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MAT 536 Problem Set 3

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. In each of the following cases, say whether the given category (a) has an initial object, (b) has a final object, (c) has a zero object, (d) has finite products, (e) has finite coproducts, (f) has arbitrary products, (g) has arbitrary coproducts, (h) has arbitrary limits (sometimes called *inverse limits*), (i) has arbitrary colimits (sometimes called *direct limits*), (j) coproducts / filtering colimits preserve monomorphisms, (k) products / cofiltering limits preserve epimorphisms.

(i) The category **Sets** whose objects are sets, whose morphisms are set maps, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(ii) The opposite category $\mathbf{Sets}^{\text{opp}}$.

(iii) For a given set S, the category whose objects are elements of the set, and where the only morphisms are the identity morphisms from an element to that same element. What if the set is the empty set? What if the set is a singleton set?

(iv) For a partially ordered set $(S, \preceq)$, the category whose objects are elements of S, and where the Hom set between two elements x, y of S is a singleton set if $x \preceq y$ and empty otherwise. What if the partially ordered set $(S, \preceq)$ is a **lattice**, i.e., every finite subset (resp. arbitrary subset) has a least upper bound and has a greatest lower bound?

(v) For a monoid $(M, \cdot, 1)$, the category with only one object whose Hom set, with its natural composition and identity, is $(M, \cdot, 1)$. What is M equals $\{1\}$?

(vi) For a monoid $(M, \cdot, 1)$ and an action of that monoid on a set, $\rho : M \times S \to S$, the category whose objects are the elements of S, and where the Hom set from x to y is the subset $M_{x,y} = \{m \in M | m \cdot x = y\}$. What if the action is both transitive and faithful, i.e., S equals M with its left regular representation?

(vii) The category **PSets** whose objects are pairs (S, s_0) of a set S and a specified element s_0 of S, i.e., *pointed sets*, whose morphisms are set maps that send the specified point of the domain to the specified point of the target, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(viii) The category **Monoids** whose objects are monoids, whose morphisms are homomorphisms of monoids, whose composition is sual composition, and whose identity morphisms are usual identity maps.

(ix) For a specified monoid $(M, \cdot, 1)$, the category whose objects are pairs (S, ρ) of a set S and an action $\rho : M \times S \to S$ of M on S, whose morphisms are set maps compatible with the action, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(x) The full subcategory **Groups** of **Monoids** whose objects are groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xi) The full subcategory $\mathbb{Z}-$mod of **Groups** whose objects are Abelian groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xii) The full subcategory **FiniteGroups** of **Groups** whose objects are finite groups. Are coproducts, resp. products, in the subcategory also coproducts, resp. products, in the larger category **Groups**? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xiii) The full subcategory $\mathbb{Z} - \text{mod}_{\text{tor}}$ of $\mathbb{Z} - \text{mod}$ consisting of torsion Abelian groups, i.e., every element has finite order (allowed to vary from element to element). Are coproducts, resp. products, preserved by the inclusion functor? Are monomorphisms, resp. epimorphisms preserved?

(xiv) The category **Rings** whose objects are associative, unital rings, whose morphisms are homomorphisms of rings (preserving the multiplicative identity), whose composition is the usual composition, and whose identity morphisms are the usual identity maps. **Hint.** For the coproduct of two associative, unital rings $(R', +, 0, \cdot', 1')$ and $(R'', +, 0, \cdot'', 1'')$, first form the coproduct $R' \oplus R''$ of $(R', +, 0)$ and $(R'', +, 0)$ as a $\mathbb{Z}$-module, then form the total tensor product ring $T^\bullet_{\mathbb{Z}}(R' \oplus R'')$ as in the previous problem set. For the two natural maps $q' : R' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ and $q'' : R'' \hookrightarrow T^1_{\mathbb{Z}}(R' \oplus R'')$ form the left-right ideal $I \subset T^\bullet_{\mathbb{Z}}(R' \oplus R'')$ generated by $q'(1') - 1$, $q''(1'') - 1$, $q'(r' \cdot' s') - q'(r') \cdot q'(s')$, and $q''(r'' \cdot'' s'') - q''(r'') \cdot q''(s'')$ for all elements $r', s' \in R'$ and $r'', s'' \in R''$. Define

$$p : T^1_{\mathbb{Z}}(R' \oplus R'') \to R,$$

to be the quotient by I. Prove that $p \circ q' : R' \to R$ and $p \circ q'' : R'' \to R$ are ring homomorphisms that make R into a coproduct of R' and R''.

(xv) The full subcategory **CommRings** of **Rings** whose objects are commutative, unital rings. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvi) The full subcategory **NilCommRings** of **CommRings** whose objects are commutative, unital rings such that every noninvertible element is nilpotent. Does the inclusion functor preserve coproducts, resp. products? (Be careful about products!) Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvii) Let R and S be associative, unital rings. Let $R-\text{mod}$, resp. $\text{mod}-S$, $R-S-\text{mod}$, be the category of left R-modules, resp. right S-modules, $R-S$-bimodules. Does the inclusion functor from $R-S-\text{mod}$ to $R-\text{mod}$, resp. to $\text{mod}-S$, preserve coproduct, products, monomorphisms and epimorphisms?

(xviii) Let $(I,\preceq)$ be a partially ordered set. Let $\mathcal{C}$ be a category. An $(I,\preceq)$-*system* in $\mathcal{C}$ is a datum

$$c = ((c_i)_{i\in I}, (f_{i,j})_{(i,j)\in I\times I, i\preceq j})$$

where every c_i is an object of $\mathcal{C}$, where for every pair $(i,j)\in I\times I$ with $i\preceq j$, $c_{i,j}$ is an element of $\text{Hom}_{\mathcal{C}}(c_i,c_j)$, and satisfying the following conditions: (a) for every $i\in I$, $c_{i,i}$ equals Id_{c_i}, and (b) for every triple $(i,j,k)\in I$ with $i\preceq j$ and $j\preceq k$, $c_{j,k}\circ c_{i,j}$ equals $c_{i,k}$. For every pair of $(I,\preceq)$-systems in $\mathcal{C}$, $c=((c_i)_{i\in I},(c_{i,j})_{i\preceq j})$ and $c'=((c'_i)_{i\in I},(c'_{i,j})_{i\preceq j})$, a morphism $g: c\to c'$ is defined to be a datum $(g_i)_{i\in I}$ of morphisms $g_i\in\text{Hom}_{\mathcal{C}}(c_i,c'_i)$ such that for every $(i,j)\in I\times I$ with $i\preceq j$, $g_j\circ c_{i,j}$ equals $c'_{i,j}\circ g_i$. Composition of morphisms g and g' is componentwise $g'_i\circ g_i$, and identities are $\text{Id}_c=(\text{Id}_{c_i})_{i\in I}$. This category is $\text{Fun}((I,\preceq),\mathcal{C})$, and is sometimes referred to as the category of $(I,\preceq)$-*presheaves*. Assuming $\mathcal{C}$ has finite coproducts, resp. finite products, arbitrary coproducts, arbitrary products, a zero object, kernels, cokernels, etc., what can you say about $\text{Fun}((I,\preceq),\mathcal{C})$?

(xix)This part is intended for students who already know about sheaves. Let $\mathcal{C}$ be a category that has arbitrary products. Let $(I,\preceq)$ be a partially ordered set whose associated category as in (iv) has finite coproducts and has arbitrary products. The main example is when $I=\mathfrak{U}$ is the collection of all open subsets U of a topology on a set X, and where $U\preceq V$ if $U\supseteq V$. Then coproduct is intersection and product is union. Motivated by this case, an *covering* of an element i of I is a collection $\underline{j}=(j_\alpha)_{\alpha\in A}$ of elements j_α of I such that for every α, $i\preceq j_\alpha$, and such that i is the product of $(j_\alpha)_{\alpha\in A}$ in the sense of (iv). In this case, for every $(\alpha,\beta)\in A\times A$, define $j_{\alpha,\beta}$ to be the element of I such that $j_\alpha\preceq j_{\alpha,\beta}$, such that $j_\beta\preceq j_{\alpha,\beta}$, and such that $j_{\alpha,\beta}$ is a coprodcut of (j_α,j_β). An $(I,\preceq)$-presheaf $c=((c_i)_{i\in I},(c_{i,j})_{i\preceq j})$ is an $(I,\preceq)$-*sheaf* if for every element i of I and for every covering $\underline{j}=(j_\alpha)_{\alpha\in A}$, the following diagram in $\mathcal{C}$ is *exact* in a sense to be made precise,

$$c_i \xrightarrow{q} \prod_{\alpha\in A} c_{j_\alpha} \overset{p'}{\rightrightarrows} p'' \prod_{(\alpha,\beta)\in A\times A} c_{j_{\alpha,\beta}}.$$

For every $\alpha\in A$, the factor of q,

$$\text{pr}_\alpha\circ q: c_i\to c_{j_\alpha},$$

is defined to be c_{i,j_α}. For every $(\alpha,\beta)\in A\times A$, the factor of p',

$$\text{pr}_{\alpha,\beta}\circ p' : \prod_{\gamma\in A} c_{j_\gamma}\to c_{j_{\alpha,\beta}},$$

is defined to be $c_{j_\alpha,j_{\alpha,\beta}}\circ\text{pr}_\alpha$. Similarly, $\text{pr}_{\alpha,\beta}\circ p''$ is defined to be $c_{j_\beta,j_{\alpha,\beta}}\circ\text{pr}_\beta$. The diagram above is *exact* in the sense that q is a monomorphism in $\mathcal{C}$ and q is a fiber product in $\mathcal{C}$ of the pair of morphisms (p',p''). The category of $(I,\preceq)$ is the full subcategoryof the category of $(I,\preceq)$-presheaves whose objects are $(I,\preceq)$-sheaves. Does this subcategory have coproducts, products, etc.? Does the inclusion functor preserve coproducts, resp. products, monomorphisms, epimorphisms? Before considering the general case, it is probably best to first consider the case that $\mathcal{C}$ is $\mathbb{Z}-\text{mod}$, and then consider the case that $\mathcal{C}$ is **Sets**.

Problem 2. Let $\mathcal{A}$ be an Abelian category, e.g., the category $\mathbb{Z}-\text{mod}$ of Abelian groups. Let $\mathcal{A}-\text{S.E.S.}$ be the category whose objects are short exact sequences in $\mathcal{A}$,

$$\Sigma:\ 0 \longrightarrow A' \xrightarrow{q_\Sigma} A \xrightarrow{p_\Sigma} A'' \longrightarrow 0,$$

i.e., a datum $(A',A,A'',q_\Sigma,p_\Sigma)$ of objects A', A, and A'' of $\mathcal{A}$, an element $q_\Sigma\in\text{Hom}_{\mathcal{A}}(A',A)$ that is a monomorphism, and an element $p_\Sigma\in\text{Hom}_{\mathcal{A}}(A,A'')$ that is an epimorphism. For a short exact sequence $\Sigma=(A',A,A'',q_\Sigma,p_\Sigma)$ and a short exact sequence $T=(B',B,B'',q_T,p_T)$, a morphism $f:\Sigma\to T$ is a triple (f',f,f'') of elements $f'\in\text{Hom}_{\mathcal{A}}(A',B')$, $f\in\text{Hom}_{\mathcal{A}}(A,B)$, and $f''\in\text{Hom}_{\mathcal{A}}(A'',B'')$ such that the following diagram commutes,

$$\begin{tikzcd}
\Sigma: \arrow[d, "f"'] & 0 \arrow[r] & A' \arrow[r, "q_\Sigma"] \arrow[d, "f'"'] & A \arrow[r, "p_\Sigma"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0, \\
T: & 0 \arrow[r] & B' \arrow[r, "q_T"] & B \arrow[r, "p_T"] & B'' \arrow[r] & 0
\end{tikzcd}.$$

Prove that $\mathcal{A}-\text{S.E.S.}$ is an additive category that has kernels and cokernels. If $\mathcal{A}$ is $\mathbb{Z}-\text{mod}$, prove that this is not an Abelian category.

Problem 3. Let $\mathcal{A}$ be an Abelian category. Carefully formulate and prove the Snake Lemma in this Abelian category **without** chasing elements. (If you are going to work much with homological algebra, you should do this exercise at least once in your career. Many mathematicians afterwards use the Freyd-Mitchell Theorem to avoid doing this exercise a second time.)

Problem 4.(Baer Sum) Let $\mathcal{A}$ be an Abelian category. For objects A' and A'', for short exact sequences in $\mathcal{A}$,

$$\Sigma:\ 0 \longrightarrow A' \xrightarrow{q_\Sigma} A \xrightarrow{p_\Sigma} A'' \longrightarrow 0,$$
$$T:\ 0 \longrightarrow A' \xrightarrow[q_T]{} B \xrightarrow[p_T]{} A'' \longrightarrow 0,$$

an (A'',A')-morphism is an element $f\in\text{Hom}_{\mathcal{A}}(A,B)$ such that the following diagram is commutative,

$$\begin{tikzcd}
\Sigma: \arrow[d, "f"'] & 0 \arrow[r] & A' \arrow[r, "q_\Sigma"] \arrow[d, "\text{Id}_{A'}"'] & A \arrow[r, "p_\Sigma"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "\text{Id}_{A''}"] & 0, \\
T: & 0 \arrow[r] & B' \arrow[r, "q_T"] & B \arrow[r, "p_T"] & B'' \arrow[r] & 0
\end{tikzcd}.$$

(a) Prove that every (A'',A')-morphism is an isomorphism. Prove that the identity Id_A is an (A'',A')-morphism. Prove that composition of (A'',A')-morphisms (in the category of short exact sequences) is an (A'',A')-morphism.

(b) Let $A' \oplus A''$ be the product and coproduct of A' and A'' with natural morphisms, $q_1 : A' \to A' \oplus A''$, $q_2 : A'' \to A' \oplus A''$, $p_1 : A' \oplus A'' \to A'$, and $p_2 : A' \oplus A'' \to A''$. The *split* short exact sequence is

$$A' \oplus A'' : \ 0 \longrightarrow A' \xrightarrow{q_1} A' \oplus A'' \xrightarrow{p_2} A'' \longrightarrow 0\cdot$$

For a short exact sequence Σ as above, prove the the following are equivalent.

(i) There exists an element $r \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ such that $r \circ q_\Sigma$ equals $\mathrm{Id}_{A'}$.

(ii) There exists an (A'', A')-isomorphism between Σ and the split short exact sequence.

(iii) There exists an element $s \in \mathrm{Hom}_{\mathcal{A}}(A'', A)$ such that $p_\Sigma \circ s$ equals $\mathrm{Id}_{A''}$.

If any of these equivalent conditions holds, the short exact sequence Σ is called *a split short exact sequence.*

(c) For every element $u \in \mathrm{Hom}_{\mathcal{A}}(A'', A')$, define

$$\widetilde{u} : A' \oplus A'' \to A' \oplus A''$$

to be the unique morphism such that $p_1 \circ \widetilde{u} \circ q_1 = \mathrm{Id}_{A'}$, $p_2 \circ \widetilde{u} \circ q_1 = 0$, $p_1 \circ \widetilde{u} \circ q_2 = u$, and $p_2 \circ \widetilde{u} \circ q_2 = \mathrm{Id}_{A''}$. Prove that $\widetilde{u}$ is an (A'', A')-morphism from the split exact sequence to itself. Prove that the rule $u \mapsto \widetilde{u}$ defines a bijection from $\mathrm{Hom}_{\mathcal{A}}(A'', A')$ to the set of (A'', A')-morphisms from the split exact sequence to itself.

(d) Let Σ and T be (A'', A') be short exact sequence as above. Define $\widetilde{C}$ to be the fiber product $A \times_{p_\Sigma, A'', p_T} B$,

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{p_1} & A \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_\Sigma} \\ B & \xrightarrow[p_T]{} & A'' \end{array},$$

and denote by $\widetilde{p}_\Upsilon : \widetilde{C} \to A''$ to be $p_\Sigma \circ p_1 = p_T \circ p_2$. Since the compositions,

$$A' \xrightarrow{q_\Sigma} A \xrightarrow{p_\Sigma} A'',$$

$$A' \xrightarrow{0} A \xrightarrow{p_\Sigma} A'',$$

$$A' \xrightarrow{q_T} B \xrightarrow{p_T} A'',$$

and

$$A' \xrightarrow{0} B \xrightarrow{p_T} A'',$$

all equal 0, there exist unique morphisms to the fiber product

$$(q_\Sigma, 0) : A' \to \widetilde{C},$$

and
$$(0, q_T) : A' \to \widetilde{C}.$$
Define the coproduct of these two maps to be $\rho : \widetilde{C} \to C$. Denote by
$$p_\Upsilon : A' \to C$$
the common composition $\rho \circ (q_\Sigma, 0) = \rho \circ (0, q_T)$. Finally, since both compositions $\widetilde{p}_\Upsilon \circ (q_\Sigma, 0)$ and $\widetilde{p}_\Upsilon \circ (0, q_T)$ equal the common value 0, there exists a unique morphism
$$p_\Upsilon : C \to A'',$$
such that $p_\Upsilon \circ \rho$ equals $\widetilde{p}_\Upsilon$. Prove that
$$\Upsilon : \ 0 \longrightarrow A' \xrightarrow{q_\Upsilon} C \xrightarrow{p_\Upsilon} A'' \longrightarrow 0,$$
is an (A'', A')-short exact sequence. This is called the *Baer sum* of the (A'', A')-short exact sequences Σ and T.

(e) Prove that there is a (canonical) (A'', A')-isomorphism between the Baer sum of Σ and T and the Baer sum of T and Σ. Thus Baer sum is "commutative". Prove that the Baer sum of any (A'', A')-short exact sequence Σ and the split exact sequence is (A'', A')-isomorphic again to Σ. Thus the split exact sequence is an identity / neutral element for Baer sum. For (A'', A')-exact sequences P, Σ and T, prove that there is a (canonical) isomorphism between the two different ways of forming the Baer sum of P, Σ and T. Thus the Baer sum is "associative".

(f) Let Σ be an (A'', A')-short exact sequence as above. Define Σ^- to be the (A'', A')-short exact sequence
$$\Sigma : \ 0 \longrightarrow A' \xrightarrow{-q_\Sigma} A \xrightarrow{p_\Sigma} A'' \longrightarrow 0.$$
Prove that the Baer sum of Σ and Σ^- is a split exact sequence.

(g) Assume that there exists a collection of (A'', A')-short exact sequences indexed by a set such that every (A'', A')-short exact sequence is (A'', A')-isomorphic to one from this collection, e.g., this is true if there exists a monomorphism from A' to an injective object of $\mathcal{A}$ (thus, this is true in every Grothendieck Abelian category). In this case, there is a well-defined set $\mathrm{Ext}_{\mathcal{A}}(A'', A')$ of of (A'', A')-isomorphism classes of (A'', A')-short exact sequences. Prove that Baer sum makes $\mathrm{Ext}_{\mathcal{A}}(A'', A')$ into an Abelian group.

Problem 5. Let $\mathcal{A}$ be a category. Recall that for every object A of $\mathcal{A}$, there is a covariant Yoneda functor,
$$h^A : \mathcal{A} \to \mathbf{Sets}, \quad h^A(Z) = \mathrm{Hom}_{\mathcal{A}}(A, Z),$$
and there is a contravariant Yoneda functor,
$$h_A : \mathcal{A}^{\mathrm{opp}} \to \mathbf{Sets}, \quad h_A(Z) = \mathrm{Hom}_{\mathcal{A}}(Z, A).$$

For every morphism $u : A \to B$, for every object Z, define

$$h^u(Z) : h^B(Z) \to h^A(Z), \quad (B \xrightarrow{v} Z) \mapsto (A \xrightarrow{v \circ u} Z).$$

Similarly, define

$$h_u(Z) : h_A(Z) \to h_B(Z), \quad (Z \xrightarrow{t} A) \mapsto (Z \xrightarrow{u \circ t} B).$$

(a) For A, B and u fixed, prove that both h^u and h_u are natural transformations of functors.

(b) Prove that u is a monomorphism if and only if, for every object Z of $\mathcal{A}$, $h_u(Z)$ is an injective set map, i.e., a monomorphism in the category of sets. Similarly, prove that u is an epimorphism if and only if, for every Z, $h^u(Z)$ is an injective set map.

(c) Now assume that $\mathcal{A}$ is an additive category, so that each Yoneda functor is enriched to a functor to Abelian groups,

$$h^A : \mathcal{A} \to \mathbb{Z} - \text{mod}, \quad h^A(Z) = \text{Hom}_{\mathcal{A}}(A, Z),$$

$$h_A : \mathcal{A}^{\text{opp}} \to \mathbb{Z} - \text{mod}, \quad h_A(Z) = \text{Hom}_{\mathcal{A}}(Z, A).$$

Prove that $h^u(Z)$ and $h_u(Z)$ are homomorphisms of Abelian groups, thus h^u and h_u are natural transformations of the enriched Yoneda functors.

(d) For a morphism $u : A \to B$ in $\mathcal{A}$, prove that a morphism $q : K \to A$ is a kernel of u if and only if, for every object Z, the following diagram of Abelian groups is exact,

$$0 \longrightarrow h_K(Z) \xrightarrow{h_q(Z)} h_A(Z) \xrightarrow{h_u(Z)} h_B(Z).$$

Similarly, prove that a morphism $p : B \to C$ is a cokernel of u if and only if for every object Z, the following diagram of Abelian groups is exact,

$$0 \longrightarrow h^C(Z) \xrightarrow{h^p(Z)} h^B(Z) \xrightarrow{h^u(Z)} h^A(Z).$$

Thus the Yoneda functors transform "abstract" exactness in $\mathcal{A}$ into "concrete" exactness in the category of Abelian groups. This gives one approach to avoid "chasing elements" and the use of the Full Embedding Theorem.

Problem 6. Let $\mathcal{A}$ be an Abelian category where for every pair of objects A and B, there is an Ext set $\text{Ext}_{\mathcal{A}}(B, A)$ as in Problem 4, and this set is an Abelian group via Baer sum. For an object B, denote the covariant Yoneda functor by h^B,

$$h^B : \mathcal{A} \to \mathbb{Z} - \text{mod}, \quad h^B(A) = \text{Hom}_{\mathcal{A}}(B, A).$$

Let Σ be a short exact sequence in $\mathcal{A}$,

$$\Sigma : \ 0 \longrightarrow A' \xrightarrow{q_\Sigma} A \xrightarrow{p_\Sigma} A'' \longrightarrow 0,$$

By the previous problem, the following sequences of Abelian groups are exact,

$$h^B(\Sigma) : \ 0 \longrightarrow \text{Hom}_{\mathcal{A}}(B, A') \xrightarrow{h^B(q_\Sigma)} \text{Hom}_{\mathcal{A}}(B, A) \xrightarrow{h^B(p_\Sigma)} \text{Hom}_{\mathcal{A}}(B, A'')$$

$$h_B(\Sigma): \ 0 \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A', B') \xrightarrow{h_B(p_\Sigma)} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{h_B(q_\Sigma)} \mathrm{Hom}_{\mathcal{A}}(A', B)$$

(a) For every triple of objects B, A and A'' of $\mathcal{A}$, for every morphism $u : A \to A''$ in $\mathcal{A}$, for every (B, A)-short exact sequence,

$$T: \ 0 \longrightarrow A \xrightarrow{e} X \xrightarrow{f} B \longrightarrow 0,$$

define the *pushout* by u to be the sequence

$$u_*T: \ 0 \longrightarrow A'' \xrightarrow{q_1} A'' \oplus_{u,A,e} X \xrightarrow{p_2} B \longrightarrow 0,$$

where $A'' \oplus_{u,A,e} X$ is a cofiber coproduct of (u, e) together with its natural morphisms,

$$q_1 : A'' \to A'' \oplus_{u,A,e} X, \ q_2 : X \to A'' \oplus_{u,A,e} X,$$

$$p_1 : A'' \oplus_{u,A,e} X \to \mathrm{Coker}(u), \ p_2 : A'' \oplus_{u,A,e} \to \mathrm{Coker}(e).$$

Prove that u_*T is a (B, A'')-short exact sequence. Denote the set map that sends every T to the pushout u_*T by

$$\mathrm{Ext}_{\mathcal{A}}(B, u) : \mathrm{Ext}_{\mathcal{A}}(B, A) \to \mathrm{Ext}_{\mathcal{A}}(B, A'').$$

Prove that this set map preserves Baer sums. Moreover, prove that the pushout u_*T is a split exact sequence if and only if there exists a morphism $\widetilde{r} : X \to A''$ such that $r \circ e$ equals u. In particular, for every splitting $r : X \to A$ of T, $\widetilde{r} = u \circ r$ gives a splitting of u_*T. So pushout sends split exact sequences to split exact sequences. Conclude that $\mathrm{Ext}_{\mathcal{A}}(B, u)$ is a group homomorphism.

(b) For a short exact sequence Σ as above, prove that the following diagram of Abelian groups is exact,

$$\mathrm{Ext}_{\mathcal{A}}(B, \Sigma): \ \mathrm{Ext}_{\mathcal{A}}(B, A') \xrightarrow{\mathrm{Ext}(B, q_\Sigma)} \mathrm{Ext}_{\mathcal{A}}(B, A) \xrightarrow{\mathrm{Ext}(B, p_\Sigma)} \mathrm{Ext}_{\mathcal{A}}(B, A'').$$

Hint. With notation as above, if $\widetilde{r} : X \to A''$ is a morphism such that $\widetilde{r} \circ e$ equals p_Σ, define Y to be $\mathrm{Ker}(\widetilde{r})$, and compute the kernel of the induced morphism $Y \to B$.

(c) For every triple of objects B, A and A' of $\mathcal{A}$, for every morphism $v : A' \to A$ in $\mathcal{A}$, for every (A, B)-short exact sequence,

$$\Pi: \ 0 \longrightarrow B \xrightarrow{g} Z \xrightarrow{h} A \longrightarrow 0,$$

define the *pullback* by v to be the sequence

$$v^*\Pi: \ 0 \longrightarrow B \xrightarrow{p_1} Z \times_{h,A,v} A' \xrightarrow{q_2} A' \longrightarrow 0,$$

where $Z \times_{h,A,v} A'$ is a fiber product of (h, v) together with its natural morphisms

$$p_1 : Z \times_{h,A,v} A' \to Z, \ p_2 : Z \times_{h,A,v} A' \to A',$$

$$q_1 : \mathrm{Ker}(h) \to Z \times_{h,A,v} A', \ q_2 : \mathrm{Ker}(v) \to Z \times_{h,A,v} A'.$$

Prove that $v^*\Pi$ is an (A', B)-short exact sequence. Denote the set map that sends every Π to the pushout v^*T by

$$\mathrm{Ext}_{\mathcal{A}}(v, B) : \mathrm{Ext}_{\mathcal{A}}(A, B) \to \mathrm{Ext}_{\mathcal{A}}(A', B).$$

Prove that this set map preserves Baer sums. Moreover, prove that the pullback $v^*\Pi$ is a split exact sequence if and only if there exists a morphism $\widetilde{s} : A' \to Z$ such that $h \circ \widetilde{s}$ equals v. In particular, for every splitting $s : A \to Z$ of Π, $\widetilde{s} = s \circ v$ gives a splitting of $v^*\Pi$. So pullback sends split exact sequences to split exact sequences. Conclude that $\mathrm{Ext}_{\mathcal{A}}(v, B)$ is a group homomorphism.

(d) For a short exact sequence Σ as above, prove that the following diagram of Abelian groups is exact,

$$\mathrm{Ext}_{\mathcal{A}}(\Sigma, B) : \ \mathrm{Ext}_{\mathcal{A}}(A'', B) \xrightarrow{\mathrm{Ext}(p_\Sigma, B)} \mathrm{Ext}_{\mathcal{A}}(A, B) \xrightarrow{\mathrm{Ext}(q_\Sigma, B)} \mathrm{Ext}_{\mathcal{A}}(A', B)\cdot$$

Hint. With notation as above, if $\widetilde{s} : A' \to Z$ is a morphism such that $h \circ \widetilde{z}$ equals q_Σ, define W to be $\mathrm{Coker}(\widetilde{s})$, and compute the cokernel of the induced morphism $B \to W$.

(e) Prove that pushout and pullback commute.

(f) Let B be an object of $\mathcal{A}$ and let Σ be a short exact sequence in $\mathcal{A}$ as above. For every morphism $t \in \mathrm{Hom}_{\mathcal{A}}(B, A'')$, define $\delta_{B,\Sigma}(t) \in \mathrm{Ext}_{\mathcal{A}}(B, A')$ to be the (B, A')-short exact sequence $t^*\Sigma$,

$$t^*\Sigma : \ 0 \longrightarrow A' \xrightarrow{p_1} A \times_{p_\Sigma, A'', t} B \xrightarrow{q_2} B \longrightarrow 0.$$

Prove that the induced set map

$$\delta_{B,\Sigma} : \mathrm{Hom}_{\mathcal{A}}(B, A'') \to \mathrm{Ext}_{\mathcal{A}}(B, A'), \quad t \mapsto \delta_{B,\Sigma}(t)$$

sends sums to Baer sums. Moreover, prove that there is a splitting of $\delta_{B,\Sigma}(t)$ if and only if there exists a morphism $\widetilde{s} \in \mathrm{Hom}_{\mathcal{A}}(B, A)$ such that $h_B(p_\Sigma)$ maps $\widetilde{s}$ to t. Finally, for a given (B, A')-short exact sequence T, prove that the pushout $q_{\Sigma,*}T$ (B, A)-short exact sequence admits a splitting if and only if there exists a morphism $t \in \mathrm{Hom}_{\mathcal{A}}(B, A'')$ such that T is (B, A')-equivalent to $\delta_{B,\Sigma}(t)$. Altogether, conclude the following sequence of Abelian groups is exact,

$$\mathrm{Hom}_{\mathcal{A}}(B, A) \xrightarrow{\mathrm{Hom}(B, p_\Sigma)} \mathrm{Hom}_{\mathcal{A}}(B, A'') \xrightarrow{\delta_{B,\Sigma}} \mathrm{Ext}_{\mathcal{A}}(B, A') \xrightarrow{\mathrm{Ext}(B, q_\Sigma)} \mathrm{Hom}_{\mathcal{A}}(B, A).$$

Finally, denoting by $\mathcal{A}_{SES}$ the category whose objects are short exact sequences in $\mathcal{A}$ and whose morphisms are commutative diagrams of short exact sequences, prove that $\Sigma \mapsto \delta_{B,\Sigma}$ is a natural transformation between the following two functors,

$$\mathrm{Hom}_{\mathcal{A}}(B, *'') : \mathcal{A}_{SES} \to \mathbb{Z} - \mathrm{mod}, \ \Sigma \mapsto \mathrm{Hom}_{\mathcal{A}}(B, A''),$$

$$\mathrm{Ext}_{\mathcal{A}}(B, *') : \mathcal{A}_{SES} \to \mathbb{Z} - \mathrm{mod}, \ \Sigma \mapsto \mathrm{Ext}_{\mathcal{A}}(B, A').$$

Finally, use opposite categories (or symmetric arguments) to prove the analogous results for $\mathrm{Hom}_{\mathcal{A}}(-, B)$ and $\mathrm{Ext}_{\mathcal{A}}(-, B)$.