MAT 543 Problem Set 8

Homework Policy. Read through and carefully consider all of the following problems. Please write up and hand-in solutions to **five** of the problems.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

Textbook Problems.

Problem 1. Problem 17.1, p. 145, Forster.

Problem 2. Problem 17.2, p. 145, Forster.

Problem 3. Problem 17.3, p. 145, Forster.

Problem 4. Problem 17.4, p. 145, Forster.

Problem 5. Problem 17.5, p. 145, Forster.

Problem 6. Problem 17.6, p. 145, Forster.

Problem 7. Problem 17.7, p. 145, Forster.

Problem 8. For a genus g hyperelliptic curve X with its degree 2 morphism $f : X \to \mathbb{P}^1$ branched over $\{p_0, \dots, p_{2g+1}\}$ and hyperelliptic involution $\iota : X \to X$, prove that ω_X is isomorphic to $f^*\mathcal{O}_{\mathbb{P}^1}(g-1)$. (Hint: For the usual presentation $w^2 = (z-p_0)\cdots(z-p_{2g+1})$, what is the divisor of dw?) By computing the $\mathbb{C}$-vector space dimension of the domain and the target, prove that the following injective pullback homomorphism is surjective,

$$H^0(f^*, \mathcal{O}_{\mathbb{P}^1}(g-1)) : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1)) \to H^0(X, \omega_X).$$

Problem 9. This problem continues the previous problem. Denote by H the divisor class (modulo principal divisors) of $f^*\underline{p}$ on X where $p \in \mathbb{P}^1$ is any point. Conclude that for every integer $\ell = 0, \dots, g-1$, also the following injective pullback map is surjective (or else multiplying by z^ℓ would contradict the previous isomorphism),

$$H^0(f^*, \mathcal{O}_{\mathbb{P}^1}(g-1-\ell)) : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1-\ell)) \to H^0(X, \omega_X(-\ell \cdot H)).$$

Next, consider the effective divisor $D = \ell \cdot H + \sum_{i=1}^r \underline{q}_i$ where $0 \leq \ell \leq g-1$ and where $\{q_1, \dots, q_r\}$ are points of X such that for every $i \neq j$, both $q_i \neq q_j$ and $q_i \neq \iota(q_j)$. Prove that $H^0(X, \omega_X(-D))$

has dimension $\max(g-\ell-r,0)$. Use Riemann-Roch and Serre duality to prove that the injective pullback homomorphism,

$$H^0(f^*,\mathcal{O}_{\mathbb{P}^1}(\ell)) : H^0(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(\ell)) \to H^0(X,\mathcal{O}_X(D)),$$

is an isomorphism if $\ell+r\leq g-1$, i.e., if there exists a nonzero global section of ω_X that vanishes on D, and otherwise the cokernel has dimension $\ell+r-g$, i.e., the index of specialty of D equals 0. Finally, for $\ell\geq g$, since D has degree $>2g-2$, conclude that $H^0(X,\mathcal{O}_X(D))$ has dimension $2\ell+r+1-g$, i.e., the index of specialty equals 0. Altogether, this computes the index of specialty of every effective divisor class on every hyperelliptic curve.

Problem 10. Let X be a non-hyperelliptic curve of genus g. Prove that for every ordered pair (p,q) of points in X (possibly the same point), the vector space $H^1(X,\omega_X(-\underline{p}-\underline{q}))$ is the zero vector space. This implies that the holomorphic map $\phi_X : X\to\mathbb{CP}^{g-1}$ of the "complete linear system" of ω_X is a holomorphic embedding of complex manifolds. This is the *canonical embedding* of a non-hyperelliptic curve.