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# MAT 536 Problem Set 8

**Homework Policy.** Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

**Problems.**

**Problem 0.**(Limits and Colimits) These notions have been implicit throughout the semester, although mostly we used the special cases of products and coproducts. The notation here is meant to emphasize the connection with operations on presheaves and sheaves such as formation of global sections, stalks, pushforward and inverse image. Let $\tau$ be a small category. Let $\mathcal{C}$ be a category. A $\tau$*-family* in $\mathcal{C}$ is a (covariant) functor,

$$\mathcal{F} : \tau \to \mathcal{C}.$$

Precisely, for every object $U$ of $\tau$, $\mathcal{F}(U)$ is a specified object of $\mathcal{C}$. For every morphism of objects of $\tau$, $r : U \to V$, $\mathcal{F}(r) : \mathcal{F}(U) \to \mathcal{F}(V)$ is a morphism of $\mathcal{C}$. Also, $\mathcal{F}(\text{Id}_U)$ equals $\text{Id}_{\mathcal{F}(U)}$. Finally, for every pair of morphisms of $\tau$, $r : U \to V$ and $s : V \to W$, $\mathcal{F}(s) \circ \mathcal{F}(r)$ equals $\mathcal{F}(s \circ r)$.

For every pair $\mathcal{F}$, $\mathcal{G}$ of $\tau$-families in $\mathcal{C}$, a *morphism* of $\tau$-families from $\mathcal{F}$ to $\mathcal{G}$ is a natural transformation of functors, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$. For every object $a$ of $\mathcal{C}$, denote by

$$\underline{a}_\tau : \tau \to \mathcal{C}$$

the functor that sends every object to $a$ and that sends every morphism to $\text{Id}_a$. For every morphism in $\mathcal{C}$, $p : a \to b$, denote by

$$\underline{p}_\tau : \underline{a}_\tau \Rightarrow \underline{b}_\tau$$

the natural transformation that assigns to every object $U$ of $\tau$ the morphism $p : a \to b$. Finally, for every object $U$ of $\tau$, denote

$$\Gamma(U, \mathcal{F}) = \mathcal{F}(U), \quad \Gamma(U, \theta) = \theta(U),$$

and for every morphism $r : U \to V$ of $\tau$, denote

$$\Gamma(r, \mathcal{F}) = \mathcal{F}(r).$$

**(a)**(Functor Categories and Section Functors) For $\tau$-families $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$, and for morphisms of $\tau$-families, $\theta : \mathcal{F} \to \mathcal{G}$ and $\eta : \mathcal{G} \to \mathcal{H}$, define the composition of $\theta$ and $\eta$ to be the composite natural transformation $\eta \circ \theta : \mathcal{F} \to \mathcal{H}$. **Prove** that with this notion, there is a category $\mathbf{Fun}(\tau, \mathcal{C})$ whose objects are $\tau$-families $\mathcal{F}$ and whose morphisms are natural transformations. **Prove** that

$$\underline{*}_\tau : \mathcal{C} \to \mathbf{Fun}(\tau, \mathcal{C}),\ a \mapsto \underline{a}_\tau,\ p \mapsto \underline{p}_\tau,$$

is a functor that preserves monomorphisms, epimorphisms and isomorphisms. For every object $U$ of $\tau$, **prove** that

$$\Gamma(U, -) : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathcal{C},\ \mathcal{F} \mapsto \Gamma(U, \mathcal{F}),\ \theta \mapsto \Gamma(U, \theta),$$

is a functor. For every morphism $r : U \to V$ of $\tau$, **prove** that $\Gamma(r, -)$ is a natural transformation $\Gamma(U, -) \Rightarrow \Gamma(V, -)$.

**(b)**(Adjointness of Constant / Diagonal Functors and the Global Sections Functor) If $\mathcal{C}$ has an initial object $X$, **prove** that $(\underline{*}_\tau, \Gamma(X, -))$ extends to an adjoint pair of functors. More generally, a *limit* of a $\tau$-family $\mathcal{F}$ (if it exists) is a natural transformation $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ that is final among all such natural transformations, i.e., for every natural transformation $\theta : \underline{b}_\tau \Rightarrow \mathcal{F}$, there exists a unique morphism $t : b \to a$ in $\mathcal{C}$ such that $\theta$ equals $\eta \circ \underline{t}_\tau$. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, for limits $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ and $\theta : \underline{b}_\tau \Rightarrow \mathcal{G}$, **prove** that there exists a unique morphism $f : a \to b$ such that $\theta \circ \underline{p}_\tau$ equals $\phi \circ \eta$. In particular, **prove** that if a limit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object $a$ of $\mathcal{C}$, **prove** that the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a limit of $\underline{a}_\tau$.

**(c)**(Adjointness of Constant / Diagonal Functors and Limits) For this part, assume that every $\tau$-family has a limit; a category $\mathcal{C}$ is said to *have all limits* if for every small category $\tau$ and for every $\tau$-family $\mathcal{F}$, there is a limit. Assume further that there is a rule $\Gamma_\tau$ that assigns to every $\mathcal{F}$ an object $\Gamma_\tau(\mathcal{F})$ and a natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \to \mathcal{F}$ that is a limit. (Typically such a rule follows from the "construction" of limits, but such a rule also follows from some form of the Axiom of Choice.) **Prove** that this extends uniquely to a functor,

$$\Gamma_\tau : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation of functors

$$\eta : \underline{*}_\tau \circ \Gamma_\tau \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau, \mathcal{C})}.$$

Moreover, **prove** that the rule sending every object $a$ of $\mathcal{C}$ to the identity natural transformation $\theta_a$ is a natural transformation $\theta : \mathrm{Id}_{\mathcal{C}} \Rightarrow \Gamma_\tau \circ \underline{*}_\tau$. **Prove** that $(\underline{*}_\tau, \Gamma, \theta, \eta)$ is an adjoint pair of functors. In particular, the limit functor $\Gamma_\tau$ preserves monomorphisms and sends injective objects of $\mathbf{Fun}(\tau, \mathcal{C})$ to injective objects of $\mathcal{C}$.

**(d)**(Adjointness of Colimits and Constant / Diagonal Functors) If $\mathcal{C}$ has a final object $O$, **prove** that $(\Gamma(O, -), \underline{*}_\tau)$ extends to an adjoint pair of functors. More generally, a *colimit* of a $\tau$-family $\mathcal{F}$ (if it exists) is a natural transformation $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ that is final among all such natural transformations, i.e., for every natural transformation $\eta : \mathcal{F} \Rightarrow \underline{b}_\tau$, there exists a unique morphism $h : a \to b$ in $\mathcal{C}$ such that $\underline{h}_\tau \circ \theta$ equals $\eta$. For a morphism $\phi : \mathcal{F} \to \mathcal{G}$, for colimits $\theta : \mathcal{F} \Rightarrow \underline{a}_\tau$ and $\eta : \mathcal{G} \Rightarrow \underline{b}_\tau$,

**prove** that there exists a unique morphism $f : a \to b$ such that $\underline{f}_\tau \circ \theta$ equals $\eta \circ \phi$. In particular, **prove** that if a colimit of $\mathcal{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object $a$ of $\mathcal{C}$, **prove** that the identity transformation $\theta_a : \underline{a}_\tau \to \underline{a}_\tau$ is a colimit of $\underline{a}_\tau$. Finally, **repeat** the previous part for colimits in place of limits. Deduce that colimits (if they exist) preserve epimorphisms and projective objects.

**(e)**(Functoriality in the Source) Let $x : \sigma \to \tau$ be a functor of small categories. For every $\tau$-family $\mathcal{F}$, define $\mathcal{F}_x$ to be the composite functor $\mathcal{F} \circ x$, which is a $\sigma$-family. For every morphism of $\tau$-families, $\phi : \mathcal{F} \to \mathcal{G}$, define $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ to be $\phi \circ x$, which is a morphism of $\sigma$-families. **Prove** that this defines a functor

$$*_x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}).$$

For the identity functor $\mathrm{Id}_\tau : \tau \to \tau$, **prove** that $*_{\mathrm{Id}_\tau}$ is the identity functor. For $y : \rho\ \sigma$ a functor of small categories, **prove** that $*_{x \circ y}$ is the composite $*_y \circ *_x$. In this sense, deduce that $*_x$ is a contravariant functor in $x$.

For two functors, $x, x_1 : \sigma \to \tau$ and for a natural transformation $n : x \Rightarrow x_1$, define $\mathcal{F}_n : \mathcal{F}_x \Rightarrow \mathcal{F}_{x_1}$ to be $\mathcal{F}(n(V)) : \mathcal{F}(x(V)) \to \mathcal{F}(x_1(V))$ for every object $V$ of $\sigma$. **Prove** that $\mathcal{F}_n$ is a morphism of $\sigma$-families. For every morphism of $\tau$-families, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $\phi_{x_1} \circ \mathcal{F}_n$ equals $\mathcal{G}_n \circ \phi_x$. In this sense, conclude that $*_n$ is a natural transformation $*_x \Rightarrow *_{x'}$. For the identity natural transformation $\mathrm{Id}_x : x \Rightarrow x$, **prove** that $*_{\mathrm{Id}_x}$ is the identity natural transformation of $*_x$. For a second natural transformation $m : x_1 \Rightarrow x_1$, **prove** that $\mathcal{F}_{m \circ n}$ equals $\mathcal{F}_m \circ \mathcal{F}_n$. In this sense, deduce that $*_x$ is also compatible with natural transformations. In particular, if $(x, y, \theta, \eta)$ is an adjoint pair of functors, **prove** that $(*_y, *_x, *_\theta, *_\eta)$ is an adjoint pair of functors.

**(f)**(Fiber Categories) The following notion of *fiber category* is a special case of the notion of *2-fiber product* of functors of categories. Let $x : \sigma \to \tau$ be a functor; this is also called a *category over* $\tau$. For every object $U$ of $\tau$, a *$\sigma_{x,U}$-object* is a pair $(V, r : x(V) \to U)$ of an object $V$ of $\sigma$ and a $\tau$-isomorphism $r : x(V) \to U$. For two objects $\sigma_{x,U}$-objects $(V, r)$ and $(V', r')$ of $\sigma_{x,U}$, a *$\sigma_{x,U}$-morphism* from $(V, r)$ to $(V', r')$ is a morphism of $\sigma$, $s : V \to V'$, such that $r' \circ x(s)$ equals $r$. **Prove** that $\mathrm{Id}_V$ is a $\sigma_{x,U}$-morphism from $(V, r)$ to itself; more generally, the $\sigma_{x,U}$-morphisms from $(V, r)$ to $(V, r)$ are precisely the $\sigma$-morphisms $s : V \to V$ such that $x(s)$ equals $\mathrm{Id}_{x(V)}$. For every pair of $\sigma_{x,U}$-morphisms, $s : (V, r) \to (V', r')$ and $s' : (V', r') \to (V'', r'')$, **prove** that $s' \circ s$ is a $\sigma_{x,U}$-morphism from $(V, r)$ to $(V'', r'')$. Conclude that these rules form a category, denoted $\sigma_{x,U}$. **Prove** that the rule $(V, r) \mapsto V$ and $s \mapsto s$ defines a faithful functor,

$$\Phi_{x,U} : \sigma_{x,U} \to \sigma,$$

and $r : x(V) \to U$ defines a natural isomorphism $\theta_{x,U} : x \circ \Phi_{x,U} \Rightarrow \underline{U}_{\sigma_{x,U}}$. Finally, for every category $\sigma'$, for every functor $\Phi' : \sigma' \to \sigma$, and for every natural isomorphism $\theta' : x \circ \Phi' \Rightarrow \underline{U}_{\sigma'}$, **prove** that there exists a unique functor $F : \sigma' \to \sigma_{x,U}$ such that $\Phi'$ equals $\Phi_{x,U} \circ F$ and $\theta'$ equals $\theta_{x,U} \circ F$. In this sense, $(\Phi_{x,U}, \theta_{x,U})$ is final among pairs $(\Phi', \theta')$ as above.

For every pair of functors $x, x_1 : \sigma \to \tau$, and for every natural *isomorphism* $n : x \Rightarrow x_1$, for every $\sigma_{x_1,U}$-object $(V, r_1 : x_1(V) \to U)$, **prove** that $(V, r_1 \circ n_V : x(V) \to U)$ is an object of $\sigma_{x,U}$. For every morphism in $\sigma_{x_1,U}$, $s : (V, r_1) \to (V', r_1')$, **prove** that $s$ is also a morphism $(V, r_1 \circ n_V) \to (V', r_1' \circ n_{V'})$. Conclude that these rules define a functor,

$$\sigma_{n,U} : \sigma_{x_1,U} \to \sigma_{x,U}.$$

**Prove** that this functor is a *strict equivalence* of categories: it is a bijection on Hom sets (as for all equivalences), but it is also a bijection on objects (rather than merely being essentially surjective). **Prove** that $\sigma_{n,U}$ is functorial in $n$, i.e., for a second natural isomorphism $m : x_1 \Rightarrow x_2$, prove that $\sigma_{m \circ n,U}$ equals $\sigma_{n,U} \circ \sigma_{m,U}$.

For every pair of functors, $x : \sigma \to \tau$ and $y : \rho \to \tau$, and for every functor $z : \sigma \to \rho$ such that $x$ equals $y \circ z$ equals $x$, for every $\sigma_{x,U}$-object $(V, r)$, **prove** that $(z(V), r)$ is a $\rho_{y,U}$-object. For every $\sigma_{x,U}$-morphism $s : (V, r) \to (V', r')$, **prove** that $z(s)$ is a $\rho_{y,U}$-morphism $(z(V), r) \to (z(V'), r')$. **Prove** that $z(\text{Id}_V)$ equals $\text{Id}_{z(V)}$, and **prove** that $z$ preserves composition. Conclude that these rules define a functor,

$$z_U : \sigma_{x,U} \to \rho_{y,U}.$$

**Prove** that this is functorial in $z$: $(\text{Id}_\sigma)_U$ equals $\text{Id}_{\sigma_{x,U}}$, and for a third functor $w : \pi \to \tau$ and functor $z' : \rho \to \pi$ such that $y$ equals $w \circ z'$, then $(z' \circ z)_U$ equals $z'_U \circ z_U$. For an object $(W, r_W)$ of $\rho_{y,U}$, for each object $((V, r_V), q : Z(V) \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, define the *associated* object of $\sigma_{z,W}$ to be $(V, q)$. For an object $((V', r_{V'}), q' : Z(V') \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, for every morphism $s : (V, r_V) \to (V', r_{V'})$ such that $q$ equals $q' \circ z(s)$, define the *associated* morphism of $\sigma_{z,W}$ to be $s$. **Prove** that this defines a functor

$$\widetilde{z}_{U,(W,r_W)} : (\sigma_{x,U})_{z_U,(W,r_W)} \to \sigma_{z,W}.$$

**Prove** that this functor is a strict equivalence of categories. **Prove** that this equivalence is functorial in $z$. Finally, for two functors $z, z_1 : \sigma \to \rho$ such that $x$ equals both $y \circ z$ and $y \circ z_1$, and for a natural transformation $m : z \Rightarrow z_1$, for every object $(V, r : x(V) \to U)$ of $\sigma_{x,U}$, **prove** that $m_V$ is a morphism in $\rho_{y,U}$ from $(z(V), r)$ to $(z_1(V), r)$. Moreover, for every morphism in $\sigma_{x,U}$, $s : (V, r) \to (V', r')$, **prove** that $m_{V'} \circ z(s)$ equals $z_1(s) \circ m_V$. Conclude that this rule is a natural transformation $m_U : z_U \Rightarrow (z_1)_U$. **Prove** that this is functorial in $m$. If $m$ is a natural isomorphism, **prove** that also $m_U$ is a natural isomorphism, and the strict equivalence $(m_U)_{(W,r_W)}$ is compatible with the strict equivalence $m_W$. Finally, **prove** that $m \mapsto m_U$ is compatible with precomposition and postcomposition of $m$ with functors of categories over $\tau$.

**(g)**(Colimits and Limits along an Essentially Surjective Functor) Let $x : \sigma \to \tau$ be a functor of small categories. **Prove** that every fiber category $\sigma_{x,U}$ is small. Next, assume that $x$ is *essentially surjective*, i.e., for every object $U$ of $\tau$, there exists a $\sigma_{x,U}$-object $(V, r)$. Let $y : \tau \to \sigma$ be a functor, and let $\alpha : \text{Id}_\sigma \Rightarrow y \circ x$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors $(x, y, \alpha, \beta)$ if and only if for every object $V$ of $\sigma$, the morphism $x(\alpha_V) : x(V) \to x(y(x(V)))$ is an isomorphism and $(y(x(V)), x(\alpha_V)^{-1})$ is a final object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists $y$ and $\alpha$ extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has a final object.) For every adjoint pair $(x, y, \alpha, \beta)$, also $(*_y, *_x, *_\alpha, *_\beta)$

is an adjoint pair. More generally, no longer assume that there exists $y$ and $\alpha$, yet let $L_x$ be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $L_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\theta_{\mathcal{F}} : \mathcal{F} \to *_x \circ L_x(\mathcal{F}),$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object $U$ of $\tau$, this defines a natural transformation

$$\theta_{\mathcal{F},x,U} : \mathcal{F} \circ \Phi_{x,U} \Rightarrow L_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(L_x(\mathcal{F})(U), \theta_{\mathcal{F},x,U})$ is a colimit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation

$$\theta_x : \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})} \Rightarrow *_x \circ L_x.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G} \circ x \circ \Phi_{x,U} \to \mathcal{G} \circ \underline{U}_{\sigma_{x,U}},$$

factors uniquely through a $\mathcal{C}$-morphism $L_x(\mathcal{G} \circ x)(U) \to \mathcal{G}(U)$. **Prove** that this defines a morphism $\eta_{\mathcal{G}} : L_x(\mathcal{G} \circ x) \to \mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that is a natural transformation,

$$\eta : L_x \circ *_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})}.$$

**Prove** that $(L_x, *_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a $\Gamma^x$ and $\theta$ as above.)

Next, as above, let $x : \sigma \to \tau$ be a functor of small catgories that is essentially surjective. Let $y : \tau \to sigma$ be a functor, and let $\beta : y \circ x \Rightarrow \mathrm{Id}_\sigma$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors $(x, y, \alpha, \beta)$ if and only if for every object $V$ of $\sigma$, the morphism $x(\beta_v) : x(y(x(V))) \to x(V)$ is an isomorphism and $(y(x(V)), x(\beta_v))$ is an initial object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists $y$ and $\beta$ extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has an initial object.) For every adjoint pair $(y, x, \alpha, beta)$ also $(*_x, *_y, *_\alpha, *_\beta)$ is an adjoint pair. More generally, no longer assume that there exists $y$ and $\beta$, yet let $R_x$ be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $R_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\eta_{\mathcal{F}} : *_x \circ R_x(\mathcal{F}) \to \mathcal{F},$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object $U$ of $\tau$, this defines a natural transformation

$$\eta_{\mathcal{F},x,U} : R_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{F} \circ \Phi_{x,U},$$

of objects in $\mathbf{Fun}(\sigma_{x,U},\mathcal{C})$. Assume that each $(R_x(\mathcal{F})(U),\eta_{\mathcal{F},x,U})$ is a limit of $\mathcal{F}\circ\Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$R_x : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}),$$

and a natural transformation,

$$\eta : *_x \circ R_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\sigma,\mathcal{C})}.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau,\mathcal{C})$, the identity morphism,

$$\mathrm{Id}_{\mathcal{G}} : \mathcal{G}\circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{G}\circ x \circ \Phi_{x,U},$$

factors uniquely through a $\mathcal{G}(U)\to\mathcal{C}$-morphism $R_x(\mathcal{G}\circ x)(U)$. **Prove** that this defines a morphism $\theta_{\mathcal{G}} : \mathcal{G} \to R_x(\mathcal{G}\circ x)$ in $\mathbf{Fun}(\tau,\mathcal{C})$. **Prove** that this is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow R_x \circ *_x.$$

**Prove** that $(*_x, R_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F}\circ\Phi_{x,U}$ admits a colimit, then there exists a $R_x$ and $\eta$ as above.)

**(h)**(Adjoints Relative to a Full, Upper Subcategory) In a complementary direction to the previous case, let $x : \sigma \to \tau$ be an embedding of a full subcategory (thus, $x$ is essentially surjective if and only if $x$ is an equivalence of categories). In this case, the functor

$$*_x : \mathbf{Fun}(\tau,\mathcal{C}) \to \mathbf{Fun}(\sigma,\mathcal{C})$$

is called *restriction*. Assume further that $\sigma$ is *upper* (a la the theory of partially ordered sets) in the sense that every morphism of $\tau$ whose source is an object of $\sigma$ also has target an object of $\sigma$. Assume that $\mathcal{C}$ has an initial object, $\odot$. Let $\mathcal{G}$ be a $\sigma$-family of objects of $\mathcal{C}$. Also, let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of $\sigma$-families. For every object $U$ of $\tau$, if $U$ is an object of $\sigma$, then define ${}_x\mathcal{G}(U)$ to be $\mathcal{G}(U)$, and define ${}_x\phi(U)$ to be $\phi(U)$. For every object $U$ of $\tau$ that is not an object of $\sigma$, define ${}_x\mathcal{G}(U)$ to be $\odot$, and define ${}_x\phi(U)$ to be $\mathrm{Id}_{\odot}$. For every morphism $r : U \to V$, if $U$ is an object of $\sigma$, then $r$ is a morphism of $\sigma$. In this case, define ${}_x\mathcal{G}(r)$ to be $\mathcal{G}(r)$. On the other hand, if $U$ is not an object of $\sigma$, then $\mathcal{G}(U)$ is the initial object $\odot$. In this case, define ${}_x\mathcal{G}(r)$ to be the unique morphism ${}_x\mathcal{G}(U) \to {}_x\mathcal{G}(V)$. **Prove** that ${}_x\mathcal{G}$ is a $\tau$-family of objects, i.e., the definitions above are compatible with composition of morphisms in $\tau$ and with identity morphisms. Also **prove** that ${}_x\phi$ is a morphism of $\tau$-families. **Prove** that ${}_x\mathrm{Id}_{\mathcal{G}}$ equals $\mathrm{Id}_{{}_x\mathcal{G}}$. Also, for a second morphism of $\sigma$-families, $\psi : \mathcal{H} \to \mathcal{I}$, **prove** that ${}_x(\psi\circ\phi)$ equals ${}_x\psi \circ {}_x\phi$. Conclude that these rules form a functor,

$${}_x* : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\tau,\mathcal{C}).$$

**Prove** that $({}_x*, *_x)$ extends to an adjoint pair of functors. In particular, conclude that $*_x$ preserves epimorphisms and ${}_x*$ preserves monomorphisms.

Next assume that $\mathcal{C}$ is an Abelian category that satisfies (AB3). For every $\tau$-family $\mathcal{F}$, for every object $U$ of $\tau$, define $\theta_{\mathcal{F}}(U) : \mathcal{F}(U) \to {}^x\mathcal{F}(U)$ to be the cokernel of $\mathcal{F}(U)$ by the direct sum of the images of

$$\mathcal{F}(s) : \mathcal{F}(T) \to \mathcal{F}(U),$$

for all morphisms $s : T \to U$ with $V$ *not* in $\sigma$ (possibly empty, in which case $\theta_{\mathcal{F}}(U)$ is the identity on $\mathcal{F}(U)$). In particular, if $U$ is not in $\sigma$, then ${}^x\mathcal{F}(U)$ is zero. For every morphism $r : U \to V$ in $\tau$, **prove** that the composition $\theta_{\mathcal{F}}(V) \circ \mathcal{F}(r)$ equals ${}^x\mathcal{F}(r) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$${}^x\mathcal{F}(r) : {}^x\mathcal{F}(U) \to {}^x\mathcal{F}(V).$$

**Prove** that ${}^x\mathcal{F}(\mathrm{Id}_U)$ is the identity morphism of ${}^x\mathcal{F}(U)$. **Prove** that $r \mapsto {}^x\mathcal{F}(r)$ is compatible with composition in $\tau$. Conclude that ${}^x\mathcal{F}$ is a $\tau$-family, and $\theta_{\mathcal{F}}$ is a morphism of $\tau$-families. For every morphism $\phi : \mathcal{F} \to \mathcal{E}$ of $\tau$-families, for every object $U$ of $\tau$, **prove** that $\theta_{\mathcal{E}}(U) \circ \phi(U)$ equals ${}^x\phi(U) \circ \theta_{\mathcal{F}}(U)$ for a unique morphism

$${}^x\phi(U) : {}^x\mathcal{F}(U) \to {}^x\mathcal{E}(U).$$

**Prove** that the rule $U \mapsto {}^x\phi(U)$ is a morphism of $\tau$-families. **Prove** that ${}^x\mathrm{Id}_{\mathcal{F}}$ is the identity on ${}^x\mathcal{F}$. Also **prove** that $\phi \mapsto {}^x\phi$ is compatible with composition. Conclude that these rules define a functor

$${}^x* : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}).$$

**Prove** that the rule $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is a natural transformation $\mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow {}^x*$. **Prove** that the natural morphism of $\tau$-families,

$${}^x\mathcal{F} \to {}_x(({}^x\mathcal{F})_x),$$

is an isomorphism. Conclude that there exists a unique functor,

$$*^x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}),$$

and a natural isomorphism ${}^x* \Rightarrow {}_x(*^x)$. **Prove** that $(*^x, {}_x*, \theta)$ extends to an adjoint pair of functors. In particular, conclude that ${}_x*$ preserves epimorphisms and $*^x$ preserves monomorphisms.

Finally, drop the assumption that $\mathcal{C}$ has an initial object, but assume that $\sigma$ is upper, assume that $\sigma$ has an initial object, $W_\sigma$, and assume that there is a functor

$$y : \tau \to \sigma$$

and a natural transformation $\theta : \mathrm{Id}_\tau \Rightarrow x \circ y$, such that for every object $U$ of $\tau$, the unique morphism $W_\sigma \to y(U)$ and the morphism $\theta_U : U \to y(U)$ make $y(U)$ into a coproduct of $W_\sigma$ and $U$ in $\tau$. For simplicity, for every object $U$ of $\sigma$, assume that $\theta_U : U \to y(U)$ is the identity $\mathrm{Id}_U$ (rather than merely being an isomorphism), and for every morphism $r : U \to V$ in $\sigma$, assume that $y(r)$ equals $r$. Thus, for every object $V$ of $\sigma$, the identity morphism $y(V) \to V$ defines a natural transformation $\eta : y \circ x \Rightarrow \mathrm{Id}_\sigma$. **Prove** that $(y, x, \theta, \eta)$ is an adjoint pair of functors. Conclude that $(*_x, *_y, *_\theta, *_\eta)$ is an adjoint pair of functors. In particular, conclude that $*_x$ preserves monomorphisms and $*_y$ preserves epimorphisms.

**(i)**(Compatibility of Limits and Colimits with Functors) Denote by 0 the "singleton category" 0 with a single object and a single morphism. **Prove** that $\Gamma(0, -)$ is an equivalence of categories. For an arbitrary category $\tau$, for the unique natural transformation $\hat{\tau} : \tau \to 0$, **prove** that $*_{\hat{\tau}}$ equals the composite $\underline{*}_\tau \circ \Gamma(0,-)$ so that $\underline{*}_\tau$ is an example of this construction. In particular, for every functor $x : \sigma \to \tau$, **prove** that $(\underline{a}_\tau)_x$ equals $\underline{a}_\sigma$. If $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ is a limit of a $\tau$-family $\mathcal{F}$, and if $\theta : \underline{b}_\sigma \Rightarrow \mathcal{F}_x$ is a limit of the associated $\sigma$-family $\mathcal{F}_x$, then **prove** that there is a unique morphism $h : a \to b$ in $\mathcal{C}$ such that $\eta_x$ equals $\theta \circ \underline{p}_\sigma$. If there are right adjoints $\Gamma_\tau$ of $\underline{*}_\tau$ and $\Gamma_\sigma$ of $\underline{*}_\sigma$, conclude that there exists a unique natural transformation

$$\Gamma_x : \Gamma_\tau \Rightarrow \Gamma_\sigma \circ *_x$$

so that $\eta_{\mathcal{F}_x} \circ \underline{\Gamma_x(\mathcal{F})}_\sigma$ equals $(\eta_{\mathcal{F}})_x$. **Repeat** this construction for colimits.

**(j)**(Limits / Colimits of a Concrete Category) Let $\sigma$ be a small category in which the only morphisms are identity morphisms: identify $\sigma$ with the underlying set of objects. Let $\mathcal{C}$ be the category **Sets**. For every $\sigma$-family $\mathcal{F}$, **prove** that the rule

$$\Gamma_\sigma(\mathcal{F}) := \prod_{U \in \Sigma} \Gamma(U, \mathcal{F})$$

together with the morphism

$$\eta_{\mathcal{F}} : \underline{\Gamma_\sigma(\mathcal{F})}_\sigma \Rightarrow \mathcal{F},$$

$$\eta_{\mathcal{F}}(V) = \mathrm{pr}_V : \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}),$$

is a limit of $\mathcal{F}$. Next, for every small category $\tau$, define $\sigma$ to be the category with the same objects as $\tau$, but with the only morphisms being identity morphisms. Define $x : \sigma \to \tau$ to be the unique functor that sends every object to itself. Define $\Gamma_\tau(\mathcal{F})$ to be the subobject of $\Gamma_\sigma(\mathcal{F}_x)$ of data $(f_U)_{U \in \Sigma}$ such that for every morphism $r : U \to V$, $\mathcal{F}(r)$ maps $f_U$ to $f_V$. **Prove** that with this definition, there exists a unique natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_\tau(\mathcal{F})}_\tau \Rightarrow \mathcal{F}$ such that the natural transformation $\underline{\Gamma_\tau(\mathcal{F})}_\sigma \Rightarrow \underline{\Gamma_\sigma(\mathcal{F}_x)} \Rightarrow \mathcal{F}_x$ equals $(\eta_{\mathcal{F}})_x$. **Prove** that $\eta_{\mathcal{F}}$ is a limit of $\mathcal{F}$. Conclude that **Sets** has all small limits. Similarly, for associative, unital rings $R$ and $S$, **prove** that the forgetful functor

$$\Phi : R - S\text{mod} \to \textbf{Sets}$$

sends products to products. Let $\mathcal{F}$ be a $\tau$-family of $R - S$-modules. **Prove** that the defining relations for $\Gamma_\tau(\Phi \circ \mathcal{F})$ as a subset of $\Gamma_\sigma(\Phi \circ \mathcal{F})$ are the simultaneous kernels of $R - S$-module homomorphisms. Conclude that there is a natural $R - S$-module structure on $\Gamma_\tau(\Phi \circ \mathcal{F})$, and use this to **prove** that $R - S$-mod has all limits.

**(k)**(Functoriality in the Target) For every functor of categories,

$$H : \mathcal{C} \to \mathcal{D},$$

for every $\tau$-family $\mathcal{F}$ in $\mathcal{C}$, **prove** that $H \circ \mathcal{F}$ is a $\tau$-family in $\mathcal{D}$. For every morphism of $\tau$-families in $\mathcal{C}$, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$, **prove** that $H \circ \phi$ is a morphism of $\tau$-families in $\mathcal{D}$. **Prove** that this defines a functor

$$H_\tau : \textbf{Fun}(\tau, \mathcal{C}) \to \textbf{Fun}(\tau, \mathcal{D}).$$

For the identity functor $\mathrm{Id}_{\mathcal{C}}$, **prove** that $(\mathrm{Id}_{\mathcal{C}})_\tau$ is the identity functor. For $I : \mathcal{D} \to \mathcal{E}$ a functor of categories, **prove** that $(I \circ H)_\tau$ is the composite $I_\tau \circ H_\tau$. In this sense, deduce that $H_\tau$ is functorial in $H$.

For two functors, $H, I : \mathcal{C} \to \mathcal{D}$, and for a natural transformation $N : H \Rightarrow I$, for every $\tau$-family $\mathcal{F}$ in $\mathcal{C}$, define $N_\tau(\mathcal{F})$ to be

$$N \circ \mathcal{F} : H \circ \mathcal{F} \Rightarrow I \circ \mathcal{F}.$$

**Prove** that $N_\tau(\mathcal{F})$ is a morphism of $\tau$-families in $\mathcal{D}$. For every morphism of $\tau$-families in $\mathcal{C}$, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $N_\tau(\mathcal{G}) \circ H_\tau(\phi)$ equals $I_\tau(\phi) \circ N_\tau(\mathcal{F})$. In this sense, conclude that $N_\tau$ is a natural transformation $H_\tau \Rightarrow I_\tau$. For the identity natural transformation $\mathrm{Id}_H : H \Rightarrow H$, **prove** that $(\mathrm{Id}_H)_\tau$ is the identity natural transformation of $H_\tau$. For a second natural transformation $M : I \Rightarrow J$, **prove** that $(M \circ N)_\tau$ equals $M_\tau \circ N_\tau$. In this sense, deduce that $(-)_\tau$ is also compatible with natural transformations.

**(l)**(Reductions of Limits to Finite Systems for Concrete Categories) A category is *cofiltering* if for every pair of objects $U$ and $V$ there exists a pair of morphisms, $r : W \to U$ and $s : W \to V$, and for every pair of morphisms, $r, s : V \to U$, there exists a morphism $t : W \to V$ such that $r \circ t$ equals $s \circ t$ (both of these are automatic if the category has an initial object $X$). Assume that the category $\mathcal{C}$ has limits for all categories $\tau$ with finitely many objects, and also for all small cofiltering categories. For an arbitrary small category $\tau$, define $\widehat{\tau}$ to be the small category whose objects are finite full subcategories $\sigma$ of $\tau$, and whose morphisms are inclusions of subcategories, $\rho \subset \sigma$, of $\tau$. **Prove** that $\widehat{\tau}$ is cofiltering. Let $\mathcal{F}$ be a $\tau$-family in $\mathcal{C}$. For every finite full subcategory $\sigma \subset \tau$, denote by $\mathcal{F}_\sigma$ the restriction as in (f) above. By hypothesis, there is a limit $\eta_\sigma : \underline{\widehat{\mathcal{F}}(\sigma)}_\sigma \Rightarrow \mathcal{F}_\sigma$. Moreover, by (g), for every inclusion of full subcategories $\rho \subset \sigma$, there is a natural morphism in $\mathcal{C}$, $\widehat{\mathcal{F}}(\rho) \to \widehat{\mathcal{F}}(\sigma)$, and this is functorial. Conclude that $\widehat{\mathcal{F}}$ is a $\widehat{\tau}$-family in $\mathcal{C}$. Since $\widehat{\tau}$ is filtering, there is a limit

$$\eta_{\widehat{\mathcal{F}}} : \underline{a}_{\widehat{\tau}} \Rightarrow \widehat{\mathcal{F}}.$$

**Prove** that this defines a limit $\eta_{\mathcal{F}} \underline{a}_\tau \Rightarrow \mathcal{F}$.

Finally, use this to **prove** that limits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where $R$ and $S$ are associative, unital rings).

**(m)**(bis, Colimits) Repeat the steps above for colimits in place of limits. Use this to **prove** that colimits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R-S$-bimodules (where $R$ and $S$ are associative, unital rings).

**Problem 1.**(Categories of Topologies on a Fixed Set) Recall from Problem 1(iv) on Problem Set 3, for every partially ordered set there is an associated category. For a set $P$, form the partially ordered set $\mathcal{P}(P)$ of subsets $S$ of $P$. Then for objects $S$, $S'$ of the category $\mathcal{P}(P)$, i.e., for subsets of $P$, the Hom set $\mathrm{Hom}_{\mathcal{P}(P)}(S, S')$ is nonempty if and only if $S' \subset S$, in which case the Hom set is a singleton set. In particular, this category has arbitrary (inverse) limits, namely unions, and it has arbitrary colimits (direct limits), namely intersections. Moreover, it has a final object, $\emptyset$, and it has an initial object, $P$.

Now let $X$ be a set, and let $P$ be $\mathcal{P}(X)$, so that $P$ is a lattice. Denote by $\text{Power}_X$ the category from the previous paragraph. Thus, objects are subsets $S \subset \mathcal{P}(X)$, and there exists a morphism from $S$ to $S'$ if and only if $S' \subset S$, and then the morphism is unique. We say that $S$ *refines* $S'$. There is a covariant functor

$$\cup : \mathcal{P}(P) \to P, \cup S = \{x \in X | \exists p \in S, x \in p\},$$

and a contravariant functor

$$\cap : \mathcal{P}(P)^{\text{opp}} \to P, \cap S = \{x \in X | \forall p \in S, x \in p\}.$$

By convention, $\cup\emptyset = \emptyset$ and $\cap\emptyset = X$.

A *topology* on $X$ is a subset $\tau \subset \mathcal{P}(X)$ such that (i) $\emptyset \in \tau$ and $X \in \tau$, (ii) for every finite subset $S \subset \tau$, also $\cap S$ is in $\tau$, and (iii) for every $S \subset \tau$ (possibly infinite), the set $\cup S$ is in $\tau$. Denote by $\text{Top}_X$ the full subcategory of $\text{Power}_X$ whose objects are topologies on $X$. A *topological basis* on $X$ is a subset $B \subset \mathcal{P}(X)$ such that for every finite subset $S$ of $B$, the set $V = \cap S$ equals $\cup B_V$, where $B_V = \{U \in B : U \subset V\}$. Denote by $\text{Basis}_X$ the full subcategory of $\text{Power}_X$ whose objects are topological bases on $X$.

**(a) Prove** that $\text{Top}_X$ is stable under colimits, i.e., for every collection of topologies, there is a topology that is refined by every topology in the collection and that refines every topology that is refined by every topology in the collection. **Prove** that $\text{Top}_X$ is a full subcategory of $\text{Basis}_X$. For every topological basis $B$ on $X$, define $\mathcal{T}(B)$ to consist of all elements $\cup S$ for $S \subset B$. **Prove** that $\mathcal{T}(B)$ is a topology on $X$. **Prove** that this uniquely extends to a functor

$$\mathcal{T} : \text{Basis}_X \to \text{Top}_X,$$

and **prove** that $\mathcal{T}$ is a right adjoint of the full embedding. Moreover, for every subset $S \subset \mathcal{P}(X)$, define $\mathcal{B}(S)$ to consist of all elements $\cap R$ for $R \subset S$ a *finite* subset. In particular, $\cap\emptyset = X$ is an element of $\mathcal{B}(S)$. **Prove** that $\mathcal{B}(S)$ is topological basis on $X$. **Prove** that this uniquely extends to a functor

$$\mathcal{B} : \text{Power}_X \to \text{Basis}_X,$$

and **prove** that $\mathcal{T} \circ \mathcal{B}$ is a right adjoint to the full embedding of $\text{Basis}_X$ in $\text{Power}_X$.

**(b) Prove** that for every adjoint pair of functors, the left adjoint functor preserves colimits (direct limits), and the right adjoint functor preserves limits (inverse limits). Conclude that $\text{Top}_X$ is stable under limits, i.e., for every collection of topologies, there is a topology that refines every topology in the collection and that is refined by every topology that refines every topology in the collection.

**(c)** Let $f : Y \to X$ be a set map. Denote by

$$\mathcal{P}^f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

the functor that associates to every subset $S$ of $X$ the preimage subset $f^{-1}(S)$ of $Y$, and denote by

$$\mathcal{P}_f : \mathcal{P}(Y) \to \mathcal{P}(X)$$

the functor that associates to every subset $T$ of $Y$ the image subset $f(T)$ of $X$. **Prove** that $(\mathcal{P}^f, \mathcal{P}_f)$ extends uniquely to an adjoint pair of functors. In particular, define

$$\text{Power}_f : \text{Power}_X \to \text{Power}_Y$$

to be $\mathcal{P}_{\mathcal{P}^f}$, i.e., for every subset $S \subset \mathcal{P}(X)$, $\text{Power}_f(S) \subset \mathcal{P}(Y)$ is the set of all subsets $f^{-1}(U) \subset Y$ for subsets $U \subset X$ that are in $S$. Similarly, define

$$\text{Power}^f : \text{Power}_Y \to \text{Power}_X,$$

to be $\mathcal{P}^{\mathcal{P}^f}$, i.e., for every subset $T \subset \mathcal{P}(Y)$, $\text{Power}^f(Y) \subset \mathcal{P}(X)$ is the set of all subsets $U \subset X$ such that the subset $f^{-1}(U) \subset Y$ is in $T$. **Prove** that $(\text{Power}^f, \text{Power}_f)$ extends uniquely to an adjoint pair of functors. **Prove** that $\text{Power}_f$ and $\text{Power}^f$ restrict to functors $\text{Top}_X \to \text{Top}_Y$. For a given topology $\sigma$ on $Y$ and $\tau$ on $X$, $f$ is *continuous* with respect to $\sigma$ and $\tau$ if $\sigma$ refines $\text{Power}_f(\tau)$, i.e., for every $\tau$-open subset $U$ of $X$, also $f^{-1}(U)$ is $\sigma$-open in $Y$. For a given topology $\tau$ on $X$, for every topology $\sigma$ on $Y$, $\sigma$ refines $\text{Power}_f(\tau)$ if and only if $f$ is continuous with respect to $\sigma$ and $\tau$. Similarly, for a given topology $\sigma$ on $Y$, for every topology $\tau$ on $X$, $\text{Power}^f(\sigma)$ refines $\tau$ if and only if $f$ is continuous with respect to $\sigma$ and $\tau$.

**Problem 2.**(The Category of Topological Spaces) A topological space is a pair $(X, \tau)$ of a set $X$ and a topology $\tau$ on $X$. For topological spaces $(X, \tau)$ and $(Y, \sigma)$, a *continuous map* is a function $f : X \to Y$ such that for every subset $V$ of $Y$ that is in $\sigma$, the inverse image subset $f^{-1}(V)$ of $X$ is in $\tau$, i.e., $\sigma$ refines $\text{Power}_f(\tau)$ and $\tau$ is refined by $\text{Power}^f(\sigma)$.

**(a) Prove** that for every topological space $(X, \tau)$, the identity function $\text{Id}_X : X \to X$ is a continuous map from $(X, \tau)$ to $(X, \tau)$. For every pair of continuous maps $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \rho)$, **prove** that the composition $g \circ f : (X, \tau) \to (Z, \rho)$ is a continuous map. With this notion of composition of continuous map, check that the topological spaces and continuous maps form a category, **Top**.

**(b)** For every topological space $(X, \tau)$, define $\Phi(X)$ to be the set $X$. For every continuous map of topological spaces, $f : (X, \tau) \to (Y, \sigma)$, define $\Phi(f) : \Phi(X) \to \Phi(Y)$ to be $f : X \to Y$. **Prove** that this defines a covariant functor,

$$\Phi : \textbf{Top} \to \textbf{Sets}.$$

**(c)** For every set $X$, define $L(X) = (X, \mathcal{P}(X))$, i.e., every subset of $X$ is open. **Prove** that $\mathcal{P}(X)$ satisfies the axioms for a topology on $X$. This is called the *discrete topology* on $X$. For every set map, $f : X \to Y$, **prove** that $f : (X, \mathcal{P}(X)) \to (Y, \mathcal{P}(Y))$ is a continuous map, denoted $L(f)$. **Prove** that this defines a functor,

$$L : \textbf{Sets} \to \textbf{Top}.$$

For every set $X$, define $\theta_X : X \to \Phi(L(X))$ to be the identity map on $X$. **Prove** that $\theta$ is a natural equivalence $\text{Id}_{\textbf{Sets}} \Rightarrow \Phi \circ L$. For every topological space $(X, \tau)$, **prove** that $\text{Id}_X$ is a continuous map $(X, \mathcal{P}(X)) \to (X, \tau)$, denoted $\eta_{(X,\tau)}$. **Prove** that $\eta$ is a natural transformation $L \circ \Phi \Rightarrow \text{Id}_{\textbf{Top}}$. **Prove** that $(L, \Phi, \theta, \eta)$ is an adjoint pair of functors. In particular, $\Phi$ preserves monomorphisms and limits (inverse limits).

**(d)** For every set $X$, define $R(X) = (X, \{\emptyset, X\})$. **Prove** that $\{\emptyset, X\}$ satisfies the axioms for a topology on $X$. This is called the *indiscrete topology* on $X$. For every set map $f : X \to Y$, **prove** that $f : R(X) \to R(Y)$ is a continuous map, denoted $R(f)$. **Prove** that this defines a functor,

$$R : \textbf{Sets} \to \textbf{Top}.$$

For every set topological space $(X, \tau)$, **prove** that $\text{Id}_X$ is a continous map $(X, \tau) \to R(\Phi(X, \tau))$, denoted $\alpha_{(X,\tau)}$. **Prove** that $\alpha$ is a natural transformation $\text{Id}_{\textbf{Top}} \Rightarrow R \circ \Phi$. For every set $S$, denote by $\beta_X : \Phi(R(X)) \to X$ the identity morphism. **Prove** that $\beta$ is a natural equivalence $\Phi \circ R \Rightarrow \text{Id}_{\textbf{Sets}}$. **Prove** that $(\Phi, R, \alpha, \beta)$ is an adjoint pair of functors. In particular, $\Phi$ preserves epimorphisms and colimits (direct limits).

**(e)** Use the method of Problem 0 to prove that **Top** has (small) limits and colimits. Finally, **prove** that the projective objects in **Top** are precisely the discrete topological spaces, and the injective objects in **Top** are precisely the nonempty indiscrete topological spaces.

**Problem 3.**(Presheaves) Let $(X, \tau_X)$ be a topological space. As above, consider $\tau_X$ as a category whose objects are open sets $U$ of the topology, and where for open sets $U$ and $V$, there is a unique morphism from $U$ to $V$ if $U \supseteq V$, and otherwise there is no morphism. Let $\mathcal{C}$ be a category. A **presheaf** on $(X, \tau_X)$ of objects of $\mathcal{C}$ is a functor,

$$A : \tau_X \to \mathcal{C},$$

i.e., a $\tau_X$-family as in Problem 0. By Problem 0, the $\tau$-families form a category $\textbf{Fun}(\tau_X, \mathcal{C})$, called the category of presheaves of objects of $\mathcal{C}$. For every continuous map $f : (Y, \tau_Y) \to (X, \tau_X)$, define

$$f^{-1} : \tau_X \to \tau_Y,$$

as in Problem 1(c), i.e., $U \mapsto f^{-1}(U)$. The corresponding functor

$$*_{f^{-1}} : \textbf{Fun}(\tau_Y, \mathcal{C}) \to \textbf{Fun}(\tau_X, \mathcal{C})$$

is called the *direct image functor* and is denoted $f_*$, i.e., for every presheaf $\mathcal{F}$ on $(Y, \tau_Y)$, $f_*\mathcal{F}$ is a presheaf on $(X, \tau_X)$ given by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.

**(a)** Denote by $\sigma_f$ the category whose objects are pairs $(U, V)$ of an object $U$ of $\tau_X$ and an object $V$ of $\tau_V$ such that $V$ is contained in $f^{-1}(U)$. For objects $(U, V)$ and $(U', V')$, there is a morphism from $(U, V)$ to $(U', V')$ if and only if there is a morphism $U \supseteq U'$ in $\tau_X$ and a morphism $V \supseteq V'$ in $\tau_Y$, and in this case the morphism for $(U, V)$ to $(U', V')$ is unique. **Prove** that this is a category. **Prove** that the map on objects,

$$x : \sigma_f \to \tau_X, \ (U, V) \mapsto U,$$

extends uniquely to a functor that is essentially surjective (in fact strictly surjective on objects). **Prove** that the following maps on objects,

$$\ell x : \tau_X \to \sigma_f,\ U \mapsto (U, f^{-1}(U)),$$

$$rx : \tau_X \to \sigma_f,\ U \mapsto (U, \emptyset)$$

extend uniquely to functors, and **prove** that $(\ell x, x)$ and $(x, rx)$ extend uniquely to adjoint functors, i.e., $(U, f^{-1}(U))$, resp. $(U, \emptyset)$, is the initial object, resp. final object, in the fiber category $(\sigma_f)_{x,U}$. **Prove** that the map on objects

$$y : \sigma_f \to \tau_Y,\ (U, V) \mapsto V$$

extends uniquely to a functor that is essentially surjective (in fact strictly surjective on objects). **Prove** that the following map on objects,

$$\ell y : \tau_Y \to \sigma_f,\ V \mapsto (X, V),$$

extends uniquely to a functor, and **prove** that $(\ell y, y)$ extends uniquely to an adjoint functor, i.e., $(X, V)$ is the initial object in the fiber category $(\sigma_f)_{y,V}$. Prove that $y \circ \ell_x$ is the functor $f^{-1} : \tau_X \to \tau_Y$ from above. **Find** an example where $y$ does not admit a right adjoint.

Assume now that $\mathcal{C}$ has colimits. Apply Problem 0(g) to conclude that there are adjoint pairs of functors $(*_x, *_{\ell x})$, $(*_{rx}, *_x)$, $(*_y, *_{\ell y})$, and $(L_y, *_y)$. Compose these adjoint pairs to obtain an adjoint pair $(L_y \circ *_x, *_{\ell x} \circ *_y)$. Also, by functoriality of $*_z$ in $z$, $*_{\ell x} \circ *_y$ equals $*_{y \circ \ell x}$, and this equals $*_{f^{-1}}$. Thus, this is an adjoint pair $(L_y \circ *_x, f_*)$. Unwind the defintions from Problem 0(g) to **check** that for every presheaf $A$ on $X$ and for every $V$ an object of $\tau_Y$, $L_y \circ *_x(A)$ on $V$ is the colimit over the fiber category $(\sigma_f)_{y,V}$ of all $U$ an object of $\tau_X$ with $V \subseteq f^{-1}(U)$ of $A(U)$. The functor $L_y \circ *_x$ is the *inverse image functor* for presheaves,

$$f^{-1} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_Y, \mathcal{C}).$$

**Problem 4.**(Constant Cosimplicial Objects and the Right Adjoint) Please read the basic definitions of cosimplicial objects in a category $\mathcal{C}$. In particular, for the small category $\Delta$ of totally ordered finite sets with nondecreasing morphisms, read the equivalent characterization of a (covariant) functor

$$C : \Delta \to \mathcal{C},$$

via the specification for every integer $r \geq 0$ of an object $C^r$ of $\mathcal{C}$, the specification for every integer $r \geq 0$ and every integer $i = 0, \dots, r+1$, of a morphism,

$$\partial_r^i : C^r \to C^{r+1},$$

and the specification for every integer $r \geq 0$ and every integer $i = 0, \dots, r$, of a morphism,

$$\sigma_{r+1}^i : C^{r+1} \to C^r,$$

satisfying the *cosimplicial identities*: for every $r \geq 0$, for every $0 \leq i < j \leq r+2$,

$$\partial_{r+1}^j \circ \partial_r^i = \partial_{r+1}^i \circ \partial_r^{j-1},$$

for every $0 \leq i \leq j \leq r$,

$$\sigma_{r+1}^j \circ \sigma_{r+2}^i = \sigma_{r+1}^i \circ \sigma_{r+2}^{j+1},$$

and for every $0 \leq i \leq r+1$ and $0 \leq j \leq r$,

$$\sigma_{r+1}^j \circ \partial_r^i = \begin{cases} \partial_{r-1}^i \circ \sigma_r^{j-1}, & i < j, \\ \mathrm{Id}_{C^r}, & i = j, i = j+1, \\ \partial_{r-1}^{i-1} \circ \sigma_r^j, & i > j+1 \end{cases}$$

Moreover, for cosimplicial objects $C^\bullet = (C^r, \partial_r^i, \sigma_{r+1}^i)$ and $\widetilde{C}^\bullet = (\widetilde{C}^r, \widetilde{\partial}_r^i, \widetilde{\sigma}_{r+1}^i)$, read about the equivalent specification of a natural transformation $\alpha^\bullet : C^\bullet \to \widetilde{C}^\bullet$ as the specification for every integer $r \geq 0$ of a $\mathcal{C}$-morphism $\alpha^r : C^r \to \widetilde{C}^r$ such that for every $r$ and $i$,

$$\widetilde{\partial}_r^i \circ \alpha^r = \alpha^{r+1} \circ \partial_r^i, \quad \widetilde{\sigma}_{r+1}^i \circ \alpha^{r+1} = \alpha^r \circ \sigma_{r+1}^i.$$

Finally, for every pair of morphisms of cosimplicial objects, $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$, a *cosimplicial homotopy* is a specification for every integer $r \geq 0$ and for every integer $i = 0, \dots, r$ of a $\mathcal{C}$-morphism,

$$g_{r+1}^i : C^{r+1} \to \widetilde{C}^r,$$

satisfying the following *cosimplicial homotopy identities*: for every $r \geq 0$,

$$g_{r+1}^0 \circ \partial_r^0 = \alpha^r, \quad g_{r+1}^r \circ \partial_r^{r+1} = \beta^r,$$

$$g_{r+1}^j \circ \partial_r^i = \begin{cases} \widetilde{\partial}_{r-1}^i \circ g_r^{j-1}, & 0 \leq i < j \leq r, \\ g_{r+1}^{i-1} \circ \partial_r^i, & 0 < i = j \leq r, \\ \widetilde{\partial}_{r-1}^{i-1} \circ g_r^j, & 1 \leq j+1 < i \leq r+1. \end{cases}$$

$$g_r^j \circ \sigma_{r+1}^i = \begin{cases} \widetilde{\sigma}_r^i \circ g_{r+1}^{j+1}, & 0 \leq i \leq j \leq r-1, \\ \widetilde{\sigma}_r^{i-1} \circ g_{r+1}^j, & 0 \leq j < i \leq r. \end{cases}$$

**(a)**(Constant Cosimplicial Objects) For every object $C$ of $\mathcal{C}$, define $\mathrm{const}(C)$ to be the rule that associates to every integer $r \geq 0$ the object $C$ of $\mathcal{C}$, and that associates to $(r, i)$ the morphisms $\partial_r^i = \mathrm{Id}_C$, $\sigma_{r+1}^i = \mathrm{Id}_C$. **Prove** that $\mathrm{const}(C)$ is a cosimplicial object of $\mathcal{C}$. For every morphism of objects $\alpha : C \to \widetilde{C}$, **prove** that the specification for every integer $r \geq 0$ of the morphism $\alpha : C \to \widetilde{C}$ defines a morphism of cosimplicial objects,

$$\mathrm{const}(\alpha) : \mathrm{const}(C) \to \mathrm{const}(\widetilde{C}).$$

**Prove** that $\mathrm{const}(\mathrm{Id}_C)$ is the identity morphism of $\mathrm{const}(C)$. For a pair of morphisms, $\alpha : C \to \widetilde{C}$ and $\beta : \widetilde{C} \to \widehat{C}$, **prove** that $\mathrm{const}(\beta \circ \alpha)$ equals $\mathrm{const}(\beta) \circ \mathrm{const}(\alpha)$. Conclude that these rules define a functor

$$\mathrm{const} : \mathcal{C} \to \mathbf{Fun}(\Delta, \mathcal{C}).$$

**Prove** that this is functorial in $\mathcal{C}$, i.e., given a functor $F : \mathcal{C} \to \mathcal{D}$, for the associated functor,

$$\mathbf{Fun}(\Delta, F) : \mathbf{Fun}(\Delta, \mathcal{C}) \to \mathbf{Fun}(\Delta, \mathcal{D}),\ (C^r, \partial^i_r, \sigma^i_{r+1}) \mapsto (F(C^r), F(\partial^i_r), F(\sigma^i_{r+1})),$$

$\mathbf{Fun}(\Delta, F) \circ \text{const}_{\mathcal{C}}$ strictly equals $\text{const}_{\mathcal{D}} \circ F$.

**(b)**(Morphisms from a Constant Cosimplicial Object) For every integer $r \geq 1$ and for every pair of distinct morphisms $[0] \to [r]$, **prove** that there exists a unique $\Delta$-morphism $F : [1] \to [r]$ such that the two morphisms are $F \circ \partial^0_0$ and $F \circ \partial^1_0$. Let $C^\bullet = (C^r, \partial^i_r, \sigma^i_{r+1})$ be a cosimplicial object in $\mathcal{C}$. For every object $A$ of $\mathcal{C}$ and for every morphism, $\alpha^\bullet : \text{const}(A) \to C^\bullet$, of cosimplicial objects, **prove** that $\alpha^0 : A \to C^0$ is a morphism such that $\partial^0_0 \circ \alpha^0$ equals $\partial^1_0 \circ \alpha^0$. **Prove** that the morphism $\alpha^\bullet$ is uniquely determined by $\alpha^0$, i.e., for every $r \geq 0$, and for every morphism $f : [0] \to [r]$, $\alpha^r : A \to C^r$ equals $C(f) \circ \alpha^0$. Conversely, for every morphism $\alpha^0 : A \to C^0$ such that $\partial^0_0 \circ \alpha^0$ equals $\partial^1_0 \circ \alpha^0$, **prove** that the morphisms $\alpha^r := C(f) \circ \alpha^0$ are well-defined and define a morphism $\alpha^\bullet : \text{const}(A) \to C^\bullet$ of cosimplicial objects. Conclude that the set map,

$$\text{Hom}_{\mathbf{Fun}(\Delta,\mathcal{C})}(\text{const}(A), C^\bullet) \to \{\alpha^0 \in \text{Hom}_{\mathcal{C}}(A, C^0) | \partial^0_0 \circ \alpha^0 = \partial^1_0 \circ \alpha^0\},\ \alpha^\bullet \mapsto \alpha^0,$$

is a bijection. **Prove** that this bijection is natural in both $A$ and in $C^\bullet$. In particular, conclude that the functor,

$$\text{const} : \mathcal{C} \to \mathbf{Fun}(\Delta, \mathcal{C}),$$

is fully faithful. Finally, for every pair of morphisms, $\alpha^0, \beta^0 : A \to C^0$ equalizing $\partial^0_0$ and $\partial^1_0$, **prove** that there exists a cosimplicial homotopy $g^i_{r+1} : A \to C^r$ from $\alpha^\bullet$ to $\beta^\bullet$ if and only if $\beta^0$ equals $\alpha^0$, and in this case there is a unique cosimplicial homotopy given by $g^i_{r+1} = \alpha^r = \beta^r$.

**(c)**(Equalizers in Cartesian Categories) Let $\Delta_{\leq 1}$ be the category of totally ordered sets of cardinality $\leq 1$. Prove that a functor $C^\bullet : \Delta_{\leq 1} \to \mathcal{C}$ is equivalent to the data of a pair of objects $C^0$, $C^1$, a pair of morphisms $\partial^0_0, \partial^1_0 : C^0 \to C^1$, and a morphism $\sigma^0_1 : C^1 \to C^0$ such that $\sigma^0_1 \circ \partial^0_0 = \sigma^0_1 \circ \partial^1_0 = \text{Id}_{C^0}$. Let,

$$Z^0 : \mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C}) \to \mathcal{C},$$

be a functor and let,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C})},$$

be a natural transformation such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors $(\text{const}, Z^0, \theta, \eta)$. **Prove** that the natural transformation $\theta$ is a natural isomorphism. **Prove** that for every object $C^\bullet$ of $\mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C})$, the morphism $\eta_{C^\bullet} : Z^0(C^\bullet) \to C^0$ satisfies $\partial^0_0 \circ \eta_{C^\bullet} = \partial^1_0 \circ \eta_{C^\bullet}$ and is final among all such morphisms. **Prove** that if $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$ are two morphisms of cosimplicial objects, and if $(g^i_{r+1} : C^{r+1} \to \widetilde{C}^r)$ is a cosimplicial homotopy from $\alpha^\bullet$ to $\beta^\bullet$, then $Z^0(\alpha^\bullet)$ equals $Z^0(\beta^\bullet)$.

Assume that $\mathcal{C}$ has finite products. For every pair of objects $N^0$ and $N^1$ of $\mathcal{C}$ and for every pair of morphisms $d^0_0, d^1_0 : N^0 \to N^1$, define $C^0 = N^0$, define $C^1 = N^0 \times N^1$, define $\partial^0_0 = (\text{Id}_{C^0}, d^0_0)$, define $\partial^1_0 = (\text{Id}_{C^0}, d^1_0)$, and define $\sigma^0_1 = \text{pr}_{N^0}$. **Prove** that $C^\bullet$ is an object of $\mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C})$, and **prove** that $\eta_{C^\bullet} : Z^0(C^\bullet) \to C^0$ is an equalizer of $d^0_0, d^1_0 : N^0 \to N^1$. In particular, if $\mathcal{C}$ has both finite products and $Z^0$, **prove** that $\mathcal{C}$ has all equalizers of a pair of morphisms. For every pair of morphisms $f_0^0 : M_0^0 \to N^1$ and $f_0^1 : M_1^0 \to N^1$ in $\mathcal{C}$, for $N^0 = M_0^0 \times M_1^0$, and for $d_0^0 = f_0^0 \circ \mathrm{pr}_{M_0^0}$ and $d_0^1 = f_0^1 \circ \mathrm{pr}_{M_1^0}$, **prove** that the equalizer of $d_0^0, d_0^1 : N^0 \to N^1$ is a fiber product of $f_0^0$ and $f_0^1$. Conclude that $\mathcal{C}$ has all finite fiber products, i.e., $\mathcal{C}$ is a *Cartesian category*. Conversely, assuming that $\mathcal{C}$ is a Cartesian category, then, up to some form of the Axiom of Choice, prove that there exists a functor $Z^0$ and a natural transformation $\eta$ such that $(\mathrm{const}, Z^0, \eta)$ extends to an adjoint pair of functors.

**(d)**(The Right Adjoint to the Constant Cosimplicial Object) Assume now that there exists a functor

$$Z^0 : \mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta : \mathrm{const} \circ Z^0 \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C})},$$

such that $(\mathrm{const}, Z^0, \eta)$ extends to an adjoint pair of functors. For every cosimplicial object $C^\bullet : \Delta \to \mathcal{C}$, for the equalizer $\eta : Z^0(C^\bullet) \to C^0$ of $\partial_0^0$ and $\partial_0^1$, use (b) above to prove that there exists a unique extension $\eta^\bullet : \mathrm{const}(Z^0) \to C^\bullet$ of $\eta$ to a morphism of cosimplicial objects of $\mathcal{C}$. **Prove** that this defines a functor,

$$Z^0 : \mathbf{Fun}(\Delta, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta^\bullet : \mathrm{const} \circ Z^0 \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\Delta, \mathcal{C})},$$

such that $(\mathrm{const}, Z^0, \eta^\bullet)$ extends uniquely to an adjoint pair of functors, $(\mathrm{const}, Z^0, \eta^\bullet, \theta)$. **Prove** that $\theta$ is a natural isomorphism. **Prove** that if $\alpha^\bullet, \beta^\bullet : C^\bullet \to \widetilde{C}^\bullet$ are two morphisms of cosimplicial objects, and if $(g_{r+1}^i : C^{r+1} \to \widetilde{C}^r)$ is a cosimplicial homotopy from $\alpha^\bullet$ to $\beta^\bullet$, then $Z^0(\alpha^\bullet)$ equals $Z^0(\beta^\bullet)$.

**Problem 5.**(Čech Cosimplicial Object of a Covering) Let $(X, \tau_X)$ be a topological space. For every object $U$ of $\tau_X$, **prove** that the topology $\tau_U$ on $U$ associated to $i : U \to X$ via Problem 1(c) is a full, upper subcategory of $\tau_X$ that has an initial object $\odot = U$. For every $U$, an *open covering* of $U$ is a set $\mathfrak{U}$ and a set map $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$ such that $\cup \mathrm{Image}(\iota_{\mathfrak{U}})$ equals $U$. Define $\sigma$ to be the category whose objects are pairs $(U, \mathfrak{U})$ of an open $U$ in $\tau_X$ and an open covering $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$. For objects $(U, \mathfrak{U})$ and $(V, \mathfrak{V})$, a $\sigma$-morphism from $(U, \mathfrak{U})$ to $(V, \mathfrak{V})$ is a pair $U \supseteq V$ of a morphism in $\tau_X$ and a *refinement* $\phi : \mathfrak{U} \succeq \mathfrak{V}$, i.e., a set function $\phi : \mathfrak{V} \to \mathfrak{U}$ such that for every $V_0$ in $\mathfrak{V}$, $\iota_{\mathfrak{U}}(\phi(V_0))$ contains $\iota_{\mathfrak{V}}(V_0)$. In particular, for every object $(U, \iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U)$ of $\sigma$, define $\mathfrak{V} = \mathrm{Image}(\iota_{\mathfrak{U}})$ with its natural inclusion $\iota_{\mathfrak{V}} : \mathfrak{V} \hookrightarrow \tau_U$ . Up to the Axiom of Choice, **prove** that there exists a refinement $\phi : (U, \mathfrak{U}) \succeq (U, \mathfrak{V})$. Thus, the open coverings with $\iota$ a monomorphism are cofinal in the category $\sigma$.

**(a)**(Category of Open Coverings) For every pair of refinements, $\phi : (U, \mathfrak{U}) \succeq (V, \mathfrak{V})$ and $\psi : (V, \mathfrak{V}) \succeq (W, \mathfrak{W})$, **prove** that the composition $\phi \circ \psi : \mathfrak{W} \to \mathfrak{U}$ is a refinement, $\phi \circ \psi : (U, \mathfrak{U}) \to (W, \mathfrak{W})$. Also **prove** that $\mathrm{Id}_{\mathfrak{U}} : \mathfrak{U} \to \mathfrak{U}$ is a refinement $(U, \mathfrak{U}) \to (U, \mathfrak{U})$. Conclude that these rules define a category $\sigma$ whose objects are open coverings $(U, \mathfrak{U})$ of opens $U$ in $\tau_X$ and whose morphisms are refinements. Define $x : \sigma \to \tau_X$ to be the rule that associates to every $(U, \mathfrak{U})$ the open $U$ and that associates to every refinement $\phi : (U,\mathfrak{U}) \succeq (V,\mathfrak{V})$ the inclusion $U \supseteq V$. **Prove** that this is a strictly surjective functor. **Prove** that the map on objects,

$$\ell x : \tau_X \to \sigma,\ U \mapsto (U, \{U\}),$$

extends uniquely to a functor, and **prove** that $(\ell x, x)$ extends uniquely to an adjoint pair of functors, i.e., $(U,\{U\})$ is the initial object in the fiber category $\sigma_{x,U}$. Typically $x$ does not admit a right adjoint.

For every open covering $\iota_{\mathfrak{U}} : \mathfrak{U} \to \tau_U$, for every integer $r \geq 0$, define the following set map,

$$\iota_{\mathfrak{U}^{r+1}} : \mathfrak{U}^{r+1} \to \tau_U,\ (U_0, U_1, \dots, U_r) \mapsto \iota_{\mathfrak{U}}(U_0) \cap \iota_{\mathfrak{U}}(U_1) \cap \dots \cap \iota_{\mathfrak{U}}(U_r).$$

Let $\mathcal{C}$ be a category, and let $A$ be an $\mathcal{C}$-presheaf on $(X,\tau_X)$. Let $(U,\mathfrak{U})$ be an object of $\sigma$. Recall that for every object $T$ of $\mathcal{C}$, there is a Yoneda functor,

$$h_T : \mathcal{C}^{\text{opp}} \to \textbf{Sets},\ S \mapsto \textbf{Hom}_{\mathcal{C}}(S,T),$$

and this is covariant in $T$. For every integer $r \geq 0$, define

$$h_{A,\mathfrak{U},r} : \mathcal{C}^{\text{opp}} \to \textbf{Sets},\ S \mapsto \prod_{(U_0,\dots,U_r)\in\mathfrak{U}^{r+1}} h_{A(\iota(U_0,\dots,U_r))}(S),$$

together with the projections,

$$\pi_{(U_0,\dots,U_r)} : h_{A,\mathfrak{U},r} \to h_{A(\iota(U_0,\dots,U_r))}.$$

For every integer $r \geq 0$, and for every integer $i = 0, \dots, r+1$, define

$$\partial_r^i : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{U},r+1},$$

to be the unique natural transformation such that for every $(U_0,\dots,U_{r+1}) \in \mathfrak{U}^{r+2}$, $\pi_{(U_0,\dots,U_{r+1})} \circ \partial_r^i$ equals the composition of the projection,

$$\pi_{(U_0,\dots,U_{i-1},U_{i+1},\dots,U_{r+1})} : h_{A,\mathfrak{U},r} \to h_{A(\iota(U_0,\dots,U_{i-1},U_{i+1}\cap,\cap U_{r+1}))},$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota(U_0)\cap\dots\cap\iota(U_{i-1})\cap\iota(U_{i+1})\cap\dots\cap\iota(U_{r+1})) \to A(\iota(U_0)\cap\dots\cap\iota(U_{r+1})).$$

Similarly, for every $i = 0, \dots, r$, define

$$\sigma_{r+1}^i : h_{A,\mathfrak{U},r+1} \to h_{A,\mathfrak{U},r}$$

to be the unique natural transformation such that for every $(U_0,\dots,U_r) \in \mathfrak{U}^{r+1}$, $\pi_{(U_0,\dots,U_{r+1})} \circ \sigma_{r+1}^i$ equals the projection $\pi_{(U_0,\dots,U_{i-1},U_i,U_i,U_{i+1},\dots,U_r)}$.

**(b)**(Cosimplicial Identities) **Prove** that these natural transformations satisfy the *cosimplicial identities*: for every $r \geq 0$, for every $0 \leq i < j \leq r+2$,

$$\partial^j_{r+1} \circ \partial^i_r = \partial^i_{r+1} \circ \partial^{j-1}_r,$$

for every $0 \leq i \leq j \leq r$,

$$\sigma^j_{r+1} \circ \sigma^i_{r+2} = \sigma^i_{r+1} \circ \sigma^{j+1}_{r+2},$$

and for every $0 \leq i \leq r+1$ and $0 \leq j \leq r$,

$$\sigma^j_{r+1} \circ \partial^i_r = \begin{cases} \partial^i_{r-1} \circ \sigma^{j-1}_r, & i < j, \\ \mathrm{Id}, & i = j, i = j+1, \\ \partial^{i-1}_{r-1} \circ \sigma^j_r, & i > j+1 \end{cases}$$

In the case that $\mathcal{C}$ is an additive category, define

$$d^r : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{U},r+1}, \quad d^r = \sum_{i=0}^{r+1} \partial^i_r.$$

**Prove** that $d^{r+1} \circ d^r$ equals 0.

**(c)**(Refinements and Cosimplicial Homotopies) For every refinement, $\phi : (U, \mathfrak{U}) \succeq (V, \mathfrak{V})$, for every integer $r \geq 0$, define

$$h_{A,\phi,r} : h_{A,\mathfrak{U},r} \to h_{A,\mathfrak{V},r}$$

to be the unique natural transformation such that for every $(V_0, \dots, V_r) \in \mathfrak{V}^{r+1}$, the composition $\pi_{(V_0,\dots,V_r)} \circ h_{A,q,r}$ equals the composition of projection

$$\pi_{(\phi(V_0),\dots,\phi(V_r))} : h_{A,\mathfrak{U},r} \to h_{A(\phi(V_0)\cap\dots\cap\phi(V_r))}$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota\phi(V_0) \cap \dots \cap \iota\phi(V_r)) \to A(\iota(V_0) \cap \dots \cap \iota(V_r)).$$

**Prove** that the natural transformations $(h_{A,\phi,r})_{r\geq 0}$ are compatible with the natural transformations $\partial^i_r$ and $\sigma^i_{r+1}$. For every pair of refinements, $\phi : (U, \mathfrak{U}) \succeq (V, \mathfrak{V})$ and $\psi : (V, \mathfrak{V}) \succeq (W, \mathfrak{W})$, for the composition refinement $\phi \circ \psi : (U, \mathfrak{U}) \succeq (W, \mathfrak{W})$, **prove** that $h_{A,\phi\circ\psi,r}$ equals $h_{A,\psi,r} \circ h_{A,\phi,r}$, and also **prove** that $h_{A,\mathrm{Id}_{\mathfrak{U}},r}$ equals $\mathrm{Id}_{h_{A,\mathfrak{U},r}}$. Thus $h_{A,\phi,r}$ is functorial in $\phi$.

Let $\phi : (U, \mathfrak{U}) \succeq (V, \mathfrak{V})$ and $\psi : (U, \mathfrak{U}) \succeq (V, \mathfrak{V})$ be refinements. For every integer $r \geq 0$, for every integer $i = 0, \dots, r$, define

$$g^i_{A,\phi,\psi,r+1} : h_{A,\mathfrak{U},r+1} \to h_{A,\mathfrak{V},r}$$

to be the unique natural transformation such that for every $(V_0, \dots, V_r) \in \mathfrak{V}^{r+1}$, $\pi_{(V_0,\dots,V_r)} \circ g^i_{A,\phi,\psi,r+1}$ equals the composition of the projection,

$$\pi_{\psi(V_0),\dots,\psi(V_i),\phi(V_i),\dots,\phi(V_r)} : h_{A,\mathfrak{U},r+1} \to h_{A(\iota(\psi(V_0),\dots,\psi(V_i),\phi(V_i),\dots,\phi(V_r)))},$$

with the natural transformation of Yoneda functors arising from the restriction morphism

$$A(\iota\psi(V_0)\cap\cdots\cap\iota\psi(V_i)\cap\iota\phi(V_i)\cap\cdots\cap\iota\phi(V_r))\to A(\iota(V_0)\cap\cdots\cap\iota(V_i)\cap\cdots\cap\iota(V_r)).$$

**Prove** the following identities (cosimplicial homotopy identities),

$$g^0_{A,\phi,\psi,r+1}\circ\partial^0_{A,\mathfrak{U},r}=h_{A,\phi,r},\quad g^r_{A,\phi,\psi,r+1}\circ\partial^{r+1}_{A,\mathfrak{U},r}=h_{A,\psi,r},$$

$$g^j_{A,\phi,\psi,r+1}\circ\partial^i_{A,\mathfrak{U},r}=\begin{cases}\partial^i_{A,\mathfrak{V},r-1}\circ g^{j-1}_{A,\phi,\psi,r}, & 0\leq i<j\leq r,\\ g^{i-1}_{A,\phi,\psi,r+1}\circ\partial^i_{A,\mathfrak{U},r}, & 0<i=j\leq r,\\ \partial^{i-1}_{A,\mathfrak{V},r-1}\circ g^j_{A,\phi,\psi,r}, & 1\leq j+1<i\leq r+1.\end{cases}$$

$$g^j_{A,\phi,\psi,r}\circ\sigma^i_{A,\mathfrak{U},r+1}=\begin{cases}\sigma^i_{A,\mathfrak{V},r}\circ g^{j+1}_{A,\phi,\psi,r+1}, & 0\leq i\leq j\leq r-1,\\ \sigma^{i=1}_{A,\mathfrak{V},r}\circ g^j_{A,\phi,\psi,r+1}, & 0\leq j<i\leq r.\end{cases}$$

For the identity refinement $\mathrm{Id}_{\mathfrak{U}}:\mathfrak{U}\succeq\mathfrak{U}$, **prove** that $g^j_{A,\mathrm{Id},\mathrm{Id},r+1}$ equals $\sigma^j_{A,\mathfrak{U},r+1}$. Also **prove** that for refinements $\chi:\mathfrak{V}\to\mathfrak{W}$ and $\xi:\mathfrak{T}\to\mathfrak{U}$, $g^j_{A,\phi\circ\chi,\psi\circ\chi,r+1}$ equals $h_{A,\chi,r}\circ g^j_{A,\phi,\psi,r+1}$ and $g^j_{A,\xi\circ\phi,\xi\circ\psi,r+1}$ equals $g^j_{A,\phi,\psi,r+1}\circ h_{A,\xi,r+1}$.

**(d)**(Functoriality in $A$) For every morphism of $\mathcal{C}$-presheaves, $\alpha:A\to A'$, define

$$h_{\alpha,\mathfrak{U},r}:h_{A,\mathfrak{U},r}\to h_{B,\mathfrak{U},r},$$

to be the unique natural transformation whose postcomposition with each projection $\pi_{B,(U_0,\dots,U_r)}$ equals the composition of $\pi_{A,(U_0,\dots,U_r)}$ with the natural transformation induced by the morphism

$$\alpha_{\iota(U_0,\dots,U_r)}:A(\iota(U_0,\dots,U_r))\to A'(\iota(U_0,\dots,U_r)).$$

**Prove** that these maps are compatible with the cosimplicial operations $\partial^i_r$ and $\sigma^i_{r+1}$, as well as the operations $h_{A,\phi,r}$ associated to a refinement $\phi:\mathfrak{U}\succeq\mathfrak{V}$, and the cosimplicial homotopies $g^i_{A,\phi,\psi,r+1}$ associated to a pair of refinements, $\phi,\psi:\mathfrak{U}\succeq\mathfrak{V}$. **Prove** that this is functorial in $\alpha$. Conclude that (up to serious set-theoretic issues), for every open cover $\mathfrak{U}$, morally these rules define a functor from the category of $\mathcal{C}$-presheaves to the "category" of cosimplicial objects in the category of contravariant functors from $\mathcal{C}$ to **Sets**. Stated differently, to every open cover $\mathfrak{U}$ there is an associated cosimplicial object in the category $\mathbf{Fun}(\mathcal{C}-\mathrm{Presh},\mathbf{Fun}(\mathcal{C},\mathbf{Sets}))$ of covariant functors from the category of $\mathcal{C}$-presheaves to the category of contravariant functors $\mathcal{C}\to\mathbf{Sets}$. This rule is covariant for refinement of open covers. Moreover, up to simplicial homotopy, it is independent of the choice of refinement.

**(e)**(Coadjunction of Sections) As a particular case, for the left adjoint $\ell x$ of $x$, observe that there is a canonical refinement

$$\eta_{U,\mathfrak{U}}:\ell x\circ x(U,\mathfrak{U})\succeq(U,\mathfrak{U}),\ \text{i.e.,}\ (U,\{U\})\succeq(U,\mathfrak{U}).$$

**Prove** that $h_{A,\{U\},r}$ is the constant / diagonal cosimplicial object that for every $r$ associates $h_{A(U)}$ and with $\partial^i$ and $\sigma^i$ equal to the identity morphism. Conclude that for every cover $(U,\mathfrak{U})$ in $\sigma$, there is a natural coaugmentation,

$$g^r_{A,\mathfrak{U}} : h_{A(U)} \to h_{A,\mathfrak{U},r},$$

that is functorial in $A$, functorial in $(U,\mathfrak{U})$ with respect to refinements, and that equalizes the simplicial homotopies associated to a pair of refinements in the sense that

$$g^j_{A,\phi,\psi,r+1} \circ g^{r+1}_{A,\mathfrak{U}} = g^r_{A,\mathfrak{V}} \circ h_{A^U_V}.$$

Define the functor

$$\text{const} : \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C})$$

that associates to a functor $B : \sigma \to \mathcal{C}$ the functor $\text{const}_B : \sigma \to \mathbf{Fun}(\Delta,\mathcal{C})$ whose value on every $(U,\mathfrak{U})$ is the constant / diagonal cosimplicial object $r \mapsto B(U,\mathfrak{U})$ for every $r$ with every $\partial^i$ and $\sigma^i$ defined to be the identity morphism. Conclude that the rule $U \mapsto (r \mapsto h_{A(U)})$ above is the Yoneda functor associated to $\text{const} \circ *_x(A)$.

**(f)**(Čech cosimplicial object) Assume now that $\mathcal{C}$ has all finite products. Thus, for every open covering $(U,\mathfrak{U})$ and for every integer $r \geq 0$, there exists an object

$$\check{C}^r(\mathfrak{U},A) = \prod_{(U_0,\dots,U_r)\in\mathfrak{U}} A(U_0 \cap \dots \cap U_r),$$

such that $h_{A,\mathfrak{U},r}$ equals $h_{\check{C}^r(\mathfrak{U},A)}$. Use the Yoneda Lemma to **prove** that there are associated morphisms in $\mathcal{C}$,

$$\partial^i_{A,\mathfrak{U},r} : \check{C}^r(\mathfrak{U},A) \to \check{C}^{r+1}(\mathfrak{U},A),$$
$$\sigma^i_{A,\mathfrak{U},r+1} : \check{C}^{r+1}_{\mathfrak{U}}(A) \to \check{C}^r_{\mathfrak{U}}(A),$$
$$\check{C}^r(\phi,A) : \check{C}^r(\mathfrak{U},A) \to \check{C}^r(\mathfrak{V},A),$$
$$\check{C}^{r+1,i}(\phi,\psi,A) : \check{C}^{r+1}(\mathfrak{U},A) \to \check{C}^r(\mathfrak{V},A),$$
$$\check{C}^r(\mathfrak{U},\alpha) : \check{C}^r(\mathfrak{U},A) \to \check{C}^r(\mathfrak{U},A'),$$

whose associated morphisms of Yoneda functors equal the morphisms defined above. Thus, in this case, $\check{C}^*(\mathfrak{U},A)$ is a cosimplicial object in $\mathcal{C}$. **Prove** that this defines a covariant functor

$$\check{C}(\mathfrak{U},-) : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\Delta,\mathcal{C}).$$

Incorporating the role of $\mathfrak{U}$, **prove** that this defines a functor

$$\check{C} : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C}).$$

**Prove** that this is, typically, *not* equivalent to the composite functor,

$$\text{const} \circ *_x : \mathbf{Fun}(\tau_X,\mathcal{C}) \to \mathbf{Fun}(\sigma,\mathcal{C}) \to \mathbf{Fun}(\Delta \times \sigma,\mathcal{C}).$$

However, **prove** that the coadjunction in the last part does give rise to a natural transformation,

$$g : \text{const} \circ *_x \Rightarrow \check{C}.$$

**(g)** Assume now that there exists a functor,

$$Z^0 : \mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C}) \to \mathcal{C},$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta_{\leq 1}, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors, i.e., assume that $\mathcal{C}$ is a Cartesian category. Use Problem 4(d) to conclude that there exists a functor,

$$Z^0 : \mathbf{Fun}(\Delta \times \sigma, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C}),$$

and a natural transformation,

$$\eta : \text{const} \circ Z^0 \Rightarrow \text{Id}_{\mathbf{Fun}(\Delta \times \sigma, \mathcal{C})},$$

such that $(\text{const}, Z^0, \eta)$ extends to an adjoint pair of functors, $(\text{const}, Z^0, \eta, \theta)$ such that $\theta$ is a natural isomorphism. Moreover, for every $A^\bullet : \Delta \times \sigma \to \mathcal{C}$, for every object $(U, \mathfrak{U})$ of $\sigma$, **prove** that $\eta : Z^0(A^\bullet(\mathfrak{U}) \to A^0(\mathfrak{U}))$ is an equalizer of $\partial_0^0, \partial_0^1 : A^0(\mathfrak{U}) \to A^1(\mathfrak{U})$. Finally, the composition of natural transformations, $(Z^0 \circ g) \circ (\theta \circ *_x)$, is a natural transformation

$$Z^0(g) : *_x \Rightarrow Z^0 \circ \text{const} \circ *_x \Rightarrow Z^0 \circ \check{C}.$$

In particular, conclude that for a refinement $\phi : (U, \mathfrak{U}) \succeq (V, \mathfrak{V})$, the induced morphism $Z^0(\check{C}^\bullet(\mathfrak{U}, A)) \to Z^0(\check{C}^\bullet(\mathfrak{V}, A))$ is independent of the choice of refinement.

**(h)** Let $(U, \iota : \mathfrak{U} \to \tau_U)$ be an object of $\sigma$. Let $\phi : (U, \mathfrak{U}) \succeq (U, \{U\})$ be a refinement, i.e., $* = \phi(U)$ is an element of $\mathfrak{U}$ such that $\iota(*)$ equals $U$. Thus, $(U, \mathfrak{U})$ admits both the identity refinement of $(U, \mathfrak{U})$ and also the composite of $\phi$ with the canonical refinment from (e), $\eta_{U,\mathfrak{U}} circ \phi$. Using (c), **prove** that the identity on $\check{C}^\bullet(\mathfrak{U}, -)$ is homotopy equivalent to $\check{C}(\eta_{U,\mathfrak{U}}, -) \circ \check{C}(\phi, -)$. On the other hand, the refinement $\phi \circ \eta_{U,\mathfrak{U}}$ of $(U, \{U\})$ is the identity refinement. Thus the composite $\check{C}(\phi, -) \circ \check{C}(\eta_{U,\mathfrak{U}}, -)$ equals the identity on $\check{C}^\bullet(\{U\}, -)$. **Prove** that $\check{C}^\bullet(\mathfrak{U}, A)$ is homotopy equivalent to the constant simplicial object $\text{const}_{A(U)}$, and these homotopy equivalences are natural in $A$ and open coverings $(U, \mathfrak{U})$ that refine to $(U, \{U\})$.

**Problem 6**(Sheaves) Let $(X, \tau_X)$ be a topological space. Let $\mathcal{C}$ be a category. A $\mathcal{C}$-*sheaf* on $(X, \tau_X)$ is a $\mathcal{C}$-presheaf $A$ such that for every open subset $U$ in $\tau_X$, for every open covering $\iota : \mathfrak{U} \to \tau_U$ of $U$, the associated sequence of Yoneda functors,

$$h_{A(U)} \xrightarrow{g^0_{A,\mathfrak{U}}} h_{A,\mathfrak{U},0} \rightrightarrows h_{A,\mathfrak{U},1},$$

is exact, where the two arrows are $\partial^0_{A,\mathfrak{U},0}$ and $\partial^1_{A,\mathfrak{U},0}$. Stated more concretely, for every object $S$ of $\mathcal{C}$, for every collection $(s_{U_0} : S \to A(\iota(U_0)))_{U_0 \in \mathfrak{U}}$ of $\mathcal{C}$-morphisms such that for every $(U_0, U_1) \in \mathfrak{U}^2$, the following two compositions are equal,

$$S \xrightarrow{s_{U_0}} A(\iota(U_0)) \xrightarrow{A^{\iota(U_0)}_{\iota(U_0)\cap\iota(U_1)}} A(\iota(U_0) \cap \iota(U_1)), \quad S \xrightarrow{s_{U_1}} A(\iota(U_1)) \xrightarrow{A^{\iota(U_1)}_{\iota(U_0)\cap\iota(U_1)}} A(\iota(U_0) \cap \iota(U_1)),$$

there exists a unique morphism $s_U : S \to A(U)$ such that for every $U_0 \in \mathfrak{U}$, $s_{U_0}$ equals $A^U_{\iota(U_0)} \circ s_U$.

**(a)**(Sheaf Axiom via Čech Objects) For simplicity, assume that $\mathcal{C}$ is a Cartesian category that has all small products. In particular, assume that the functors $\check{C}$ and $Z^0$ of the previous exercise are defined. **Prove** that a $\mathcal{C}$-presheaf on $(X, \tau_X)$ is a sheaf if and only if the morphism

$$Z^0(g) : *_x(A) \to Z^0(\check{C}(A))$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$ is an isomorphism.

**(b)**(Associated Sheaf / Sheafification Functor) Now assume that $\mathcal{C}$ has all small colimits. In particular, assume that there exists a functor

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C}),$$

such that $(L_x, *_x)$ extends to an adjoint pair of functors. Using Exercise 0(g), **prove** that for every open $U$ in $\tau_X$ and for every functor,

$$B : \sigma \to \mathcal{C},$$

$L_x(B)(U)$ is the colimit of the restriction of $B$ to the fiber category $\sigma_{x,U}$. In particular, since open coverings $(U, \iota : \mathfrak{U} \to U)$ such that $\iota$ is a monomorphism are cofinal in the category $\sigma_{x,U}$, it suffices to compute the colimit over such open coverings. For every functor,

$$A : \tau_X \to \mathcal{C},$$

**prove** that $L_x \circ *_x(A) \to A$ is a natural isomorphism. Denote by $\mathrm{Sh} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_X, \mathcal{C})$ the composite functor,

$$L_x \circ Z^0 \circ \check{C} : \mathbf{Fun}(\tau_X, \mathcal{C}) \to \mathbf{Fun}(\tau_x, \mathcal{C}).$$

**Prove** that there exists a unique natural transformation,

$$\mathrm{sh} : \mathrm{Id}_{\mathbf{Fun}(\tau_x, \mathcal{C})} \Rightarrow \mathrm{Sh},$$

whose composition with the natural isomorphism above equals $L_x(Z^0(g))$. For every sheaf $A$, **prove** that

$$\mathrm{sh} : A \to \mathrm{Sh}(A)$$

is an isomorphism.

**(c)**(The Associated Sheaf is a Sheaf) Let $(U, \iota : \mathfrak{U} \to \tau_U)$ an object of $\sigma$, and let,

$$(\iota(U_0), \kappa_{U_0} : \mathfrak{V}_{U_0} \to \tau_{\iota(U_0)}),$$

be a collection of open coverings of each $\iota(U_0)$. For every pair $(U_0, U_1) \in \mathfrak{U}^2$, let

$$(\iota(U_0, U_1), \kappa_{U_0,U_1} : \mathfrak{V}_{U_0,U_1} \to \tau_{\iota(U_0,U_1)}),$$

be an open covering together with refinements

$$\phi_0^0 : (\iota(U_0), \mathfrak{V}_{U_0}) \succeq (\iota(U_0, U_1), \mathfrak{V}_{U_0,U_1}), \quad \phi_0^1 : (\iota(U_1), \mathfrak{V}_{U_1}) \succeq (\iota(U_0, U_1), \mathfrak{V}_{U_0,U_1}).$$

Define

$$\mathfrak{V} := (\sqcup_{U_0 \in \mathfrak{U}} \mathfrak{V}_{U_0}) \sqcup \left(\sqcup (U_0, U_1) \in \mathfrak{U}^2 \mathfrak{V}_{U_0,U_1}\right),$$

define

$$\kappa : \mathfrak{V} \to \tau_U,$$

to be the unique set map whose restriction to every $\mathfrak{V}_{U_0}$ equals $\kappa_{U_0}$ and whose restriction to every $\mathfrak{V}_{U_0,U_1}$ equals $\kappa_{U_0,U_1}$. For every $U_0 \in \mathfrak{U}$, define

$$\phi_{U_0} : (U, \kappa : \mathfrak{V} \to \tau_U) \succeq (\iota(U_0), \kappa_{U_0} : \mathfrak{V}_{U_0} \to \tau_{\iota(U_0)}),$$

to be the obvious refinement. For every $U_0 \in \mathfrak{U}$, define $Z(U_0, A) = Z^0(\check{C}^\bullet(\mathfrak{V}_{U_0}, A))$. For every $(U_0, U_1) \in \mathfrak{U}^2$, define $Z^0(U_0, U_1, A) = Z^0(\check{C}^\bullet(\mathfrak{V}_{U_0,U_1}, A))$. Define

$$Z^0(\mathfrak{U}, A) := \prod_{U_0 \in \mathfrak{U}} Z^0(U_0, A),$$

$$Z^1(\mathfrak{U}, A) := \prod_{(U_0,U_1) \in \mathfrak{U}^2} Z^0(U_0, U_1, A),$$

$$\partial_0^0 : Z^0(\mathfrak{U}, A) \to Z^1(\mathfrak{U}, A), \ \partial_0^i(z_{U_0}) = (A_{U_0 \cap U_1}^{U_i}(z_{U_i}))_{U_0,U_1}.$$

**Prove** that the restriction morphism,

$$Z^0(\phi^\bullet) : Z^0(\mathfrak{V}, A) \to Z^0(Z^\bullet(\mathfrak{U}, A)),$$

is a $\mathfrak{C}$-isomorphism. Conclude that $\text{Sh}(A)$ is a sheaf. Denote by,

$$\Phi : \mathcal{C} - \text{Sh}_{(X,\tau_X)} \to \mathcal{C} - \text{Presh}_{(X,\tau_X)},$$

the full embedding of the category of sheaves in the category of presheaves. Thus, Sh is a functor,

$$\text{Sh} : \mathcal{C} - \text{Presh}_{(X,\tau_X)} \to \mathcal{C} - \text{Sh}_{(X,\tau_X)},$$

and sh is a natural transformation $\text{Id}_{\mathcal{C}-\text{Presh}_X} \Rightarrow \Phi \circ \text{Sh}$. Conclude that $(\text{Sh}, \Phi, \text{sh})$ extends to an adjoint pair of functors.

**(d)**(Pushforward and Inverse Image) For a continuous map $f : (X, \tau_X) \to (Y, \tau_Y)$, **prove** that the composite functor,

$$\mathcal{C} - \text{Sh}_{(X,\tau_X)} \xrightarrow{\Phi} \mathcal{C} - \text{Presh}_{(X,\tau_X)} \xrightarrow{f_*} \mathcal{C} - \text{Presh}_{(Y,\tau_Y)},$$

factors uniquely through $\Phi : \mathcal{C} - \text{Sh}_{(Y,\tau_Y)} \to \mathcal{C} - \text{Presh}_{(Y,\tau_Y)}$, i.e., there is a functor

$$f_* : \mathcal{C} - \text{Sh}_{(X,\tau_X)} \to \mathcal{C} - \text{Sh}_{(Y,\tau_Y)},$$

such that $f_* \circ \Phi$ equals $\Phi \circ f_*$. On the other hand, **prove** by example that the composite

$$\mathcal{C} - \text{Sh}_{(Y,\tau_Y)} \xrightarrow{\Phi} \mathcal{C} - \text{Presh}_{(Y,\tau_Y)} \xrightarrow{f^{-1}} \mathcal{C} - \text{Presh}_{(X,\tau_X)}$$

need not factor through $\Phi$. Define

$$f^{-1} : \mathcal{C} - \text{Sh}_{(Y,\tau_Y)} \to \mathcal{C} - \text{Sh}_{(X,\tau_X)},$$

to be the composite of the previous functor with $\text{Sh} : \mathcal{C} - \text{Presh}_{(X,\tau_X)} \to \mathcal{C} - \text{Sh}_{(X,\tau_X)}$. **Prove** that the functors $(f^{-1}, f_*)$ extend to an adjoint pair of functors between $\mathcal{C} - \text{Sh}_{(X,\tau_X)}$ and $\mathcal{C} - \text{Sh}_{(Y,\tau_Y)}$.