MAT 543 Problem Set 4

Homework Policy. Read through and carefully consider all of the following problems. Please write up and hand-in solutions to **five** of the problems.

Each student is encouraged to work with other students, but submitted problem sets must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource.

Textbook Problems.

Problem 1. Problem 12.1, p. 103, Forster.

Problem 2. Problem 12.2, p. 103, Forster.

Problem 3. Problem 12.3, p. 104, Forster.

Problem 4. Let X be a compact Riemann surface (or any Hausdorff, connected, paracompact differentiable manifold with finitely generated fundamental group). For the Abelianization of the fundamental group, $\pi_1(X, x_0) \to H_1^{\text{sing}}(X; \mathbb{Z})$, let $\pi : (X_{\text{Ab}}, y_0) \to (X, x_0)$ be an unbranched cover that is Galois with Galois group equal to the Abelianization (as a quotient group of the fundamental group). Prove that on X_{Ab} there is a differentiable primitive, resp. holomorphic primitive, for the pullback of every $\mathbb{C}$-valued differentiable closed 1-form, resp. for every "Abelian differential of the first kind", i.e., holomorphic $(1, 0)$-form on X. For the $\mathbb{C}$-vector space $Z^1_{\text{dR}}(X; \mathbb{C})$, resp. the $\mathbb{C}$-vector subspace $\Omega^{1,0}(X)$, of $\mathbb{C}$-valued differentiable closed 1-forms, resp. of holomorphic $(1, 0)$-forms, interpret the periods / summands of automorphy as a function,

$$Z^1_{\text{dR}}(X) \times H_1^{\text{sing}}(X; \mathbb{Z}) \to \mathbb{C}.$$

Prove that this binary operation is $\mathbb{Z}$-bilinear, prove that it is $\mathbb{C}$-linear in the first argument, and prove that it is zero whenever the first argument is an exact 1-form. Deduce the existence of a $\mathbb{C}$-linear transformation, the *period map*,

$$\text{per}_X : H^1_{\text{dR}}(X; \mathbb{C}) \to \text{Hom}_{\mathbb{Z}}(H_1^{\text{sing}}(X; \mathbb{Z}), \mathbb{C}) = H^1_{\text{sing}}(X; \mathbb{C}).$$

Problem 5. Continuing the previous problem, Use Theorem 10.15, p. 75, to prove that the period map is injective. In fact, this map is also surjective by the **de Rham Theorem**, that holds for general smooth manifolds. In this exercise, for a compact Riemann surface with $H_1^{\text{sing}}(X; \mathbb{Z})$ isomorphic to $\mathbb{Z}^g$ (a Riemann surface of *genus* equal to g) you will reduce this to a corollary of the (much earlier) **Riemann-Roch Theorem**. Let $\alpha = f(z)dz$ be a holomorphic $(1,0)$-form. Let $\beta = \overline{g(z)}d\overline{z}$ be a complex conjugate of a holomorphic $(1,0)$-form. If $\alpha + \beta$ is exact, say df, use the holomorphic, resp. conjugate holomorphic, hypothesis of α, resp. β, to conclude that f is harmonic. By usual single-variable complex analysis, conclude that the real and imaginary parts of the $\mathbb{C}$-valued function f each equal the real part of a *holomorphic* function on X. Since X is compact, use the maximum modulus principle to conclude that f is constant, i.e., α and β are zero. Using injectivity above, conclude that for the image of $\Omega^{1,0}(X)$ under the period map, this $\mathbb{C}$-vector subspace of the complexification of $H^1_{\text{sing}}(X;\mathbb{R})$ has zero intersection with the complex conjugate $\mathbb{C}$-vector subspace. Since $H^1_{\text{sing}}(X;\mathbb{C})$ has dimension $2g$ as a $\mathbb{C}$-vector space, assuming that $\Omega^{1,0}(X)$ has dimension g as a $\mathbb{C}$-vector space, prove that the period map is surjective, and in fact $H^1_{\text{sing}}(X;\mathbb{C})$ equals the direct sum of its $\mathbb{C}$-vector subspaces $\Omega^{1,0}(X)$ and $\overline{\Omega^{1,0}(X)}$.

N.B. The **Torelli theorem** says that the compact Riemann surface X is determined up to (not necessarily unique) biholomorphism by the data above of its *weight one Hodge structure*, i.e., a pair (H,Ω) of a finitely free Abelian group H of rank $2g$, i.e., $H \cong \mathbb{Z}^{2g}$, and a $\mathbb{C}$-vector subspace $\Omega \subset H\otimes_{\mathbb{Z}}\mathbb{C}$ satisfying $H\otimes_{\mathbb{Z}}\mathbb{C} = \Omega\oplus\overline{\Omega}$. The **Schottky problem** asks which (principally polarized) weight one Hodge structures are isomorphic to the Hodge structure of a compact Riemann surface.

Problem 6. Compute the weight one Hodge structure of the Riemann sphere. For the compact Riemann surfaces $\mathbb{C}/\Gamma$ of Exercises 1.4 and 1.5, deduce already from the previous exercise (even without Riemann-Roch) that $\Omega^{1,0}(X)$ is the 1-dimensional $\mathbb{C}$-vector subspace spanned by the unique holomorphic $(1,0)$-form whose pullback to $\mathbb{C}$ equals dz. In this case, what is the data of the weight one Hodge structure? Use Exercises 1.4 and 1.5 to prove the Torelli theorem for these compact Riemann surfaces.

Problem 7. For some Riemann surface X, write an explicit example of a germ ϕ of a holomorphic function at a point $x_0 \in X$ such that for the maximal analytic continuation, $(\pi : (Y,y_0) \to (X,x_0), f : Y \to \mathbb{C})$, the induced local biholomorphism $\pi : Y \to \pi(Y)$ is not a covering map.

Problem 8. Let $(a_0,\dots,a_{2g+1})$ be distinct points of $\mathbb{C}$, and denote $A = \{a_0,\dots,a_{2g+1}\} \subset \mathbb{C}$. Denote by $f(z)$ the meromorphic function on $\mathbb{P}^1$,

$$f(z) = (z-a_0)(z-a_1)\cdots(z-a_{2g+1}).$$

For any point $x_0 \in U = \mathbb{C}\backslash A$, let ϕ be one of the two germs $\sqrt{f(z)}$ at x_0. Denote by $(\pi_U : V \to U, g)$ the corresponding maximal analytic continuation of ϕ. Show that g satisfies the monic polynomial equation,

$$F(z,T) = T^2 - f(z).$$

Conclude that π_U is a degree 2 unbranched cover, and denote the unique extension of π_U to a branched cover of (compact) Riemann surfaces by

$$\pi : Y \to X.$$

Show that $-g$ also satisfies the monic polynomial equation. Conclude that π is a Galois cover with Deck transformation group (Id, i), where $g\circ i$ equals $-g$. Also, compute the order of zero / pole of

g at each of the unique points $b_0, \dots, b_{2g+1} \in Y$ with $\pi(b_i) = a_i$ and at the two points $y_\infty, i(y_\infty) \in Y$ that map to $\infty \in \mathbb{P}^1$. Verify directly the corollary of the Residue Theorem that the sum over all zeroes of g of the order of zero equals the sum over all poles of g of the pole order.

Problem 9. Continuing the previous problem, assume that $0 \notin A$, denote by $y_0, i(y_0)$ the two points of Y mapping to $0 \in \mathbb{P}^1$, and compute the zeroes and poles of each meromorphic function $\pi^*(z^\ell)/g$ at each of the points of $B = \{b_0, \dots, b_{2g+1}\}$, $\{y_\infty, i(y_\infty)\}$, and $\{y_0, i(y_0)\}$. Since π is a covering map away from $B \cup \{y_\infty, i(y_\infty)\}$, conclude that π^*dz is a meromorphic differential whose zeroes and poles are contained in this finite set. In fact, check that g is a local coordinate at each point of B, and check that $\pi^*(1/z)$ is a local coordinate at $\{y_\infty, i(y_\infty)\}$. Use these local coordinates to compute the order of zero / pole of π^*dz at each of these points. Put the pieces together to conclude that each of the meromorphic differentials $\omega_\ell = \pi^*(z^\ell dz)/g$ for $\ell = 0, \dots, g-1$, are $\mathbb{C}$-linearly independent global holomorphic $(1,0)$-forms on Y. Since we can confirm by topology that Y has $H_1(Y;\mathbb{Z})$ isomorphic to $\mathbb{Z}^{2g}$ (e.g., by using excision for the description as a branched cover), this proves that $\Omega^{1,0}(X)$ does have the maximal possible dimension g.

Problem 9. For a Laurent tail ψ at one of the points of B, $\{y_\infty, i(y_\infty)\}$ and $\{y_0, i(y_\infty)\}$, what are the residues of $\psi \cdot \omega_\ell$ at that point? From the Residue Theorem, what is the explicit form of the necessary condition on a collection of Laurent tails at these points for the collection to be the collection of Laurent tails (at all poles) of a meromorphic function on Y. How does this compare to Theorem 8.12, p. 57, that every meromorphic function on Y has a representation $\pi^*(a(z)) + g \cdot \pi^*(b(z))$ for unique mermomorphic functions a and b on $\mathbb{P}^1$?