Jason Starr
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MAT 536 Problem Set 2

Homework Policy. Please read through all the problems. Please solve 5 of the problems. I will be happy to discuss the solutions during office hours.

Problems.

Problem 1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let

$$L : \mathcal{A} \to \mathcal{B},\ R : \mathcal{B} \to \mathcal{A},$$

be (covariant) functors. Let

$$\theta : \mathrm{Id}_{\mathcal{A}} \Rightarrow RL,\ \theta(a) : a \to R(L(a)),$$

$$\eta : LR \Rightarrow \mathrm{Id}_{\mathcal{B}},\ \eta(b) : L(R(b)) \to b,$$

be natural transformations of functors. This data is an *adjoint pair* of (covariant) functors if the following compositions of natural transformations equal Id_R, resp. Id_L,

$$(*_R) : R \stackrel{\theta \circ R}{\Rightarrow} RLR \stackrel{R \circ \eta}{\Rightarrow} R,$$

$$(*_L) : L \stackrel{L \circ \theta}{\Rightarrow} LRL \stackrel{\eta \circ L}{\Rightarrow} R.$$

For every object a of $\mathcal{A}$ and for every object b of $\mathcal{B}$, define set maps,

$$H_R^L(a,b) : \mathrm{Hom}_{\mathcal{B}}(L(a),b) \to \mathrm{Hom}_{\mathcal{A}}(a,R(b)),$$

$$(L(a) \xrightarrow{\phi} b) \mapsto \left(a \xrightarrow{\theta(a)} R(L(a)) \xrightarrow{R(\phi)} R(b)\right),$$

and

$$H_L^R(a,b) : \mathrm{Hom}_{\mathcal{A}}(a,R(b)) \to \mathrm{Hom}_{\mathcal{B}}(L(a),b),$$

$$(a \xrightarrow{\psi} R(b)) \mapsto \left(L(a) \xrightarrow{L(\psi)} L(R(b)) \xrightarrow{\eta(b)} b\right).$$

(i) For L, R, θ and η as above, the conditions $(*_R)$ and $(*_L)$ hold if and only if for every object a of $\mathcal{A}$ and every object b of $\mathcal{B}$, $H_R^L(a,b)$ and $H_L^R(a,b)$ are inverse bijections.

(ii) Prove that both $H_R^L(a,b)$ and $H_L^R(a,b)$ are binatural in a and b.

(iii) For functors L and R, and for binatural inverse bijections $H^L_R(a,b)$ and $H^R_L(a,b)$ between the bifunctors
$$\text{Hom}_{\mathcal{B}}(L(a),b), \text{Hom}_{\mathcal{A}}(a,R(b)) : \mathcal{A}\times\mathcal{B} \to \textbf{Sets},$$
prove that there exist unique θ and η extending L and R to an adjoint pair such that H^L_R and H^R_L agree with the binatural inverse bijections defined above.

(iv) Let (L,R,θ,η) be an adjoint pair. Let a (covariant) functor
$$\widetilde{R} : \mathcal{B} \to \mathcal{A},$$
and natural transformations,
$$\widetilde{\theta} : \text{Id}_{\mathcal{A}} \Rightarrow \widetilde{R}\circ L, \widetilde{\eta} : L\circ\widetilde{R} \Rightarrow \text{Id}_{\mathcal{B}},$$
be natural transformations such that $(L,\widetilde{R},\widetilde{\theta},\widetilde{\eta})$ is also an adjoint pair. For every object b of B, define $I(b)$ in $\text{Hom}_{\mathcal{B}}(R(b),\widetilde{R}(b))$ to be the image of Id_b under the composition,
$$\text{Hom}_{\mathcal{B}}(b,b) \xrightarrow{\text{Hom}_{\mathcal{B}}(\theta(b),b)} \text{Hom}_{\mathcal{B}}(L(R(b)),b) \xrightarrow{H^{\widetilde{R}}_L(R(b),b)} \text{Hom}_{\mathcal{B}}(R(b),\widetilde{R}(b)).$$
Similarly, define $J(b)$ in $\text{Hom}_{\mathcal{B}}(\widetilde{R}(b),R(b))$, to be the image of Id_b under the composition,
$$\text{Hom}_{\mathcal{B}}(b,b) \xrightarrow{\text{Hom}_{\mathcal{B}}(\widetilde{\theta}(b),b)} \text{Hom}_{\mathcal{B}}(L(\widetilde{R}(b)),b) \xrightarrow{H^R_L(\widetilde{R}(b),b)} \text{Hom}_{\mathcal{B}}(\widetilde{R}(b),R(b)).$$
Prove that I and J are the unique natural transformations of functors,
$$I : R \Rightarrow \widetilde{R},\ J : \widetilde{R} \Rightarrow R,$$
such that $\widetilde{\theta}$ equals $(I\circ L)\circ\theta$, θ equals $(J\circ L)\circ\widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta\circ(L\circ I)$, and η equals $\widetilde{\eta}\circ(L\circ J)$. Moreover, prove that I and J are inverse natural isomorphisms. In this sense, every extension of a functor L to an adjoint pair (L,R,θ,η) is unique up to unique natural isomorphisms (I,J). Formulate and prove the symmetric statement for all extensions of a functor R to an adjoint pair (L,R,θ,η).

(v) Formulate the corresponding notions of adjoint pairs when L and R are contravariant functors (just replace one of the categories by its opposite category).

Problem 2. Let R be a unital, associative ring. Let Σ be a finite set. For every function
$$f : \Sigma \to R,\ \sigma \mapsto f_\sigma,$$
the *support* of f is the subset $f^{-1}(R\setminus\{0\})$, i.e., those σ with $f_\sigma \neq 0$. Let F_Σ be the subset of $\text{Hom}_{\textbf{Sets}}(\Sigma,R)$ consisting of functions with finite support. Define addition componentwise,
$$f+g : \Sigma \to R,\ (f+g)_\sigma = f_\sigma + g_\sigma,$$

and define left-right R-actions componentwise,

$$r \cdot f : \Sigma \to R, \ (r \cdot f)_\sigma = r \cdot (f_\sigma),$$

$$f \cdot r : \Sigma \to R, \ (f \cdot r)_\sigma = (f_\sigma) \cdot r.$$

(i) Prove that with these operations, F_Σ is an R-bimodule.

(ii) Define a set map $i_\Sigma : \Sigma \to F_\sigma$ by the Kronecker delta function,

$$i_\Sigma(\sigma) : \Sigma \to R, \ \tau \mapsto \begin{cases} 1, & \tau = \sigma, \\ 0, & \tau \neq \sigma \end{cases}$$

Prove that this is well-defined, and prove that every element of F_Σ can be expressed as an R-linear combination (right or left) of elements in the image of i_Σ. Assuming that $1 \neq 0$ in R, prove that i_Σ is injective, and prove that every nontrivial (finite) R-linear combination of (distinct) elements of Σ is nonzero.

(iii) Let N be any left R-module. Let $j : \Sigma \to N$ be any set map. Prove that there exists a unique homomorphism of left R-modules,

$$\widetilde{j} : F_\Sigma \to N,$$

such that $\widetilde{j} \circ i_\Sigma$ equals j. Repeat this for right R-modules.

(iv) For every set map $u : \Sigma \to \Xi$, the composition

$$i_\Xi \circ u : \Sigma \to F_\Xi,$$

is a set map that gives a unique R-bimodule homomorphism by (iii). Denote $\widetilde{i_\xi \circ u}$ by

$$F_u : F_\Sigma \to F_\xi.$$

Prove that F_{Id_Σ} is the identity on F_Σ. Also, for a set map $v : \Xi \to \Theta$, prove that $F_{v \circ u}$ equals $F_v \circ F_u$. Conclude that F is a covariant functor,

$$F : \mathbf{Sets} \to R - \mathrm{bimod}.$$

(v) Denote by Φ the "forgetful functor",

$$\Phi : R - \mathrm{mod} \to \mathbf{Sets},$$

that sends each left R-module M to the underlying set of M. Prove that Φ is faithful. Show that $\Sigma \mapsto i_\Sigma$ is a natural transformation of functors from $\mathbf{Sets}$ to $\mathbf{Sets}$,

$$i : \mathrm{Id}_{\mathbf{Sets}} \Rightarrow \Phi \circ F.$$

Similarly, for every left R-module M, for the identity map $j = \mathrm{Id}_M$,

$$j : M \to M,\ m \mapsto m,$$

by (iii) there is a unique left R-module homomorphism $\widetilde{j}$, which we will denote by η_M,

$$\widetilde{j} : F_M \to M,$$

such that $\widetilde{j} \circ i_M$ equals j, i.e., $\eta_M \circ i_M$ equals Id_M. Prove that $M \mapsto \eta_M$ is a natural transformation of functors from $R-\mathrm{mod}$ to $R-\mathrm{mod}$,

$$\eta : F \circ \Phi \Rightarrow \mathrm{Id}_{R-\mathrm{mod}}.$$

Finally, prove that for every set Σ and every left R-module M, the natural transformations i and η make (F, Φ) into an adjoint pair of functors, i.e., they establish a binatural equivalence

$$\mathrm{Hom}_{R-\mathrm{Mod}}(F_\Sigma, M) = \mathrm{Hom}_{\mathbf{Sets}}(\Sigma, \Phi(M)).$$

Repeat all of this for right R-modules.

Problem 3. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let

$$L' : \mathcal{A} \to \mathcal{B}, R' : \mathcal{B} \to \mathcal{A},$$

be (covariant) functors, and let

$$\theta' : \mathrm{Id}_\mathcal{A} \Rightarrow R'L',\ \eta' : L'R' \Rightarrow \mathrm{Id}_\mathcal{B},$$

be natural transformations that are an adjoint pair of functors. Also let

$$L'' : \mathcal{B} \to \mathcal{C}, R'' : \mathcal{C} \to \mathcal{B},$$

be (covariant) functors, and let

$$\theta'' : \mathrm{Id}_\mathcal{B} \Rightarrow R''L'',\ \eta'' : L''R'' \Rightarrow \mathrm{Id}_\mathcal{C},$$

be natural transformations that are an adjoint pair of functors. Define functors

$$L : \mathcal{A} \to \mathcal{C},\ R : \mathcal{C} \to \mathcal{A}$$

by $L = L'' \circ L'$, $R = R' \circ R''$. Define the natural transformation,

$$\theta : \mathrm{Id}_\mathcal{A} \Rightarrow R \circ L,$$

to be the composition of natural transformations,

$$\mathrm{Id}_\mathcal{A} \stackrel{\theta'}{\Rightarrow} R' \circ L' \stackrel{R' \circ \theta'' \circ L'}{\Rightarrow} R' \circ R'' \circ L'' \circ L'.$$

Similarly, define the natural transformation,

$$\eta : L \circ R \Rightarrow \mathrm{Id}_{\mathcal{C}},$$

to be the composition of natural transformations,

$$L'' \circ L' \circ R' \circ R'' \overset{L''\circ\eta'\circ R''}{\Rightarrow} L'' \circ R'' \overset{\eta''}{\Rightarrow} \mathrm{Id}_{\mathcal{C}}.$$

Prove that L, R, θ and η form an adjoint pair of functors. This is the **composition** of (L', R', θ', η') and $(L'', R'', \theta'', \eta'')$. If $\mathcal{A}$ equals $\mathcal{B}$, if L' and R' are the identity functors, and if θ' and η' are the identity natural transformations, prove that (L, R, θ, η) equals $(L'', R'', \theta'', \eta'')$. Similarly, if $\mathcal{B}$ equals $\mathcal{C}$, if L'' and R'' are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that (L, R, θ, η) equals (L', R', θ', η'). Finally, prove that composition of three adjoint pairs is associative.

Problem 4. A **monoid** is a triple (G, m, e) of a set, G, a binary relation,

$$m : G \times G \to G,$$

and an element, e, of G such that m is associative, i.e., the following diagram commutes,

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{Id}_G} & G \times G \\ {\scriptstyle \mathrm{Id}_G \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow[m]{} & G \end{array},$$

and such that e is a two-sided inverse to m, i.e., for every g in G, $m(g, e)$ and $m(e, g)$ both equal g. For monoids (G, m, e) and (G', m', e') a morphism of monoids is a set map

$$u : G \to G',$$

such that $u(e)$ equals e' and such that the following diagram commutes,

$$\begin{array}{ccc} G \times G & \xrightarrow{u \times u} & G' \times G' \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m'} \\ G & \xrightarrow[u]{} & G' \end{array}.$$

(i) Prove that the identity set map of every monoid is a morphism of monoids. For monoids (G, m, e), (G', m', e') and (G'', m'', e''), for morphisms of monoids,

$$u' : G \to G',\ u'' : G' \to G'',$$

prove that the composition $u = u'' \circ u'$

$$u : G \to G'',$$

is a morphism of monoids. If (G', m', e) equals (G, m, e) and if u' equals Id_G, prove that u equals u''. Similarly, if (G'', m'', e'') equals (G', m', e) and if u'' equals $\text{Id}_{G''}$, prove that u equals u'. Conclude that these operations define a category **Monoids** whose objects are monoids and whose morphisms are morphisms of monoids.

(ii) Denote by

$$\Phi' : \textbf{Monoids} \to \textbf{Sets},$$

the "forgetful functor" that associates to every monoid (G, m, e) the underlying set G, and that associates to every morphism of monoids $u : (G, m, e) \to (G', m', e')$ the set map $u : G \to G'$. Prove that Φ' is a faithful functor. Prove that there exists a functor

$$L' : \textbf{Sets} \to \textbf{Monoids},$$

and prove that there exist natural transformations,

$$\theta' : \text{Id}_{\textbf{Sets}} \Rightarrow \Phi' \circ L', \ \eta' : L' \circ \Phi' \Rightarrow \text{Id}_{\textbf{Monoids}},$$

such that $(L', \Phi', \theta', \eta')$ is an adjoint pair of functors. For every set Σ, the pair $(L'(\Sigma), \theta'(\Sigma) : \Sigma \to L'(\Sigma))$ is called a *free monoid* associated to Σ. One construction of $L'(\Sigma)$ is the set of all finite, ordered tuples of elements in Σ ("words") with the empty word being the identity, and with m given by concatenation of words.

(iii) Denote by **Groups** the category whose objects are groups and whose morphisms are group homomorphisms. Denote by

$$\Phi : \textbf{Groups} \to \textbf{Sets}$$

the "forgetful functor" that associates to every group (G, m, e) the underlying set G, and that associates to every morphism of groups $u : (G, m, e) \to (G', m', e')$ the set map $u : G \to G'$. Prove that Φ is a faithful functor.

(iv) For every set Σ, let Σ_+ and Σ_- be disjoint copies of Σ, e.g., $\Sigma \times \{+1\}$ and $\Sigma \times \{-1\}$. Let

$$(L'(\Sigma_+ \sqcup \Sigma_-), \cdot, e), i : (\Sigma_+ \sqcup \Sigma_-) \to (L'(\Sigma_+ \sqcup \Sigma_-), \cdot, e))$$

be a free monoid associated to $\Sigma_+ \sqcup \Sigma_-$. Define $\sim'$ to be the binary relation on $L'(\Sigma_+ \sqcup \Sigma_-)$ such that for every $\sigma \in \Sigma$, and for every $f_L, f_R \in L'(\Sigma_+ \sqcup \Sigma_-)$,

$$f_L \cdot i(\sigma_+) \cdot i(\sigma_-) \cdot f_R \sim' f_L \cdot f_R \sim' f_L \cdot i(\sigma_-) \cdot i(sigma_+) \cdot f_R.$$

For every triple $f, g, h \in L'(\Sigma_+ \sqcup \Sigma_-)$, prove that $f \cdot h \sim' g \cdot h$ if and only if $f \sim' g$ if and only if $h \cdot f \sim' h \cdot g$. Let $\sim$ be the weakest equivalence relation on $L'(\Sigma_+ \sqcup \Sigma_-)$ generated by $\sim'$. Conclude that $f \cdot h \sim g \cdot h$ if and only if $f \sim g$ if and only if $h \cdot f \sim h \cdot f$. Denote by $q : L'(\Sigma_+ \sqcup \Sigma_-) \to L(\Sigma)$ the quotient of F by the equivalence relation $\sim$. Prove that there exists a unique binary operation,

$$\cdot : L(\Sigma) \times L(\Sigma) \to L(\Sigma),$$

such that $q : (F, \cdot, e) \to (L(\Sigma), \cdot, q(e))$ is a morphism of monoids. Denote by $j_\Sigma : \Sigma \to L(\Sigma)$ the set map $\sigma \mapsto q(i(\sigma_+))$. Prove that $q(i(\sigma_-))$ is a left-right inverse of $j_\Sigma(\sigma)$ in $L(\Sigma)$. Conclude that every element of $L(\Sigma)$ admits a left-right inverse, i.e., $L(\Sigma)$ is a group.

(v) Continuing the notation above, for every group (G, m, e) and for every set map $k : \Sigma \to G$, define $h_+ : \Sigma_+ \to G$ to be h, and define $h_- : \Sigma_- \to G$ to be $h_-(\sigma_-) = (h_+(\sigma_+))^{-1}$. Associated to the set map

$$h_+ \sqcup h_- : \Sigma_+ \sqcup \Sigma_- \to G,$$

there exists a unique morphism of monoids, $\widetilde{h} = H_{\Phi'}^{L'}(h_+ \sqcup h_-)$,

$$\widetilde{h} : L'(\Sigma_+ \sqcup \Sigma_-) \to G,$$

such that $\widetilde{h} \circ i$ equals $h_+ \sqcup h_-$. Prove that for every pair of elements $f, g \in L'(\Sigma_+ \sqcup \Sigma_-)$ if $f \sim' g$, then $\widetilde{h}(f)$ equals $\widetilde{h}(g)$. Conclude that if $f \sim g$, then $\widetilde{h}(f)$ equals $\widetilde{h}(g)$. Therefore there exists a unique set map,

$$H_\Phi^L(h) : L(\Sigma) \to G,$$

such that $H_\Phi^L(h) \circ q$ equals $\widetilde{h} = H_{\Phi'}^{L'}(h_+ \sqcup h_-)$. Prove that $H_\Phi^L(h)$ is a group homomorphism. Prove that $H_\Phi^L(h)$ is the unique group homomorphism such that $H_\Phi^L(h) \circ j_\Sigma$ equals h. Prove that the set map,

$$H_\Phi^L(\Sigma, G) : \mathrm{Hom}_{\mathrm{Sets}}(\Sigma, \Phi(G)) \to \mathrm{Hom}_{\mathrm{Groups}}(L(\Sigma), G),$$

is a bijection. In particular, for every set map $u : \Sigma \to \Sigma'$, associated to the set map

$$j_{\Sigma'} \circ u : \Sigma \to L(\Sigma'),$$

there exists a unique group homomorphism $L(u) = H_\Phi^L(j_{\Sigma'} \circ u)$,

$$L(u) : L(\Sigma) \to L(\Sigma'),$$

such that $L(u) \circ j_\Sigma$ equals $j_{\Sigma'} \circ u$. Prove that the rule $\Sigma \mapsto L(\Sigma)$ and $u \mapsto L(u)$ define a functor,

$$L : \mathbf{Sets} \to \mathbf{Groups}.$$

(vi) Prove that $\Sigma \mapsto j_\Sigma$ is a natural transformation,

$$j_\Sigma : \mathrm{Id}_{\mathbf{Sets}} \Rightarrow \Phi \circ L.$$

For every group G, associated to the set map Id_G, there exists a unique group homomorphism $\eta_G = H_\Phi^L(\mathrm{Id}_G)$,

$$\eta_G : L(\Phi(G)) \to G.$$

Prove that $G \mapsto \eta_G$ is a natural transformation,

$$\eta : L \circ \Phi \Rightarrow \mathrm{Id}_{\mathbf{Groups}}.$$

Prove that (L, Φ, j, η) is an adjoint pair of functors and that the bijections H^L_Φ defined above are the bijections naturally associated to this adjoint pair.

(vii) Denote by
$$\Phi'' : \mathbf{Groups} \to \mathbf{Monoids}$$
the "forgetful functor" that associates to every group (G, m, e) that same datum (G, m, e) considered as a monoid. Prove that Φ'' is a fully faithful functor. Prove that there exists a functor
$$L'' : \mathbf{Monoids} \to \mathbf{Groups},$$
and prove that there exist natural transformations,
$$\theta'' : \mathrm{Id}_{\mathbf{Monoids}} \Rightarrow \Phi'' \circ L'',\ \ \eta'' : L'' \circ \Phi'' \Rightarrow \mathrm{Id}_{\mathbf{Groups}},$$
such that $(L'', \Phi'', \theta'', \eta'')$ is an adjoint pair of functors. For every monoid (G, m, e), the pair
$$(L''(G, m, e), \theta''(G, m, e) : (G, m, e) \to L''(G, m, e))$$
is called a *group completion* associated to the monoid (G, m, e). Moreover, prove that the composition of the adjoint pairs (L', Φ', i, η') and $(L'', \Phi'', \theta'', \eta'')$ is naturally isomorphic to the adjoint pair (L, Φ, θ, η).

Hint. The condition on compositions uniquely determines L''. In particular, for every monoid (G, m, e), associated to the set map,
$$i_G : \Phi'(G) \to L(\Phi'(G)),$$
there is a group homomorphism,
$$\widetilde{\theta}''_G : L(\Phi'(G)) \to L''(G),$$
such that the composition $\widetilde{\theta}''_G \circ i_G$, denoted θ''_G, is a morphism of monoids and such that θ''_G is initial among all group homomorphisms,
$$t : L(\Phi'(G)) \to T,$$
such that the composition $t \circ i_G$ is a morphisms of monoids, i.e., the set map,
$$H^{L''}_{R''}(G, T) : \mathrm{Hom}_{\mathbf{Groups}}(L''(G), T) \to \mathrm{Hom}_{\mathbf{Monoids}}(G, \Phi''(T)),\ u \mapsto u \circ \theta''_G,$$
is a bijection. Indeed, let $L''(G)$ be the group quotient of the group $L(\Phi'(G))$ by the normal subgroup generated by all elements of the form $i_G(g)^{-1} i_G(g \cdot g') i_G(g')^{-1}$ for $g, g' \in G$. Now use the universal property of $\widetilde{\theta}''$ to extend L'' to a functor, observe that $G \mapsto \theta''_G$ is a natural transformation,
$$\theta'' : \mathrm{Id}_{\mathbf{Monoids}} \Rightarrow L'' \circ \Phi'',$$
and for every group (T, m, e), define
$$\eta''_T : L''(\Phi''(T)) \to T,$$

to be the unique group homomorphism such that $H^{L''}_{R''}(\eta''_T)$ equals $\text{Id}_{\Phi''(T)}$. Then $T \mapsto \eta''_T$ is a natural transformation,

$$\eta'' : L'' \circ \Phi'' \Rightarrow \text{Id}_{\mathbf{Groups}},$$

that makes $(L'', \Phi'', \theta'', \eta'')$ an adjoint pair.

(viii) For categories $\mathcal{B}$, $\mathcal{C}$, for functors

$$L'' : \mathcal{B} \to \mathcal{C},\ R'' : \mathcal{C} \to \mathcal{B},$$

and for natural transformations

$$\theta'' : \text{Id}_{\mathcal{B}} \Rightarrow R'' \circ L'',\ \eta'' : L'' \circ R'' \Rightarrow \text{Id}_{\mathcal{C}},$$

such that $(L'', R'', \theta'', \eta'')$ is an adjoint pair, the adjoint pair is *reflective* if R'' is fully faithful. In this case, prove that there exists a unique binatural transformation

$$\widetilde{H}^{L''}_{R''}(b, b') : \text{Hom}_{\mathcal{C}}(L''(R''(b)), b') \to \text{Hom}_{\mathcal{C}}(b, b'),$$

such that the composition with R'',

$$\text{Hom}_{\mathcal{C}}(L''(R''(b)), b') \xrightarrow{\widetilde{H}^{L''}_{R''}(b,b')} \text{Hom}_{\mathcal{C}}(b, b') \xrightarrow{R''} \text{Hom}_{\mathcal{B}}(R''(b), R''(b')),$$

equals $H^{L''}_{R''}(R(b), b')$. In particular, taking $b' = L''(R''(b))$, denote the image of $\text{Id}_{b'}$ by

$$\widetilde{\eta}''_b : b \to L''(R''(b)).$$

Prove that $\widetilde{\eta}''_b$ is an inverse to $\eta''_b : L''(R''(b)) \to b$. Thus, for a reflective adjoint pair, η'' is a natural isomorphism. Conversely, if η'' is a natural isomorphism, prove that the adjoint pair is reflective, i.e., R'' is fully faithful. In particular, for the group completion, conclude that the group completion of the monoid underlying a group is naturally isomorphic to that group.

Problem 5 Denote by

$$\Phi : \mathbb{Z} - \text{mod} \to \mathbf{Groups}$$

the full subcategory of **Groups** whose objects are Abelian groups. For every group $(G, \cdot, e)$, denote by $[G, G]$ the normal subgroup of G generated by all commutators

$$[g, h] = g \cdot h \cdot g^{-1} \cdot h^{-1}$$

for pairs $g, h \in G$. Denote by

$$\theta_G : G \to L(G),$$

the group quotient associated to the normal subgroup $[G, G]$ of G. Prove that $L(G)$ is an Abelian group. Moreover, for every Abelian group $(A, \cdot, e)$, prove that the set map

$$H^L_\Phi : \text{Hom}_{\mathbb{Z}-\text{mod}}(L(G), A) \to \text{Hom}_{\mathbf{Groups}}(G, \Phi(A)),\ v \mapsto v \circ \theta_G,$$

is a bijection. In particular, for every group homomorphism,

$$u : G \to G',$$

the composition $\theta_{G'} \circ u : G \to L(G')$ is a group homomorphism, and thus there exists a unique group homomorphism,

$$L(u) : L(G) \to L(G'),$$

such that $H^L_\Phi(L(u)) \circ \theta_G$ equals $\theta_{G'} \circ u$. Prove that the rule $G \mapsto L(G)$, $u \mapsto L(u)$ defines a functor,

$$L : \mathbf{Groups} \to \mathbb{Z} - \text{mod}.$$

This functor is called *Abelianization*. Prove that $G \mapsto \theta_G$ is a natural transformation,

$$\theta : \text{Id}_{\mathbf{Groups}} \Rightarrow \Phi \circ L.$$

For every Abelian group A, prove that $[A, A]$ is the identity subgroup, and thus the quotient homomorphism,

$$\theta_{\Phi(A)} : \Phi(A) \to \Phi(L(\Phi(A))),$$

is an isomorphism. Thus there exists a unique group homomorphism, just the inverse isomorphism of $\theta_{\Phi(A)}$,

$$\eta_A : L(\Phi(A)) \to A,$$

such that $\theta_{\Phi(A)} \circ \Phi(\eta_A)$ equals the $\text{Id}_{\Phi(A)}$. Prove that $A \mapsto \eta_A$ is a natural isomorphism,

$$\eta : L \circ \Phi \to \text{Id}_{\mathbb{Z}-\text{mod}}.$$

Prove that (L, Φ, θ, η) is an adjoint pair.

Problem 6 Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be categories. Let

$$R' : \mathcal{B} \to \mathcal{A},\ \ R'' : \mathcal{C} \to \mathcal{B},$$

be fully faithful functors. Denote the composition $R' \circ R''$ by

$$R : \mathcal{C} \to \mathcal{A}.$$

(i) If there exist extensions to reflective adjoint pairs (L', R', θ', η'), $(L'', R'', \theta'', \eta'')$, prove that there is also an extension to a reflective adjoint pair (L, R, θ, η).

(ii) If there exists an extension of R to a reflective adjoint pair (L, R, θ, η), prove that there exists an extension $(L'', R'', \theta'', \eta'')$. Give an example demonstrating that R' need not extend to a reflective adjoint pair (for instance, consider the full subcategory of Abelian groups in the full subcategory of solvable groups in the category of all groups).

(iii) A monoid $(G, \cdot, e)$ is called **left cancellative**, resp. **right cancellative**, if for every f, g, h in G, if $f \cdot g$ equals $f \cdot h$, resp. if $g \cdot f$ equals $h \cdot f$, then g equals h. A monoid is **cancellative** if it is both left cancellative and right cancellative. A monoid is **commutative** if for every $f, g \in G$, $f \cdot g$ equals $g \cdot f$. A commutative monoid is left cancellative if and only if it is right cancellative if and only if it is cancellative. Denote by

$$\mathbf{LCanMonoids}, \mathbf{RCanMonoids}, \mathbf{CanMonoids}, \mathbf{CommMonoids}, \mathbf{CommCanMonoids} \subseteq \mathbf{Monoids}$$

the full subcategories of the category of all monoids whose objects are left cancellative monoids, resp. right cancellative monoids, cancellative monoids, commutative monoids, commutative cancellative monoids. In each of these cases, prove that the fully faithful inclusion functor R extends to a reflective adjoint pair. Use (ii) to conclude that for every inclusion functor among the full subcategories listed above, there is an extension of the inclusion functor to a reflective adjoint pair.

(iv) In particular, prove that the group completion adjoint pair

$$(L : \mathbf{Monoids} \to \mathbf{Groups}, R : \mathbf{Groups} \to \mathbf{Monoids}, \theta, \eta)$$

factors as the composition of the reflective adjoint pair

$$(L' : \mathbf{Monoids} \to \mathbf{CanMonoids}, R' : \mathbf{CanMonoids} \to \mathbf{Monoids}, \theta', \eta'),$$

and the restriction to **CanMonoids** of the group completion adjoint pair

$$(L'' = L \circ R', R'', \theta'', \eta'').$$

Similarly, prove that the composition of the Abelianization functor and the group completion functor

$$(L : \mathbf{Monoids} \to \mathbb{Z} - \text{mod}, R : \mathbb{Z} - \text{mod} \to \mathbf{Monoids}, \theta, \eta),$$

factors through the reflection to the full subcategory of commutative, cancellative monoids,

$$(L' : \mathbf{Monoids} \to \mathbf{CommCanMonoids}, R' : \mathbf{CommCanMonoids} \to \mathbf{Monoids}, \theta', \eta').$$

Problem 7 Let A and B be unital, associative rings, and let $\phi : A \to B$ be a morphism of unital, associative rings.

(i) For every left B-module,

$$(N, m_{B,N} : B \times N \to N),$$

prove that the composition

$$A \times N \xrightarrow{\phi \times \mathrm{Id}_N} B \times N \xrightarrow{m_{B,N}} N,$$

makes the datum

$$(N, m_{B,N} \circ (\phi \times \mathrm{Id}_N) : A \times N \to N),$$

an R-module. For every morphism of left B-modules,

$$u : (N, m_{B,N}) \to (N', m_{B,N'}),$$

prove that also

$$u : (N, m_{B,N} \circ (\phi \times \mathrm{Id}_N)) \to (N', m_{B,N'} \circ (\phi \times \mathrm{Id}_{N'}))$$

is a morphism of left A-modules. Altogether, prove that the association $(N, m_{B,N}) \mapsto (N, m_{B,N} \circ (\phi \times \mathrm{Id}_N))$ and $u \mapsto u$ is a faithful functor

$$R_\phi : B - \mathrm{mod} \to A - \mathrm{mod}.$$

In particular, in the usual manner, for every unital, associative ring C and for every $B-C$-bimodule N, prove that $R_\phi(N)$ is naturally an $A - C$-bimodule.

(ii) Formulate and prove the analogous results for right modules, giving a faithful functor

$$R^\phi : \mathrm{mod} - B \to \mathrm{mod} - A.$$

For every $C - B$-bimodule N, prove that $R^\phi(N)$ is naturally a $C - A$-bimodule. In particular for the $B - B$-bimodule $N = B$, $R^\phi(B)$ is naturally a $B - A$-bimodule.

For every left A-module M, denote $L_\phi(M) = R^\phi(B) \otimes_A M$. For every morphism of left A-modules,

$$u : M \to M',$$

denote by $L_\phi(u) = \mathrm{Id}_{R^\phi(B)} \otimes u$,

$$L_\phi(u) : L_\phi(M) \to L_\phi(M'),$$

the associated morphism of left B-modules. Prove that the associations $M \mapsto L_\phi(M)$ and $u \mapsto L_\phi(u)$ define a functor

$$L_\phi : A - \mathrm{mod} \to B - \mathrm{mod}.$$

(iv) Denote by 1_B the multiplicative unit in B. For every left A-module M, prove that the composition

$$M \xrightarrow{1_B \times \mathrm{Id}_M} B \times M \xrightarrow{\beta_{B,M}} B \otimes_A M,$$

is a morphism of left A-modules,

$$\theta_M : M \to R_\phi(L_\phi(M)),$$

i.e., for every $a \in A$ and for every $m \in M$,

$$\beta_{B,M}(1_B, a \cdot m) = \beta_{B,M}(1_B \cdot \phi(a), m) = \beta_{B,M}(\phi(a) \cdot 1_B, m).$$

Prove that the association $M \mapsto \theta_M$ defines a natural transformation

$$\theta : \mathrm{Id}_{A-\mathrm{mod}} \Rightarrow R_\phi \circ L_\phi.$$

(v) For every left B-module $(N, m_{B,N})$, for the induced right A-module structure on $R^\phi(B)$ and left A-module structure on N, prove that

$$m_{B,N} : B \times N \to N$$

is A-bilinear, i.e., for every $a \in A$, for every $b \in B$, and for every $n \in N$,

$$m_{B,N}(b, \phi(a) \cdot n) = m_{B,N}(b \cdot \phi(a), n).$$

Thus, by the universal property of tensor product, there exists a unique homomorphism of Abelian groups,

$$m_N : B \otimes_A N \to N,$$

such that $m_N \circ \beta_{B,N}$ equals $m_{B,N}$. Prove that m_N is a morphism of left B-modules, i.e., for every $b, b' \in B$ and for every $n \in N$,

$$m_N(b \cdot \beta_{B,N}(b', n)) = m_N(\beta_{B,N}(b \cdot b', n)) = m_{B,N}(b \cdot b', n) = m_{B,N}(b, m_{B,N}(b', n)).$$

Prove that the association $N \mapsto m_N$ defines a natural transformation

$$m : R_\phi \circ L_\phi \Rightarrow \mathrm{Id}_{B-\mathrm{mod}}.$$

(vi) Prove that $(L_\phi, R_\phi, \theta, m)$ is an adjoint pair of functors. In particular, even though R_ϕ is faithful, the natural transformation m is typically not a natural isomorphism. Conclude that one cannot weaken the definition of reflective adjoint pair from "fully faithful" to "faithful".

(vii) Prove the analogues of the above for right modules. Also, taking A to be $\mathbb{Z}$, and taking $\phi : \mathbb{Z} \to B$ to be the unique ring homomorphism, obtain an adjoint pair

$$(L'' : \mathbb{Z} - \mathrm{mod} \to B - \mathrm{mod}, R'' : B - \mathrm{mod} \to \mathbb{Z} - \mathrm{mod}, \theta'', \eta'')$$

whose composition with the adjoint pair

$$(L' : \mathbf{CommCanMonoids} \to \mathbb{Z} - \mathrm{mod}, R' : \mathbb{Z} - \mathrm{mod} \to \mathbf{CommCanMonoids}, \theta', \eta')$$

is an adjoint pair (L, R, θ, η) extending the forgetful functor

$$R : B - \mathrm{mod} \to \mathbf{CommCanMonoids}.$$

Composing this adjoint pair further with the other adjoint pairs above gives, in particular, an adjoint pair (F, Φ, i, η) extending the forgetful functor

$$\Phi : B - \mathrm{mod} \to \mathbf{Sets}.$$

The functor $F : \mathbf{Set} \to B - \mathrm{mod}$ and the natural transformation i is called the "free B-module". Use the usual functorial properties to conclude that F naturally maps to the category of $B - B$-bimodules.

Problem 8 Let A be an associative, unital ring that is commutative. A (central) A**-algebra** is a pair (B, ϕ) of an associative, unital ring B and a morphism of associative, unital rings, $\phi : A \to B$, such that for every $a \in A$ and every $b \in B$, $\phi(a) \cdot b$ equals $b \cdot \phi(a)$, i.e., $\phi(A)$ is contained in the center of B. In particular, the identity map

$$\mathrm{Id}_B : R^{\phi}(B) \to R_{\phi}(B),$$

is an isomorphism of $A-A$-bimodules making B into a left-right A-module.

For A-algebras (B, ϕ) and (B', ϕ'), a morphism of A-algebras is a morphism of associative, unital rings, $\psi : B \to B'$, such that $\psi \circ \phi$ equals ϕ'. In particular, ψ is a morphism of left-right A-modules.

(i) Prove that the usual composition and the usual identity maps define a faithful (but not full!) subcategory

$$R : A - \text{algebra} \to A - \text{mod}$$

whose objects are A-algebras and whose morphisms are morphisms of A-algebras. The rest of this problem extends this to an adjoint pair that is a composition of two other (more elementary) adjoint pairs.

(ii) Let $n \geq 2$ be an integer. Let $M_1, \ldots, M_n$ be (left-right) A-modules. For every A-module U, a map

$$\gamma : M_1 \times \cdots \times M_n \to U,$$

is an n-A-**multilinear** map if for every $i = 1, \ldots, n$, for every choice of

$$\overline{m}_i = (m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n) \in M_1 \times \cdots \times M_{i-1} \times M_{i+1} \times \cdots \times M_n,$$

the induced map

$$\gamma_{\overline{m}_i} : M_i \to U, \; m_i \mapsto \gamma(m_1, \ldots, m_{i-1}, m_i, m_{i+1}, \ldots, m_n),$$

is a morphism of A-modules. Prove that there exists a pair $(T(M_1, \ldots, M_n), \beta_{M_1,\ldots,M_n})$ of an A-module $T(M_1, \ldots, M_n)$ and an n-A-multilinear map

$$\beta_{M_1,\ldots,M_N} : M_1 \times \cdots \times M_n \to T(M_1, \ldots, M_n),$$

such that for every n-A-multilinear map γ as above, there exists a unique A-module homomorphism,

$$u : T(M_1, \ldots, M_n) \to U,$$

such that $u \circ \beta_{M_1,\ldots,M_n}$ equals γ. For $n = 3$, prove that β_{M_1,M_2,M_3} factors through

$$\beta_{M_1,M_2} \times \mathrm{Id}_{M_3} : M_1 \times M_2 \times M_3 \to (M_1 \otimes_A M_2) \times M_3.$$

Prove that the induced map

$$\beta_{M_1 \otimes M_2, M_3} : (M_1 \otimes_A M_2) \times M_3 \to T(M_1, M_2, M_3),$$

is A-bilinear. Conclude that there exists a unique A-module homomorphism,

$$u : (M_1 \otimes_A M_2) \otimes_A M_3 \to T(M_1, M_2, M_3).$$

Prove that this is an isomorphism of A-modules. Similarly, prove that there is a natural isomorphism of A-modules,

$$M_1 \otimes_A (M_2 \otimes_A M_3) \to T(M_1, M_2, M_3).$$

Conclude that there is a natural isomorphism of A-modules,

$$(M_1 \otimes_A M_2) \otimes_A M_3 \cong M_1 \otimes_A (M_2 \otimes_A M_3),$$

i.e., tensor product is associative for A-modules. Iterate this to conclude that there are natural isomorphisms between all the different interpretations of $M_1 \otimes_A \cdots \otimes_A M_n$, and each of these is naturally isomorphic to $T(M_1, \dots, M_n)$. (All of this is also true in the case of M_i that are $A_{i-1} - A_i$-bimodules with n-$(A_i)_i$-multilinearity defined appropriately.)

(iii) Let B be an A-algebra. A $\mathbb{Z}_+$-grading of B is a direct sum decomposition as an A-module,

$$B = \oplus_{n \geq 0} B_n,$$

such that for every pair of integers $n, p \geq 0$, the restriction to the summands B_n and B_p of the multiplication map,

$$m_B : B_n \times B_p \to B$$

factors through B_{n+p}. The induced A-bilinear map is denoted

$$m_{B,n,p} : B_n \times B_p \to B_{n+p}.$$

In particular, notice that this means that B_0 is an A-subalgebra of B, and every direct summand B_n is a $B_0 - B_0$-bimodule. Finally, for every triple of integers $n, p, r \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} B_n \times B_p \times B_r & \xrightarrow{m_{B,n,p} \times \mathrm{Id}_{B_r}} & B_{n+p} \times B_r \\ {\scriptstyle \mathrm{Id}_{B_n} \times m_{B,p,r}} \downarrow & & \downarrow {\scriptstyle m_{B,n+p,r}} \\ B_n \times B_{p+r} & \xrightarrow[m_{B,n,p+r}]{} & B_{n+p+r} \end{array}$$

Prove that a $\mathbb{Z}_+$-graded A-algebra is equivalent to the data $((B_n)_{n\in\mathbb{Z}_+}, (m_{B,n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$ satisfying the conditions above.

(iv) For $\mathbb{Z}_+$-graded A-algebras $((B_n)_{n\in\mathbb{Z}_+}, (m_{B,n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$ and $((B'_n)_{n\in\mathbb{Z}_+}, (m_{B',n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$, a morphism of $\mathbb{Z}_+$-graded A-algebras is a morphism of A-algebras,

$$\psi : B \to B',$$

such that for every integer $n \geq 0$, $\psi(B_n)$ is contained in B'_n. The induced A-linear map is denoted

$$\psi_n : B_n \to B'_n.$$

In particular, ψ_0 is a morphism of A-algebras. Relative to ψ_0, every map ψ_n is a morphism of $B_0 - B_0$-bimodules. Finally, for every pair of integers $n, p \geq 0$, the following diagram commutes,

$$\begin{tikzcd} B_n \times B_p \arrow[r, "\psi_n \times \psi_p"] \arrow[d, "m_{B,n,p}"'] & B'_n \times B'_p \arrow[d, "m_{B',n,p}"] \\ B_{n+p} \arrow[r, "\psi_{n+p}"'] & B'_{n+p} \end{tikzcd}$$

Prove that a morphism of $\mathbb{Z}_+$-graded A-algebras is equivalent to the data $(\psi_n)_{n\in\mathbb{Z}_+}$ satisfying the conditions above. Prove that composition of morphisms of $\mathbb{Z}_+$-graded A-algebras is a morphism of $\mathbb{Z}_+$-graded A-algebras. Prove that identity maps are morphisms of $\mathbb{Z}_+$-graded A-algebras. Conclude that there is a faithful (but not full!) subcategory,

$$L'' : \mathbb{Z}_+ - A - \text{algebra} \to A - \text{algebra},$$

whose objects are $\mathbb{Z}_+$=graded A-algebras and whose morphisms are morphisms of $\mathbb{Z}_+$-graded A-algebras. Prove that this extends to an adjoint pair $(L'', R'', \theta'', \eta'')$ where

$$R'' : A - \text{algebra} \to \mathbb{Z}_+ - A - \text{algebra},$$

associates to an associative, unital A-algebra (C, m_C) the $\mathbb{Z}_+$-graded A-algebra,

$$((C_n)_{n\in\mathbb{Z}_+}, (m_{n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+}) = ((C)_{n\in\mathbb{Z}_+}, (m)_{(n,p)}).$$

Thus C_0 equals C as an A-algebra, and the C_0-algebra $\oplus_n C_n$ is equivalent as a $\mathbb{Z}_+$-graded C-algebra to $C[t] = C \otimes_{\mathbb{Z}} \mathbb{Z}[t]$, where $\mathbb{Z}[t]$ is graded in the usual way.

(v) Let M be an A-module. For every integer $n \geq 1$, denote

$$T^n_A(M) = T(M_1, \dots, M_n) = M^{\otimes n} = M \otimes_A \cdots \otimes_A M,$$

with the universal n-A-multilinear map,

$$\beta^n_M : M^n \to T^n_A(M).$$

Similarly, denote $T^0_A(M) = A$. For every pair of integers $n, p \geq 0$, the composition,

$$M^n \times M^p \xrightarrow{=} M^{n+p} \xrightarrow{\beta^{n+p}} T^{n+p}_A(M),$$

is n-A-multilinear, resp. p-A-multilinear in the two arguments separately. Thus the composition factors as

$$M^n \times M^p \xrightarrow{\beta^n_M \times \beta^p_M} T^n_A(M) \times T^p_A(M) \xrightarrow{\mu^{n,p}_M} T^{n+p}_A(M),$$

where $\mu_M^{n,p}$ is A-bilinear. Finally, for every triple of integers $n, p, r \geq 0$, associativity of tensor products implies that the following diagram commutes,

$$\begin{array}{ccc} T_A^n(M) \times T_A^p(M) \times T_A^r(M) & \xrightarrow{\mu_M^{n,p} \times \mathrm{Id}_{T_A^r(M)}} & T_A^{n+p}(M) \times T_A^r(M) \\ \mathrm{Id}_{T_A^n(M)} \times \mu_M^{p,r} \downarrow & & \downarrow \mu_M^{n+p,r} \\ T_A^n(M) \times T_A^{p+r}(M) & \xrightarrow[\mu_M^{n,p+r}]{} & T_A^{n+p+r}(M) \end{array}.$$

Thus, the data $((T_A^n(M))_{m\in\mathbb{Z}_+}, (\mu_M^{n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$ defines a $\mathbb{Z}_+$-graded A-algebra, denoted $T_A(M)$ and called the *tensor algebra* associated to M. For every $\mathbb{Z}_+$-graded A-algebra

$$B = ((B_n)_{n\in\mathbb{Z}_+}, (m_{B,n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+}),$$

for every integer n, inductively define the A-module morphism

$$\eta'_{B,n} : T_A^n(B_1) \to B_n,$$

by $\eta'_{B,0} : A \to B_0$ is the usual A-algebra structure map ϕ, $\eta'_{B,1} : T_A^1(B_1) \to B_1$ is the usual identity morphism on B_1, and for every $n \geq 0$, assuming that $\eta'_{B,n}$ is defined,

$$\eta'_{B,n+1} : T_A^{n+1}(B_1) = B_1 \otimes_A T_A^n(B) \to B_{n+1},$$

is the unique A-module homomorphism whose composition with the universal A-bilinear map,

$$\beta_M : B_1 \times T_A^n(B) \to B_A \otimes_A T_A^n(B),$$

equals the A-bilinear composition

$$B_1 \times T_A^n(B_1) \xrightarrow{\mathrm{Id}_{B_1}\times\eta_{B,n}} B_1 \times B_n \xrightarrow{m_{B,1,n}} B_{n+1}.$$

Use associativity of tensor product (and induction) to prove that for every pair of integers $n, p \geq 0$, the following diagram commutes,

$$\begin{array}{ccc} T_A^n(B_1) \times T_A^p(B_1) & \xrightarrow{\eta'_{B,n}\times\eta'_{B,p}} & B_n \times B_p \\ \mu_{B_1}^{n,p} \downarrow & & \downarrow m_{B,n,p} \\ T_A^{n+p}(B_1) & \xrightarrow[\eta'_{B,n+p}]{} & B_{n+p} \end{array}$$

Conclude that $(\eta'_{B,n})_{n\in\mathbb{Z}_+}$ is a morphism of $\mathbb{Z}_+$-graded A-algebras,

$$\eta'_B : T_A(B_1) \to B.$$

(vi) Denote by
$$R' : \mathbb{Z}_+ - A - \text{algebra} \to A - \text{mod}$$
the functor that associates to a $\mathbb{Z}_+$-graded A-algebra $((B_n)_{n\in\mathbb{Z}_+}, (m_{B,n,p})_{(n,p)\in\mathbb{Z}_+\times\mathbb{Z}_+})$ the A-module B_1 and that associates to a morphism $(\psi_n)_{n\in\mathbb{Z}_+}$ of $\mathbb{Z}_+$-graded A-algebras the A-module ψ_1. For every A-module M, denote by
$$\theta'_M : M \to R'(T_A(M))$$
the identity morphism $M \to T^1_A(M)$. Prove that this defines an adjoint pair $(T_A, R', \theta', \eta')$. Composing with the adjoint pair $(L'', R'', \theta'', \eta'')$ gives an adjoint pair $(L'' \circ T_A, R, \theta, \eta)$ extending the faithful (but not full!) forgetful functor
$$R : A - \text{Alg} \to A - \text{mod},\ B \mapsto B.$$