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MAT 535 Cochain Complexes

1 Introduction

These are notes on cochain complexes and δ-functors supplementing the material from our textbook. Some of the notes are cut-and-pasted from previous courses I taught. Much of the notes are exercises working through the basic results about these notions.

2 The Abelian category of complexes

Definition 2.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, a **C-cochain complex** is an ordered pair $A = ((A^n)_{n\in\mathbb{Z}}, (d_A^n)_{n\in\mathbb{Z}})$ of a sequence of $\mathbf{C}$-objects A^n and of a sequence of $\mathbf{C}$-morphisms d_A^n from A^n to A^{n+1} such that $d_A^{n+1} \circ d_A^n$ equals 0 for every n in $\mathbb{Z}$. For all $\mathbf{C}$-cochain complexes $((A^n)_{n\in\mathbb{Z}}, (d_A^n)_{n\in\mathbb{Z}})$ and $((\widetilde{A}^n)_{n\in\mathbb{Z}}, (d_{\widetilde{A}}^n)_{n\in\mathbb{Z}})$, a $\mathbf{C}$-cochain morphism from $((A^n)_{n\in\mathbb{Z}}, (d_A^n)_{n\in\mathbb{Z}})$ to $((\widetilde{A}^n)_{n\in\mathbb{Z}}, (d_{\widetilde{A}}^n)_{n\in\mathbb{Z}})$ is a sequence $(u^n)_{n\in\mathbb{Z}}$ of $\mathbf{C}$-morphisms u^n from A^n to $\widetilde{A}^n$ such that $u^{n+1} \circ d_A^n$ equals $d_{\widetilde{A}}^n \circ u^n$ for every n in $\mathbb{Z}$. This defines a category, denoted $\mathsf{Ch}(\mathbf{C})$. Moreover, the addition operation $\mathsf{add}_{\mathbf{C}}$ induces an addition operation $\mathsf{add}_{\mathsf{Ch}(\mathbf{C})}$.

Exercise 2.2. For every $\mathbf{C}$-cochain morphism u from a $\mathbf{C}$-cochain complex A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, for every integer n let $\mathrm{Ker}(u)^n$ with its monomorphism q_u^n to A^n be the kernel of u^n, Prove that there is a unique sequence $(d_{\mathrm{Ker}(u)}^n)_{n\in\mathbb{Z}}$ of $\mathbf{C}$-morphisms $d_{\mathrm{Ker}(u)}^n$ from $\mathrm{Ker}(u)^n$ to $\mathrm{Ker}(u)^{n+1}$ such that $q_u^{n+1} \circ d_{\mathrm{Ker}(u)}^n$ equals $d_A^n \circ q_u^n$ for every integer n. Deduce that $d_{\mathrm{Ker}(u)}^{n+1} \circ d_{\mathrm{Ker}(u)}^n$ equals the zero morphism for every integer n, hence $((\mathrm{Ker}(u)^n)_{n\in\mathbb{Z}}, (d_{\mathrm{Ker}(u)}^n)_{n\in\mathbb{Z}})$ is a $\mathbf{C}$-cochain complex $\mathrm{Ker}(u)$, and $(q_u^n)_{n\in\mathbb{Z}}$ is a $\mathbf{C}$-cochain morphism q_u from $\mathrm{Ker}(u)$ to A. Prove that this is a kernel of u in the additive category $\mathsf{Ch}(\mathbf{C})$.

Exercise 2.3. Continuing the previous exercise, let $\mathrm{Coker}(u)^n$ with the epimorphism p_u^n from $\widetilde{A}^n$ to $\mathrm{Coker}(u)^n$ be the cokernel of u^n for each integer n. Prove that there is a unique sequence $(d_{\mathrm{Coker}(u)}^n)_{n\in\mathbb{Z}}$ of $\mathbf{C}$-morphisms $d_{\mathrm{Coker}(u)}^n$ from $\widetilde{A}^n$ to $\mathrm{Coker}(u)^n$ such that both $((\mathrm{Coker}(u)^n)_{n\in\mathbb{Z}}, (d_{\mathrm{Coker}(u)}^n)_{n\in\mathbb{Z}})$

is a $\mathbf{C}$-cochain complex $\text{Coker}(u)$ and $(p_u^n)_{n\in\mathbb{Z}}$ is a $\mathbf{C}$-cochain morphism from $\widetilde{A}$ to $\text{Cokern}(u)$. Prove that this is a cokernel of u in the additive category $\mathsf{Ch}(\mathbf{C})$.

Exercise 2.4. Iterate kernel and cokernel to construct images and coimages. Finally, use the (AB2) axioms for $\mathbf{C}$ to deduce the (AB2) axioms for $\mathsf{Ch}(\mathbf{C})$. Altogether, this proves that the additive category $(\mathsf{Ch}(\mathbf{C}), \mathsf{add}_{\mathsf{Ch}(\mathbf{C})})$ is an Abelian category.

Definition 2.5. For every additive functor $\mathbf{F}$ from an Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, for every $\mathbf{C}$-cochain complex A, the **$\mathbf{F}$-associated $\mathbf{D}$**-cochain complex $\mathsf{Ch}(\mathbf{F})(A)$ has $\mathsf{Ch}(\mathbf{F})(A)^n$ equal to $\mathbf{F}(A^n)$ for every integer n and has $d^n_{\mathsf{Ch}(\mathbf{F})(A)}$ equal to $\mathbf{F}^{A^n}_{A^{n+1}}(d^n_A)$ for every integer n. For every $\mathbf{C}$-cochain morphism u from a $\mathbf{C}$-cochain complex A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the **$\mathsf{Ch}(\mathbf{F})$-associated $\mathbf{D}$**-cochain morphism $\mathsf{Ch}(\mathbf{F})^A_{\widetilde{A}}(u)$ from $\mathsf{Ch}(\mathbf{F})(A)$ to $\mathsf{Ch}(\mathbf{F})(\widetilde{A})$ has $\mathsf{Ch}(\mathbf{F})(u)^n$ equal to $\mathfrak{F}(u^n)$ for every integer n. Altogether this defines a functor $\mathsf{Ch}(\mathbf{F})$ from $\mathsf{Ch}(\mathbf{C})$ to $\mathsf{Ch}(\mathbf{D})$.

Exercise 2.6. Prove that $\mathsf{Ch}(\mathbf{F})(A)$ is a $\mathbf{D}$-cochain complex, prove that $\mathsf{Ch}(\mathbf{F})(u)$ is a $\mathbf{D}$-cochain morphism, and prove that $\mathsf{Ch}(\mathbf{F})$ is an additive functor of Abelian categories.

Definition 2.7. For additive functors $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ from an Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, for every natural transformation θ from $\mathbf{F}$ to $\widetilde{\mathbf{F}}$, for every $\mathbf{C}$-cochain complex A, the **$\mathsf{Ch}(\theta)$-associated $\mathbf{D}$**-cochain morphism $\mathsf{Ch}(\theta)_A$ from $\mathsf{Ch}(\mathbf{F})(A)$ to $\mathsf{Ch}(\widetilde{\mathbf{F}})(A)$ has $\mathsf{Ch}(\theta)^n_A$ equal to θ_{A^n} for every integer n.

Exercise 2.8. Prove that $\mathsf{Ch}(\theta)_A$ is a $\mathbf{D}$-cochain morphism. Prove that $\mathsf{Ch}(\theta)$ is a natural transformation from $\mathsf{Ch}(\mathbf{F})$ to $\mathsf{Ch}(\widetilde{\mathbf{F}})$.

Exercise 2.9. For the identity functor $\text{Id}_{\mathbf{C}}$ from $\mathbf{C}$ to itself, prove that also $\mathsf{Ch}(\text{Id}_{\mathbf{C}})$ is the identity functor from $\mathsf{Ch}(\mathbf{C})$ to itself. Prove that for additive functors $\mathbf{F}$ from an Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$ and $\mathbf{G}$ from $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$ to an Abelian category $(\mathbf{E}, \mathsf{add}_{\mathbf{E}})$, the composition functor $\mathsf{Ch}(\mathbf{G}) \circ \mathsf{Ch}(\mathbf{F})$ equals $\mathsf{Ch}(\mathbf{G} \circ \mathbf{F})$. Also for the identity natural transformation $\text{Id}_{\mathbf{F}}$ from $\mathbf{F}$ to itself, prove that $\mathsf{Ch}(\text{Id}_{\mathbf{F}})$ is the identity natural transformation from $\mathsf{Ch}(\mathbf{F})$ to itself. Finally, prove that Ch is compatible with the various notions of composition for natural transformations.

Definition 2.10. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, the **embedding** from $\mathbf{C}$ to $\mathsf{Ch}(\mathbf{C})$, denoted $-[0]$, associates to every object a of $\mathbf{C}$ the $\mathbf{C}$-cochain complex $a[0]$ such that $a[0]^0$ equals a and $a[0]^n$ equals 0 for every $n \neq 0$. For every $\mathbf{C}$-morphism u from an $\mathbf{C}$-object a to an $\mathbf{C}$-object b, the associated $\mathbf{C}$-cochain morphism $u[0]$ has $u[0]^0$ equal to u.

Exercise 2.11. Prove that this is a Serre embedding from $\mathbf{C}$ to $\mathsf{Ch}(\mathbf{C})$, i.e., an exact, fully faithful functor that is an equivalence to a full subcategory that is stable for extensions (in short exact sequences). Moreover, for every additive functor of Abelian categories, $\mathbf{F}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\widetilde{\mathbf{C}}, \mathsf{add}_{\widetilde{\mathbf{C}}})$, prove that $\mathsf{Ch}(\mathbf{F})$ composed with $-[0]$ equals the composition of $-[0]$ and $\mathbf{F}$. Finally, for additive functors $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, for every natural transformation θ from $\mathbf{F}$ to $\widetilde{\mathbf{F}}$, prove that the pullback by $-[0]$ of the natural transformation $\mathsf{Ch}(\theta)$ equals θ.

Exercise 2.12. If you know about 2-categories, deduce that Ch is a 2-functor from the 2-category of strictly small Abelian categories to itself, and the embedding is a natural transformation from the identity functor of this 2-category to Ch.

3 Some autoequivalences of the cochain category

Definition 3.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every integer m, the m-**translation** functor from $\mathsf{Ch}(\mathbf{C})$ to itself, denoted $T^m_{\mathbf{C}}$ or $-[m]$, associates to every $\mathbf{C}$-cochain complex A the $\mathbf{C}$-cochain complex $T^m_{\mathbf{C}}(A)$ with $T^m_{\mathbf{C}}(A)^n$ equals A^{m+n} and with $d^n_{T^m(A)}$ equal to $(-1)^m d^{m+n}_A$ for every integer n. For every $\mathbf{C}$-cochain morphism u from a $\mathbf{C}$-cochain complex A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the $\mathbf{C}$-cochain morphism $T^m_{\mathbf{C}}(u)$ from $T^m_{\mathbf{C}}(A)$ to $T^m_{\mathbf{C}}(\widetilde{A})$ has $T^m_{\mathbf{C}}(u)^n$ equal to u^{m+n} for every integer n.

Exercise 3.2. Check that the composition functor $T^m_{\mathbf{C}} \circ T^n_{\mathbf{C}}$ equals $T^{m+n}_{\mathbf{C}}$ for all integers m and n. In particular, both $T^m_{\mathbf{C}} \circ T^{-m}_{\mathbf{C}}$ and $T^{-m}_{\mathbf{C}} \circ T^m_{\mathbf{C}}$ equal the identity functor. Deduce that $T^m_{\mathbf{C}}$ is an exact autoequivalence of $\mathsf{Ch}(\mathbf{C})$ for every integer m.

Exercise 3.3. Prove that for every additive functor $\mathbf{F}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, the composition of $T^m_{\mathbf{D}}$ with $\mathsf{Ch}(\mathbf{F})$ equals the composition of $\mathsf{Ch}(\mathbf{F})$ with $T^m_{\mathbf{C}}$. Similarly, for additive functors $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, prove that the $T^m_{\mathbf{C}}$-pullback of $\mathsf{Ch}(\theta)$ equals the $T^m_{\mathbf{D}}$-pushforward of $\mathsf{Ch}(\theta)$. If you know about 2-categories, deduce that T^m is an autoequivalence of the 2-functor Ch for every integer m. In particular, this defines a morphism of Abelian groups from $\mathbb{Z}$ to the "group" of autoequivalences of the 2-functor (not really a group, since the class of autoequivalences is not naturally a set).

Definition 3.4. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, the **braid** $\mathbb{Z}[-1] \otimes_{\mathbb{Z}} A \otimes_{\mathbb{Z}} \mathbb{Z}[1]$ of A is the $\mathbf{C}$-cochain complex $((A^n)_{n\in\mathbb{Z}}, (-d^n_A)_{n\in\mathbb{Z}})$. This extends to an exact functor from $\mathsf{Ch}(\mathbf{C})$ to itself by acting as the identity on $\mathbf{C}$-cochain morphisms. The composition of this functor with itself is the identity functor. There is a natural equivalence of the identity functor with this functor by associating to every $\mathbf{C}$-cochain complex A the $\mathbf{C}$-cochain isomorphism $((-1)^n \mathrm{Id}_{A^n})_{n\in\mathbb{Z}}$ between A and its braid.

Exercise 3.5. As with translation, show that the braid is compatible with additive functors between Abelian categories and with natural transformations between additive functors. Deduce that this gives another autoequivalence of the 2-functor Ch.

4 Two adjoints of the embedding in the cochain category

Definition 4.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, the **$\mathbf{C}$-cocycles** in degree 0 of A is the $\mathbf{C}$-object $Z^0(A)$ with its $\mathbf{C}$-monomorphism $q^0_{Z,A}$ to A^0 that is the kernel of the $\mathbf{C}$-morphism d^0_A from A^0 to A^1. For every $\mathbf{C}$-cochain morphism u from A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, since the composition of $d^0_{\widetilde{A}}$ with $u^0 \circ q^0_{Z,A}$ equals u^1 composed with $d^0_A \circ q^0_{Z,A}$, this composition is zero. Thus, by the universal property of the kernel, there is a unique $\mathbf{C}$-morphism $Z^0(u)$ from $Z^0(A)$ to $Z^0(\widetilde{A})$ such that $q^0_{Z,\widetilde{A}} \circ Z^0(u)$ equals $u^0 \circ q^0_{Z,A}$. The $\mathbf{C}$-morphism $Z^0(u)$ is the **$\mathbf{C}$-cocyles morphism** associated to u.

Exercise 4.2. Prove that for the identity $\mathbf{C}$-cochain morphism Id_A from A to itself, also $Z^0(\mathrm{Id}_A)$ is the identity morphism of $Z^0(A)$. Also check that Z^0 is compatible with composition. Altogether, this defines a functor Z^0 from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$. Check that this functor is additive and half exact (it is neither left exact nor right exact).

Exercise 4.3. Check that the composition functor $Z^0 \circ (-[0])$ equals the identity functor from $\mathbf{C}$ to itself. Also check that q^0_Z is a natural transformation from the composition $(-[0]) \circ Z^0$ to the identity functor on $\mathsf{Ch}(\mathbf{C})$. Altogether, this defines an adjoint pair of functors, i.e., it defines a bifunctorial bijection of $\mathrm{Hom}_{\mathbf{C}}(a, Z^0(A))$ and $\mathrm{Hom}_{\mathsf{Ch}(\mathbf{C})}(a[0], A)$ for all $\mathbf{C}$-objects a and all $\mathbf{C}$-cochain complexes A. Thus Z^0 is right adjoint to the exact embedding of $\mathbf{C}$ in $\mathsf{Ch}(\mathbf{C})$.

Definition 4.4. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, the **$\mathbf{C}$-coboundary quotient** in degree 0 of A is the $\mathbf{C}$-object $\overline{A}^0$ with its $\mathbf{C}$-epimorphism $p^0_{\overline{:},A}$ to $\overline{A}^0$ that is the cokernel of the $\mathbf{C}$-morphism d^{-1}_A from A^{-1} to A^0. For every $\mathbf{C}$-cochain morphism u from A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, since the composition of $p^0_{\overline{:},\widetilde{A}} \circ u^0$ with d^{-1}_A equals $p^0_{\overline{:},\widetilde{A}}$ composed with $d^{-1}_{\widetilde{A}} \circ u^{-1}$, this composition is zero. Thus, by the universal property of the cokernel, there is a unique $\mathbf{C}$-morphism $\overline{u}^0$ from $\overline{A}^0$ to $\overline{\widetilde{A}}^0$ such that $p^0_{\overline{:},\widetilde{A}} \circ u^0$ equals $\overline{u}^0 \circ p^0_{\overline{:},A}$. The $\mathbf{C}$-morphism $\overline{u}^0$ is the **$\mathbf{C}$-coboundary quotient morphism** associated to u.

Exercise 4.5. Prove that for the identity $\mathbf{C}$-cochain morphism Id_A from A to itself, also $\overline{\mathrm{Id}_A}^0$ is the identity morphism of $\overline{A}^0$. Also check that $\overline{:}^0$ is compatible with composition. Altogether, this defines a functor $\overline{:}^0$ from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$. Check that this functor is additive and right exact.

Exercise 4.6. Check that the composition functor $\overline{\cdot}^0 \circ (-[0])$ equals the identity functor from $\mathbf{C}$ to itself. Also check that $p^0_{\cdot}$ is a natural transformation from the identity functor on $\mathsf{Ch}(\mathbf{C})$ to the composition functor $(-[0]) \circ \overline{\cdot}^0$. Altogether, this defines an adjoint pair of functors, i.e., it defines a bifunctorial bijection of $\mathrm{Hom}_{\mathbf{C}}(\overline{A}^0, a)$ and $\mathrm{Hom}_{\mathsf{Ch}(\mathbf{C})}(A, a[0])$ for all $\mathbf{C}$-objects a and all $\mathbf{C}$-cochain complexes A. Thus $\overline{\cdot}^0$ is left adjoint to the exact embedding of $\mathbf{C}$ in $\mathsf{Ch}(\mathbf{C})$.

Of course we can also incorporate shifts.

Definition 4.7. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every integer m, the functor Z^m from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$ is the composition functor $Z^0 \circ T^m$. In particular, this is right adjoint to the exact embedding $-[m]$ of $\mathbf{C}$ in $\mathsf{Ch}(\mathbf{C})$.

Now we can incorporate all shifts into a single $\mathbf{C}$-cochain complex.

Definition 4.8. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, the objects of the exact, full subcategory $\mathbf{C}^{\mathbb{Z}}$ of $\mathrm{Ch}(\mathbf{C})$ are those $\mathbf{C}$-cochain complexes such that every differential $\mathbf{C}$-morphism is a zero morphism, i.e., the $\mathbf{C}$-cochain complex is $((A^n)_{n\in\mathbb{Z}}, (0)_{n\in\mathbb{Z}})$.

Exercise 4.9. Check that this is an exact, full subcategory, but that it is not (typically) stable under extensions (so it is not a Serre subcategory). Check that for every additive functor $\mathbf{F}$ from an Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\widetilde{\mathbf{C}}, \mathsf{add}_{\widetilde{\mathbf{C}}})$, the restriction of the additive functor $\mathsf{Ch}(\mathbf{F})$ to the full subcategory $\mathbf{C}^{\mathbb{Z}}$ factors uniquely as an additive functor $\mathbf{F}^{\mathbb{Z}}$ from $\mathbf{C}^{\mathbb{Z}}$ to the exact, full subcategory $\mathbf{D}^{\mathbb{Z}}$ of $\mathsf{Ch}(\mathbf{D})$. Deduce that $(\cdot)^{\mathbb{Z}}$ is a 2-functor from the 2-category of strictly small Abelian categories to itself, and the exact, fully faithful embeddings are a natural transformation from this 2-functor to Ch.

Definition 4.10. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, the **$\mathbf{C}$-cocycles complex** is the object $Z(A) := ((Z^n(A))_{n\in\mathbb{Z}}, (0)_{n\in\mathbb{Z}})$ of the exact, full subcategory $\mathbf{C}^{\mathbb{Z}}$ of $\mathsf{Ch}(\mathbf{C})$, together with the $\mathbf{C}$-cochain morphism $q_{Z,A} := (q^n_{Z,A})_{n\in\mathbb{Z}}$ from $Z(A)$ to A. For every $\mathbf{C}$-cochain morphism u from A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the **$\mathbf{C}$-cocycles morphism** is the $\mathbf{C}$-cochain morphism $Z(u)$ from $Z(A)$ to $Z(\widetilde{A})$ with $Z(u)^n$ equal to $Z^n(u)$ for every integer n.

Exercise 4.11. Prove that $q_{Z,A}$ is a $\mathbf{C}$-cochain morphism from $Z(A)$ to A. Prove that Z is an additive, left-exact functor from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}^{\mathbb{Z}}$. Prove that the restriction of Z to the full subcategory $\mathbf{C}^{\mathbb{Z}}$ of $\mathsf{Ch}(\mathbf{C})$ is the identity functor on $\mathbf{C}^{\mathbb{Z}}$. Prove that q_Z is a natural transformation from the composition functor of the embedding of $\mathbf{C}^{\mathbb{Z}}$ in $\mathsf{Ch}(\mathbf{C})$ with the functor Z to the identity functor of $\mathsf{Ch}(\mathbf{C})$. Finally, prove that this defines an adjoint pair of functors, i.e., $\mathrm{Hom}_{\mathbf{C}^{\mathbb{Z}}}(\widetilde{A}, Z(A))$ is bifunctorially bijective to $\mathrm{Hom}_{\mathsf{Ch}(\mathbf{C})}(\widetilde{A}, A)$ for every object A of $\mathsf{Ch}(\mathbf{C})$ and for every object $\widetilde{A}$ of the exact, full subcategory $\mathbf{C}^{\mathbb{Z}}$.

Exercise 4.12. Check that Z commutes with the translation functor and the braid. Check that Z commutes with additive functors between Abelian categories.

Of course we have the "opposite" results for the functor $\overline{\cdot}^0$.

Definition 4.13. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every integer m, the functor $\overline{\cdot}^m$ from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$ is the composition functor $\overline{\cdot}^0 \circ T^m$. In particular, this is left adjoint to the exact embedding $-[m]$ of $\mathbf{C}$ in $\mathsf{Ch}(\mathbf{C})$.

Definition 4.14. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, the **$\mathbf{C}$-coboundaries quotient complex** is the object $\overline{A} := ((\overline{A}^n)_{n\in\mathbb{Z}}, (0)_{n\in\mathbb{Z}})$ of the exact, full subcategory $\mathbf{C}^{\mathbb{Z}}$ of $\mathsf{Ch}(\mathbf{C})$, together with the $\mathbf{C}$-cochain morphism $p_{\overline{\cdot},A} := (p^n_{\overline{\cdot},A})_{n\in\mathbb{Z}}$ from A to $\overline{A}$. For every $\mathbf{C}$-cochain morphism u from A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the **$\mathbf{C}$-coboundaries quotient morphism** is the $\mathbf{C}$-cochain morphism $\overline{u} := (\overline{u}^n)_{n\in\mathbb{Z}}$ from $\overline{A}$ to $\overline{\widetilde{A}}$.

Exercise 4.15. Prove that $p_{\overline{\cdot},A}$ is a $\mathbf{C}$-cochain morphism from A to $\overline{A}$. Prove that $\overline{\cdot}$ is an additive, right-exact functor from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}^{\mathbb{Z}}$. Prove that the restriction of $\overline{\cdot}$ to the full subcategory $\mathbf{C}^{\mathbb{Z}}$ of $\mathsf{Ch}(\mathbf{C})$ is the identity functor on $\mathbf{C}^{\mathbb{Z}}$. Prove that $p_{\overline{\cdot}}$ is a natural transformation from the identity functor of $\mathsf{Ch}(\mathbf{C})$ to the composition functor of the embedding of $\mathbf{C}^{\mathbb{Z}}$ in $\mathsf{Ch}(\mathbf{C})$ with the functor $\overline{\cdot}$. Finally, prove that this defines an adjoint pair of functors, i.e., $\mathrm{Hom}_{\mathbf{C}^{\mathbb{Z}}}(\overline{A}, \widetilde{A})$ is bifunctorially bijective to $\mathrm{Hom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$ for every object A of $\mathsf{Ch}(\mathbf{C})$ and for every object $\widetilde{A}$ of the exact, full subcategory $\mathbf{C}^{\mathbb{Z}}$.

Exercise 4.16. Check that Z commutes with the translation functor and the braid. Check that Z commutes with additive functors between Abelian categories.

Finally, this brings us to our main functor on cochain complexes.

Definition 4.17. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A and for every integer n, the **$\mathbf{C}$-cohomology object** in degree n is the cokernel $H^n(A)$ of d_A^{n-1} from $\overline{A}^{n-1}$ to $Z^n(A)$ with its natural epimorphism $p^n_{H,A}$ from $Z^n(A)$ to $H^n(A)$. Because coimages equal images in the Abelian category, also $H^n(A)$ is the kernel of d^n_A from $\overline{A}^n$ to $Z^{n+1}(A)$ with its natural monomorphism $q^n_{H,A}$ from $H^n(A)$ to $\overline{A}^n$. Because this is defined in terms of kernels and cokernels of natural transformations, this extends automatically to a functor from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$. The **$\mathbf{C}$-cohomology complex** of A is the object $((H^n(A))_{n\in\mathbb{Z}}, (0)_{n\in\mathbb{Z}})$ of $\mathbf{C}^{\mathbb{Z}}$. This is a functor H from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}^{\mathbb{Z}}$. For every morphism u of $\mathbf{C}$-cochain complexes from a $\mathbf{C}$-cochain complex A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the morphism u is a **quasi-isomorphism** if (and only if) the induced $\mathbf{C}$-morphism $H^n(u)$ from $H^n(A)$ to $H^n(\widetilde{A})$ is an isomorphism for every integer n.

Exercise 4.18. Check that H^n and H are additive, half exact functors. Check that q_H and p_H are natural transformations. Check that these

Exercise 4.19. Check that H commutes with the translation functor and the braid. Check that H commutes with additive functors between Abelian categories.

5 Truncations of complexes

Definition 5.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, for every integer n, the **brutal truncation** of A in degrees $\le n$, respectively, in degrees $\ge n$, is the $\mathbf{C}$-cochain complex $\sigma^{\le n}(A)$, resp. $\sigma^{\ge n}(A)$, with $\sigma^{\le n}(A)^m$ equals A for all $m \le n$, resp. $\sigma^{\ge n}(A)^m$ equals A for all $m \ge n$, and $\sigma^{\le n}(A)^m$ equals 0 for all $m > n$, resp. $\sigma^{\ge n}(A)^m$ equals 0 for all $m < n$. For all differentials between nonzero terms of the truncation, the differentials are the same as the differentials of A. For every morphism u of $\mathbf{C}$-cochain complexes from A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the **brutal truncation** of u in degrees $\le n$, resp. in degrees $\ge n$, is the morphism of $\mathbf{C}$-cochain complexes $\sigma^{\le n}(u)$ from $\sigma^{\le n}(A)$ to $\sigma^{\le n}(\widetilde{A})$, resp. $\sigma^{\ge n}(u)$ from $\sigma^{\ge n}(A)$ to $\sigma^{\ge n}(\widetilde{A})$ whose $\mathbf{C}$-morphism between nonzero objects of the truncations are the same as u if the objects are both nonzero.

Exercise 5.2. Check that both $\sigma^{\le n}(A)$ and $\sigma^{\ge n}(A)$ are $\mathbf{C}$-cochain complexes, and check that both $\sigma^{\le n}(u)$ and $\sigma^{\ge n}(u)$ are morphisms of $\mathbf{C}$-cochain complexes. Check that $\sigma^{\le n}$ and $\sigma^{\ge n}$ are exact, additive functors from $\mathsf{Ch}(\mathbf{C})$ to itself. Even better, $\sigma^{\le n}$ is a strictly surjective, exact, additive functor (that is neither full nor faithful) from $\mathsf{Ch}(\mathbf{C})$ to the Serre subcategory $\mathsf{Ch}^{\le n}(\mathbf{C})$ of complexes concentrated in degrees $\le n$. Similarly, $\sigma^{\ge n}$ is a strictly surjective, exact, additive functor (that is neither full nor faithful) from $\mathsf{Ch}(\mathbf{C})$ to the Serre subcategory $\mathsf{Ch}^{\ge n}(\mathbf{C})$ of complexes concentrated in degrees $\ge n$.

Definition 5.3. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain complex A, for every integer n, the **good truncation** of A in degrees $\le n$, respectively, in degrees $\ge n$, is the $\mathbf{C}$-cochain complex $\tau^{\le n}(A)$, resp. $\tau^{\ge n}(A)$, that is the same as $\sigma^{\le n}(A)$, resp. as $\sigma^{\ge n}(A)$, except in degree n, where $\sigma^{\le n}(A)^n$ equals the subobject $Z^n(A)$ of A^n, resp., where $\sigma^{\ge n}(A)^n$ equals the quotient object $\overline{A}^n$ of A^n. For every morphism u of $\mathbf{C}$-cochain complexes from A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, the **good truncation** of u in degrees $\le n$, resp. in degrees $\ge n$, is the morphism of $\mathbf{C}$-cochain complexes $\tau^{\le n}(u)$ from $\tau^{\le n}(A)$ to $\tau^{\le n}(\widetilde{A})$, resp. $\tau^{\ge n}(u)$ from $\tau^{\ge n}(A)$ to $\tau^{\ge n}(\widetilde{A})$, that is the same as $\sigma^{\le n}(u)$, resp. as $\sigma^{\ge n}(u)$, except in degree n, where $\tau^{\le n}(u)^n$ equals $Z^n(u)$, resp. where $\tau^{\ge n}(u)^n$ equals $\overline{u}^n$. The **good monomorphism** from $\tau^{\le n}(A)$ to A is the morphism of $\mathbf{C}$-cochain complexes that is the identity in degrees $< n$, is the zero map in degrees $> n$, and is the natural transformation from $Z^n(A)$ to A^n in degree n. The **good epimorphism** from A to $\tau^{\ge n}(A)$ is the morphism of $\mathbf{C}$-cochain complexes that is the identity in degrees $> n$, is the zero map in degrees $< n$, and is the natural transformation from A^n to the quotient $\overline{A}^n$ in degree n.

Exercise 5.4. Check that both $\tau^{\le n}(A)$ and $\sigma^{\ge n}(A)$ are $\mathbf{C}$-cochain complexes, and check that both $\tau^{\le n}(u)$ and $\tau^{\ge n}(u)$ are morphisms of $\mathbf{C}$-cochain complexes. Check that $\tau^{\le n}$ and $\tau^{\ge n}$ are exact, additive functors from $\mathsf{Ch}(\mathbf{C})$ to itself. Check that the good monomorphism, resp. the good epimorphism, defines a natural transformation of functors. Check that $\tau^{\le n}$ is a strictly surjective, left exact, additive functor (that is neither full nor faithful) from $\mathsf{Ch}(\mathbf{C})$ to the Serre subcategory $\mathsf{Ch}^{\le n}(\mathbf{C})$ of complexes concentrated in degrees $\le n$. Similarly, check that $\tau^{\ge n}$ is a strictly surjective, right exact, additive functor (that is neither full nor faithful) from $\mathsf{Ch}(\mathbf{C})$ to the Serre subcategory $\mathsf{Ch}^{\ge n}(\mathbf{C})$ of complexes concentrated in degrees $\ge n$.

6 Cochain homotopy

Definition 6.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-cochain morphism u from a $\mathbf{C}$-cochain complex A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, a **$\mathbf{C}$-homotopy** from u to 0, or a **$\mathbf{C}$-null homotopy** of u, is a sequence $(s^n)_{n\in\mathbb{Z}}$ of $\mathbf{C}$-morphisms s^n from A^n to $\widetilde{A}^{n-1}$ such that, for every integer n,

$$d_{\widetilde{A}}^{n-1} \circ s^n + s^{n+1} \circ d_A^n = u^n.$$

If such exists then u is **$\mathbf{C}$-null homotopic** or **$\mathbf{C}$-homotopic to** 0. More generally, for $\mathbf{C}$-cochain morphisms u and v from A to $\widetilde{A}$, a **$\mathbf{C}$-homotopy** from u to v is a $\mathbf{C}$-homotopy from $u - v$ to 0. If such exists then u is **$\mathbf{C}$-homotopic** to v.

Exercise 6.2. For $\mathbf{C}$-morphisms u and v from A to $\widetilde{A}$, check that for every $\mathbf{C}$-null homotopy s of u and for every $\mathbf{C}$-null homotopy t of v, the sum $s + t$ is a $\mathbf{C}$-null homotopy of $u + v$. Thus, the null homotopic $\mathbf{C}$-cochain morphisms form an Abelian subgroup $\mathsf{NullHom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$ of $\mathsf{Hom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$, and $\mathbf{C}$-homotopy is an equivalence relation on $\mathsf{Hom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$. Check that also precomposing and postcomposing by $\mathbf{C}$-cochain morphisms sends $\mathbf{C}$-null homotopies to $\mathbf{C}$-null homotopies. Thus, the null homotopic $\mathbf{C}$-cochain morphisms are an "ideal" for both precomposition and postcomposition. Use this to prove that there exists a (strictly) surjective, full (but not faithful) additive functor $\mathsf{quot}_{\mathsf{K}(\mathbf{C})}^{\mathsf{Ch}(\mathbf{C})}$ from $\mathsf{Ch}(\mathbf{C})$ to an additive category $\mathsf{K}(\mathbf{C})$ whose objects class is the same as the objects class of $\mathsf{Ch}(\mathbf{C})$, and such that $\mathsf{Hom}_{\mathsf{K}(\mathbf{C})}(A, \widetilde{A})$ equals the quotient Abelian group of $\mathsf{Hom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$ by the subgroup $\mathsf{NullHom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$ for all objects A and $\widetilde{A}$.

Definition 6.3. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for all $\mathbf{C}$-cochain complexes A and $\widetilde{A}$, a **$\mathbf{C}$-homotopy equivalence** of A and $\widetilde{A}$ is an ordered pair (u, v) of morphisms of $\mathbf{C}$-cochain complexes u from A to $\widetilde{A}$ and v from $\widetilde{A}$ to A such that $v \circ u$ is homotopic to Id_A and $u \circ v$ is homotopic to $\mathrm{Id}_{\widetilde{A}}$, i.e., u and v give an inverse pair of isomorphisms in the homotopy category $\mathsf{K}(\mathbf{C})$.

Exercise 6.4. Prove that for every additive functor $\mathbf{F}$ from an Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, the additive functor $\mathsf{Ch}(\mathbf{F})$ maps every Abelian subgroup $\mathsf{NullHom}_{\mathsf{Ch}(\mathbf{C}}(A, \widetilde{A})$ of $\mathsf{Hom}_{\mathsf{Ch}(\mathbf{C})}(A, \widetilde{A})$ to the Abelian subgroup $\mathsf{NullHom}_{\mathsf{Ch}(\mathbf{D})}(\mathsf{Ch}(\mathbf{F})(A), \mathsf{Ch}(\mathbf{F})(\widetilde{A}))$ of $\mathsf{Hom}_{\mathsf{Ch}(\mathbf{D})}(\mathsf{Ch}(\mathbf{F})(A), \mathsf{Ch}(\mathbf{F}$

Exercise 6.5. For additive functors $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, for every $\mathbf{C}$-null homotopy s of a $\mathbf{C}$-cochain morphism u from A to $\widetilde{A}$, prove that $\mathrm{Ch}(\theta) \circ \mathrm{Ch}(F)(s)$ equals $\mathrm{Ch}(G)(s) \circ \mathrm{Ch}(\theta)$ as a $\mathbf{D}$-null homotopy of the $\mathbf{D}$-cochain morphism $\mathrm{Ch}(\theta)_{\widetilde{A}} \circ \mathrm{Ch}(\mathbf{F})(u) = \mathrm{Ch}(\mathbf{G})(u) \circ \mathrm{Ch}(\theta)_A$ from $\mathrm{Ch}(\mathbf{F})(A)$ to $\mathrm{Ch}(\mathbf{G}(\widetilde{A})$.

Exercise 6.6. For every $\mathbf{C}$-null homotopy s of a $\mathbf{C}$-cochain morphism u from a $\mathbf{C}$-cochain complex A to a $\mathbf{C}$-cochain complex $\widetilde{A}$, for every integer n, prove that s^n restricts to a $\mathbf{C}$-morphism from $Z(A)^n$ to $\widetilde{A}^{n-1}$ such that $Z(u)^n$ equals $d_{\widetilde{A}}^{n-1} \circ s^n$. Similarly, prove that the composition of s^{n+1} with the projection to $\overline{\widetilde{A}}^n$ is a $\mathbf{C}$-morphism from A^{n+1} to $\overline{\widetilde{A}}^n$ whose precomposition with d_A^n equals $\overline{u}^n$. Finally, prove that the induced morphism $H(u)^n$ from $H^n(A)$ to $H^n(\widetilde{A})$ is a zero map. Deduce that the cohomology functor H from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}^{\mathbb{Z}}$ factors uniquely through an additive functor from $\mathsf{K}(\mathbf{C})$ to $\mathbf{C}^{\mathbb{Z}}$.

7 The snake lemma

For each Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ and each morphism of $\mathbf{C}$-short exact sequences,

$$\begin{array}{ccccccccccc} \Sigma: & 0 & \longrightarrow & a' & \xrightarrow{q_\Sigma} & a & \xrightarrow{p_\sigma} & a'' & \longrightarrow & 0 & \\ {\scriptstyle\phi}\downarrow & & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & & , \\ \widetilde{\Sigma}: & 0 & \longrightarrow & \widetilde{a}' & \xrightarrow{\widetilde{q}_\Sigma} & \widetilde{a} & \xrightarrow{\widetilde{p}_\sigma} & \widetilde{a}'' & \longrightarrow & 0 & \end{array}$$

denote the kernels of f', respectively f, f'', by

$$i' : K'_\phi \to a', \text{ resp. } i : K_\phi \to a, \; i'' : K''_\phi \to a'',$$

and denote the cokernels of f', resp. f, f'', by

$$s' : \widetilde{a}' \to C'_\phi, \text{ resp. } s : \widetilde{a} \to C_\phi, \; s'' : \widetilde{a}'' \to C''_\phi.$$

Because $\widetilde{q} \circ f'$ equals $f \circ q$, also $f \circ (q \circ i')$ equals $\widetilde{q} \circ (f' \circ i')$, which equals $\widetilde{q} \circ 0$, i.e., it equals 0. Thus, by the universal property of the kernel, there is a unique morphism

$$q_K : K'_\phi \to K_\phi$$

such that $i \circ q_K$ equals $q \circ i'$. For a similar reason, there is a unique morphism

$$p_K : K_\phi \to K''_\phi$$

such that $i'' \circ p_K$ equals $p \circ i$. And by analogous arguments there are unique morphisms

$$q_C : C'_\phi \to C_\phi, \quad p_C : C_\phi \to C''_\phi$$

such that $q_C \circ s'$ equals $s \circ \widetilde{q}$, and $p_C \circ s$ equals $s'' \circ \widetilde{p}$. To summarize, we have that the following diagram is commutative.

$$\begin{array}{ccccccccc}
 & & K'_\phi & \xrightarrow{q_K} & K_\phi & \xrightarrow{p_K} & K''_\phi & & \\
 & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\
\Sigma : & 0 & \longrightarrow & a' & \xrightarrow{q} & a & \xrightarrow{p} & a'' & \longrightarrow 0 \\
{\scriptstyle \phi}\downarrow & & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & \\
\widetilde{\Sigma} : & 0 & \longrightarrow & \widetilde{a}' & \xrightarrow[\widetilde{q}]{} & \widetilde{a} & \xrightarrow[\widetilde{p}]{} & \widetilde{b}'' & \longrightarrow 0 \\
 & & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s''} & \\
 & & & C'_\phi & \xrightarrow[q_C]{} & C_\phi & \xrightarrow[p_C]{} & C''_\phi &
\end{array}$$

By hypothesis, both $f'' \circ p$ and $\widetilde{p} \circ f$ are equal. Denote by t this common morphism

$$t : a \to \widetilde{a}''.$$

Denote the kernel of t by

$$j : K_t \to a.$$

Now $f'' \circ (p \circ j)$ equals $t \circ j$, which is 0. By the universal property of the kernel of f'', there is a unique morphism

$$\overline{p} : K_t \to K''_{\Sigma_f}$$

such that $i'' \circ \overline{p}$ equals $p \circ j$. Similarly, $\widetilde{p} \circ (f \circ j)$ equals $t \circ j$, which is 0. By the universal property of the kernel of $\widetilde{p}$, there is a unique morphism

$$\overline{f} : K_t \to \widetilde{a}'$$

such that $\widetilde{q} \circ \overline{f}$ equals $f \circ j$.

Lemma 7.1 (The Snake Lemma). *For every Abelian category* $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$*, for every morphism* ϕ *of* $\mathbf{C}$*-short exact sequences as above, all of the following hold.*

(i) The morphism q_K *is a monomorphism, and the morphism* p_C *is an epimorphism.*

(ii) The image of q_K *equals the kernel of* p_K*, and the kernel of* p_C *equals the image of* q_C*.*

(iii) There is a unique morphism $\delta_\phi : K''_{\Sigma_f} \to C'_{\Sigma_f}$ *such that* $\delta_\phi \circ \overline{p}$ *equals* $s' \circ \overline{f}$ *as morphisms from* K_t *to* C'_ϕ*.*

(iv) The image of p_K *equals the kernel of* δ_ϕ*, and the kernel of* q_C *equals the image of* δ_ϕ*.*

In summary, the following long sequence is exact,

$$0 \longrightarrow K'_\phi \xrightarrow{q_K} K_\phi \xrightarrow{p_K} K''_\phi \xrightarrow{\delta_\phi} \dots$$

$$\dots \xrightarrow[\delta_\phi]{} C'_\phi \xrightarrow[q_C]{} C_\phi \xrightarrow[p_C]{} C''_\phi \longrightarrow 0.$$

This entire situation is often summarized with the following large diagram.

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \longrightarrow & K'_\phi & \xrightarrow{q_K} & K_\phi & \xrightarrow{p_K} & K''_\phi & \xrightarrow{\delta_\phi} \dots \\
 & & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & \\
\Sigma: & 0 & \longrightarrow & a' & \xrightarrow{q} & a & \xrightarrow{p} & a'' & \longrightarrow 0 \\
{\scriptstyle \phi}\downarrow & & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & \\
\widetilde{\Sigma}: & 0 & \longrightarrow & \widetilde{a}' & \xrightarrow[\widetilde{q}]{} & \widetilde{a} & \xrightarrow[\widetilde{p}]{} & \widetilde{a}'' & \longrightarrow 0 \\
 & & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s''} & \\
 & \dots & \xrightarrow[\delta_\phi]{} & C'_\phi & \xrightarrow[q_C]{} & C_\phi & \xrightarrow[p_C]{} & C''_\phi & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 & & 0 &
\end{array}$$

There are many variants of the snake lemma. Here is one. Consider a commutative diagram with exact rows, but where the top row is not left exact, and where the bottom row is not right exact.

$$
\begin{array}{ccccccccc}
\Pi : & & b' & \xrightarrow{Q} & a & \xrightarrow{p} & a'' & \longrightarrow & 0 \\
\psi \downarrow & & F' \downarrow & & \downarrow f & & \downarrow F'' & & \\
\widetilde{\Pi} : & 0 \longrightarrow & \widetilde{a}' & \xrightarrow[\widetilde{q}]{} & \widetilde{a} & \xrightarrow[\widetilde{P}]{} & \widetilde{b}'' & &
\end{array}
$$

Define a' to be the image of the morphism Q from b' to a, so that the induced morphism q from a' to a is a monomorphism. Define f' from a' to $\widetilde{a}'$ to be the morphism induced by F'. Define $\widetilde{a}''$ to be the image of the morphism $\widetilde{P}$ from $\widetilde{a}$ to $\widetilde{b}''$, so that the induced morphism $\widetilde{p}$ from $\widetilde{a}$ to $\widetilde{a}''$ is an epimorphism. Define f'' to be the morphism from a'' to $\widetilde{a}''$ induced by F''.

With these substitutions, we are again in the setting of the snake lemma. Also, the induced morphism from $\text{Ker}(F')$ to $\text{Ker}(f')$ is an epimorphism, since the morphism from b' to a' is an epimorphism. Similarly, the morphism from $\text{Coker}(f'')$ to $\text{Coker}(F'')$ is a monomorphism, since the morphism from $\widetilde{a}''$ to $\widetilde{b}''$ is a monomorphism. Altogether this gives the following.

Corollary 7.2. *For every Abelian category $(\mathbf{C}, \mathit{add}_{\mathbf{C}})$, for every commutative diagram of $\mathbf{C}$-objects with exact rows, $\Pi \xrightarrow{\psi} \widetilde{\Pi}$, as above, there is a long exact sequence,*

$$\mathit{Ker}(F') \to \mathit{Ker}(f) \to \mathit{Ker}(F'') \xrightarrow{\delta} \mathit{Coker}(F') \to \mathit{Coker}(f) \to \mathit{Coker}(F'').$$

Now let $\Sigma = (A' \xrightarrow{q} A, A \xrightarrow{p} A'')$ be a short exact sequence in the Abelian category $\mathsf{Ch}(\mathbf{C})$. Then for every integer n we have a $\mathbf{C}$-commutative diagram with exact rows as follows.

$$
\begin{array}{ccccccccc}
\overline{\Sigma}^n : & & (\overline{A'})^n & \xrightarrow{\overline{q}^n} & \overline{A}^n & \xrightarrow{\overline{p}^n} & (\overline{A''})^n & \longrightarrow & 0 \\
d^n_{\Sigma} \downarrow & & d^n_{A'} \downarrow & & \downarrow d^n_A & & \downarrow d^n_{A''} & & \\
Z^n(\Sigma) : & 0 \longrightarrow & Z^n(A') & \xrightarrow[Z^n(q)]{} & Z^n(A) & \xrightarrow[Z^n(p)]{} & Z^n(A'') & &
\end{array}
$$

Applying the previous corollary gives the following.

Corollary 7.3. *For every Abelian category $(\mathbf{C}, \mathit{add}_{\mathbf{C}})$, for every short sequence $(A' \xrightarrow{q} A, A \xrightarrow{p} A'')$ in $\mathit{Ch}(\mathbf{C})$, for every integer n, there is an exact sequence, functorial in Σ,*

$$H^n(A') \xrightarrow{H^n(q)} H^n(A) \xrightarrow{H^n(p)} H^n(A'') \xrightarrow{\delta^n_{\Sigma}} H^{n+1}(A') \xrightarrow{H^{n+1}(q)} H^{n+1}(A) \xrightarrow{H^{n+1}(p)} H^{n+1}(A'').$$

8 Delta functors

Definition 8.1. For Abelian categories $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ and $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, a δ-**functor** from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$ is an ordered pair $R = ((R^n)_{n\in\mathbb{Z}}, (\delta_R^n)_{n\in\mathbb{Z}})$ of a sequence of additive, half exact functors R^n from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$ and a sequence of natural transformations δ^n from the composite functor $R^n \circ \pi''$ to the composite functor $R^{n+1} \circ \pi'$ from $\mathbf{C}_{\text{ses}}$ to $\mathbf{D}$ such that for every $\mathbf{C}$-short exact sequence $\Sigma = (a' \xrightarrow{q} a, a \xrightarrow{p} a'')$ and every integer n, the following complex is exact:

$$R^n(a') \xrightarrow{R^n(q)} R^n(a) \xrightarrow{R^n(p)} R^n(a'') \xrightarrow{\delta_\Sigma^n} R^{n+1}(a') \xrightarrow{R^{n+1}(q)} R^{n+1}(a').$$

For δ-functors R and S, a **morphism** of δ-functors from R to S is a sequence $(\theta^n)_{n\in\mathbb{Z}}$ of natural transformations θ^n from R^n to S^n such that $\delta_S^n \circ \theta^n$ equals $\theta^{n+1} \circ \delta_R^n$ for every integer n. For every integer m, for every δ-functor R, if R^n is a zero functor for all integers $n < m$, then the δ-functor is **concentrated** in degrees $\geq m$. Similarly, for every integer m, if R^n is a zero functor for all integers $n > m$, then the δ-functor is **concentrated** in degrees $\leq m$.

Example 8.2. By the previous corollary, for every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, the ordered pair $((H^n)_{n\in\mathbb{Z}}, (\delta^n)_{n\in\mathbb{Z}})$ is a δ-functor from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$. More generally, for every integer m, let R^n be the zero functor for $n < m$, let R^m be the functor Z^m, and let R^n be H^n for $n \geq m$. This also gives a δ-functor, usually denoted $R^n = H^n \circ \tau_{\geq m}$, concentrated in degrees $\geq m$. Similarly, if we define R^n to be the zero functor for $n > m$, define R^m to be $\div^m$, and define R^n to be H^n for $n < m$, then this gives a δ-functor, usually denoted $R^n = H^n \circ \tau_{\leq m}$, concentrated in degrees $\leq m$.

Historically, there were many important known examples of δ-functors before any thorough study of all δ-functors, cf. *Homological algebra* by Cartan and Eilenberg and *Homology* by MacLane. The key unifying notion, introduced by Grothendieck in his Tohoku article, is as follows.

Definition 8.3. A δ-functor R concentrated in degrees ≥ 0 is **universal** if (and only if) for every δ-functor S concentrated in degrees ≥ 0 and for every natural transformation θ^0 from R^0 to S^0, there exists a unique morphism θ of δ-functors from R to S extending θ^0. A δ-functor R concentrated in degrees ≥ 0 is **effaceable** if (and only if), for every object a of $\mathbf{C}$ and for every strictly positive integer $n > 0$, there exists a monomorphism $a \xrightarrow{\iota} b$ such that $R^n(\iota)$ is a zero morphism.

Similarly, a δ-functor R concentrated in degrees ≤ 0 is **universal** if (and only if) for every δ-functor S concentrated in degrees ≤ 0 and for every natural transformation θ^0 from S^0 to R^0, there exists a unique morphism θ of δ-functors from S to R extending θ^0. A δ-functor R concentrated in degrees ≤ 0 is **coeffaceable** if (and only if), for every object a of $\mathbf{C}$ and for every strictly negative integer $n < 0$, there exists an epimorphism $b \xrightarrow{\pi} a$ such that $R^n(\pi)$ is a zero morphism.

Lemma 8.4 (Grothendieck's criterion)**.** *Every δ-functor concentrated in degrees ≥ 0 that is effaceable is universal. Every δ-functor concentrated in degrees ≤ 0 that is coeffaceable is universal.*

Proof. Let R and S be δ-functor from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$ concentrated in degrees ≥ 0. Let θ^0 be a natural transformation from R^0 to S^0. Assume that R is effaceable. By way of induction, let n be a nonnegative integer, and assume that there exists a unique sequence $(\theta^0, \dots, \theta^n)$ of natural transformations θ^m from R^m to S^m for $0 \leq m \leq n$ that are compatible with the connecting maps $\delta^0, \dots, \delta^{n-1}$. Since R^{n+1} is effaceable, for every object a' there exists a monomorphism, $q : a' \hookrightarrow a$, such that $R^{n+1}(q)$ is zero. Denote the cokernel of q by $p : a \twoheadrightarrow a''$. Then $\Sigma = (q, p)$ is a $\mathbf{C}$-short exact sequence. Thus, there exists a commutative diagram,

$$\begin{array}{ccccccc} R^n(a'') & \xrightarrow{R^n(p)} & R^n(a) & \xrightarrow{\delta^n_{R,\Sigma}} & R^{n+1}(a') & \xrightarrow{0} & R^{n+1}(a) \\ {\scriptstyle \theta^n_{a''}}\downarrow & & {\scriptstyle \theta^n_{a}}\downarrow & & & & \\ S^n(a'') & \xrightarrow[S^n(p)]{} & S^n(a) & \xrightarrow[\delta^n_{S,\Sigma}]{} & S^{n+1}(a') & \xrightarrow[S^{n+1}(q)]{} & S^{n+1}(a) \end{array}$$

Because $\delta^n_{R,\Sigma}$ is a cokernel of $R^n(p)$, there is a unique $\mathbf{D}$-morphism $\theta^{n+1}_{a'}$ from $R^{n+1}(a')$ to $S^{n+1}(a')$ such that $\theta^{n+1}_{a'} \circ \delta^n_{R,\Sigma}$ equals $\delta^n_{S,\sigma} \circ \theta^n_{a'}$. It remains to show that $\theta^{n+1}_{a'}$ is independent of the choice of monomorphism, and the $\mathbf{D}$-morphisms $\theta^{n+1}_{a'}$ form a natural transformation from R^{n+1} to S^{n+1}.

For $\mathbf{C}$-monomorphisms $q_i : a' \hookrightarrow a_i$ such that $R^{n+1}(q_i)$ is zero, for $i = 1, 2$, define q to be the monomorphism (q_1, q_2) from a' to $a = a_1 \oplus a_2$. Then for the projection pr_i from a to a_i, we have $\mathrm{pr}_i \circ q$ equals q_i. Denote by $p : a \twoheadrightarrow a''$ and $p_i : a_i \twoheadrightarrow a''_i$ the cokernels of q and q_i, and denote by pr''_i the unique epimorphism from a'' to a''_i such that $\mathrm{pr}''_i \circ p$ equals $p_i \circ \mathrm{pr}_i$, for $i = 1, 2$. This defines $\mathbf{C}$-short exact sequences $\Sigma = (q, p)$ and $\Sigma_i = (q_i, p_i)$ for $i = 1, 2$, as well as morphisms $(\mathrm{Id}_{a'}, \mathrm{pr}_i, \mathrm{pr}''_i)$ of $\mathbf{C}$-short exact sequences from Σ to Σ_i for $i = 1, 2$. Clearly it suffices to check that the induced morphism $\theta^{n+1}_{a'}$ from $R^{n+1}(a')$ to $S^{n+1}(a')$ equals each morphism $\theta^{n+1}_{a',i}$ induced by q_i.

By construction, both $\theta^{n+1}_{a'} \circ \delta^n_{R,\Sigma}$ equals $\delta^n_{S,\sigma} \circ \theta^n_{a''}$, and $\theta^{n+1}_{a',i} \circ \delta^n_{R,\Sigma_i}$ equals $\delta^n_{S,\sigma_i} \circ \theta^n_{a''_i}$, for $i = 1, 2$. Via the naturality of the connecting morphisms in Σ, we have $\delta^n_{R,\Sigma_i} \circ R^n(\mathrm{pr}''_i)$ equals $\delta^n_{R,\Sigma}$ and $\delta^n_{S,\Sigma_i} \circ S^n(\mathrm{pr}''_i)$ equals $\delta^n_{S,\Sigma}$, for $i = 1, 2$. Precomposing with $R^n(\mathrm{pr}''_i)$ the identity $\theta^{n+1}_{a',i} \circ \delta^n_{R,\Sigma_i} = \delta^n_{S,\Sigma_i} \circ \theta^n_{a''_i}$ gives the identity $\theta^{n+1}_{a',i} \circ \delta^n_{R,\Sigma_i} = \delta^n_{S,\Sigma_i} \circ \theta^n_{a''_i} \circ R^n(\mathrm{pr}''_i)$. Since θ^n is a natural transformation from R^n to S^n, we also have $\theta^n_{a''_i} \circ R^n(\mathrm{pr}''_i)$ equals $S^n(\mathrm{pr}''_i) \circ \theta^n_{a''}$, so that $\delta^n_{S,\Sigma_i} \circ \theta^n_{a''_i} \circ R^n(\mathrm{pr}''_i)$ equals $\delta^n_{S,\Sigma_i} \circ S^n(\mathrm{pr}''_i) \circ \theta^n_{a''}$. Since also $\delta_{S,\Sigma_i} \circ S^n(\mathrm{pr}''_i$ equals $\delta_{S,\Sigma}$, this finally gives that $\theta^{n+1}_{a',i} \circ \delta^n_{R,\Sigma}$ equals $\delta_{S,\Sigma} \circ \theta^n_{a''}$. Therefore $\theta^{n+1}_{a'} \circ \delta^n_{R,\Sigma}$ equals $\theta^{n+1}_{a',i} \circ \delta^n_{R,\Sigma}$ for $i = 1, 2$. Since $\delta^n_{R,\Sigma}$ is an epimorphism, it follows that $\theta^{n+1}_{a',1}$ equals $\theta^{n+1}_{a'}$ equals $\theta^{n+1}_{a',2}$, i.e., the $\mathbf{D}$-morphism $\theta^{n+1}_{a'}$ is independent of the choice of monomorphism q from a' to a such that $R^{n+1}(q)$ equals a zero morphism.

A similar "diagram chasing" argument proves that θ^{n+1} is a natural transformation. □

Corollary 8.5. *The δ-functor $(Z^0, H^1, H^2, \dots)$ concentrated in degrees ≥ 0 from $\mathsf{Ch}(\mathbf{C})$ to $\mathbf{C}$ is universal. Similarly, the δ-functor $(\dots, H^{-2}, H^{-1}, \div^0)$ concentrated in degrees ≤ 0 is universal.*

Proof. For every $\mathbf{C}$-complex A', for every nonnegative integer n, define A to be the same complex as A' except in degree n, where A^n equals the direct sum $(A')^n \oplus Z^n(A')$. The differentials of A equal the differentials of A' except for d_A^{n-1}, which equals the morphism $(d_{A'}^{n-1}, 0)$ from $(A')^{n-1}$ to $(A')^n \oplus Z^n(A')$, and for d_A^n, which equals $d_{A'}^n$ on the summand $(A')^n$ and which equals the inclusion on the summand $Z^n(A')$. Thus, A' is naturally a subcomplex of A, with cokernel complex equal to $Z^n(A')[n]$. By construction, $H^{n+1}(A)$ is zero. Thus, the δ-functor $(Z^0, H^1, H^2, \dots)$ is effaceable.

A similar argument applies for $(\dots, H^{-2}, H^{-1}, \div^0)$, or one can formally deduce this case from the previous case by passing to opposite Abelian categories. $\square$

9 Mapping cone complexes

Definition 9.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every morphism u of $\mathbf{C}$-cochain complexes from A to $\widetilde{A}$, the **mapping cone complex** $\mathsf{Cone}(u)$ of u is the $\mathbf{C}$-cochain complex with $\mathsf{Cone}(u)^n = \widetilde{A}^n \oplus A^{\oplus(n+1)}$ for every integer n, and with differential $d^n_{\mathsf{Cone}(u)}$ from $\widetilde{A}^n \oplus A^{\oplus(n+1)}$ to $\widetilde{A}^{n+1} \oplus A^{\oplus(n+2)}$ equal to the following 2×2-matrix of $\mathbf{C}$-morphisms,

$$\begin{bmatrix} d^n_{\widetilde{A}} & u^{n+1} \\ 0 & -d^{n+1}_A \end{bmatrix}.$$

Exercise 9.2. Prove that the composition $d^{n+1}_{\mathsf{Cone}(u)} \circ d^n_{\mathsf{Cone}(u)}$ is a zero morphism for every integer n, so that $\mathsf{Cone}(u)$ is a $\mathbf{C}$-cochain complex. Prove that the sequence of epimorphism / projections $p^n_{\mathsf{Cone}(u)}$ from $\widetilde{A}^n \oplus A^{\oplus(n+1)}$ to $A^{\oplus(n+1)}$ defines a morphism $p_{\mathsf{Cone}(u)}$ from $\mathsf{Cone}(u)$ to the translate $T(A) = A[+1]$. Similarly, prove that the sequence of monomorphisms $q^n_{\mathsf{Cone}(u)}$ from $\widetilde{A}^n$ to $\widetilde{A}^n \oplus A^{\oplus(n+1)}$ defines a morphism $q_{\mathsf{Cone}(u)}$ from $\widetilde{A}$ to $\mathsf{Cone}(u)$. Altogether, this defines a short exact sequence $\Sigma_u = (q_{\mathsf{Cone}(u)}, p_{\mathsf{Cone}(u)})$ of $\mathbf{C}$-cochain complexes that is term-by-term split, but typically not split as a short exact sequence in the category $\mathsf{Ch}(\mathbf{C})$. Finally, for the associated long exact sequence of cohomology, check that the connecting $\mathbf{C}$-morphism δ^n_Σ from $H^n(T(A)) = H^{n+1}(A)$ to $H^{n+1}(\widetilde{A})$ equals $H^{n+1}(u)$ for every integer n.

Exercise 9.3. Prove that the mapping cone complex is functorial in u: for every morphism v of $\mathbf{C}$-cochain complexes from B to $\widetilde{B}$, for all morphisms of $\mathbf{C}$-cochain complexes f from A to B and $\widetilde{f}$ from $\widetilde{A}$ to $\widetilde{B}$ such that $\widetilde{f} \circ u$ equals $v \circ f$, then there is a unique morphism $\mathsf{Cone}(f, \widetilde{f})$ of $\mathbf{C}$-cochain complexes from $\mathsf{Cone}(u)$ to $\mathsf{Cone}(v)$ such that $(\widetilde{f}, \mathsf{Cone}(f, \widetilde{f}), T(f))$ is a morphism of short exact sequences in $\mathsf{Ch}(\mathbf{C})$ from Σ_u to Σ_v.

Definition 9.4. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every morphism u of $\mathbf{C}$-cochain complexes from A to $\widetilde{A}$, the **mapping cone null homotopy** of $T(u) \circ p_{\mathsf{Cone}(u)}$ is $(s^n)_{n \in \mathbb{Z}}$ from $\mathsf{Cone}(u)$ to $T(\widetilde{A})$ with s^n from $\widetilde{A}^n \oplus A^{\oplus(n+1)}$ to $\widetilde{A}^n$ equal to the first projection.

Exercise 9.5. Check that this is a null homotopy of $T(u) \circ p_{\mathsf{Cone}(u)}$. Check that this null homotopy is functorial in u. Better, check that for every $\mathbf{C}$-cochain complex B, for every morphism of $\mathbf{C}$-cochain complexes f from B to $T(A)$, and for every null homotopy t of $T(u) \circ f$, there exists a unique morphism (t, f) of $\mathbf{C}$-cochain complexes from B to $\mathsf{Cone}(u)$ such that both $p_{\mathsf{Cone}(u)} \circ (t, f)$ equals f and $s \circ (t, f)$ equals t.

10 Resolutions

Definition 10.1. For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-object a, an **injective complex** for a is a $\mathbf{C}$-cochain complex I concentrated in degrees ≥ 0 and a morphism e of $\mathbf{C}$-cochain complexes from $a[0]$ to I. A **resolution** of a is a $\mathbf{C}$-cochain complex A and a quasi-isomorphism e from $a[0]$ to A. A **injective resolution** for a is a resolution for a that is also an injective complex.

Lemma 10.2. *For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every resolution of a $\mathbf{C}$-object, $a[0] \xrightarrow{e} A$, respectively, for every injective complex for a $\mathbf{C}$-object, $a[0] \xrightarrow{e} I$, also the shifted brutal truncation gives a resolution, $A^0/q(a) \xrightarrow{d_A^0} T(\sigma^{\geq} A)$, resp. the shifted brutal truncation gives an injective complex, $I^0/q(a) \xrightarrow{d_I^0} T(\sigma^{\geq I})$.*

Proof. This follows from the definitions. □

Lemma 10.3. *For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-morphism u from an $\mathbf{C}$-object a to a $\mathbf{C}$-object $\widetilde{a}$, for every resolution of a, say $a[0] \xrightarrow{e} A$, for every injective complex for $\widetilde{a}$, say $\widetilde{a}[0] \xrightarrow{\widetilde{e}} I$, there exists a morphism v of $\mathbf{C}$-cochain complexes from A to I such that $v \circ e$ equals $\widetilde{e} \circ u$. If $\widetilde{e}$ is an injective resolution of $\widetilde{a}$, then v is unique up to null homotopy.*

Proof. Since e from a to A^0 is a monomorphism, and since I^0 is an injective object of $\mathbf{C}$, for the $\mathbf{C}$-morphism $\widetilde{e} \circ u$ from a to I^0 there exists a $\mathbf{C}$-morphism v^0 from A^0 to I^0 such that $v^0 \circ q$ equals $\widetilde{q} \circ u$.

By way of induction, let n be a nonnegative integer such that there exist a sequence $(v^0, \dots, v^n)$ of $\mathbf{C}$-morphisms v^m from A^m to I^m that commute with the differentials and extending the given morphism u. The precomposition of the morphism $d_I^n \circ v^n$ from A^n to I^{n+1} by the morphism d_A^{n-1} from A^{n-1} to A^n equals $d_I^n \circ d_I^{n-1} \circ v^{n-1}$ by the hypothesis on the sequence, i.e., the precomposition is zero. Thus, $d_I^n \circ v^n$ factors uniquely through a $\mathbf{C}$-morphism from $A^n/d_A^{n-1}(A^{n-1})$ to I^{n+1}. Since

e is a resolution, the natural map from $A^n/d_A^{n-1}(A^{n-1})$ to the subobject $Z(A)^{n+1}$ of A^{n+1} is an isomorphism, i.e., the morphism d_A^n from $A^n/d_A^{n-1}(A^{n-1})$ to A^{n+1} is a monomorphism. Thus, since I^{n+1} is an injective object, there exists a $\mathbf{C}$-morphism v^{n+1} from A^{n+1} to I^{n+1} such that $v^{n+1}\circ d_A^n$ equals $d_I^n \circ v^n$. Thus, by induction on n (and by the countable variant of the Axiom of Choice), there exists a $\mathbf{C}$-cochain morphism v from A to I such that $v\circ e$ equals $\widetilde{e}\circ u$.

Let v_1 and v_2 be $\mathbf{C}$-cochain morphisms from A to I such that both $v_1^0\circ e$ and $v_2^0\circ e$ equal $\widetilde{e}\circ u$. Assume further that $\widetilde{e}$ is an injective resolution of $\widetilde{a}$. Then $v_2^0 - v_1^0$ is zero on the subobject $e(a)$ of A^0, hence $v_2 - v_1$ factors through the quotient $A^0/e(a)$. Since q is a resolution of a, the $\mathbf{C}$-morphism d_A^0 from $A^0/e(a)$ to A^1 is a monomorphism. Since I^0 is an injective object, there exists a $\mathbf{C}$-morphism s^1 from A^1 to I^0 such that $v_2^0 - v_1^0$ equals $s^1\circ d_A^0$.

By way of induction, let n be a nonnegative integer such that s^1 extends to a sequence $(s^1,\dots,s^{n+1})$ of $\mathbf{C}$-morphisms s^{m+1} from A^{m+1} to I^m with $v_2^m - v_1^m = d_I^{m-1}\circ s^m + s^{m+1}\circ d_A^m$ for $m=0,\dots,n$. The composition $(v_2^{n+1}-v_1^{n+1})\circ d_A^n$ equals $d_I^n\circ(v_2^n - v_1^n)$ since v_1 and v_2 are morphisms of $\mathbf{C}$-cochain complexes. By the hypothesis on the sequence, $v_2^n - v_1^n$ equals $d_I^{n-1}\circ s^n + s^{n+1}\circ d_A^n$. Thus, the composite $d_I^n\circ(v_2^n - v_1^n)$ equals $d_I^n\circ s^{n+1}\circ d_A^n$. Therefore the difference, $(v_2^n - v_1^n) - d_I^n\circ s^{n+1}$ is zero on the subobject $d_A^n(A^n)$ of A^{n+1}. So it factors through a $\mathbf{C}$-morphism from the quotient $A^{n+1}/d_A^n(A^n)$ to I^{n+1}. By the hypothesis that q is a resolution of a, the differential d_A^{n+1} from $A^{n+1}/d_A^n(A^n)$ to A^{n+2} is a monomorphism. Since I^{n+1} is an injective object, there exists a $\mathbf{C}$-morphism s^{n+2} from A^{n+2} to I^{n+1} such that $v_2^n - v_1^n$ equals $d_I^n\circ s^{n+1} + s^{n+2}\circ d_A^{n+1}$. Thus, by way of induction, there exists a $\mathbf{C}$-null homotopy from v_1 to v_2. □

Corollary 10.4. *For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-object a, for all injective resolutions $a[0]\xrightarrow{e} I$ and $a[0]\xrightarrow{\widetilde{e}}\widetilde{I}$ of a, there exists a $\mathbf{C}$-homotopy equivalence from I to $\widetilde{I}$ commuting with e and $\widetilde{e}$. In particular, this is a quasi-isomorphism.*

Lemma 10.5 (Horseshoe Lemma)**.** *For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$, for every $\mathbf{C}$-short exact sequence $(a'\xrightarrow{q} a, a\xrightarrow{p} a'')$, for every injective resolution $a'[0]\xrightarrow{e'} I'$, and for every injective resolution $a''[0]\xrightarrow{e''} I''$, there exists a morphism v of $\mathbf{C}$-cochain complexes from I'' to the translated injective resolution $I'[+1]$ and a null homotopy t of $v\circ e''\circ p$ from $a''[0]$ to I' such that for the induced morphism of $\mathbf{C}$-cochain complexes $e=(t, e''\circ p)$ from $a[0]$ to $\mathsf{Cone}(v)[-1]$, the triple (e', e, e'') is a morphism of short exact sequences in $\mathsf{Ch}(\mathbf{C})$ from $(q[0], p[0])$ to the mapping cone short exact sequence $(I'\xrightarrow{q_{\mathsf{Cone}(v)}[-1]}\mathsf{Cone}(v)[-1], \mathsf{Cone}(v)[-1]\xrightarrow{p_{\mathsf{Cone}(v)}[-1]} I'')$. In particular, $a[0]\xrightarrow{e}\mathsf{Cone}(v)[-1]$ is an injective resolution.*

Proof. Since q is a monomorphism, and since $(I')^0$ is an injective object, associated to the $\mathbf{C}$-morphism e' from a' to $(I')^0$ there exists a $\mathbf{C}$-morphism t^1 from a to $(I')^0$ with $t^1\circ q$ equal to e'. Since e' is an injective resolution of a', also the translated brutal truncation gives an injective resolution, $(I')^0/e'(a')[0] \xrightarrow{d^1_{I'}} \sigma^{\geq 1}(I')[+1])$. Now t^1 induces a morphism from $a/q(a')$ to $(I')^0/e'(a')$, i.e., a $\mathbf{C}$-morphism u from a'' to $(I')^0/e'(a')$. Thus, by the previous lemma, there exists a morphism of $\mathsf{Ch}(\mathbf{C})$-cochain complexes, well-defined up to null homotopy, from the injective resolution I'' to the translated brutal truncation $\sigma^{\geq 1}(I')[+1]$. Since the brutal truncation is a subcomplex, this also defines a morphism u from I'' to $I'[+1]$. By construction, t is a null homotopy of $u \circ e'' \circ p$ from $a[0]$ to $I'[+1]$. Thus, there is an induced morphism of $\mathbf{C}$-cochain complexes, $e = (t, e'' \circ p)$, from $a[0]$ to $\mathsf{Cone}(u)[-1]$. Altogether, this defines a morphism of short exact sequences (e', e, e'') from $(q[0], p[0])$ to the mapping cone short exact sequence. In particular, since both e' and e'' are resolutions, by the long exact sequence of cohomology associated to the δ-functor H, also e is a resolution of a. Finally, since the terms of the mapping cone are, term-by-term, direct sums of the terms of I' and I'', and since a direct sum of two injective objects is an injective object, also e is an injective resolution of a. □

By passing to opposite categories, we get all of the analogous results for projective resolutions as well.

11 Derived functors

The slogan is that, if $\mathbf{C}$ has enough injective objects (so that we can form injective resolutions), then every object has an injective resolution that is unique up to homotopy equivalence, and for every short exact sequence of objects, there is an associated mapping cone short exact sequence of injective resolutions. Thus, if we modify our Abelian category $\mathbf{C}$ by first passing to the Abelian category $\mathsf{Ch}(\mathbf{C})$, then passing to the additive category $\mathsf{K}^+(\mathbf{C})$ (the full subcategory of $\mathsf{K}(\mathbf{C})$ whose objects are those complexes that are bounded below), and then passing to the additive category $\mathsf{D}^+(\mathbf{C})$ where we "localize" at all quasi-isomorphisms, then injective resolutions form a "functor" from $\mathbf{C}$ to $\mathsf{D}^+(\mathbf{C})$ that sends short exact sequences to mapping cone sequences.

There are set-theoretic issues when we localize a category at a non-small multiplicatively closed class of morphisms (the quasi-isomorphisms in this case), but these can be resolved. More seriously, the category $\mathsf{D}^+(\mathbf{C})$ is not typically an Abelian category. However, the mapping cone sequences gives it a structure called a *triangulated category*, and the cohomology object functors are still well-defined on this category (since quasi-isomorphisms preserve cohomology objects by definition). The injective resolutions "functor" sends short exact sequences to *distinguished triangles*, i.e., triangles that are quasi-isomorphic to mapping cone triangles. For every left exact additive functor $\mathbf{F}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to another Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{C}})$, there exists an associated *triangulated functor / exact functor* $R\mathbf{F}$ from $\mathsf{D}^+(\mathbf{C})$ to $\mathsf{D}^+(\mathbf{D})$ that satisfies an appropriate adjointness condition. All of this was worked out by Jean-Louis Verdier in his thesis (supervised by Grothendieck). This was generalized by a to more general context by Daniel Quillen (and many others, particularly Michel André).

Note that the cohomology objects of $R\mathbf{F}$ give a well-defined sequence of functors. Since the cohomology objects of complexes form a δ-functor, the cohomology objects of $R\mathbf{F}$ extend to a δ-functor. These equal the δ-functors that were originally studied by Saunders MacLane and Samuel Eilenberg (using a very different approach), and then investigated more fully by Cartan and Eilenberg. This approach was finally put in the form above by Grothendieck in his Tohoku paper. This δ-functor is enough of $R\mathbf{F}$ for many purposes, especially when combined with other techniques such as the Grothendieck spectral sequence.

Theorem 11.1 (Right Derived Functors)**.** *For every Abelian category $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ that has enough injective objects, for every left exact, additive functor $\mathbf{F}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to an Abelian category $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$, there exists a universal δ-functor $R^{\bullet}\mathbf{F}$ from $(\mathbf{C}, \mathsf{add}_{\mathbf{C}})$ to $(\mathbf{D}, \mathsf{add}_{\mathbf{D}})$ extending $R^0\mathbf{F} = \mathbf{F}$. Moreover, for every object a of $\mathbf{C}$ and for every injective resolution $a[0] \xrightarrow{e} I$, the $\mathbf{D}$-object $R^n\mathbf{F}(a)$ equals the cohomology object $H^n(\mathsf{Ch}(\mathbf{F})(I))$, and for every morphism (u,v) from an injective resolution $a[0] \xrightarrow{e} I$ to an injective resolution $\widetilde{a}[0] \xrightarrow{\widetilde{e}} \widetilde{I}$, the $\mathbf{D}$-morphism $R^n\mathbf{F}(u)$ from $R^n\mathbf{F}(a) = H^n(\mathsf{Ch}(\mathbf{F})(I))$ to $R^n\mathbf{F}(\widetilde{a}) = H^n(\mathsf{Ch}(\mathbf{F})(\widetilde{I}))$ equals $H^n(\mathsf{Ch}(\mathbf{F})(v))$.*

Proof. The statement of the theorem dictates the definition of $R^n\mathbf{F}$. It remains to prove that this is well-defined for each object a independent of the choice of injective resolution $a[0] \xrightarrow{e} I$, that this gives a δ-functor, and that this δ-functor is universal.

Since injective resolutions are unique up to homotopy equivalence, and since homotopy equivalence is preserved by $\mathsf{Ch}(\mathbf{F})$, also the $\mathbf{D}$-cochain complex $\mathsf{Ch}(\mathbf{F})(I)$ is well-defined up to homotopy equivalence. Since homotopy equivalences are quasi-isomorphisms, the cohomology objects of $\mathsf{Ch}(\mathbf{F})(I)$ are well-defined for each object a independent of the choice of injective resolution. Similarly, for every morphism u from a to $\widetilde{a}$, by the lemma there exists an induced morphism v from the injective resolution I to the injective resolution $\widetilde{I}$ such that $v \circ e$ equals $e' \circ u$. Moreover, v is well-defined up to null homotopy. Since $\mathsf{Ch}(\mathbf{F})$ preserves null homotopies, also $\mathsf{Ch}(\mathbf{F})(u)$ is well-defined up to null homotopy. Since null homotopies induce zero morphisms of cohomology objects, the induced morphisms $H^n(\mathsf{Ch}(\mathbf{F})(u))$ are well-defined in terms of v, independent of the choice of u. Altogether, the functors $R^n\mathbf{F}$ are well-defined.

For every $\mathbf{C}$-short exact sequence, by the Horseshoe Lemma, there is an associated mapping cone short exact sequence of injective resolutions. Since $\mathsf{Ch}(\mathbf{F})$ preserves mapping cones, this gives a mapping cone short exact sequence in $\mathsf{Ch}(\mathbf{D})$, in fact even in the Serre subcategory $\mathsf{Ch}^{\geq 0}(\mathbf{D})$. Since H is a δ-functor concentrated in degrees ≥ 0 on $\mathsf{Ch}^{\geq 0}(\mathbf{D})$ (even a universal δ-functor concentrated in degrees ≥ 0), we get well-defined connecting maps $\delta^n_{R\mathbf{F}}$ that extend the sequence $(R^n\mathbf{F})_{n\in\mathbb{Z}_{\geq 0}}$ to a δ-functor concentrated in degrees ≥ 0.

Finally, by construction, for every injective object I^0 of $\mathbf{C}$, we can choose the injective resolution to be $I^0[0] \xrightarrow{\mathrm{Id}} I^0[0]$, so that $R^n\mathbf{F}(I^0)$ is zero for every strictly positive integer n. Since there are enough injective objects in $\mathbf{C}$, for every object a, there exists a monomorphism $a \xrightarrow{q} I^0$. For every strictly positive integer n, the induced morphism $R^n\mathbf{F}(q)$ from $R^n\mathbf{F}(a)$ to $R^n\mathbf{F}(I^0)$ is a zero morphism since $R^n\mathbf{F}(I^0)$ is a zero object. Thus, by Grothendieck's criterion, the δ-functor $R^\bullet\mathbf{F}$ concentrated in degrees ≥ 0 is universal. $\square$

Of course we can develop all of this instead using projective resolutions and right exact functors. In fact this follows formally for the development above by passing to opposite categories.

Theorem 11.2 (Left Derived Functors). *For every Abelian category $(\mathbf{C}, \mathit{add}_\mathbf{C})$ that has enough projective objects, for every right exact, additive functor $\mathbf{F}$ from $(\mathbf{C}, \mathit{add}_\mathbf{C})$ to an Abelian category $(\mathbf{D}, \mathit{add}_\mathbf{D})$, there exists a universal δ-functor $L_\bullet\mathbf{F}$ concentrated in cohomological degrees ≤ 0, hence in homological degrees ≥ 0, from $(\mathbf{C}, \mathit{add}_\mathbf{C})$ to $(\mathbf{D}, \mathit{add}_\mathbf{D})$ extending $L_0\mathbf{F} = \mathbf{F}$. Moreover, for every object a of $\mathbf{C}$ and for every projective resolution $P \xrightarrow{\pi} a[0]$, the $\mathbf{D}$-object $L_n\mathbf{F}(a)$ equals the cohomology object $H^{-n}(\mathit{Ch}(\mathbf{F})(P))$, and for every morphism (u,v) from a projective resolution $P \xrightarrow{\pi} a[0]$ to an projective resolution $\widetilde{P} \xrightarrow{\widetilde{\pi}} \widetilde{a}[0]$, the $\mathbf{D}$-morphism $L_n\mathbf{F}(u)$ from $L_n\mathbf{F}(a) = H^{-n}(\mathit{Ch}(\mathbf{F})(P))$ to $L_n\mathbf{F}(\widetilde{a}) = H^{-n}(\mathit{Ch}(\mathbf{F})(\widetilde{P}))$ equals $H^{-n}(\mathit{Ch}(\mathbf{F})(v))$.*

It is crucial to recognize that, both in the early investigations by Eilenberg-MacLane, the subsequent study by Cartan-Eilenberg, and especially in the applications by Grothendieck, constantly it is necessary to prove existence and universality of δ-functors for Abelian categories that do not have enough injective objects or do not have enough projective objects (or where the categories do have enough such objects, but these are unsuitable for proving certain properties of the δ-functors). In most such settings, we can still use Grothendieck's criterion. For some applications, the best solution is to instead pass to triangulated categories and the *total derived functors*, since these exist under weaker hypotheses. Moreover, many theorems in the past fifty years have shown that the derived category itself, with its many structures (that we have barely touched on), is an important object in all areas of mathematics.