., every finite subset (resp. arbitrary subset) has a least upper bound and has a greatest lower bound?

(v) For a monoid $(M, \cdot, 1)$, the category with only one object whose Hom set, with its natural composition and identity, is $(M, \cdot, 1)$. What is M equals $\{1\}$?

(vi) For a monoid $(M, \cdot, 1)$ and an action of that monoid on a set, $\rho : M \times S \to S$, the category whose objects are the elements of S, and where the Hom set from x to y is the subset $M_{x,y} = \{m \in M | m \cdot x = y\}$. What if the action is both transitive and faithful, i.e., S equals M with its left regular representation?

(vii) The category **PtdSets** whose objects are pairs (S, s_0) of a set S and a specified element s_0 of S, i.e., *pointed sets*, whose morphisms are set maps that send the specified point of the domain to the specified point of the target, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(viii) The category **Monoids** whose objects are monoids, whose morphisms are homomorphisms of monoids, whose composition is sual composition, and whose identity morphisms are usual identity maps.

(ix) For a specified monoid $(M, \cdot, 1)$, the category whose objects are pairs (S, ρ) of a set S and an action $\rho : M \times S \to S$ of M on S, whose morphisms are set maps compatible with the action, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(x) The full subcategory **Groups** of **Monoids** whose objects are groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xi) The full subcategory $\mathbb{Z}-\text{mod}$ of **Groups** whose objects are Abelian groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xii) The full subcategory **FiniteGroups** of **Groups** whose objects are finite groups. Are coproducts, resp. products, in the subcategory also coproducts, resp. products, in the larger category **Groups**? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xiii) The full subcategory $\mathbb{Z}-\text{mod}_{\text{tor}}$ of $\mathbb{Z}-\text{mod}$ consisting of torsion Abelian groups, i.e., every element has finite order (allowed to vary from element to element). Are coproducts, resp. products, preserved by the inclusion functor? Are monomorphisms, resp. epimorphisms preserved?

(xiv) The category **Rings** whose objects are associative, unital rings, whose morphisms are homomorphisms of rings (preserving the multiplicative identity), whose composition is the usual composition, and whose identity morphisms are the usual identity maps. **Hint.** For the coproduct of two associative, unital rings $(R', +, 0, \cdot', 1')$ and $(R'', +, 0, \cdot'', 1'')$, first form the coproduct $R' \oplus R''$ of $(R', +, 0)$ and $(R'', +, 0)$ as a $\mathbb{Z}$-module, then form the total tensor product ring $T^\bullet_{\mathbb{Z}}(R' \oplus R'')$ as in the previous problem set. For the two natural maps $q' : R' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ and $q'' : R'' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ form the left-right ideal $I \subset T^\bullet_{\mathbb{Z}}(R' \oplus R'')$ generated by $q'(1') - 1$, $q''(1'') - 1$, $q'(r' \cdot' s') - q'(r') \cdot q'(s')$, and $q''(r'' \cdot'' s'') - q''(r'') \cdot q''(s'')$ for all elements $r', s' \in R'$ and $r'', s'' \in R''$. Define

$$p : T^1_{\mathbb{Z}}(R' \oplus R'') \to R,$$

to be the quotient by I. Prove that $p \circ q' : R' \to R$ and $p \circ q'' : R'' \to R$ are ring homomorphisms that make R into a coproduct of R' and R''.

(xv) The full subcategory **CommRings** of **Rings** whose objects are commutative, unital rings. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvi) The full subcategory **NilCommRings** of **CommRings** whose objects are commutative, unital rings such that every noninvertible element is nilpotent. Does the inclusion functor preserve coproducts, resp. products? (Be careful about products!) Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvii) Let R and S be associative, unital rings. Let $R-\text{mod}$, resp. $\text{mod}-S$, $R-S-\text{mod}$, be the category of left R-modules, resp. right S-modules, $R-S$-bimodules. Does the inclusion functor from $R-S-\text{mod}$ to $R-\text{mod}$, resp. to $\text{mod}-S$, preserve coproduct, products, monomorphisms and epimorphisms?

(xviii) Let $(I, \le)$ be a partially ordered set. Let $\mathcal{C}$ be a category. An $(I, \le)$-*system* in $\mathcal{C}$ is a datum

$$c = ((c_i)_{i \in I}, (f_{i,j})_{(i,j) \in I \times I, i \le j})$$

where every c_i is an object of $\mathcal{C}$, where for every pair $(i, j) \in I \times I$ with $i \le j$, $c_{i,j}$ is an element of $\text{Hom}_{\mathcal{C}}(c_i, c_j)$, and satisfying the following conditions: (a) for every $i \in I$, $c_{i,i}$ equals Id_{c_i}, and (b) for every triple $(i, j, k) \in I$ with $i \le j$ and $j \le k$, $c_{j,k} \circ c_{i,j}$ equals $c_{i,k}$. For every pair of $(I, \le)$-systems in $\mathcal{C}$, $c = ((c_i)_{i \in I}, (c_{i,j})_{i \le j})$ and $c' = ((c'_i)_{i \in I}, (c'_{i,j})_{i \le j})$, a morphism $g : c \to c'$ is defined to be a datum $(g_i)_{i \in I}$ of morphisms $g_i \in \text{Hom}_{\mathcal{C}}(c_i, c'_i)$ such that for every $(i, j) \in I \times I$ with $i \le j$, $g_j \circ c_{i,j}$ equals $c'_{i,j} \circ g_i$. Composition of morphisms g and g' is componentwise $g'_i \circ g_i$, and identities are $\text{Id}_c = (\text{Id}_{c_i})_{i \in I}$. This category is $\text{Fun}((I, \le), \mathcal{C})$, and is sometimes referred to as the category of $(I, \le)$-*presheaves*. Assuming $\mathcal{C}$ has finite coproducts, resp. finite products, arbitrary coproducts, arbitrary products, a zero object, kernels, cokernels, etc., what can you say about $\text{Fun}((I, \le), \mathcal{C})$?

(xix) Let $\mathcal{C}$ be a category that has arbitrary products. Let $(I, \le)$ be a partially ordered set whose associated category as in (iv) has finite coproducts and has arbitrary products. The main example is when $I = \mathfrak{U}$ is the collection of all open subsets U of a topology on a set X, and where $U \le V$ if $U \supseteq V$. Then coproduct is intersection and product is union. Motivated by this case, an *covering* of an element i of I is a collection $\underline{j} = (j_\alpha)_{\alpha \in A}$ of elements j_α of I such that for every α, $i \le j_\alpha$, and such that i is the product of $(j_\alpha)_{\alpha \in A}$ in the sense of (iv). In this case, for every $(\alpha, \beta) \in A \times A$, define $j_{\alpha,\beta}$ to be the element of I such that $j_\alpha \le j_{\alpha,\beta}$, such that $j_\beta \le j_{\alpha,\beta}$, and such that $j_{\alpha,\beta}$ is a coprodcut of (j_α, j_β). An $(I,\le)$-presheaf $c = ((c_i)_{i\in I}, (c_{i,j})_{i\le j})$ is an $(I,\le)$-*sheaf* if for every element i of I and for every covering $\underline{j} = (j_\alpha)_{\alpha\in A}$, the following diagram in $\mathcal{C}$ is *exact* in a sense to be made precise,

$$c_i \xrightarrow{q} \prod_{\alpha\in A} c_{j_\alpha} \overset{p'}{\rightrightarrows} p'' \prod_{(\alpha,\beta)\in A\times A} c_{j_{\alpha,\beta}}.$$

For every $\alpha \in A$, the factor of q,

$$\mathrm{pr}_\alpha \circ q : c_i \to c_{j_\alpha},$$

is defined to be c_{i,j_α}. For every $(\alpha,\beta) \in A \times A$, the factor of p',

$$\mathrm{pr}_{\alpha,\beta} \circ p' : \prod_{\gamma\in A} c_{j_\gamma} \to c_{j_{\alpha,\beta}},$$

is defined to be $c_{j_\alpha, j_{\alpha,\beta}} \circ \mathrm{pr}_\alpha$. Similarly, $\mathrm{pr}_{\alpha,\beta} \circ p''$ is defined to be $c_{j_\beta, j_{\alpha,\beta}} \circ \mathrm{pr}_\beta$. The diagram above is *exact* in the sense that q is a monomorphism in $\mathcal{C}$ and q is a fiber product in $\mathcal{C}$ of the pair of morphisms (p', p''). The category of $(I,\le)$ is the full subcategoryof the category of $(I,\le)$-presheaves whose objects are $(I,\le)$-sheaves. Does this subcategory have coproducts, products, etc.? Does the inclusion functor preserve coproducts, resp. products, monomorphisms, epimorphisms? Before considering the general case, it is probably best to first consider the case that $\mathcal{C}$ is $\mathbb{Z}-\mathrm{mod}$, and then consider the case that $\mathcal{C}$ is **Sets**.