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MAT 536 Problem Set 5

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. For every additive functor,

$$F : \mathcal{A} \to \mathcal{B},$$

there is an associated additive functor,

$$\mathrm{Ch}(F) : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{B}),$$

that associates to every cochain complex in $\mathrm{Ch}(\mathcal{A})$,

$$A^\bullet = ((A^n)_{n\in\mathbb{Z}}, (d_A^n)_{n\in\mathbb{Z}})$$

the cochain complex in $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(F)(A^\bullet) = ((F(A^n))_{n\in\mathbb{Z}}, (F(d_A^n))_{n\in\mathbb{Z}}),$$

and that associates to every morphism of cochain complexes in $\mathrm{Ch}(\mathcal{A})$,

$$u^\bullet : C^\bullet \to A^\bullet, \quad (u^n : C^n \to A^n)_{n\in\mathbb{Z}},$$

the morphism of cochain complexes in $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(F)(u^\bullet) = (F(u^n) : F(C^n) \to F(A^n))_{n\in\mathbb{Z}}.$$

In particular, for every homotopy

$$s^\bullet = (s^n : C^n \to A^{n-1})_{n\in\mathbb{Z}},$$

from $u^\bullet$ to 0, also

$$\mathrm{Ch}(F)(s^\bullet) := (F(s^n) : F(C^n) \to F(A^{n-1}))_{n\in\mathbb{Z}},$$

is a homotopy from $\mathrm{Ch}(F)(u^\bullet)$ to 0.

(a) For additive functors,

$$F, G : \mathcal{A} \to \mathcal{B},$$

let

$$\alpha : F \Rightarrow G,$$

be a natural transformation. For every cochain complex $A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, prove that

$$(\alpha_{A^n} : F(A^n) \to G(A^n))_{n\in\mathbb{Z}}$$

is a morphism of cochain complexes in $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(\alpha)(A^\bullet) : \mathrm{Ch}(F)(A^\bullet) \to \mathrm{Ch}(G)(A^\bullet).$$

(b) Prove that the rule $A^\bullet \mapsto \mathrm{Ch}(\alpha)(A^\bullet)$ is a natural transformation

$$\mathrm{Ch}(\alpha) : \mathrm{Ch}(F) \Rightarrow \mathrm{Ch}(G).$$

Moreover, for every morphism $u^\bullet : C^\bullet \to A^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, and for every homotopy $(s^n : C^n \to A^{n-1})_{n\in\mathbb{Z}}$ from $u^\bullet$ to 0, prove that also $\mathrm{Ch}(\alpha)(A^\bullet) \circ \mathrm{Ch}(F)(s^\bullet)$ equals $\mathrm{Ch}(G)(s^\bullet) \circ \mathrm{Ch}(\alpha)(C^\bullet)$.

(c) For the identity natural transformation $\mathrm{Id}_F : F \Rightarrow F$, prove that $\mathrm{Ch}(\mathrm{Id}_F)$ is the identity natural transformation $\mathrm{Ch}(F) \Rightarrow \mathrm{Ch}(F)$. Also, for every pair of natural transformations of additive functors $\mathcal{A} \to \mathcal{B}$,

$$\alpha : F \Rightarrow G, \quad \beta : E \Rightarrow F,$$

for the composite natural transformation $\alpha \circ \beta$, prove that $\mathrm{Ch}(\alpha\circ\beta)$ equals $\mathrm{Ch}(\alpha) \circ \mathrm{Ch}(\beta)$. In this sense, Ch is a "functor" from the "2-category" of Abelian categories to the "2-category" of Abelian categories.

Problem 2. Let $\mathcal{A}$ and $\mathcal{B}$ be Abelian categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Assume that $\mathcal{A}$ has enough injective objects. Thus, every object A admits an injective resolution in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccc}
A[0] : & \dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \\
{\scriptstyle \epsilon_A}\downarrow & & & \downarrow & & {\scriptstyle \epsilon}\downarrow & & \downarrow & & \\
I_A^\bullet : & \dots & \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{d_I^0} & I^1 & \xrightarrow{d_I^1} & \dots
\end{array},$$

which is functorial up to null homotopies (in particular, any two injective resolutions are homotopy equivalent). Moreover, for every short exact sequence in $\mathcal{A}$,

$$\Sigma : \ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

there exists a diagram of injective resolutions with rows being short exact sequences in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{ccccccccccc}
\Sigma[0] : & 0 & \longrightarrow & K[0] & \xrightarrow{q[0]} & A[0] & \xrightarrow{p[0]} & Q[0] & \longrightarrow & 0 \\
{\scriptstyle \epsilon_\Sigma}\downarrow & & & {\scriptstyle \epsilon_K}\downarrow & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_Q} & & \\
I_\Sigma : & 0 & \longrightarrow & I_K^\bullet & \xrightarrow{q^\bullet} & I_A^\bullet & \xrightarrow{p^\bullet} & I_Q^\bullet & \longrightarrow & 0
\end{array}$$

whose associated short exact sequences in $\mathcal{A}$,

$$I^n_\Sigma : \ 0 \longrightarrow I^n_K \xrightarrow{q^n} I^n_A \xrightarrow{p^n} I^n_Q \longrightarrow 0,$$

are automatically split. Moreover, this diagram of injective resolutions is functorial up to homotopy, i.e., for every commutative diagram of short exact sequences in $\mathcal{A}$,

$$\begin{array}{ccccccccccc} \Sigma : & 0 & \longrightarrow & K & \xrightarrow{q} & A & \xrightarrow{p} & Q & \longrightarrow & 0 & \\ u\downarrow & & & \downarrow u_K & & \downarrow u_A & & \downarrow u_Q & & & , \\ \widetilde{\Sigma} : & 0 & \longrightarrow & \widetilde{K} & \xrightarrow{\widetilde{q}} & \widetilde{A} & \xrightarrow{\widetilde{p}} & \widetilde{Q} & \longrightarrow & 0 & \end{array}$$

there exists a commutative diagram in $\mathrm{Ch}(\mathcal{A})$,

$$\begin{array}{cccccccccc} I_\Sigma : & 0 & \longrightarrow & I_K & \xrightarrow{q^\bullet} & I_A & \xrightarrow{p^\bullet} & I_Q & \longrightarrow & 0 \\ u^\bullet\downarrow & & & \downarrow u^\bullet_K & & \downarrow u^\bullet_A & & \downarrow u^\bullet_Q & & \\ I_{\widetilde{\Sigma}} : & 0 & \longrightarrow & I_{\widetilde{K}} & \xrightarrow{\widetilde{q}^\bullet} & I_{\widetilde{A}} & \xrightarrow{\widetilde{p}^\bullet} & I_{\widetilde{Q}} & \longrightarrow & 0 \end{array}$$

compatible with the morphisms ϵ_-, and the cochain morphisms $u^\bullet$ making all diagrams commute are unique up to homotopy.

As proved in lecture, there is an associated cohomological δ-functor in degrees ≥ 0, $R^\bullet F$, with

$$R^n F : \mathcal{A} \to \mathcal{B}, \quad R^n F(A) = H^n(\mathrm{Ch}(F)(A^\bullet)).$$

For every short exact sequence in $\mathcal{A}$,

$$\Sigma : \ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

the corresponding complex in $\mathcal{B}$, $\mathrm{Ch}(\mathcal{B})$,

$$\mathrm{Ch}(F)(I_\Sigma) : \ 0 \longrightarrow \mathrm{Ch}(F)(I^\bullet_K) \xrightarrow{\mathrm{Ch}(F)(q^\bullet)} \mathrm{Ch}(F)(I^\bullet_A) \xrightarrow{\mathrm{Ch}(F)(p^\bullet)} \mathrm{Ch}(F)(I^\bullet_Q) \longrightarrow 0,$$

has associated complexes in $\mathcal{B}$,

$$\mathrm{Ch}(F)(I_\Sigma)^n : \ 0 \longrightarrow F(I^n_K) \xrightarrow{F(q^n)} F(I^n_A) \xrightarrow{F(p^n)} F(I^n_Q) \longrightarrow 0,$$

being split exact sequences (since the additive functor F preserves split exact sequences), and hence $\mathrm{Ch}(F)(I_\Sigma)$ is a short exact sequence in $\mathcal{B}$. The maps $\delta^n_{R^\bullet F,\Sigma}$ are the connecting maps determined by the Snake Lemma for this short exact sequence,

$$\delta^n_{\mathrm{Ch}(F)(I_\Sigma)} : H^n(\mathrm{Ch}(F)(I^\bullet_Q)) \to H^{n+1}(\mathrm{Ch}(F)(I^\bullet_K)).$$

Associated to ϵ, there are morphisms in $\mathcal{B}$

$$F(\epsilon_A) : F(A) \to R^0F(A).$$

(a) Let $G : \mathcal{A} \to \mathcal{B}$ be an additive functor. Let

$$\alpha : F \Rightarrow G,$$

be a natural transformation. For every object A of $\mathcal{A}$ and for every injective resolution $\epsilon : A[0] \to I^\bullet_A$, there is an induced morphism in $textCh(\mathcal{B})$,

$$\mathrm{Ch}(\alpha)(I^\bullet_A) : \mathrm{Ch}(F)(I^\bullet_A) \to \mathrm{Ch}(G)(I^\bullet_A).$$

This induces morphisms,

$$R^n\alpha(A) : R^nF(A) \to R^nG(A),$$

given by,

$$H^n(\mathrm{Ch}(\alpha)(I^\bullet_A)) : H^n(\mathrm{Ch}(F)(I^\bullet_A)) \to H^n(\mathrm{Ch}(G)(I^\bullet_A)).$$

For every n, prove that $A \mapsto R^n\alpha(A)$ defines a natural transformation

$$R^n\alpha : R^nF \Rightarrow R^nG.$$

Moreover, prove that this natural transformation is a morphism of δ-functors, i.e., for every short exact sequence,

$$\Sigma : \ 0 \longrightarrow K \xrightarrow{q} A \xrightarrow{p} Q \longrightarrow 0,$$

for every integer n, the following diagram commutes,

$$\begin{array}{ccc} R^nF(Q) & \xrightarrow{\delta^n_{R^\bullet F,\Sigma}} & R^{n+1}F(K) \\ {\scriptstyle R^n\alpha(Q)}\big\downarrow & & \big\downarrow{\scriptstyle R^{n+1}\alpha(K)} \\ R^nG(Q) & \xrightarrow[\delta^n_{R^\bullet G,\Sigma}]{} & R^{n+1}G(K) \end{array}$$

(b) Prove that the morphisms $F(\epsilon_A)$ form a natural transformation, $\rho_F : F \to R^0F$.

(c) Prove that R^0F is a left-exact functor. Assuming that F is left-exact, prove that ρ_F is a natural equivalence of funcors. In particular, conclude that $\rho_{R^0F} : R^0F \to R^0(R^0F)$ is a natural equivalence of functors.

(d) For every half-exact functor,

$$G : \mathcal{A} \to \mathcal{B},$$

and for every natural transformation,

$$\gamma : F \Rightarrow G,$$

prove that the two natural transformations,

$$R^0\gamma \circ \rho_F, \rho_G \circ \gamma : F \Rightarrow R^0 G,$$

are equal. In particular, if G is left-exact, so that ρ_G is a natural equivalence, conclude that there exists a unique natural transformation,

$$\widetilde{\gamma} : R^0 F \Rightarrow G,$$

such that γ equals $\widetilde{\gamma} \circ \rho_F$.

(e) Now assume that $\mathcal{A}$ and $\mathcal{B}$ are small Abelian categories. Thus, functors from $\mathcal{A}$ to $\mathcal{B}$ are well-defined in the usual axiomatization of set theory. Let $\text{Fun}(\mathcal{A}, \mathcal{B})$ be the category whose objects are functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are natural transformations of functors. Let $\text{AddFun}(\mathcal{A}, \mathcal{B})$ be the full subcategory of additive functors. Let

$$e : \text{LExactFun}(\mathcal{A}, \mathcal{B}) \to \text{AddFun}(\mathcal{A}, \mathcal{B}),$$

be the full subcategory whose objects are left-exact additive functors from $\mathcal{A}$ to $\mathcal{B}$. Prove that the rule associating to F the left-exact functor $R^0 F$ and associating to every natural transformation $\alpha : F \Rightarrow G$ the natural transformation $R^0\alpha : R^0 F \Rightarrow R^0 G$ is a left adjoint to e.

(f) With the same hypotheses as above, denote by $\text{Fun}_\delta^{\geq 0}(\mathcal{A}, \mathcal{B})$ the category whose objects are cohomological δ-functors from $\mathcal{A}$ to $\mathcal{B}$ concentrated in degrees ≥ 0,

$$T^\bullet = ((T^n : \mathcal{A} \to \mathcal{B})_{n\in\mathbb{Z}}, (\delta_T^n)_{n\in\mathbb{Z}}),$$

and whose morphisms are natural transformations of δ-functors,

$$\alpha^\bullet : S^\bullet \to T^\bullet, \quad (\alpha^n : S^n \Rightarrow T^n)_{n\in\mathbb{Z}}.$$

Denote by

$$(-)^0 : \text{Fun}_\delta^{\geq 0}(\mathcal{A}, \mathcal{B}) \to \text{LExactFun}(\mathcal{A}, \mathcal{B}),$$

the functor that associates to every cohomological δ-functor, $T^\bullet$, the functor, T^0, and that associates to every natural transformation of cohomological δ-functors, $u^\bullet : S^\bullet \to T^\bullet$, the natural transformation $u^0 : S^0 \to T^0$. Denote by

$$R : \text{LExactFun}(\mathcal{A}, \mathcal{B}) \to \text{Fun}_\delta^{\geq 0}(\mathcal{A}, \mathcal{B}),$$

the functor that associates to every left-exact functor, F, the cohomological δ-functor, $R^\bullet F$, and that associates to the natural transformation, $\alpha : F \Rightarrow G$, the natural transformation of cohomological δ-functors, $R^\bullet\alpha : R^\bullet F \Rightarrow R^\bullet G$. Prove that R is left adjoint to $(-)^0$.

(g) In particular, for $n > 0$, prove that $R^0(R^n F)$ is the zero functor. Thus, for every $m \geq n$, $R^m(R^n F)$ is the zero functor.

Problem 3.(Enough Projective and Injective Objects) Recall that for a category $\mathcal{C}$, for every object X of $\mathcal{C}$, there is a covariant Yoneda functor,

$$h^X : \mathcal{C} \to \mathbf{Sets}, \quad B \mapsto \mathrm{Hom}_{\mathcal{C}}(X, B),$$

and for every object Y of $\mathcal{C}$, there is a contravariant Yoneda functor,

$$h_Y : \mathcal{C}^{\mathrm{opp}} \to \mathbf{Sets}, \quad A \mapsto \mathrm{Hom}_{\mathcal{C}}(A, Y).$$

An object X of $\mathcal{C}$ is **projective** if the Yoneda functor h^X sends epimorphisms to epimorphisms. An object Y of $\mathcal{C}$ is **injective** if the Yoneda functor h_Y sends monomorphisms to epimorphisms. The category has **enough projectives** if for every object B there exists a projective object X and an epimorphism $X \to B$. The category has **enough injectives** if for every object A there exists an injective object Y and a monomorphism from A to Y.

(a) Check that this notion agrees with the usual definition of projective and injective for objects in an Abelian category.

(b) For the category **Sets**, assuming the Axioms of Choice, prove that every object is both projective and injective. Deduce the same for the opposite category, $\mathbf{Sets}^{\mathrm{opp}}$.

(c) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Let (L, R, θ, η) be an adjoint pair of covariant functors,

$$L : \mathcal{C} \to \mathcal{D}, \quad R : \mathcal{D} \to \mathcal{C}.$$

For every object d of $\mathcal{D}$, prove that

$$\eta(d) : L(R(d)) \to d,$$

is an epimorphism. For every object c ov $\mathcal{C}$, prove that

$$\theta : c \to R(L(c)),$$

is a monomorphism. Thus, if every $L(R(d))$ is a projective object, then $\mathcal{C}$ has enough projective objects. Similarly, if every $R(L(c))$ is an injective object, then $\mathcal{C}$ has enough injective objects.

(d) Assuming that R sends epimorphisms to epimorphisms, prove that L sends projective objects of $\mathcal{C}$ to projective objects of $\mathcal{D}$. Thus, if every object of $\mathcal{C}$ is projective, conclude that $\mathcal{D}$ has enough projective objects. More generally, assume further that R is **faithful**, i.e., R sends distinct morphisms to distinct morphisms. Then conclude for every epimorphism $X \to R(D)$ in $\mathcal{C}$, the associated morphism $L(X) \to D$ in $\mathcal{D}$ is an epimorphism. Thus, if $\mathcal{C}$ has enough projective objects, also $\mathcal{D}$ has enough projective objects.

Similarly, assuming that L sends monomorphisms to monomorphisms, prove that R sends injective objects of $\mathcal{D}$ to injective objects of $\mathcal{C}$. Thus, if every object of $\mathcal{D}$ is injective, conclude that there are enough injective objects of $\mathcal{C}$. More generally, assume further that L is faithful. Then conclude for every monomorphism $L(C) \to Y$ in $\mathcal{D}$, the associated morphism $C \to R(Y)$ in $\mathcal{C}$ is a monomorphism. Thus, if $\mathcal{D}$ has enough injective objects, also $\mathcal{C}$ has enough injective objects.

(e) Let S and T be associative, unital algebras. Let $\mathcal{C}$ be the category **Sets**. Let $\mathcal{D}$ be the category $S-T-\text{mod}$ of $S-T$-bimodules. Let

$$R : S-T-\text{mod} \to \textbf{Sets}$$

be the forgetful functor that sends every $S-T$-bimodule to the underlying set of elements of the bimodule. Prove that R sends epimorphisms to epimorphisms and R is faithful. Prove that there exists a left adjoint functor,

$$L : \textbf{Sets} \to S-T-\text{mod},$$

that sends every set Σ to the corresponding $S-T$-bimodule, $L(\Sigma)$ of functions $f : \Sigma \to S \otimes_{\mathbb{Z}} T$ that are zero except on finitely many elements of Σ. Since **Sets** has enough projective objects (in fact every object is projective), conclude that $S-T-\text{mod}$ has enough projective objects.

(e) Let S, T and U be associative, unital rings. Let B be a $T-U$-bimodule. Let $\mathcal{C}$ be the Abelian category of $S-T$-bimodules, let $\mathcal{D}$ be the Abelian category of $S-U$-bimodules, let L be the exact, additive functor,

$$L : S-T-\text{mod} \to S-U-\text{mod}, \quad L(A) = A \otimes_T B,$$

and let R be the right adjoint functor,

$$R : S-U-\text{mod} \to S-T-\text{mod}, \quad R(C) = \text{Hom}_{\text{mod}-U}(B, C).$$

Prove that if B is a flat (left) T-module, resp. a faithfully flat (left) T-module, then L sends monomorphisms to monomorphisms, resp. L sends monomorphism to monomorphisms and is faithful. Conclude, then, that R sends injective objects of $S-U-\text{mod}$ to injective objects of $S-T-\text{mod}$, resp. if $S-U-\text{mod}$ has enough injective objects then also $S-T-\text{mod}$ has enough injective objects.

(f) Continuing as above, for every ring homomorphism $U \to T$, prove that the induced $T-U$-module structure on T is faithfully flat as a left T-module. Thus, given rings Λ and Π, define $S = \Lambda$, define $T = \Pi$, and define U to be $\mathbb{Z}$ with its unique ring homomorphism to T. Conclude that if there exist enough injective objects in $\Lambda-\text{mod}$, then there exist enough injective objects in $\Lambda-\Pi-\text{mod}$.

(g) For the next step, define T and U to be Λ, define B to be Λ as a left-right T-module, and define S to be $\mathbb{Z}$. Conclude that if there are enough injective $\mathbb{Z}$-modules, then there are enough injective Λ-modules, and hence there are enough injective $\Lambda-\Pi$-bimodules. Thus, to prove that there are enough $\Lambda-\Pi$-bimodules, it is enough to prove that there are enough $\mathbb{Z}$-modules.

Problem 4.(Enough Abelian Groups.) Let $\mathcal{A}$ be an Abelian category that has all small products. An object Y of $\mathcal{A}$ is an **injective cogenerator** if Y is injective and for every pair of distinct morphisms,

$$u, v : A' \to A,$$

in $\mathcal{A}$, there exists a morphism $w : A \to Y$ such that $w \circ u$ and $w \circ v$ are also distinct.

(a) Let $\mathcal{C}$ be the category $\textbf{Sets}^{\text{opp}}$. For an object Y of $\mathcal{A}$, define L to be the Yoneda functor

$$h_Y : \mathcal{A} \to \textbf{Sets}^{\text{opp}}, \quad h_Y(A) = \text{Hom}_{\mathcal{A}}(A, Y).$$

Similarly, define the functor,

$$R : \textbf{Sets}^{\text{opp}} \to \mathcal{A}, \quad L(\Sigma) = \text{"Hom}_{\textbf{Sets}}(\Sigma, Y)\text{"},$$

that sends every set Σ to the object $R(\Sigma)$ in $\mathcal{A}$ that is the direct product of copies of Y indexed by elements of Σ. Prove that L and R are an adjoint pair of functors.

(b) Assuming that $\mathcal{A}$ has an injective cogenerator Y, prove that L sends monomorphisms to monomorphisms, and prove that L is faithful. Conclude that $\mathcal{A}$ has enough injective objects.

(c) Now let S be an associative, unital ring (it suffices to consider the special case that S is $\mathbb{Z}$). Let $\mathcal{A}$ be $\text{mod} - S$. Use the Axiom of Choice to prove Baer's criterion: a right S-module Y is injective if and only if for every right ideal J of S, the induced set map

$$\text{Hom}_{\text{mod}-S}(S, Y) \to \text{Hom}_{\text{mod}-S}(J, Y)$$

is surjective. In particular, if S is a principal ideal domain, conclude that Y is injective if and only if Y is divisible.

(d) Finally, defining S to be $\mathbb{Z}$, conclude that $Y = \mathbb{Q}/\mathbb{Z}$ is injective, since it is divisible. Finally, for every Abelian group A and for every nonzero element a of A, conclude that there is a nonzero $\mathbb{Z}$-module homomorphism $\mathbb{Z} \cdot a \to \mathbb{Q}/\mathbb{Z}$. Thus, for every pair of elements $a', a'' \in A$ such that $a = a' - a''$ is nonzero, conclude that there exists a $\mathbb{Z}$-module homomorphisms $w : A \to \mathbb{Q}/\mathbb{Z}$ such that $w(a') - w(a'')$ is nonzero. Conclude that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of $\mathbb{Z}$. Thus $\text{mod} - \mathbb{Z}$ has enough injective objects. Thus, for every pair of associative, unital rings Λ, Π, the Abelian category $\Lambda - \Pi - \text{mod}$ has enough injective objects.

Problem 5. Let S be an associative, unital ring. Prove that $\text{Ch}^{\geq 0}(S)$ has enough injective objects, and prove that $\text{Ch}^{\leq 0}(S)$ has enough projective objects.

Problem 6. Let R be an associative, unital ring, and let $J \subset R$ be a right ideal. For every left R-module M, prove that there is a natural isomorphism,

$$\text{Tor}_1^R(R/J, M) \cong \text{Ker}(J \otimes_R M \to M),$$

and for every $q > 0$, there are isomorphisms,

$$\text{Tor}_q^R(J, M) \cong \text{Tor}_{q+1}^R(R/J, M).$$

In particular, if J is a principal ideal generated by a nonzerodivisor, say $J = sR$ for some nonzerodivisor s of R, conclude that

$$\text{Tor}_1^R(R/sR, M) \cong \{m \in M : sm = 0\},$$

and $\mathrm{Tor}_{q+1}(R/sR, M)$ is zero for all $q > 0$. In every case, conclude that for every left ideal I of R, $\mathrm{Tor}_1^R(R/J, R/I)$ is the same whether R/I is held fixed or whether R/J is held fixed.

Problem 7. Let R be a commutative, unital ring that is a principal ideal domain. Review the structure theorem of finitely generated modules over a principal ideal domain. Prove that for all finitely generated R-modules M and N, $\mathrm{Tor}_q^R(M, N)$ is zero for all $q \geq 2$. By realizing every R-module as a colimit of finitely generated R-modules, conclude that for every pair M, N of R-modules (whether or not finitely generated), $\mathrm{Tor}_q^R(M, N)$ is zero for all $q \geq 2$. Finally, for every pair s, t of nonzerodivisors in R, compute that $\mathrm{Tor}_1^R(R/sR, R/tR)$ is R/uR, where $sR + tR$ equals uR as a principal ideal in R.

Problem 8. Let R and T be commutative, unital rings. Let $f : R \to T$ be a ring homomorphism such that T is flat as an R-module. Prove that for every R-module M and for every T-module N, there are natural isomorphisms,

$$\mathrm{Tor}_q^R(M, N) \otimes_R T \to \mathrm{Tor}_q^T(M \otimes_R T, N).$$

In particular, if T is the ring of fractions $T = S^{-1}R$ for a multiplicatively closed subset S of R, prove that for every pair of R-modules M and N, the induced $S^{-1}R$-module homomorphism,

$$S^{-1}\mathrm{Tor}_q^R(M, N) \to \mathrm{Tor}_q^T(S^{-1}M, S^{-1}N),$$

is an isomorphism.

Problem 9. Let R and T be commutative, unital rings. Let $f : R \to T$ be a ring homomorphism. For every R-module M and for every T-module N, there is a binatural isomorphism,

$$\mathrm{Hom}_{R-\mathrm{mod}}(M, N) \cong \mathrm{Hom}_{T-\mathrm{mod}}(M \otimes_R T, N).$$

If M is a finitely presented R-module, conclude that also $M \otimes_R T$ is a finitely presented T-module. If also T is a flat R-module, conclude that for every R-module L,

$$\mathrm{Hom}_{R-\mathrm{mod}}(M, L) \otimes_R T \to \mathrm{Hom}_{R-\mathrm{mod}}(M, L \otimes_R T),$$

is an isomorphism. Finally, conclude that the natural map

$$\mathrm{Hom}_{R-\mathrm{mod}}(M, L) \otimes_R T \to \mathrm{Hom}_{T-\mathrm{mod}}(M \otimes_R T, L \otimes_R T)$$

is an isomorphism. Give a counterexamples when M is not finitely presented.

Problem 10. Continuing the previous problem, if M is a finitely presented R-module and if T is R-flat, prove that for every $q \geq 0$, the natural map

$$\mathrm{Ext}_R^q(M, L) \otimes_R T \to \mathrm{Ext}_T^q(M \otimes_R T, L \otimes_R T)$$

is an isomorphism.