MAT 536 Algebra III
Stony Brook University
Problem Set 7

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# MAT 536 Problem Set 7

**Homework Policy.** Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

**Problems.**

**Problem 0.**(The Additive Category of Spectral Sequences) Let $\mathcal{A}$ be an Abelian category. Recall that a *spectral sequence* in $\mathcal{A}$ is a collection

$$(E_r^{p,q}, d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}, \phi_{r+1}^{p,q} : \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q+r-1}) \xrightarrow{\cong} E_{r+1}^{p,q})_{p,q\in\mathbb{Z},r\in\mathbb{Z}_{\geq 0}}.$$

of objects $E_r^{p,q}$ of $\mathcal{A}$, of morphisms $d_r^{p,q}$ such that $d_r^{p+r,q+1-r} \circ d_r^{p,q}$ is zero, and isomorphisms $\phi_{r+1}^{p,q}$. For spectral sequences $E = (E_r^{p,q}, d_r^{p,q}, \phi_{r+1}^{p,q})_{p,q,r}$ and $\widetilde{E} = (\widetilde{E}_r^{p,q}, \widetilde{d}_r^{p,q}, \widetilde{\phi}_{r+1}^{p,q})_{p,q,r}$, a morphism $\theta : E \to \widetilde{E}$ is a collection of morphisms,

$$(\theta_r^{p,q} : E_r^{p,q} \to \widetilde{E}_r^{p,q})_{p,q\in\mathbb{Z},r\in\mathbb{Z}_{\geq 0}},$$

such that for every $p$, $q$ and $r$ the following diagram commutes,

$$\begin{tikzcd}
E_r^{p,q} \arrow[r, "d_r^{p,q}"] \arrow[d, "\theta_r^{p,q}"'] & E_r^{p+r,q+1-r} \arrow[d, "\theta_r^{p+r,q+1-r}"] \\
\widetilde{E}_r^{p,q} \arrow[r, "\widetilde{d}_r^{p,q}"'] & \widetilde{E}_r^{p,q}
\end{tikzcd},$$

and the following diagram of induced morphisms also commutes,

$$\begin{tikzcd}
E_{r+1}^{p,q} \arrow[r, "\phi_{r+1}^{p,q}"] \arrow[d, "\theta_{r+1}^{p,q}"'] & \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d_r^{p-r,q-1-r}) \arrow[d, "\theta_r^{p,q}"] \\
\widetilde{E}_{r+1}^{p,q} \arrow[r, "\widetilde{\phi}_r^{p,q}"'] & \mathrm{Ker}(\widetilde{d}_r^{p,q})/\mathrm{Im}(\widetilde{d}_r^{p-r,q-1-r})
\end{tikzcd}.$$

**(a)** Prove that component-wise composition of morphisms of spectral sequences is again a morphism of spectral sequence. Prove that the identity morphisms $\mathrm{Id}_{E_r^{p,q}}$ define a morphism $E \to E$ that is a left and right identity for composition. Conclude that with these definitions of morphism, composition and identity, spectral sequences form a category, $\mathrm{SS}(\mathcal{A})$.

**(b)** Prove that the zero morphisms $0 : E_r^{p,q} \to \widetilde{E}_r^{p,q}$ define a morphism of spectral sequences, called the *zero morphism*. For two morphism from $E$ to $\widetilde{E}$, $\theta$ and $\eta$, prove that the difference $\theta - \eta = (\theta_r^{p,q} - \eta_r^{p,q})_{p,q,r}$ is also a morphism of spectral sequences from $E$ to $\widetilde{E}$. Use these operations to make $\mathrm{Hom}_{\mathrm{SS}(\mathcal{A})}(E, \widetilde{E})$ into an Abelian group. Finally, prove that composition of morphisms distributes with respect to addition. With these Abelian group structures, prove that $\mathrm{SS}(\mathcal{A})$ is an **Ab**-category.

**(c)** Prove that the spectral sequence with all objects $E_r^{p,q}$ being zero objects is a zero object in the category $\mathcal{A} - \mathrm{SS}$. For spectral sequences $E$ and $\widetilde{E}$, prove that the collection,

$$E \oplus \widetilde{E} = (E_r^{p,q} \oplus \widetilde{E}_r^{p,q}, d_r^{p,q} \oplus \widetilde{d}_r^{p,q}, \phi_r^{p,q} \oplus \widetilde{\phi}_r^{p,q})_{p,q \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}},$$

together with the natural morphisms

$$E \to E \oplus \widetilde{E}, \widetilde{E} \to E \oplus \widetilde{E}, E \oplus \widetilde{E} \to E, E \oplus \widetilde{E} \to \widetilde{E},$$

is a direct sum / direct product in $\mathrm{SS}(\mathcal{A})$. Conclude that $\mathrm{SS}(\mathcal{A})$ is an additive category.

**(d)** For every $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 0}$, for every spectral sequence $E$, for every integer $n$, define

$$\Phi_r^{p,q}(E)^n = E_r^{p+nr, q+n-nr},$$

and define

$$d^n_{\Phi_r^{p,q}(E)} : E_r^{p+nr,q+n-nr} \to E_r^{p+(n+1)r, q+(n+1)-(n+1)r}$$

to be $(-1)^{p+q} d_r^{p+nr,q+n-nr}$. Prove that $\Phi_r^{p,q}(E)$ is a cochain complex of objects in $\mathcal{A}$. For every morphism $\theta : E \to \widetilde{E}$, for every integer $n$, define

$$\Phi_r^{p,q}(\theta)^n : E_r^{p+nr,q+n-nr} \to \widetilde{E}_r^{p+nr,q+n-nr},$$

to be $\theta_r^{p+nr,q+n-nr}$. Prove that $\Phi_r^{p,q}(\theta)$ is a morphism of cochain complexes. Prove that this defines a functor

$$\Phi_r^{p,q} : \mathrm{SS}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A}).$$

Moreover, prove that this is an additive functor.

**(e)** For every additive functor of Abelian groups, $F : \mathcal{A} \to \mathcal{B}$, for every spectral sequence $E$ of objects in $\mathcal{A}$, prove that the collection

$$\mathrm{SS}(F)(E) := (F(E_r^{p,q}), F(d_r^{p,q}), F(\phi_{r+1}^{p,q}))_{p,q \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}},$$

is a spectral sequence of objects in $\mathcal{B}$. For every morphism $\theta : E \to \widetilde{E}$ of spectral sequences of objects in $\mathcal{A}$, prove that the collection

$$\mathrm{SS}(F)(\theta) := (F(\theta_r^{p,q}) : F(E_r^{p,q}) \to F(\widetilde{E}_r^{p,q}))_{p,q \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}}$$

is a morphism of spectral sequences of objects in $\mathcal{B}$. Prove that these rules define a functor

$$\mathrm{SS}(F) : \mathrm{SS}(\mathcal{A}) \to \mathrm{SS}(\mathcal{B}).$$

Prove that this is an additive functor. Check that SS is associative (in the appropriate sense) for composition of additive functors of Abelian groups. Finally, for every $p, q \in \mathbb{Z}$ and $r \in \mathbb{Z}_{\geq 0}$, construct a natural isomorphism of functors $\mathrm{SS}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B})$

$$\Phi_r^{p,q} \circ \mathrm{SS}(F) \Rightarrow \mathrm{Ch}(F) \circ \Phi_r^{p,q}.$$

For composition of additive functors of Abelian categories, check that the associativity natural isomorphisms are compatible with the natural isomorphisms in the previous sentence.

**(f)** For every $p_0, q_0 \in \mathbb{Z}$, for every spectral sequence $E$, define $T^{p_0,q_0}(E)$ to be the collection

$$T^{p_0,q_0}(E) := (E_r^{p+p_0,q+q_0}, (-1)^{p_0+q_0} d_r^{p+p_0,q+q_0}, \phi_r^{p+p_0,q+q_0})_{p,q\in\mathbb{Z}, r\in\mathbb{Z}_{\geq 0}}.$$

Prove that $T^{p_0,q_0}(E)$ is again a spectral sequence. For every morphism of spectral sequences $\theta : E \to \widetilde{E}$, define $T^{p_0,q_0}(\theta)$ to be the collection

$$T^{p_0,q_0}(\theta) = (\theta_r^{p+p_0,q+q_0})_{p,q\in\mathbb{Z}, r\in\mathbb{Z}_{\geq 0}}.$$

Prove that $T^{p_0,q_0}(\theta) : T^{p_0,q_0}(E) \to T^{p_0,q_0}(\widetilde{E})$ is a morphism of spectral sequences. Prove that this defines a functor,

$$T^{p_0,q_0} : \mathrm{SS}(\mathcal{A}) \to \mathrm{SS}(\mathcal{A}).$$

Prove that this is an additive functor. Prove that $T^{0,0}$ is the identity functor. Prove that $T^{p_1,q_1} \circ T^{p_0,q_0}$ equals $T^{p_0+p_1,q_0+q_1}$, so that every functor $T^{p_0,q_0}$ is an equivalence of functors (in particular, it is exact). For every additive functor $F : \mathcal{A} \to \mathcal{B}$, prove that $T^{p_0,q_0} \circ \mathrm{SS}(F)$ equals $\mathrm{SS}(F) \circ T^{p_0,q_0}$. Finally, prove that $\Phi_r^{p,q} \circ T^{p_0,q_0}$ equals $\Phi_r^{p+p_0,q+q_0}$.

**(g)** For every $r_0 \in \mathbb{Z}$, for every $(p, q) \in \mathbb{Z}^{\oplus 2}$, define

$$t_{r_0}(p, q) = (P_{r_0}(p, q), Q_{r_0}(p, q)) = ((r_0 + 1)p + r_0 q, -r_0 p - (r_0 - 1)q).$$

Check that $t_0$ is the identity, and $t_{r_0} \circ t_{r_1}$ equals $t_{r_0+r_1}$, so $t$ is a group homomorphism from the additive group over $\mathbb{Z}$ to $\mathbf{SL}_2$ over $\mathbb{Z}$. Now assume that $r_0$ is nonnegative. For every spectral sequence $E$, define $\Theta_{r_0}(E)$ to be the collection

$$\Theta_{r_0}(E) = (E_{r+r_0}^{t_{r_0}(p,q)}, d_{r+r_0}^{t_{r_0}(p,q)}, \phi_{r+r_0+1}^{t_{r_0}(p,q)})_{p,q\in\mathbb{Z}, r\in\mathbb{Z}_{\geq 0}}.$$

Check that $\Theta_{r_0}(E)$ is a spectral sequence. For every morphism of spectral sequences $\theta : E \to \widetilde{E}$, define $\Theta_{r_0}(\theta)$ to be the collection

$$\Theta_{r_0}(\theta) = (\theta_{r+r_0}^{t_{r_0}(p,q)})_{p,q\in\mathbb{Z}, r\in\mathbb{Z}_{\geq 0}}.$$

Prove that $\Theta_{r_0}(\theta)$ is a morphism of spectral sequences. Prove that these rules define a functor

$$\Theta_{r_0} : \mathrm{SS}(\mathcal{A}) \to \mathrm{SS}(\mathcal{A}).$$

Prove that this is an additive functor. Prove that $\Theta_0$ is the identity functor. Prove that $\Theta_{r_0} \circ \Theta_{r_1}$ equals $\Theta_{r_0+r_1}$. Prove that $\Theta_{r_0} \circ T^{p_0,q_0}$ equals $T^{t_{r_0}(p,q)} \circ \Theta_{r_0}$. For every additive functor $F : \mathcal{A} \to \mathcal{B}$, prove that $\Theta_{r_0} \circ \mathrm{SS}(F)$ equals $\mathrm{SS}(F) \circ \Theta_{r_0}$. Finally, prove that $\Phi_r^{p,q} \circ \Theta_{r_0}$ equals $\Phi_{r+r_0}^{t_{r_0}(p,q)}$.

**(h)** Let $\theta : E \to \widetilde{E}$ be a morphism of spectral sequences. Let $r_0$ be a nonnegative integer. If for every $p, q \in \mathbb{Z}$ the morphism $\theta_{r_0}^{p,q}$ is an isomorphism, then prove by induction on $r$ that for every $r \in \mathbb{Z}_{\geq 0}$, the morphism $\theta_{r+r_0}^{p,q}$ is an isomorphism.

**Problem 1.**(Alternative Formulation of Spectral Sequences) Let $\mathcal{A}$ be an Abelian category. A *spectral sequence datum* $G$ is a collection

$$G = ((G^{p,q})_{p,q\in\mathbb{Z}}, (Z_r^{p,q}, B_r^{p,q}, \mathbf{d}_r^{p,q})_{p,q\in\mathbb{Z}, r\in\mathbb{Z}_{\geq 0}})$$

of objects $G^{p,q}$, and for all integers $r \geq 0$, a decreasing sequence of subobjects,

$$G^{p,p+q} = Z_0^{p,q} \supseteq \cdots \supseteq Z_r^{p,q} \supseteq \dots,$$

an increasing sequence,

$$\underline{0} = B_0^{p,q} \subseteq \cdots \subseteq B_r^{p,q} \subseteq \dots,$$

of subobjects $B_r^{p,q} \subset Z_r^{p,q}$, and morphisms

$$\mathbf{d}_r^{p,q} : Z_r^{p,q} \to Z_r^{p+r,q+1-r}/B_r^{p+r,q+1-r},$$

such that $Z_{r+1}^{p,q} \subset Z_r^{p,q}$ is the kernel of $\mathbf{d}_r^{p,q}$, such that $B_{r+1}^{p,q} \subset Z_r^{p,q}$ is the inverse image in $Z_r^{p,q}$ of the image of $\mathbf{d}_r^{p-r,q-1-r}$ in $Z_r^{p,q}/B_r^{p,q}$, and such that the induced morphisms,

$$d_r^{p,q} : Z_r^{p,q}/B_r^{p,q} \to Z_r^{p+r,q+1-r}/B_r^{p+r,q+1-r},$$

form a complex, i.e., $d_r^{p+r,q+1-r} \circ d_r^{p,q}$ equals 0.

For spectral sequence data $G$ and $\widetilde{G}$, a *morphism* $\gamma$ of spectral sequence data from $G$ to $\widetilde{G}$ is a collection

$$\gamma = (\gamma^{p,q} : G^{p,p+q} \to \widetilde{G}^{p,p+q})_{p,q\in\mathbb{Z}}$$

such that for every $p, q \in \mathbb{Z}$ and for every $r \in \mathbb{Z}_{\geq 0}$, $\gamma^{p,q}(Z_r^{p,q})$ is contained in $\widetilde{Z}_r^{p,q}$, $\gamma^{p,q}(B_r^{p,q})$ is contained in $\widetilde{B}_r^{p,q}$, and for the induced morphism,

$$\gamma_r^{p,q} : Z_r^{p,q}/B_r^{p,q} \to \widetilde{Z}_r^{p,q}/\widetilde{B}_r^{p,q},$$

the following diagram commutes,

$$\begin{array}{ccc} Z_r^{p,q} & \xrightarrow{\mathbf{d}_r^{p,q}} & Z_r^{p+r,q+1-r}/B_r^{p+r,q+1-r} \\ {\scriptstyle\gamma^{p,q}}\downarrow & & \downarrow{\scriptstyle\gamma_r^{p,q}} \\ \widetilde{Z}_r^{p,q} & \xrightarrow[\widetilde{\mathbf{d}}_r^{p,q}]{} & \widetilde{Z}_r^{p+r,q+1-r}/\widetilde{B}_r^{p+r,q+1-r} \end{array}.$$

**(a)** Prove that component-wise composition of morphisms of spectral sequence data is again a morphism of spectral sequence data. Prove that the identity maps $G^{p,p+q} \to G^{p,p+q}$ give a morphism of spectral sequence data that is a left and right identity for composition. Conclude that with these definitions of morphism, composition and identity, spectral sequence data form a category $\text{SS}'(\mathcal{A})$.

**(b)** Prove that the zero morphisms $0 : G^{p,p+q} \to G^{p,p+q}$ define a morphism of spectral sequence data, called the *zero morphism*. For two morphisms from $G$ to $\widetilde{G}$, $\gamma$ and $\beta$, prove that the difference $\gamma - \beta = (\gamma^{p,q} - \beta^{p,q})_{p,q\in\mathbb{Z}}$ is again a morphism of spectral sequence data. Use these operations to make $\text{Hom}_{\text{SS}'(\mathcal{A})}(G, \widetilde{G})$ into an Abelian group. Finally, prove that composition of morphisms distributes with respect to addition. With these Abelian group structures, prove that $\text{SS}(\mathcal{A})$ is an **Ab**-category.

**(c)** For every spectral sequence datum $G$, for every $p, q \in \mathbb{Z}$ and for every $r \in \mathbb{Z}_{\geq 0}$, define $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$ with the induced morphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}$, and define

$$\phi_{r+1}^{p,q} : E_{r+1}^{p,q} \xrightarrow{\cong} \text{Ker}(d_r^{p,q})/\text{Im}(d_r^{p-r,q-1+r}),$$

to be the natural isomorphism induced by the hypotheses on $Z_{r+1}^{p,q}$ and $B_{r+1}^{p,q}$ above,

$$Z_{r+1}^{p,q}/B_{r+1}^{p,q} \cong \text{Ker}(d_r^{p,q})/\text{Im}(d_r^{p-r,q-1-r}).$$

Prove that $(E_r^{p,q}, d_r^{p,q}, \phi_{r+1}^{p,q})_{p,q\in\mathbb{Z},r\in\mathbb{Z}_{\geq 0}}$ is a spectral sequence of objects in $\mathcal{A}$. Also, for every morphism of spectral sequence data $\gamma : G \to \widetilde{G}$, prove that the induced collection $(\gamma_r^{p,q})_{p,q\in\mathbb{Z},r\in\mathbb{Z}_{\geq 0}}$ is a morphism of the associated spectral sequences. Prove that these rules define a functor

$$\mathcal{E} : \text{SS}'(\mathcal{A}) \to \text{SS}(\mathcal{A}).$$

Prove that this functor is additive.

**(d)** Next, let $E$ be a spectral sequence of objects in $\mathcal{A}$. For every $p, \widetilde{q} \in \mathbb{Z}$, define $G^{p,\widetilde{q}} = E_0^{p,\widetilde{q}-p}$. Define $Z_0^{p,q}$ to be $G^{p,p+q}$ and define $B_0^{p,q}$ to be the zero subobject of $Z_0^{p,q}$. Thus, by definition, there is an identity isomorphism,

$$\chi_0^{p,q} : E_0^{p,q} \to Z_0^{p,q}/B_0^{p,q}.$$

For a given integer $s \geq 0$, assume for $r = 0, \dots, s$, we constructed a descending sequence of subobjects,

$$G^{p,p+q} = Z_0^{p,q} \supseteq \cdots \supseteq Z_r^{p,q} \supseteq \cdots \supseteq Z_s^{p,q},$$

an increasing sequence,

$$\underline{0} = B_0^{p,q} \subseteq \cdots \subseteq B_r^{p,q} \subseteq \cdots \subseteq B_s^{p,q},$$

of subobjects $B_r^{p,q} \subset Z_r^{p,q}$, and isomorphisms,

$$\chi_r^{p,q} : E_r^{p,q} \to Z_r^{p,q}/B_r^{p,q},$$

such that for $r = 0, \dots, s-1$, $(\chi_r^{p,q})(\text{Ker}(d_r^{p,q}))$ equals $Z_{r+1}^{p,q}$, $(\chi_r^{p,q})(\text{Im}(d_r^{p-r,q-1-r}))$ equals $B_{r+1}^{p,q}$, and $\chi_{r+1}^{p,q}$ equals the composition,

$$E_{r+1}^{p,q} \xrightarrow{\phi_{r+1}^{p,q}} \text{Ker}(d_r^{p,q})/\text{Im}(d_r^{p-r,q-1-r}) \xrightarrow{\widetilde{\chi}_r^{p,q}} Z_{r+1}^{p,q}/B_{r+1}^{p,q}.$$

Define $Z_{s+1}^{p,q}$ to be the unique subobject of $Z_s^{p,q}$ containing $B_s^{p,q}$ and such that

$$\chi_s^{p,q}(\mathrm{Ker}(d_s^{p,q})) = Z_{s+1}^{p,q}/B_s^{p,q},$$

as subobjects of $Z_s^{p,q}/B_s^{p,q}$. Similarly, define $B_{s+1}^{p,q}$ to be the unique subobject of $Z_s^{p,q}$ containing $B_s^{p,q}$ and such that

$$\chi_s^{p,q}(\mathrm{Im}(d_s^{p-s,q-1-s})) = B_{s+1}^{p,q}/B_s^{p,q},$$

as subobjects of $Z_s^{p,q}/B_s^{p,q}$. Prove that $\chi_s^{p,q}$ induces an isomorphism,

$$\widetilde{\chi_s^{p,q}} : \mathrm{Ker}(d_s^{p,q})/\mathrm{Im}(d_s^{p-s,q-1-s}) \to Z_{s+1}^{p,q}/B_{s+1}^{p,q}.$$

Defining $\chi_{s+1}^{p,q}$ to be the composition $\widetilde{\chi}_s^{p,q} \circ \phi_{s+1}^{p,q}$. Prove that this is an isomorphism.

**(e)** Conclude, by induction on $r$, that there exists for all integers $r \geq 0$ a decreasing sequence of subobjects,

$$G^{p,p+q} = Z_0^{p,q} \supseteq \cdots \supseteq Z_r^{p,q} \supseteq \dots,$$

an increasing sequence,

$$\underline{0} = B_0^{p,q} \subseteq \cdots \subseteq B_r^{p,q} \subseteq \dots,$$

of subobjects $B_r^{p,q} \subset Z_r^{p,q}$, and isomorphisms,

$$\chi_r^{p,q} : Z_r^{p,q}/B_r^{p,q} \to E_r^{p,q},$$

such that for all $r \geq 0$, $(\chi_r^{p,q})^{-1}(\mathrm{Ker}(d_s^{p,q}))$ equals $Z_{s+1}^{p,q}$, $(\chi_s^{p,q})^{-1}(\mathrm{Im}(d_s^{p-s,q-1-s}))$ equals $B_{s+1}^{p,q}$, and $\chi_{s+1}^{p,q}$ equals the composition,

$$Z_{s+1}^{p,q}/B_{s+1}^{p,q} \xrightarrow{\chi_s^{p,q}} \mathrm{Ker}(d_s^{p,q})/\mathrm{Im}(d_s^{p-s,q-1-s}) \xrightarrow{\phi_{s+1}^{p,q}} E_{s+1}^{p,q}.$$

**(f)** With the construction as above, for every $p, q \in \mathbb{Z}$ and for every $r \in \mathbb{Z}_{\geq 0}$, prove that there exists a unique morphism

$$\mathbf{d}_r^{p,q} : Z_r^{p,q} \to Z_r^{p+r,q+1-r}/B_r^{p+r,q+1-r}$$

such that $\chi_r^{p+r,q+1-r} \circ \mathbf{d}_r^{p,q}$ equals the composition

$$Z_r^{p,q} \twoheadrightarrow Z_r^{p,q}/B_r^{p,q} \xrightarrow{\chi_r^{p,q}} E_r^{p,q} \xrightarrow{d_r^{p,q}} E_r^{p+r,q+1-r}.$$

Prove that $Z_{r+1}^{p,q}$ is the kernel of $\mathbf{d}_r^{p,q}$. Conclude that $B_r^{p,q}$ is in the kernel of $\mathbf{d}_r^{p,q}$. Prove that $B_{r+1}^{p,q}$ is the unique subobject of $Z_{r+1}^{p,q}$ containing $B_r^{p,q}$ and such that $B_{r+1}^{p,q}/B_r^{p,q}$ maps isomorphically to $\mathrm{Im}(\mathbf{d}_r^{p-r,q-1-r})$. Conclude that this datum is a spectral sequence datum.

**(g)** For every morphism of spectral sequences $\theta : E \to \widetilde{E}$, prove that the collection $(\theta_0^{p,q} : E_0^{p,q} \to \widetilde{E}_0^{p,q})$ is a morphism of the associated spectral sequence data. Prove that these rules define a functor

$$\mathcal{G} : \mathrm{SS}(\mathcal{A}) \to \mathrm{SS}'(\mathcal{A}).$$

Prove that this is an additive functor.

**(h)** Prove that $\mathcal{G} \circ \mathcal{E}$ is the identity functor $\mathrm{Id}_{\mathrm{SS}'(\mathcal{A})}$. Prove that the isomorphisms $\chi_r^{p,q}$ define a natural isomorphism of functors $\mathrm{Id}_{\mathrm{SS}(\mathcal{A})} \Rightarrow \mathcal{E} \circ \mathcal{G}$. Conclude that $\mathcal{E}$ and $\mathcal{G}$ form an equivalence of additive categories. For this reason, we usually think of spectral sequences as equivalent information as spectral sequence data.

**(i)** Give descriptions on the level of spectral sequence data of the following notions for spectral sequences: the zero spectral sequence, the direct sum / direct product of spectral sequences, the functor $\mathrm{SS}(F)$ associated to an additive functor of Abelian categories $F : \mathcal{A} \to \mathcal{B}$, the forgetful functors $\Phi_r^{p,q}$, the shift endofunctors $T^{p_0,q_0}$, and the shift endofunctors $\Theta_{r_0}$.