MAT 535 Notes on Logic, Sets, and Classes

1 Propositional calculus

Category theory is most often formalized using classes. Classes can be formalized as a second-order theory using the first-order theory of predicate calculus and Zermelo – Fraenkel set theory. The zeroth-order theory is propositional calculus.

1.1 Formal languages

Every formal language has an **alphabet**, A. In our case, A is a nonempty finite set. The **Kleene star** (or **Kleene closure**), A^*, is the set of all **strings** of elements of A, i.e., the elements of A^* are those ordered pairs whose second entry is a nonnegative integer n, the **length** of the string, and whose first entry is itself an ordered n-tuple of elements of A, sometimes called a **literal** (the empty set is the unique 0-tuple). Thus, the second projection is a function from A^* to the set $\mathbb{Z}_{\geq 0}$ whose fiber over each n is the set A^n.

Every formal language also has a specified subset of A^* whose elements are called **well-formed formulas**. In formalizing mathematics, a **formal language** is usually defined to be an ordered pair whose first entry is an alphabet A and whose second entry is this subset of A^*. Most often this subset is specified by a subset of **atomic strings** and a collection of **production rules** for producing new well-formed formulas from existing well-formed formulas. The well-formed formulas are all strings obtained by iteratively applying the production rules to the atomic strings. **Automata theory** is the mathematical study of such formal languages via the Chomsky hierarchy, the Chomsky-Schützenberger theorem, etc.

For every finite nonempty set A with a specified total order, the set A^* with the lexicographic order is equivalent to the set of nonnegative integers with its total order. Thus, any two such alphabets are equally expressive, and the study of formal languages is equivalent to the study of subsets of the set of nonnegative integers via automata. Here our interest is not in the study of all formal languages, but rather in the small number of formal languages relevant to formalize sets, classes, and categories.

1.2 The formal language of propositional calculus

In the formal language for propositional logic, the alphabet includes one symbol for the propositional variable, say "p", as well as symbols for the usual logical connectives (we use the pipe to separate items in a list).

Alphabet for propositional calculus.

$$p \mid \Rightarrow \mid \neg \mid (\mid) \mid \top \mid \bot \mid \wedge \mid \vee \mid \Leftarrow \mid \Leftrightarrow$$

The symbol "$\top$" represents "true," and the symbol "$\bot$" represents "false." For each positive integer n, a consecutive substring of n entries of p, where the symbol p is neither directly preceding nor succeeding the substring, is interpreted as a propositional variable p_n. Thus, the alphabet expresses denumerably many propositional variables.

A string in propositional logic is a well-formed formula if and only if it can be obtained, starting from $\top$, $\bot$ or the propositional variables p_n for all positive integers n, by iterated application of the following production rules. For all well-formed formulas f and g, also the following are well-formed formulas.

Traditional form of production rules of propositional calculus.

$$(f) \Rightarrow (g) \mid \neg(f) \mid (f) \wedge (g) \mid (f) \vee (g) \mid (f) \Leftarrow (g) \mid (f) \Leftrightarrow (g)$$

In the language of a context-free grammar, all of the above symbols of the alphabet are **terminal symbols**, and we introduce two new **nonterminal symbols**: a symbol "B" for "begin" and a symbol "P" (for producing the propositional variables). The automaton begins with the length-1 string with literal "B" and then performs any of the following substitutions iteratively (in any order) until it reaches an output string consisting of only terminal symbols (the symbols other than "B" and "P").

Context-free grammar of propositional calculus.

$$B \to P \mid (B) \Rightarrow (B) \mid \neg(B) \mid \top \mid \bot \mid (B) \vee (B) \mid (B) \wedge (B) \mid (B) \Leftarrow (B) \mid (B) \Leftrightarrow (B)$$

$$P \to p \mid pP$$

The well-formed formulas are all output strings. The set of these is denoted $\mathcal{L}_{\mathsf{Prop}}$ or $\mathcal{L}$.

1.3 Deductive system of propositional calculus

This formal language becomes a **Hilbert system** by introducing a second list of production rules – called **axioms** (if they have arity 0) and **inference rules** (if they have arity > 0). One common Hilbert system, the Łukasieweicz system, is obtained by first adopting *modus ponens*, i.e., the production rule that associates to each pair of well-formed formulas of the form f and $(f) \Rightarrow (g)$ the well-formed formula g. In symbols, this inference rule is as follows.

Modus Ponens for f and g. $f, (f) \Rightarrow (g) \vdash g$

We also have three additional axiom schemata for the Łukasieweicz system, where f, g and h are substituted with all triples of well-formed formulas.

Ł1 for f and g. $\vdash (f) \Rightarrow ((g) \Rightarrow (f))$

Ł2 for f, g and h. $\vdash ((f) \Rightarrow ((g) \Rightarrow (h))) \Rightarrow (((f) \Rightarrow (g)) \Rightarrow ((f) \Rightarrow (h)))$

Ł3 for f and g. $\vdash ((\neg(f)) \Rightarrow (\neg(g))) \Rightarrow ((g) \Rightarrow (f))$

Since we are also using other logical connectives than just $\Rightarrow$ and $\neg$, we add as axioms the definitions of those logical connectives in terms of $\Rightarrow$ and $\neg$.

Conjunction. $\vdash \neg((f) \Rightarrow (\neg(g))) \Rightarrow ((f) \wedge (g))$

$\vdash ((f) \wedge (g)) \Rightarrow \neg((f) \Rightarrow (\neg(g)))$

Disjunction. $\vdash ((\neg(f)) \Rightarrow (g)) \Rightarrow ((f) \vee (g))$

$\vdash ((f) \vee (g)) \Rightarrow ((\neg(f)) \Rightarrow (g))$

Reverse Implication. $\vdash ((f) \Rightarrow (g)) \Rightarrow ((g) \Leftarrow (f))$,

$\vdash ((g) \Leftarrow (f)) \Rightarrow ((f) \Rightarrow (g))$

Logical Equivalence. $\vdash (((f) \Rightarrow (g)) \wedge ((g) \Rightarrow (f))) \Rightarrow ((f) \Leftrightarrow (g))$

$\vdash ((f) \Leftrightarrow (g)) \Rightarrow (((f) \Rightarrow (g)) \wedge ((g) \Rightarrow (f)))$

A **theorem** of this Hilbert system is a well-formed formula obtained by iteratively applying modus ponens beginning with the axioms above. For instance one algorithmic scheme producing such iterative proofs gives the *Deduction Theorem*: for every finite collection $\Gamma = \{g_1, \ldots, g_n\}$ of well-formed formulas, for every well-formed formula f, if there is a finite sequence of applications of the inference rules to the well-formed formulas in Γ and to the Łukasieweicz axioms that leads to a proof of the well-formed formula f, then we also have a finite sequence of applications of the inference rules to the Łukasieweicz axioms that leads to a proof of the well-formed formula

$$(g_1 \wedge (g_2 \wedge (\dots (g_{n-1} \wedge g_n)\dots))) \Rightarrow f.$$

Conversely, Modus Ponens applied to Γ and this well-formed formula gives f, so that the Deduction Theorem becomes an "if and only if" statement.

The **theory** of the Łukasieweicz deductive system is the set of all theorems, denoted $\mathcal{T}_{\mathsf{Prop}}$ or $\mathcal{T}$. For every set Γ of well-formed formulas, for every well-formed formula f, we write $\Gamma \vdash f$ if we can deduce f from Γ using the inference rules. In this case we say that f **derives** from Γ. We denote by $\mathcal{T}_{\mathsf{Prop},\Gamma}$ or $\mathcal{T}_\Gamma$ the set of all well-formed formulas that derive from Γ. By Post's Completeness Theorem, a well-formed formula f derives from a set Γ of well-formed formulas if and only if Γ *entails* f, i.e., f is valid in every model that makes Γ valid.

1.4 Semantics of propositional calculus

As explained above, the set of propositional variables is naturally bijective to the set of positive integers via $n \mapsto p_n$. A **model** of propositional calculus is (specified by) a subset S of $\mathbb{Z}_{\geq 1}$. Thus, the set of all models is identified with the power set $\mathcal{P}(\mathbb{Z}_{\geq 1})$. Associated to each subset S of $\mathbb{Z}_{\geq 1}$ is the **indicator function** 1_S: the unique function from $\mathbb{Z}_{\geq 1}$ to the binary set $\mathbb{B} = \{\top, \bot\}$ such that the fiber of 1_S over $\top$ equals S.

By Tarski's recursive scheme, each function 1_S extends uniquely to a **valuation function** α_S from the set $\mathcal{L}$ of all propositions to $\mathbb{B}$ satisfying all of the following.

(i) Both $\alpha_S(\top)$ equals $\top$, and $\alpha_S(\bot)$ equals $\bot$.

(ii) For every element n of $\mathbb{Z}_{\geq 1}$, the value $\alpha_S(p_n)$ equals $\top$ if and only if n is an element of A.

(iii) For every proposition f, the value $\alpha_S(\neg(f))$ equals $\bot$ if and only if $\alpha_S(f)$ equals $\top$.

(iv) For all propositions f and g, the value $\alpha_S((f) \Rightarrow (g))$ equals $\bot$ if and only if both $\alpha_S(f)$ equals $\top$ and $\alpha_S(g)$ equals $\bot$.

Since $\neg$ and $\Rightarrow$ are a functionally complete system of connectives, these rules alone force the rest of the recursive scheme.

(v) For all propositions f and g, the value $\alpha_S((f) \wedge (g))$ equals $\top$ if and only if both $\alpha_S(f)$ equals $\top$ and $\alpha_S(g)$ equals $\top$.

(vi) For all propositions f and g, the value $\alpha_S((f) \vee (g))$ equals $\perp$ if and only if both $\alpha_S(f)$ equals $\perp$ and $\alpha_S(g)$ equals $\perp$.

(vii) For all propositions f and g, the value $\alpha_S((f) \Leftarrow (g))$ equals $\perp$ if and only if both $\alpha_S(f)$ equals $\perp$ and $\alpha_S(g)$ equals $\top$.

(vii) For all propositions f and g, the value $\alpha_S((f) \Leftrightarrow (g))$ equals $\top$ if and only if $\alpha_S(f)$ equals $\alpha_S(g)$.

For a given model S, for every well-formed formula f, we write $\vDash_S f$ if (and only if) $\alpha_S(f)$ equals $\top$. In this case, f is **valid** in S; also f is a **validity** of S. More generally, for a set Σ of models, we write $\vDash_\Sigma f$ if, for every element S of Σ, we have $\vDash_S f$; then f is a **validity** of Σ. In particular, we write $\vDash f$ if (and only if), for every model S we have $\vDash_S f$, and then we say f is a **validity** (of propositional calculus). We denote the set of all validities by $\mathcal{V}_{\mathsf{Prop}}$ or $\mathcal{V}$. More generally, for every set Γ of well-formed formulas, we write $\vDash_S \Gamma$, respectively $\vDash \Gamma$, if (and only if), for every element g of Γ we have $\vDash_S g$, resp. $\vDash g$. Similarly, for every set Γ of well-formed formulas, for every well-formed formula f, we write $\Gamma \vDash f$ if (and only if), for every model S such that $\vDash_S \Gamma$, also $\vDash_S f$. This is called **entailment**. We denote the set of all well-formed formulas entailed by Γ as $\mathcal{V}_{\mathsf{Prop},\Gamma}$ or $\mathcal{V}_\Gamma$. Please note that the deductive system above is **sound**: for every set Γ of well-formed formulas, for every well-formed formula f that can be deduced from Γ using the axioms and inference rules, we also have Γ entails f, i.e., $\Gamma \vDash f$. Post's Completeness Theorem is the converse, i.e., $\mathcal{T}_\Gamma$ equals $\mathcal{V}_\Gamma$ for propositional calculus.

2 Predicate calculus

The alphabet of predicate calculus appends a few symbols to the alphabet of propositional calculus.

Alphabet for predicate calculus.

$$p \mid \Rightarrow \mid \neg \mid (\mid) \mid \top \mid \perp \mid \wedge \mid \vee \mid \Leftarrow \mid \Leftrightarrow \mid$$
$$t \mid , \mid = \mid \forall \mid \exists$$

The symbol t produces **term variables**. As with the propositional variables, this allows to express denumerably many term variables: for each positive integer ℓ, the string t_ℓ is ℓ consecutive instances of t neither immediately preceded nor succeeded by t. Thus, the set $\{t_\ell\}_{\ell \in \mathbb{Z}_{\geq 1}}$ of all term variables can be identified with $\mathbb{Z}_{\geq 1}$. The alphabet also includes a symbol – the comma "," – for separating term variables in a list (this is the reason we prefer the pipe to separate items in our metalanguage). Every well-formed formula f, or **predicate**, in predicate calculus has a specified finite set $\text{Free}(f)$ of **free variables** whose cardinality is a nonnegative integer called the **arity**. It also has a specified finite set $\text{Bound}(f)$ of **bound variables**. The set $\text{Var}(f)$ of all variables of f is the union $\text{Free}(f) \sqcup \text{Bound}(f)$. For each positive integer n, for each each nonnegative integer m, for each ordered m-tuple of positive integers $(\ell_1, \ldots, \ell_m)$, we interpret the string $p_n(t_{\ell_1}, \ldots, t_{\ell_m})$ of total length $(1+n)+(1+\ell_1)+\cdots+(1+\ell_m)$ as an **atomic predicate** denoted $p_{n,m}(t_{\ell_1}, \ldots, t_{\ell_m})$ whose set of free variables is $\{t_{\ell_1}, \ldots, t_{\ell_m}\}$ and whose arity equals the cardinality of this set (an integer from 1 to m). The set of bound variables is empty. Thus, for each nonnegative integer m, the alphabet expresses denumerably many arity-m predicate variables, $p_{n,m}$ for each positive integer n, into which we may insert any ordered m-tuple of term variables, and then the set of free variables equals the set of that m-tuple. By convention, we identify the propositional variables p_n of propositional calculus with the arity-0 predicate variables $p_{n,0}$, i.e., p_n succeeded by no term variables in parentheses.

The alphabet also includes a symbol – "=" – for equality. This behaves as a predicate variable of arity 2, but written in infix notation. For all positive integers ℓ and m, the string $(t_\ell = t_m)$ is an atomic predicate whose list of free variables is $\{t_\ell, t_m\}$ of arity equal to the cardinality of $\{\ell, m\}$ (2 unless $\ell = m$, in which case 1) and with empty set of bound variables.

To avoid lengthy discussion of disambiguating predicates where bound variables of subpredicates "collide" with variables of a different subpredicate, we formulate the production rules only in cases where this does not happen. The main production rule allowing this is **term substitution**. For every predicate f of arity $m \geq 1$, for every term variable t_ℓ, for every term variable t_m that is not one of the bound variables of f, the term substitution $f[t_m/t_\ell]$ replaces each instance of the substring t_ℓ in f with the string t_m.

Term substitution. $f \to f[t_m/t_\ell]$ for $t_m \notin \text{Bound}(f)$

In particular, we can use this to replace a bound variable t_ℓ of f with some term variable t_m with m arbitrarily positive (to avoid the variables of other expressions that we combine with f).

There are also quantifier symbols: "∀" and "∃". Each quantifier in a predicate is immediately succeeded by a term variable t_ℓ followed by a predicate in parentheses f where t_ℓ is not already one of the bound variables of f.

Universal quantifiers. $f \to \forall\ t_m\ (f)$ for $t_m \notin \text{Bound}(f)$

Existential quantifiers. $f \to \exists\ t_m\ (f)$ for $t_m \notin \text{Bound}(f)$

For these new quantified predicates, the set of free variables is $\text{Free}(f) \setminus \{t_m\}$, and the set of bound variables is $\text{Bound}(f) \sqcup \{t_m\}$. Thus, the arity goes down by 1 (if t_m is a free variable of f) or by 0 (if t_m is not a free variable of f).

We also allow the traditional production rules of propositional calculus, but with the proviso that when combining two predicates via a logical connective we require that the set of bound variables of each predicate is disjoint from the set of variables (both free and bound) of the other predicate. Because we have a denumerable set of term variables, by first applying term substitutions to any "colliding" bound variables in the two predicates, we can always achieve this (and such term substitutions will not affect the deductive or semantic meaning of the predicates). For such a combined predicate, the set of free variables is the union of the sets of free variables of each component predicate, and the set of bound variables is the (disjoint) union of the sets of bound variables of each component predicate.

The predicates are all output strings beginning with the atomic predicates and iteratively applying the production rules. The set of all predicates is denoted $\mathcal{L}_{\mathsf{Pred}}$ or $\mathcal{L}$. There is a context-free grammar producing $\mathcal{L}_{\mathsf{Pred}}$, but we do not record it here.

2.1 Deductive system of predicate calculus

As with propositional calculus, the formal language of predicate calculus becomes a Hilbert system by introducing a second list of production rules: the axioms and inference rules of predicate calculus. Beginning from the axioms, iterated application of the inference rules produces all theorems of predicate calculus. The set of all theorems is denoted $\mathcal{T}_{\mathsf{Pred}}$ or $\mathcal{T}$.

The inference rule of modus ponens and the axioms of propositional calculus are part of the deductive system of predicate calculus, but only in those cases where the strings involved are predicates, i.e., only when no bound variable of a component subpredicate collides with a variable of some other subpredicate. We can always achieve this using an axiom schemata of bound substitution.

For every predicate f, for every bound variable t_ℓ of f, for every term variable t_m that is not a variable of f, we have the following axiom.

Bound variable substitution. $\vdash (f) \Rightarrow (f[t_m/t_\ell])$ for $t_\ell \in \mathrm{Bound}(f)$ and $t_m \notin \mathrm{Var}(f)$.

Axioms of Equality. For all positive integers ℓ, m and n, we have the following axioms.

Reflexivity. $t_\ell = t_\ell$

Symmetry. $(t_\ell = t_m) \Rightarrow (t_m = t_\ell)$

Transitivity. $((t_\ell = t_m) \wedge (t_m = t_n)) \Rightarrow (t_\ell = t_n)$

Also, for every predicate f and for all term variables t_ℓ, and t_m that are not bound variables of f, we add the following axiom.

Substitution. $(t_\ell = t_m) \Rightarrow (f \Rightarrow f[t_m/t_\ell])$ for $t_\ell, t_m \notin \mathrm{Bound}(f)$.

Deduction for quantifiers. The deductive system for predicate calculus is the weakest deductive system (allowing the fewest axioms and inference rules) that includes the rules above and that satisfies the following conditions regarding quantifiers.

Universal Generalization. For every positive integer ℓ and for all predicates f and g such that t_ℓ is not a bound variable of f and is not a variable of g (neither bound nor free), if g proves f, then also g proves $\forall t_\ell\ (f)$.

$$\textbf{UG.}\quad \text{If } g \vdash f \text{ then } g \vdash \forall t_\ell\ (f) \text{ for } t_\ell \notin \text{Bound}(f) \cup \text{Var}(g).$$

Universal Instantiation. For all positive integers ℓ and m, and for every predicate f such that t_ℓ and t_m are not bound variables of f, we have the following axiom.

$$\textbf{UI.}\quad (\forall t_\ell\ (f)) \Rightarrow (f[t_m/t_\ell]) \text{ for } t_\ell, t_m \notin \text{Bound}(f).$$

Existential Generalization. For all positive integers ℓ and m, and for every predicate f such that t_ℓ and t_m are not bound variables of f, we have the following axiom.

$$\textbf{EG.}\quad (f[t_m/t_\ell]) \Rightarrow (\exists t_\ell\ (f)) \text{ for } t_\ell, t_m \notin \text{Bound}(f).$$

Existential Instantiation. For every positive integer ℓ, and for all predicates f and g such that t_ℓ is not a bound variable of f and is not a variable of g (neither bound nor free), if g can be derived from f, then also g can be derived from $\exists t_\ell\ (f)$.

$$\textbf{EI.}\quad \text{If } f \vdash g \text{ then } \exists t_\ell\ (f) \vdash g \text{ for } t_\ell \notin \text{Bound}(f) \cup \text{Var}(g).$$

As with propositional calculus, each predicate that can be deduced from the axioms via iterated application of the inference rules is a **theorem**. The set of all theorems is denoted $\mathcal{T}_{\mathsf{Pred}}$, or just $\mathcal{T}$ when confusion is unlikely. Similarly, for every set Γ of predicates, we denote $\mathcal{T}_{\mathsf{Pred},\Gamma}$, or just $\mathcal{T}_\Gamma$, is the set of all predicates that derive from Γ. By Gödel's Completeness Theorem, a predicate f derives from a set Γ of predicates if and only if Γ entails f, i.e., f is valid in every model that makes Γ valid.

2.2 Semantics of predicate calculus

The semantics of predicate calculus uses a theory of sets. A **model** S of predicate calculus consists of an ordered pair (A, π) of a nonempty set A, called a **universe**, together with an assignment $\pi = (\pi_{n,m})_{n,m}$ for each nonnegative integer n and each positive integer m of a function $\pi_{n,m}$ from

A^n to the binary set $\mathbb{B} = \{\top, \bot\}$. For determining validity, it will suffice to restrict A to finite sets, say $\{1, \dots, k\}$ as k varies among all nonnegative integers. For this model, there is a **valuation function** α_S that assigns to every predicate f of arity n and free variable set $(t_{\ell_1}, \dots, t_{\ell_n})$ for $\ell_1 < \dots < \ell_n$ a function $\alpha_S(f)$ from A^n to $\mathbb{B}$. The valuation function assigns to each atomic predicate $p_n(t_{\ell_1}, \dots, t_{\ell_n})$ the function $\pi_{n,m}$. The valuation assigns to $t_\ell = t_\ell$ the constant function on A with value $\top$. The valuation assigns to $t_\ell = t_m$ with $\ell \neq m$ the function on A^2 whose fiber over $\top$ is the diagonal.

Under term variable substitution, the valuation changes by substitution of variables on A^n (with appropriate shuffling of components if the order of the variables changes). For every predicate f of arity $n \geq 1$ and free variables $\{t_{\ell_1}, \dots, t_{\ell_{n-1}}, t_{\ell_n}\}$, the valuation of $\forall t_n\ (f)$ is the function on A^{n-1} whose value on each input $(a_1, \dots, a_{n-1})$ is $\bot$ precisely if there exists a_n such that the value of $\alpha_S(f)$ on $(a_1, \dots, a_{n-1}, a_n)$ is $\bot$. Similarly, the valuation of $\exists t_n\ (f)$ is the function on A^{n-1} whose value on each input $(a_1, \dots, a_{n-1})$ is $\top$ precisely if there exists a_n such that the value of $\alpha_S(f)$ on $(a_1, \dots, a_{n-1}, a_n)$ is $\top$. Similarly, for each term variable t_m that is not a variable of f, so that the free variable set of f equals the free variable set of $\forall t_m\ (f)$ and $\exists t_m\ (f)$, then valuation of f equals the valuation of both $\forall t_m\ (f)$ and $\exists t_m\ (f)$. Together with Tarski's recursive scheme, this uniquely determines the valuation of every predicate. Of course the valuation of f gives a $\mathbb{B}$-valued function on the set A^ω of countable sequences of elements of A via projection to A^n through the coordinates of $\ell_1 < \dots < \ell_n$, and then the valuation is the unique morphism of Boolean algebras that maps $\forall t_m$ to minimimum over t_m (thinking of $\top$ as 1 and $\bot$ as 0) and maps $\exists t_m$ to maximum over t_m.

In particular, the valuation of each predicate f of arity 0 is equivalent to an element of $\mathbb{B}$. A predicate of arity 0 is a **closed formula**. For a given model S, for every closed formula f, we write $\vDash_S f$ if (and only if) $\alpha_S(f)$ equals $\top$. In this case, f is **valid** in S; also f is a **validity** of S. More generally, for a set Σ of models, we write $\vDash_\Sigma f$ if, for every element S of Σ, we have $\vDash_S f$; then f is a **validity** of Σ. In particular, we write $\vDash f$ if (and only if), for every model S we have $\vDash_S f$, and then we say f is a **validity** (of predicate calculus). Please note, if the number of variables of f equals k, then it suffices to consider only models where A equals $\{1, \dots, k\}$, and we only need to refer to the functions $\pi_{n,m}$ for $m, n \leq k$ (so validation is inherently finite and computable). We denote the set of all validities by $\mathcal{V}_{\mathsf{Pred}}$ or $\mathcal{V}$. More generally, for every set Γ of closed formulae, we write $\vDash_S \Gamma$, respectively $\vDash \Gamma$, if (and only if), for every element g of Γ we have $\vDash_S g$, resp. $\vDash g$. Similarly, for every set Γ of closed formulae, for every closed formula f, we write $\Gamma \vDash f$ if (and only if), for every model S such that $\vDash_S \Gamma$, also $\vDash_S f$. This is called **entailment**. We denote the set of all well-formed formulas entailed by Γ as $\mathcal{V}_{\mathsf{Pred},\Gamma}$ or just $\mathcal{V}_\Gamma$. The deductive system above is **sound**: for every set Γ of closed formulas, for every closed formula f that can be deduced from Γ using the axioms and inference rules, we also have Γ entails f, i.e., $\Gamma \vDash f$. Gödel's Completeness Theorem is the converse, i.e., $\mathcal{T}_\Gamma$ equals $\mathcal{V}_\Gamma$ for predicate calculus.

3 Zermelo-Fraenkel axioms

The only additional symbol in Zermelo-Fraenkel set theory is an arity-2 predicate written in infix notation, $x \in y$, read "x is an element of y" or "y contains x as an element." Of course we could identify this predicate with the atomic predicate $p_{2,1}(x, y)$ in outfix notation to jibe with the previous section, but we trust no confusion will arise from the traditional infix notation. Adding this predicate, the production rules produce the Zermelo – Fraenkel predicates. To the axioms and inference rules of predicate calculus, we also add the following axioms of Zermelo – Fraenkel set theory (which can be recursively enumerated). Also we relax our conventions about naming of term variables and atomic predicates so as to write the axioms in their conventional formulation.

Axiom 3.1 (Axiom of Extensionality)**.** For every set a and for every set b, the set a equals the set b if and only if, for every set x, the set x is an element of a if and only if x is an element of b.

$$\forall a \ (\forall b \ (\forall x \ ((x \in a) \Leftrightarrow (x \in b))) \Leftrightarrow (a = b))$$

Axiom 3.2 (Axiom of Regularity)**.** For every set a such that there exists a set x that is an element of a, there exists an element y of a such that every element of y is not an element of a.

$$\forall a \ ((\exists x \ (x \in a)) \Rightarrow (\exists y \ (y \in a) \land (\forall z \ (z \in y) \Rightarrow \neg(z \in a))))$$

Together with the other axioms, the axiom of regularity implies a strong form of foundation: there does not exist a sequence of sets $(a_n)_{n \in \mathbb{Z}_{\geq 0}}$ such that for every element n of $\mathbb{Z}_{\geq 0}$ the set a_{n+1} is an element of the set a_n (every formalization of this requires first formalizing natural numbers).

The next axiom is sometimes also called the "Axiom of Separation." It is an axiom schema: there is one axiom for each predicate $f(s, t)$ in the first-order language of set theory together with an ordered pair (s, t) of (all of) the free variables of the predicate (and an arbitrary set of bound variables that includes neither s nor t).

Axiom 3.3 (Axiom Schema of Specification)**.** For every set b, for every set c, there exists a set a whose elements are precisely those elements x of b such that the predicate $f(c, x)$ is true.

$$\forall b \ \forall c \ \exists a (\forall x \ ((x \in a) \Leftrightarrow ((x \in b) \land p(c, x)))$$

In particular, assuming that the universe of sets has at least one member (which we do assume), for the predicate $p(s, t)$ of s equals s and t does **not** equal t, for each set $a = \varnothing$ produced by the axiom (for any set b and for any set c), for every set x, the set x is **not** an element of $\varnothing$. The Axiom of Extensionality guarantees that this **empty set** is unique; we denote this set by $\varnothing$. So (together with the tacit axiom that the universe of sets has at least one member), the Axiom Schema of Specification gives the existence of a unique empty set.

Please note, we certainly do need to guard the quantifier of x in the Axiom Schema of Specification, restricting x to an element of the specified set b, to avoid asserting that there exists a "set whose elements are all sets that do not include themselves as an element" (which leads to Russell's Paradox about whether the set is an element of itself). Also note, we do not claim that we can recover the predicate $p(s,t)$ from the subset of b. For one thing, different predicates can be logically equivalent, so the best we could hope for is to recover the truth-valued function whose domain equals b determined by the predicate. A subset a of b is equivalent to such a truth-valued function, and every such subset arises from substitution of a for s in the specific predicate p: $t \in s$. So this axiom schema is producing "every" subset that it should. Even though predicates are specified via a finite string of symbols from an (at most) countable alphabet, this certainly does not imply that we have (at most) countably many distinct subsets of b (in a given model of Zermelo – Fraenkel set theory), since the subset c can range freely. As Cantor proved, for every set b, there does not exist a surjective function from b to the set of all subsets of b.

Axiom 3.4 (Axiom of Pairing)**.** For every set a, for every set b, there exists a set $\{a,b\}$ whose elements are precisely a and b.

$$\forall a\ \forall b\ \exists c\ \forall x\ ((x \in c) \Leftrightarrow ((x = a) \vee (x = b)))$$

Please note, for every set a and for every set b, the set $\{a,b\}$ equals the singleton set $\{a\}$ if (and only if) b equals a. Thus, this axiom also gives the existence of the Kuratowski ordered pair, $(a,b) := \{\{a\},\{a,b\}\}$, by applying the axiom to the singleton set $\{a\}$ and the doubleton set $\{a,b\}$. By the Axiom of Extensionality, for every set a, for every set b, for every set a', for every set b', the ordered pair (a,b) equals the ordered pair (a',b') if and only if both a equals a' and b equals b'.

Axiom 3.5 (Axiom of Union)**.** For every set a, there exists a set b such that, for every set x, the set x is an element of b if and only if there exists an element y of a such that x is an element of y.

$$\forall a\ \exists b\ \forall x\ ((x \in b) \Leftrightarrow (\exists y\ ((x \in y) \wedge (y \in a))))$$

By the Axiom of Extensionality, the *union set* produced by this axiom is unique. In particular, for every set a, for every set b, the Axiom of Union applied to the set $\{a,b\}$ guarantees the existence of a set, denoted $a \cup b$, such that, for every set x, the set x is an element of $a \cup b$ if (and only if) either x is an element of a or x is an element of b (or both).

Similar to the Axiom Schema of Specification, the next axiom schema has one axiom for each predicate $f(x,b,y)$ in the first-order language of set theory together with an ordered triple (x,b,y) of (all of) the free variables of the predicate.

Axiom 3.6 (Axiom Schema of Replacement)**.** For every set b and for every set d such that, for every element x of d there exists a unique set y satisfying $f(x,b,y)$, then there exists a set c whose elements are precisely those sets y such that there exists an element x of d such that $f(x,b,y)$ holds.

$$\forall b\ \forall d\ ((\forall x\ ((x \in d) \Rightarrow (\exists y\ p(x,b,y)) \wedge (\forall z\ \forall w((p(x,b,z) \wedge p(x,b,w)) \Rightarrow (y = z)))) \Rightarrow$$

$$(\exists c\ \forall y'\ ((y' \in c) \Leftrightarrow (\exists x\ (x \in d) \wedge p(x,b,y')))))$$

Consider the predicate f with an ordered triple of free variables (x,b,y): the set y equals (x,b), i.e., y equals $\{\{x\},\{x,b\}\}$. By the Axiom of Pairing, for every set a, for every set b, and for every element x of a, there exists a unique set y satisfying the predicate $p(x,b,y)$. Thus, the Axiom Schema of Replacement guarantees the existence of a set, denoted $a \times \{b\}$, such that for every set y, the set y is an element of $a \times \{b\}$ if (and only if) there exists an element x of a such that y equals (x,b). Moreover, by the Axiom of Extensionality, this set $a \times \{b\}$ is unique.

Next, consider the predicate f' with an ordered triple of free variables (x',b',y'): y' equals $b' \times \{x'\}$. By the previous paragraph, for every set a', for every set a, and for every element x of a', there exists a unique set $a \times \{x\}$ satisfying the predicate $f'(x',a,y')$. Thus, the Axiom Schema of Replacement and the Axiom of Union guarantees the existence of a set, denoted $a \times a'$, such that for every set x'', the set x'' is an element of $a \times a'$ if (and only if) there exists an element x of a and there exists an element x' of a' such that x'' equals (a,a'). Therefore, for every set a and for every set a', the Axiom Schema of Replacement (together with the earlier axioms) guarantees the existence of a Cartesian product set $a \times a'$. By the Axiom of Extensionality, the Cartesian product set $a \times a'$ is unique.

This, finally, leads to the essential meaning of the Axiom Schema of Replacement. For every ordered triple $(b,d,g(x,z,y))$ of a set b, of a domain set d, and of a "function" predicate $g(x,z,y)$ for (b,d), i.e., such that for every element x of d there exists a unique set y such that $g(x,b,y)$ holds, there exists an *image set* c for $(b,d,g(x,z,y))$, and also there exists a Cartesian product set $d \times c$. Finally, by the Axiom Schema of Specification, the predicate $g(x,z,y)$ and the set b (substituted for z) determines a subset $\text{graph}(g(x,b,y))$ of $d \times c$ that equals the graph of a unique set function from d onto c. Therefore, for every domain set d, for every "parameter" set b, and for every predicate $g(x,b,y)$ that determines a function in the "traditional" sense on the domain set d, there exists a unique image set $c = \mathrm{cod}_{d,b,g}$ and a unique surjective set function $\mathrm{func}_{d,b,g}$ from d to $\mathrm{cod}_{d,b,g}$ such that for every element x of d, for every set y, the predicate $g(x,b,y)$ holds if and only if both y is an element of $\mathrm{cod}_{d,b,g}$ and y equals the value of $\mathrm{func}_{d,b,g}$ on x. Thus, to every function in the "traditional" sense on the domain set d, there exists a function in the set-theoretical sense of a subset of a Cartesian product $d \times c$ satisfying the "vertical line test."

As with the Axiom Schema of Specification, the Axiom Schema of Replacement is producing all the set-theoretical functions from d to c, since we can let $g(x,b,y)$ be the predicate that x is an element of d, that y is an element of c, that (x,y) is an element of b, and that b is a subset of $c \times d$ such that for every element x of d, there exists a unique element y of c for which (x,y) is an element of b (i.e., b is a subset of $c \times d$ that satisfies the "vertical line test").

Axiom 3.7 (Axiom of Infinity)**.** There exists a set $\mathbb{Z}_{\geq 0}$ such that (i) the empty set, $\varnothing$, is one element of $\mathbb{Z}_{\geq 0}$, such that (ii) for every element $n \in \mathbb{Z}_{\geq 0}$ the set $n \cup \{n\}$ is an element in $\mathbb{Z}_{\geq 0}$, and such that (iii) the set $\mathbb{Z}_{\geq 0}$ is a subset of every set that satisfies both (i) and (ii).

$$\exists z\ ((\varnothing \in z) \wedge (\forall n\ ((n \in z) \Rightarrow (n \cup \{n\} \in z))))\wedge$$
$$(\forall z'\ (((\varnothing \in z') \wedge (\forall n'\ ((n' \in z') \Rightarrow (n' \cup \{n'\} \in z')))) \Rightarrow (\forall n''\ (n'' \in z) \Rightarrow (n'' \in z'))))$$

Consider the predicate $g(x,b,y)$ with three free variables: b equals b and y equals $x \cup \{x\}$. This is a predicate as in the Axiom Schema of Replacement, i.e., it can be used to define a set function, succ (for "successor"), in the "traditional" sense for each specification of domain set. Since the empty set contains no element $\{n\}$, the empty set can never be an element of the image set of such a function. The empty set *can* be an element of the domain set, i.e., $\{\varnothing\}$ can be a subset of the domain that is disjoint from the image set. The Axiom of Infinity guarantees the existence of a domain set for this function such that the domain set equals the disjoint union of the image set and the singleton set $\{\varnothing\}$.

For each such domain set, the intersection of all subsets of the domain set satisfying these conditions is a unique subset, by the Axiom Schema of Specification and the Axiom of Extensionality. So, up to replacing any domain set as above by this unique subset, there exists a unique domain set $\mathbb{Z}_{\geq 0}$ for succ that equals the disjoint union of the image set and the singleton set $\{\varnothing\}$, and such that every domain set satisfying these conditions contains $\mathbb{Z}_{\geq 0}$ as a subset. For every model of Zermelo – Fraenkel set theory, the set $\mathbb{Z}_{\geq 0}$, the element $\varnothing$ of $\mathbb{Z}_{\geq 0}$ (interpreted as the element 0), and the associated set function succ from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$ is a model of the (second order) axiom schema of Peano arithmetic. So the Axiom of Infinity intepprets Peano arithmetic within Zermelo-Fraenkel set theory. This addresses the difficulty, mentioned earlier, that many of the metamathematical notions about this axiomatization of set theory implicitly use some formalization of the natural numbers.

Axiom 3.8 (Axiom of Power Sets)**.** For every set b, there exists a set, denoted $\mathcal{P}(b)$, such that for every set a, the set a is an element of $\mathcal{P}(b)$ if and only if the set a is a subset of b, i.e., if and only if, for every set x, if x is an element of a then a is an element of b.

$$\forall b\ \exists b'\ \forall a\ ((a \in b') \Leftrightarrow (\forall x\ ((x \in a) \Rightarrow (x \in b))))$$

Really the axiom of power sets is only the first in a continuing list of axioms (e.g., "large cardinal" axioms) considered by set theorists that allow more and more of the operations on sets that are relevant in both mathematics and metamathematics.

The following axiom, the Axiom of Choice, is **not** part of the Zermelo – Fraenkel axiom system, but it is accepted by most current mathematicians. Assuming the consistency of the Zermelo – Fraenkel axiom system, Cohen and Gödel proved the independence of the Axiom of Choice: the Zermelo – Fraenkel axiom system remains consistent if we add the Axiom of Choice, and the Zermelo – Fraenkel axiom system remains consistent if we add the negation of the Axiom of Choice (obviously it is not consistent if we add both simultaneously).

Axiom 3.9 (Axiom of Choice)**.** For every set a, for every set b, for every set c, if c is a subset of $a \times b$ such that for every element y of b there exists an element (x, y) of c, then there exists a subset d of c such that for every element y of b there exists a unique element (x, y) of d.

$$\forall a\ \forall b\ \forall c\ ((\forall y\ ((y \in b) \Rightarrow (\exists x\ ((x, y) \in c)))) \Rightarrow$$
$$(\exists d\ (\forall w\ ((w \in b) \Rightarrow ((\exists z\ ((z, w) \in d)) \wedge (\forall v\ \forall u\ ((((v, w) \in d) \wedge ((u, w) \in d)) \Rightarrow (v = u))))))))$$

As discussed in all books on set theory, in the presence of the Zermelo – Fraenkel axioms, the Axiom of Choice is equivalent to the Well-Order Principle (every set has a well-order), it is equivalent to Zorn's lemma, etc.

4 Classes

The definition of category uses the notion of a class. Classes can be axiomatized as a first-order theory, as done by von Neumann – Bernays – Gödel or by Morse – Kelley. The approach here is a second-order theory using the metalanguage of (first-order) Zermelo – Fraenkel set theory. This can be formalized, for instance, by using a Gödel numbering of the well-formed formulas of (first-order) Zermelo – Fraenkel set theory, but we prefer the verbose alternative of writing out the predicates of Zermelo – Fraenkel set theory. The classes produced in this way are the *parametrically definable* classes. For every model of Zermelo – Fraenkel set theory, the parameterically definable classes in that model form a model of class theory (the model most often intended in analysis, algebra, geometry, and topology). In particular, the (Kuratowski) ordered pair $(a,b) := \{\{a\},\{a,b\}\}$ converts predicates of higher arity into predicates of lower arity, i.e., every predicate $p(t_1,t_2,\ldots,t_{n-1},t_n)$ of arity $n \geq 1$ (with n a "true" natural number) in the first-order language of Zermelo – Fraenkel set theory converts to the following predicate $\widetilde{p}(t)$ of arity 1 with unique free variable t,

$$\exists t_1\ \exists t_2\ \ldots \exists t_{n-1}\ \exists t_n\ ((t_1,(t_2,\ldots,(t_{n-1},t_n)\ldots)) = t) \wedge p(t_1,t_2,\ldots,t_{n-1},t_n).$$

Definition 4.1 (Parametrically definable classes)**.** For every ordered pair $((p(s,t),a),(p'(s',t'),a'))$ of (first-order, Zermelo – Fraenkel) predicates p, respectively p', with a specified ordered pair (s,t), resp. (s',t'), of (all of) its free variables and of a set a, resp. a', the ordered pair $(p(s,t),a)$ is **Lindenbaum-Tarski equivalent** to $(p'(s',t'),a')$ if (and only if)

$$\forall b\ \ (p'(a',b) \Leftrightarrow p(a,b)).$$

Because logical equivalence is reflexive, transitive and symmetric, also Lindenbaum-Tarski equivalence is reflexive, transitive and symmetric. A parametrically definable **class** is a Lindenbaum-Tarski equivalence class $[p(s,t),a]$ (i.e., we are extending the usual equality predicate $a = a'$ to a predicate $[p(s,t),a] = [p'(s',t'),a']$ via Lindenbaum-Tarski equivalence). For every class $[p(s,t),a]$, a set b is a **member** of $[p(s,t),a]$ if (and only if) $p(a,b)$ holds (i.e., we are extending the set membership predicate $b \in a$ to a predicate of membership of b in the class $[p(s,t),a]$ as above). For every class $\mathbf{C}$, a class $\mathbf{B}$ is a **subclass** of $\mathbf{C}$ if (and only if) every member of $\mathbf{B}$ is a member of $\mathbf{C}$ (i.e., we are extending the subset predicate $b \subseteq c$ to a subclass predicate).

With this definition, we have a variant of extensionality for classes.

Lemma 4.2 (Extensionality)**.** *For every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, the class* $\mathbf{B}$ *equals the class* $\mathbf{B}'$ *if and only if, for every set* x*, the set* x *is a member of* $\mathbf{B}$ *if and only if* x *is a member of* $\mathbf{B}'$*.*

Proof. This is just a restatement of Lindenbaum-Tarski equivalence. □

By construction we also have the axiom of class formation.

Lemma 4.3 (Class Formation)**.** *For every (first-order, Zermelo – Fraenkel) predicate* $p(s,t)$ *with an ordered pair* (s,t) *of (all of) its free variables, for every set* a*, there exists a unique class* $\mathbf{C}$ *such that, for every set* b*, the set* b *is a member of* $\mathbf{C}$ *if and only if* $p(a,b)$ *holds.*

Proof. The class $\mathbf{C} := [p(s,t),a]$ is one such class. By the previous lemma, this is unique. □

In particular, we have a universal class.

Lemma 4.4. *There exists a unique class* $\mathbf{V}$ *such that every set is a member of* $\mathbf{V}$.

Proof. Let $p(s,t)$ be tautological, e.g., $(s=s)\wedge(t=t)$. Then for every set a, say $a=\varnothing$, every set is a member of the class $\mathbf{V} := [(s=s)\wedge(t=t), a]$. By Lemma 4.2, this class is unique. □

Also, we have a class for each set (including for the empty set). In most axiomatizations of class theory, each set is identified with its associated class (but we prefer not to do this).

Lemma 4.5. *For every set* a*, there exists a unique class whose members are the elements of* a*. In particular, for* a *equal to the empty set, the associated class has no members. Two sets are equal if and only if their associated classes are equal.*

Proof. The members of the class $[t \in s, a]$ are precisely the sets the elements of a. By Lemma 4.2, this class is unique. By the Axiom of Extensionality, two sets are equal if and only if their associated classes are equal. □

We also have a variant for classes of the axiom of foundation.

Lemma 4.6 (Foundation)**.** *For every class* $\mathbf{C}$*, there does not exist a sequence* $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ *of members of* $\mathbf{C}$ *such that, for every element* n *of* $\mathbb{Z}_{\geq 0}$*, the set* a_{n+1} *is an element of the set* a_n*. In particular, for every class* $\mathbf{C}$ *that has at least one member, there exists a member* a *of* $\mathbf{C}$ *such that for every element of* a*, that element is not a member of* $\mathbf{C}$.

Proof. By foundation for Zermelo – Fraenkel set theory, there does not exist any sequence $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ of sets such that, for every element n of $\mathbb{Z}_{\geq 0}$, the set a_{n+1} is an element of the set a_n. Thus, there exists no such sequence satisfying the additional condition that every set a_n is a member of $\mathbf{C}$.

For every class $\mathbf{C}$ that has a member, there exists a set a_0 that is a member of $\mathbf{C}$. If there exists an element a_1 of a_0 that is also a member of $\mathbf{C}$, then this gives a finite sequence (a_0, a_1) of members of $\mathbf{C}$ such that a_1 is an element of a_0. If there exists an element a_2 of a_1 that is also a member of $\mathbf{C}$, then this gives a finite sequence (a_0, a_1, a_2) of member of $\mathbf{C}$ such that a_1 is an element of a_0 and a_2 is an element of a_1. Continuing inductively, either there exists a sequence $(a_0, a_1, \dots, a_n)$ of members of $\mathbf{C}$ such that a_1 is an element of a_0, etc., a_n is an element of a_{n-1} and every element of a_n is not a member of $\mathbf{C}$, or there exists a sequence $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ of members of $\mathbf{C}$ such that, for every element n of $\mathbb{Z}_{\geq 0}$, the member a_{n+1} is an element of a_n. This second case is forbidden by the previous paragraph. Thus, there exists a member a_n of $\mathbf{C}$ such that every element of a_n is not a member of $\mathbf{C}$. □

The axioms in the previous section define Zermelo – Fraenkel set theory, i.e., ZF, but do not include the Axiom of Choice that gives ZFC set theory. The lemmas above verify the axioms of NBG, von Neumman – Bernays – Gödel class theory, for the model of parameterically definable classes in each model of ZF set theory, except for the Axiom of Limitation of Size, which is essentially a global analogue of the Axiom of Choice.

Of course there are many additional results about classes. Many of these are the analogues for classes of well-known results for sets.

Lemma 4.7. *For every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, the class* $\mathbf{B}$ *equals the class* $\mathbf{B}'$ *if and only if both* $\mathbf{B}$ *is a subclass of* $\mathbf{B}'$ *and* $\mathbf{B}'$ *is a subclass of* $\mathbf{B}$*.*

Proof. Of course if $\mathbf{B}$ equals $\mathbf{B}'$, then every member of $\mathbf{B}$ is a member of $\mathbf{B}'$, i.e., $\mathbf{B}$ is a subclass of $\mathbf{B}'$, and every member of $\mathbf{B}'$ is a member of $\mathbf{B}$, i.e., $\mathbf{B}'$ is a subclass of $\mathbf{B}$.

Conversely, if both $\mathbf{B}$ is a subclass of $\mathbf{B}'$ and $\mathbf{B}'$ is a subclass of $\mathbf{B}$, then for every set x that is a member of $\mathbf{B}$, also x is a member of $\mathbf{B}'$, and for every set x that is a member of $\mathbf{B}'$, also x is a member of $\mathbf{B}$. By Lemma 4.2, the class $\mathbf{B}$ equals the class $\mathbf{B}'$. □

Definition 4.8. The class that has every set as a member is the **von Neumann class**, sometimes called the **von Neumann universe** or the **universal class**, denoted $\mathbf{V}$ or $\text{ob}_{\mathbf{Set}}$. For every set a, the class that has as members precisely the elements of a is the **class** of the set a, denoted $\mathbf{Cl}_a$.

Lemma 4.9. *The von Neumann class* $\mathbf{V}$ *is the unique class such that, for every class* $\mathbf{B}$*, the class* $\mathbf{B}$ *is a subclass of* $\mathbf{V}$*. For every set* a*, the class* $\mathbf{Cl}_a$ *is the unique class such that, for every class* $\mathbf{B}$*, the class* $\mathbf{Cl}_a$ *is a subclass of* $\mathbf{B}$ *if and only if* x *is a member of* $\mathbf{B}$ *for every element* x *of* a*.*

Proof. By definition of $\mathbf{V}$, every set is a member of $\mathbf{V}$. Thus, every class is a subclass of $\mathbf{V}$. For every class $\mathbf{B}$, if also $\mathbf{V}$ is a subclass of $\mathbf{B}$, then $\mathbf{B}$ equals $\mathbf{V}$ by Lemma 4.7. Thus, if every class is a subclass of $\mathbf{B}$, so that $\mathbf{V}$ is a subclass of $\mathbf{B}$ in particular, then $\mathbf{B}$ equals $\mathbf{V}$. Therefore $\mathbf{V}$ is the unique class such that every class is a subclass of $\mathbf{V}$.

For every set a, for every class $\mathbf{B}$, by the definition of subclass, the class $\mathbf{Cl}_a$ is a subclass of $\mathbf{B}$ if and only if, for every set x that is a member of $\mathbf{Cl}_a$, also x is a member of $\mathbf{B}$. By the definition of $\mathbf{Cl}_a$, this holds if and only if, for every set x that is an element of a, also x is a member of $\mathbf{B}$. □

Lemma 4.10. *For every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, there exists a unique class* $\mathbf{B} \wedge \mathbf{B}'$ *whose members are those sets that are simultaneously members of* $\mathbf{B}$ *and members of* $\mathbf{B}'$*. The subclasses of* $\mathbf{B} \wedge \mathbf{B}'$ *are precisely the classes that are simultaneously subclasses of both* $\mathbf{B}$ *and* $\mathbf{B}'$*. For every ordered pair* (b, b') *of sets, the class* $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ *equals* $\mathbf{Cl}_{b \cap b'}$*. Finally, for every class* $\mathbf{B}$ *there exists a class* $\cap\mathbf{B}$ *whose members are all sets* x *such that for every member* b *of* $\mathbf{B}$*, the set* x *is an element of* b*. In particular, for every set* c*, the class* $\cap\mathbf{Cl}_c$ *equals* $\mathbf{Cl}_{\cap c}$*.*

Proof. For every class $\mathbf{B} = [p(s,t),a]$, for every class $\mathbf{B}' = [p'(s',t'),a']$ the class $[p''(s'',t''),(a,a')]$ for the following predicate has as members precisely those sets that are simultaneously members of $\mathbf{B}$ and members of $\mathbf{B}'$.

$$\exists s\ \exists s'\ (p(s,t'') \wedge p'(s',t'')) \wedge (s'' = (s,s')).$$

By Lemma 4.2, the class $\mathbf{B} \wedge \mathbf{B}' = [p''(s'',t''),(a,a')]$ is the unique class whose members are precisely those sets that are simultaneously members of $\mathbf{B}$ and members of $\mathbf{B}'$.

By definition, a class $\mathbf{C}$ is a subclass of $\mathbf{B} \wedge \mathbf{B}'$ if and only if, for every member x of $\mathbf{C}$, also x is a member of $\mathbf{B} \wedge \mathbf{B}'$. By the definition of $\mathbf{B} \wedge \mathbf{B}'$, for every set x, a set x is a member of $\mathbf{B} \wedge \mathbf{B}'$ if and only if both x is a member of $\mathbf{B}$ and x is a member of $\mathbf{B}'$. Thus, $\mathbf{C}$ is a subclass of $\mathbf{B} \wedge \mathbf{B}'$ if and only if, for every member x of $\mathbf{C}$, both x is a member of $\mathbf{B}$ and x is a member of $\mathbf{B}'$. By the definition of subclass, $\mathbf{C}$ is a subclass of $\mathbf{B} \wedge \mathbf{B}'$ if and only if both $\mathbf{C}$ is a subclass of $\mathbf{B}$ and $\mathbf{C}$ is a subclass of $\mathbf{B}'$.

For every ordered pair (b,b') of sets, by the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b$ if and only if x is an element of b, and the set x is a member of $\mathbf{Cl}_{b'}$ if and only if x is an element of b'. Thus, for every set x, the set x is a member of $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ if and only if both x is an element of b and x is an element of b'. By the definition of intersection, for every set x, the set x is both an element of b and an element of b' if and only if x is an element of $b \cap b'$. Thus, again using the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ if and only x is a member of $\mathbf{Cl}_{b\cap b'}$. By Lemma 4.2, the class $\mathbf{Cl}_b \wedge \mathbf{Cl}_{b'}$ equals $\mathbf{Cl}_{b\cap b'}$.

Finally, for every class $\mathbf{B} = [p(s,t),a]$, for the class $\cap\mathbf{B} := [\forall t\ (p(s,t) \Rightarrow (t' \in t)),a]$ with the ordered pair of free variables (s,t'), for every set x, the set x is a member of $\cap\mathbf{B}$ if and only if, for every member b of $\mathbf{B}$, the set x is an element of b. By Lemma 4.2, the class $\cap\mathbf{B}$ is the unique class such that, for every set x, the set x is a member of $\cap\mathbf{B}$ if and only if, for every member b of $\mathbf{B}$, the set x is an element of b. In particular, for every set c, the class $\cap\mathbf{Cl}_c$ equals $\mathbf{Cl}_{\cap c}$. ☐

Lemma 4.11. *For every class $\mathbf{B}$, for every class $\mathbf{B}'$, there exists a unique class $\mathbf{B} \vee \mathbf{B}'$ whose members are those sets that are either members of $\mathbf{B}$ or members of $\mathbf{B}'$ (or both). The classes that have $\mathbf{B} \vee \mathbf{B}'$ as a subclass are precisely the classes that both have $\mathbf{B}$ as a subclass and have $\mathbf{B}'$ as a subclass. For every ordered pair (b,b') of sets, the class $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ equals $\mathbf{Cl}_{b\cup b'}$. Finally, for every class $\mathbf{B}$ there exists a class $\cup\mathbf{B}$ whose members are all sets x such that there exists a member b of $\mathbf{B}$ with x an element of b. In particular, for every set c, the class $\cup\mathbf{Cl}_c$ equals $\mathbf{Cl}_{\cup c}$.*

Proof. For every class $\mathbf{B} = [p(s,t),a]$, for every class $\mathbf{B}' = [p'(s',t'),a']$, the class $[p''(s'',t''),a'']$ with $a'' = (a,a'')$ and with the following predicate has as members precisely those sets that are either members of $\mathbf{B}$ or members of $\mathbf{B}'$.

$$\exists s\ \exists s'\ (p(s,t'') \vee p'(s',t'')) \wedge (s'' = (s,s')).$$

By Lemma 4.2, the class $\mathbf{B} \vee \mathbf{B}' = [p''(s'', t''), (a, a')]$ is the unique class whose members are precisely those sets that are either members of $\mathbf{B}$ or members of $\mathbf{B}'$.

By definition, a class $\mathbf{C}$ has $\mathbf{B} \vee \mathbf{B}'$ as a subclass if and only if, for every member x of $\mathbf{B} \vee \mathbf{B}'$, also x is a member of $\mathbf{C}$. By the definition of $\mathbf{B} \vee \mathbf{B}'$, for every set x, a set x is a member of $\mathbf{B} \vee \mathbf{B}'$ if and only if either x is a member of $\mathbf{B}$ or x is a member of $\mathbf{B}'$. Thus, $\mathbf{B} \wedge \mathbf{B}'$ is a subclass of $\mathbf{C}$ if and only if, both every member x of $\mathbf{B}$ is a member of $\mathbf{C}$ and every member x of $\mathbf{B}'$ is a member of $\mathbf{C}$. By the definition of subclass, $\mathbf{B} \wedge \mathbf{B}'$ is a subclass of $\mathbf{C}$ if and only if both $\mathbf{B}$ is a subclass of $\mathbf{C}$ and $\mathbf{B}'$ is a subclass of $\mathbf{C}$.

For every ordered pair (b, b') of sets, by the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b$ if and only if x is an element of b, and the set x is a member of $\mathbf{Cl}_{b'}$ if and only if x is an element of b'. Thus, for every set x, the set x is a member of $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ if and only if either x is an element of b or x is an element of b'. By the definition of union, for every set x, the set x is either an element of b or an element of b' if and only if x is an element of $b \cup b'$. Thus, again using the definition of $\mathbf{Cl}$, for every set x, the set x is a member of $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ if and only x is a member of $\mathbf{Cl}_{b \cup b'}$. By Lemma 4.2, the class $\mathbf{Cl}_b \vee \mathbf{Cl}_{b'}$ equals $\mathbf{Cl}_{b \cup b'}$.

Finally, for every class $\mathbf{B} = [p(s,t), a]$, for the class $\cup\mathbf{B} := [\exists t\ (p(s,t) \Rightarrow (t' \in t)), a]$ with the ordered pair of free variables (s, t'), for every set x, the set x is a member of $\cup\mathbf{B}$ if and only if there exists a member b of $\mathbf{B}$ that has x as an element. By Lemma 4.2, the class $\cup\mathbf{B}$ is the unique class such that, for every set x, the set x is a member of $\cup\mathbf{B}$ if and only if there exists a member b of $\mathbf{B}$ that has x as an element. In particular, for every set c, the class $\cup\mathbf{Cl}_c$ equals $\mathbf{Cl}_{\cup c}$. □

Lemma 4.12. *For every class $\mathbf{B}$, there exists a unique class $\neg\mathbf{B}$ whose members are those sets that are not members of $\mathbf{B}$. A class is a subclasses of $\neg\mathbf{B}$ if and only if every member of the class is not a member of $\mathbf{B}$. The class $\neg(\neg\mathbf{B})$ equals $\mathbf{B}$. For every class $\mathbf{B}'$, both $\neg(\mathbf{B} \wedge \mathbf{B}')$ equals $(\neg\mathbf{B}) \vee (\neg\mathbf{B}')$ and $\neg(\mathbf{B} \vee \mathbf{B}')$ equals $(\neg\mathbf{B}) \wedge (\neg\mathbf{B}')$. Also $\neg(\cap\mathbf{B})$ equals $\cup(\neg\mathbf{B})$, and $\neg(\cup\mathbf{B})$ equals $\cap(\neg\mathbf{B})$. For every set b, for every set b', the class $\mathbf{Cl}_b \wedge (\neg\mathbf{Cl}_{b'})$ equals $\mathbf{Cl}_{b \setminus b'}$.*

Proof. For every class $\mathbf{B} = [p(s,t), a]$, the members of $[\neg p(s,t), a]$ are precisely the sets that are not members of $\mathbf{B}$, and this class is unique by Lemma 4.2.

By definition, a class $\mathbf{C}$ is a subclass of $\neg\mathbf{B}$ if and only if, for every member x of $\mathbf{C}$, also x is a member of $\neg\mathbf{B}$. By definition, for every set x, the set x is a member of $\neg\mathbf{B}$ if and only if x is not a member of $\mathbf{B}$. Therefore, $\mathbf{C}$ is a subclass of $\neg\mathbf{B}$ if and only if every member x of $\mathbf{C}$ is not a member of $\mathbf{B}$. In particular, a class $\mathbf{C}$ is a subclass of $\neg(\neg\mathbf{B})$ if and only if every member x of $\mathbf{C}$ is not a member of $\neg\mathbf{B}$, i.e., if and only if every member x of $\mathbf{C}$ is a member of $\mathbf{C}$. By Lemma 4.7, the class $\neg(\neg\mathbf{B})$ equals $\mathbf{B}$.

For every class $\mathbf{B} = [p(s,t), a]$ and for every class $\mathbf{B}' = [p'(s', t'), a']$, since $\neg(p(a, t'') \wedge p'(a', t''))$ is logically equivalent to $(\neg p(a, t'')) \vee (\neg p'(a', t''))$, also the class $\neg(\mathbf{B} \wedge \mathbf{B}')$ equals $(\neg\mathbf{B}) \vee (\neg\mathbf{B}')$. Similarly, the class $\neg(\mathbf{B} \vee \mathbf{B}')$ equals the class $(\neg\mathbf{B}) \wedge (\neg\mathbf{B}')$.

Since the following two predicates are logically equivalent,

$$\neg\left(\exists t\ (x \in t) \wedge p(a,t)\right),$$

$$\forall t\ (x \in t) \Rightarrow \neg p(a,t),$$

the class $\neg(\cup\mathbf{B})$ equals $\cap(\neg\mathbf{B})$. Similarly, the class $\neg(\cap\mathbf{B})$ equals $\cup(\neg\mathbf{B})$.

Finally, for every set b, for every set b', for every set x, the set x is a member of $\mathbf{Cl}_b \wedge (\neg\mathbf{Cl}_{b'})$ if and only if both x is an element of b and x is not an element of b', i.e., if and only if x is an element of $b \setminus b'$. Thus, the class $\mathbf{Cl}_b \wedge (\neg\mathbf{Cl}_{b'})$ equals $\mathbf{Cl}_{b\setminus b'}$ by Lemma 4.2. □

Lemma 4.13. *For every class $\mathbf{B}$, for every class $\mathbf{B}'$, there exists a unique class $\mathbf{B}\times\mathbf{B}'$ whose members are ordered pairs (b,b') of a member b of $\mathbf{B}$ and a member b' of $\mathbf{B}'$. A class is a subclass of $\mathbf{B}\times\mathbf{B}'$ if and only if every member of the class is of the form (b,b') for a member b of $\mathbf{B}$ and a member b' of $\mathbf{B}'$. For every set c, for every set c', the class $\mathbf{Cl}_c \times \mathbf{Cl}_{c'}$ equals $\mathbf{Cl}_{c\times c'}$.*

Proof. For every class $\mathbf{B} = [p(s,t),a]$, for every class $\mathbf{B}' = [p'(s',t'),a']$, the class $[p''(s'',t''),a'']$ with $a'' = (a,a'')$ and with the following predicate has as members precisely those sets (b,b') such that b is a member of $\mathbf{B}$ and such that b' is a member of $\mathbf{B}'$.

$$\exists s\ \exists s'\ \exists t\ \exists t'\ (p(s,t) \vee p'(s',t')) \wedge (s'' = (s,s')) \wedge (t'' = (t,t')).$$

By Lemma 4.2, the class $\mathbf{B}\times\mathbf{B}' = [p''(s'',t''),(a,a')]$ is the unique class whose members are precisely those sets (b,b') such that b is a member of $\mathbf{B}$ and such that b' is a member of $\mathbf{B}'$.

By definition, a class $\mathbf{C}$ is a subclass of $\mathbf{B}\times\mathbf{B}'$ if and only if, for every member b'' of $\mathbf{C}$ is also a member of $\mathbf{B}\times\mathbf{B}'$. By the definition of $\mathbf{B}\times\mathbf{B}'$, a set b'' is a member of $\mathbf{B}\times\mathbf{B}'$ if and only if b'' equals (b,b') for a member b' of $\mathbf{B}$ and for a member b' of $\mathbf{B}'$. Thus, $\mathbf{C}$ is a subclass of $\mathbf{B}\times\mathbf{B}'$ if and only if every member of $\mathbf{C}$ equals (b,b') for a member b of $\mathbf{B}$ and for a member b' of $\mathbf{B}'$.

For every set c, for every set c', by the definition of $\mathbf{Cl}$, a set is an element of $\mathbf{Cl}_c \times \mathbf{Cl}_{c'}$ if and only if the set equals (b,b') for an element b of c and for an element b' of c', i.e., if and only if the set is an element of $c\times c'$. Therefore, by Lemma 4.2, the class $\mathbf{Cl}_c \times \mathbf{Cl}_{c'}$ equals $\mathbf{Cl}_{c\times c'}$. □

Lemma 4.14. *For every class $\mathbf{R}$, there exists a unique subclass rel$(\mathbf{R})$ of $\mathbf{R}$ whose members are those members of $\mathbf{R}$ of the form (b,c) for a set b and for a set c. In particular, for every set r, the class rel$(\mathbf{Cl}_r)$ is the class of the unique maximal subset rel(r) of r such that rel(r) is a binary relation.*

Proof. For every class $\mathbf{R} = [p(s,t),a]$, the class of the following predicate $\mathrm{rel}(p)(s,t)$, the members of the class $[\mathrm{rel}(p)(s,t),a]$ are those members of $\mathbf{R}$ of the form (b,c) for a set b and for a set c.

$$\exists u\ \exists v\ (t = (u,v)) \wedge p(s,(u,v)).$$

By Lemma 4.2, this subclass of $\mathbf{R}$ is unique.

For every set r, by the Axiom Schema of Specification, there exists a unique subset $\text{rel}(r)$ of r consisting of those elements of r of the form (b,c) for some set b and for some set c. By the Axiom Schema of Replacement, there exists a unique set $\text{active}(\text{rel}(r))$ and a unique set $\text{image}(\text{rel}(r))$ such that $\text{rel}(r)$ is a subset of $\text{active}(\text{rel}(r))\times\text{image}(\text{rel}(r))$ and each of the two projection functions are surjective. ☐

Definition 4.15. For every class $\mathbf{R}$, the class $\mathbf{R}$ is a **class relation** if (and only if) the subclass $\text{rel}(\mathbf{R})$ equals $\mathbf{R}$, i.e., if (and only if) every member of $\mathbf{R}$ is of the form (b,c) for a set b and for a set c.

Lemma 4.16. *For every class $\mathbf{R}$, for every subclass of $\mathbf{R}$, the subclass is a class relation if and only if it is a subclass of rel$(\mathbf{R})$. For every class relation $\mathbf{R}$, there exists a unique class relation $\mathbf{R}^{opp}$ whose members are those sets of the form (c,b) such that (b,c) is a member of $\mathbf{R}$. Also there exists a unique class image$(\mathbf{R})$ whose members are all sets c such that (b,c) is a member of $\mathbf{R}$ for some set b. Similarly, there exists a unique class active$(\mathbf{R})$ = image$((\mathbf{R})^{opp})$ whose members are all sets b such that (b,c) is a member of $\mathbf{R}$ for some set c. More generally, for every class relation $\mathbf{R}$ and for every class $\mathbf{B}$, there exists a unique class $\mathbf{R}[\mathbf{B}]$ whose members are those sets c such that there exists a member b of $\mathbf{B}$ for which (b,c) is a member of $\mathbf{R}$. Similarly, for every class $\mathbf{R}$ and for every class $\mathbf{C}$, there exists a unique class $\mathbf{R}^{opp}[\mathbf{C}]$ whose members are those sets b such that there exists a member c of $\mathbf{C}$ for which (b,c) is a member of $\mathbf{R}$.*

Proof. For every class $\mathbf{R} = [p(s,t),a]$, the class of the following predicate $\text{rel}(p)(s,t)$, the members of the class $[\text{rel}(p)(s,t),a]$ are those members of $\mathbf{R}$ of the form (b,c) for a set b and for a set c.

$$\exists u\ \exists v\ (t=(u,v))\wedge p(s,(u,v)).$$

By Lemma 4.2, this subclass of $\mathbf{R}$ is unique.

Similarly, for the following predicate $\text{rel}(p)^{\text{opp}}(s,t)$, the members of the class $[\text{rel}(p)^{\text{opp}}(s,t),a]$ are those sets of the form (c,b) such that (b,c) is a member of $\mathbf{R}$.

$$\exists u\ \exists v\ (t=(v,u))\wedge p(s,(u,v)).$$

By Lemma 4.2, this class is unique.

For the following predicate $\text{image}(\text{rel}(p))(s,t)$, the members of the class $[\text{image}(\text{rel}(p))(s,t),a]$ are those sets c such that (b,c) is a member of $\mathbf{R}$ for some set b.

$$\exists u\ p(s,(u,t)).$$

Similarly, for the following predicate active(rel(p))(s,t), the members of the class [active(rel(p))(s,t), a] are those sets b such that (b,c) is a member of $\mathbf{R}$ for some set c.

$$\exists v \; p(s,(t,v)).$$

By Lemma 4.2, this class is unique.

For every class $\mathbf{R}$, for every class $\mathbf{B}$, the members of the class image(rel($\mathbf{R}$)) $\wedge$ $\mathbf{B}$ are those sets c such that there exists a member b of $\mathbf{B}$ for which (b,c) is a member of $\mathbf{R}$. By Lemma 4.2, this class rel($\mathbf{R}$)[$\mathbf{B}$] is unique.

Similarly, for every class $\mathbf{R}$, for every class $\mathbf{C}$, the members of the class $\mathbf{C}\wedge$active(rel($\mathbf{R}$)) are those sets b such that there exists a member c of $\mathbf{C}$ for which (b,c) is a member of $\mathbf{R}$. By Lemma 4.2, this class [$\mathbf{C}$]rel($\mathbf{R}$) is unique. □

Lemma 4.17. *For every class $\mathbf{R}$, for every set b, there exists a unique class $\mathbf{R}_b$ whose members are those sets c such that (b,c) is a member of $\mathbf{R}$.*

Proof. For every class $\mathbf{R} = [p(s,t),a]$, for every set b, for following predicate rel(p)(s,t), the members of the class [rel(p)(s,t), (a,b)] are those sets c such that (b,c) is a member of $\mathbf{R}$.

$$\exists u \; \exists v \; (s = (u,v)) \wedge p(u,(v,t)).$$

By Lemma 4.2, this subclass of $\mathbf{R}$ is unique. □

Definition 4.18. For every class $\mathbf{B}$ and for every class $\mathbf{C}$, a subclass $\mathbf{R}$ of $\mathbf{B}\times\mathbf{C}$ is a **relation** from $\mathbf{B}$ to $\mathbf{C}$. In particular, for every class $\mathbf{B}$, a **$\mathbf{B}$-class** is a relation from $\mathbf{B}$ to the von Neumann class $\mathbf{V}$. For every $\mathbf{B}$-class $\mathbf{R}$, for every member b of $\mathbf{B}$, the **fiber class** $\mathbf{R}_b$ of $\mathbf{R}$ over b is the class whose members are all sets c such that (b,c) is a member of $\mathbf{R}$.

Lemma 4.19. *For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{R}$, for every $\mathbf{B}$-class $\mathbf{R}'$, the $\mathbf{B}$-class $\mathbf{R}$ equals $\mathbf{R}'$ if and only if, for every member b of $\mathbf{B}$, the fiber class $\mathbf{R}_b$ equals $\mathbf{R}'_b$.*

Proof. By Lemma 4.2, the class $\mathbf{R}$ equals $\mathbf{R}'$ if and only if, for every set x, the set x is a member of $\mathbf{R}$ if and only if x is a member of $\mathbf{R}'$. Since $\mathbf{R}$ is a $\mathbf{B}$-class, every member x of $\mathbf{R}$ is of the form (b,c) for a unique member b of $\mathbf{B}$ and for a unique set c. Since $\mathbf{R}'$ is a $\mathbf{B}$-class, every member x' of $\mathbf{R}'$ is of the form (b',c') for a unique member b' of $\mathbf{B}$ and for a unique set c. By the defining property of Kuratowski ordered pairs, the Kuratowski ordered pair (b,c) equals (b',c') if and only if both b equals b' and c equals c'.

Thus, the following two conditions are equivalent: (i) for every set x, the set x is a member of $\mathbf{R}$ if and only if x is a member of $\mathbf{R}'$; (ii) for every member b of $\mathbf{B}$, for every set c, the Kuratowski ordered pair (b,c) is a member of $\mathbf{R}$ if and only if (b,c) is a member of $\mathbf{R}'$. Therefore $\mathbf{R}$ equals $\mathbf{R}'$ if and only if, for every member b of $\mathbf{B}$, the class $\mathbf{R}_b$ equals $\mathbf{R}'_b$. □

Lemma 4.20. *For every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, there exists a unique* $\mathbf{Cl}_{\{0,1\}}$*-class* $(\mathbf{B},\mathbf{B}')$ *whose* 0*-fiber equals* $\mathbf{B}$ *and whose* 1*-fiber equals* $\mathbf{B}'$*, where* 0 *is* $\varnothing$ *and* 1 *is* $\{\varnothing\}$*. For every* $\mathbf{Cl}_{\{0,1\}}$*-class* $\mathbf{R}$*, for every class* $\mathbf{B}$*, for every class* $\mathbf{B}'$*, the* $\mathbf{Cl}_{\{0,1\}}$*-class* $(\mathbf{B},\mathbf{B}')$ *equals* $\mathbf{R}$ *if and only if both* $\mathbf{B}$ *equals the fiber class* $\mathbf{R}_0$ *and* $\mathbf{B}'$ *equals the fiber class* $\mathbf{R}_1$*. In particular, for every class* $\mathbf{C}$*, for every class* $\mathbf{C}'$*, the* $\mathbf{Cl}_{\{0,1\}}$*-class* $(\mathbf{C},\mathbf{C}')$ *equals* $(\mathbf{B},\mathbf{B}')$ *if and only if both* $\mathbf{B}$ *equals* $\mathbf{C}$ *and* $\mathbf{B}'$ *equals* $\mathbf{C}'$*.*

Proof. For every class $\mathbf{B} = [p(s,t),a]$ and for every class $\mathbf{B}' = [p'(s',t'),a']$, for the following predicate $p''(s'',t'')$, the members of the class $[p''(s'',t''),(a,a')]$ are those sets of the form $(0,b)$ for a member b of $\mathbf{B}$ and those sets of the form $(1,c)$ for a member c of $\mathbf{C}$.

$$\exists s\ \exists s'\ \exists u\ \exists v\ (s'' = (s,s')) \wedge (t'' = (u,v)) \wedge (((u=0)\wedge p(s,v)) \vee ((u=1)\wedge p'(s',v))).$$

By Lemma 4.2, this subclass of $\mathbf{R}$ is unique.

For every $\mathbf{Cl}_{\{0,1\}}$-class $\mathbf{R}$, by the previous lemma, the $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{B},\mathbf{B}')$ equals $\mathbf{R}$ if and only if both the 0-fiber class $\mathbf{B}$ equals $\mathbf{R}_0$ and $\mathbf{B}'$ equals $\mathbf{R}_1$. In particular, for every class $\mathbf{C}$, for every class $\mathbf{C}'$, the $\mathbf{Cl}_{\{0,1\}}$-class $(\mathbf{B},\mathbf{B}')$ equals $(\mathbf{C},\mathbf{C}')$ if and only if both the 0-fiber $\mathbf{B}$ equals $\mathbf{C}$ and the 1-fiber $\mathbf{B}'$ equals $\mathbf{C}'$. □

5 Morphisms and spans between classes

For defining categories, a bit more useful than class morphisms or relations is the notion of spans.

Definition 5.1. For every class $\mathbf{B}$, for every class $\mathbf{C}$, a $(\mathbf{B},\mathbf{C})$**-span** $\mathbf{M}$ is a $\mathbf{B}\times\mathbf{C}$-class. For every member b of $\mathbf{B}$, for every member c of $\mathbf{C}$, the **fiber class** $\mathbf{M}^b_c$ of $\mathbf{M}$ over (b,c) is the fiber class $\mathbf{M}_{(b,c)}$. A $\mathbf{B}$-class is a $\mathbf{B}$**-set** if (and only if) every fiber class is a class of a set. Similarly, a $(\mathbf{B},\mathbf{C})$-span is a $(\mathbf{B},\mathbf{C})$**-set** if (and only if) it is a $\mathbf{B}\times\mathbf{C}$-set. Finally, for every class $\mathbf{O}$, an $\mathbf{O}$**-Hom span** is an $(\mathbf{O},\mathbf{O})$-set $\mathbf{M}$, i.e., for every ordered pair (b,c) of members of $\mathbf{O}$, the fiber class $\mathbf{M}^b_c$ is the class of a set.

Example 5.2. For every class $\mathbf{B}$, the **identity relation** $\mathrm{Id}_{\mathbf{B}}$ from $\mathbf{B}$ to itself is the class whose members are all ordered pairs (b,b) such that b is a member of $\mathbf{B}$. In particular, for every set a, $\mathrm{Id}_{\mathbf{Cl}_a}$ equals $\mathbf{Cl}_{\mathrm{Id}_a}$ for the usual identity set relation Id_a whose elements are all ordered pairs (b,b) such that b is an element of a. For every class $\mathbf{O}$, the **identity** $\mathbf{O}$**-Hom span** $\mathbf{Id}_{\mathbf{O}}$ is the class whose members are all ordered pairs $((b,b),\mathrm{Id}_b)$ such that b is a member of $\mathbf{O}$. In particular, the identity $\mathbf{Cl}_a$-Hom span is the $\mathbf{Cl}_{\mathrm{Id}_a}$-class whose fiber class $(\mathbf{Id}_{\mathbf{Cl}_a})^b_c$ has a unique member Id_b if c equals b is an element of a and otherwise has no member.

Definition 5.3. For every class $\mathbf{B}$, a $\mathbf{B}$-class $\mathbf{F}$ is a **class morphism** from $\mathbf{B}$ if (and only if), for every member b of $\mathbf{B}$, the fiber class $\mathbf{F}_b$ is the class of a singleton set, i.e., there exists a unique set c such that (b, c) is a member of $\mathbf{F}$. For every class $\mathbf{B}$, for every class $\mathbf{C}$, a **class morphism** from $\mathbf{B}$ to $\mathbf{C}$ is a relation from $\mathbf{B}$ to $\mathbf{C}$ that is also a class morphism from $\mathbf{B}$. The class morphism is a **class isomorphism** if also, for every member c of $\mathbf{C}$, there exists a unique member b of $\mathbf{B}$ such that (b, c) is a member of the class morphism.

Example 5.4. For every class $\mathbf{B}$, the identity $\mathrm{Id}_{\mathbf{B}}$ is a class isomorphism from $\mathbf{B}$ to itself.

Example 5.5. For every class $\mathbf{B}$, for every morphism of classes $\mathbf{F}$ from $\mathbf{B}$, there is a $\mathbf{B}$-class $\mathrm{cl}_{\mathbf{B},\mathbf{F}}$ whose members are all ordered pairs (b, c) of a member b of $\mathbf{B}$ and of an element c of the set $\mathbf{F}(b)$.

Exercise 5.6. For every class $\mathbf{B}$, for every morphism of classes $\mathbf{F}$ from $\mathbf{B}$, check that $\mathrm{cl}_{\mathbf{B},\mathbf{F}}$ is a $\mathbf{B}$-set. Conversely, for every $\mathbf{B}$-set $\mathbf{D}$, check that there is a unique morphism of classes $\mathrm{fun}_{\mathbf{B},\mathbf{D}}$ from $\mathbf{B}$ associating to every member b of $\mathbf{B}$ the unique set whose associated class is the fiber class $\mathbf{D}_b$. Check that these two operations determine an equivalence between $\mathbf{B}$-sets and morphisms of classes from $\mathbf{B}$.

Definition 5.7. For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{Q}$, for every $\mathbf{B}$-class $\mathbf{R}$, a **$\mathbf{B}$-class morphism** from $\mathbf{Q}$ to $\mathbf{R}$ is a class morphism $\mathbf{F}$ from $\mathbf{Q}$ to $\mathbf{R}$ such that, for every member b of $\mathbf{B}$, for every member c of $\mathbf{Q}_b$, there exists a unique member d of $\mathbf{R}_b$ such that $((b, c), (b, d))$ is a member of $\mathbf{F}$. In this case, the **fiber class morphism** $\mathbf{F}_b$ from $\mathbf{Q}_b$ to $\mathbf{R}_b$ associated to $\mathbf{F}$ is the class morphism whose members are all ordered pairs (c, d) such that $((b, c), (b, d))$ is a member of $\mathbf{F}$. A $\mathbf{B}$-class morphism $\mathbf{F}$ from $\mathbf{Q}$ to $\mathbf{R}$ is a **$\mathbf{B}$-class isomorphism** if and only if $\mathbf{F}$ is a class isomorphism from $\mathbf{R}$ to $\mathbf{Q}$.

In particular, for every class $\mathbf{B}$, for every class $\mathbf{C}$, for every $(\mathbf{B}, \mathbf{C})$-span $\mathbf{M}$, for every $(\mathbf{B}, \mathbf{C})$-span $\mathbf{N}$, a **$(\mathbf{B}, \mathbf{C})$-span morphism** from $\mathbf{M}$ to $\mathbf{N}$ is a $\mathbf{B} \times \mathbf{C}$-class morphism from $\mathbf{M}$ to $\mathbf{N}$. This is a **$(\mathbf{B}, \mathbf{C})$-span isomorphism $\mathbf{F}$** from $\mathbf{M}$ to $\mathbf{N}$ if (and only if) it is a $\mathbf{B} \times \mathbf{C}$-class isomorphism from $\mathbf{M}$ to $\mathbf{N}$.

Example 5.8. For every class $\mathbf{B}$, for every $\mathbf{B}$-class $\mathbf{Q}$, the identity class isomorphism $\mathrm{Id}_{\mathbf{Q}}$ is a $\mathbf{B}$-class isomorphism from $\mathbf{Q}$ to itself such that $(\mathrm{Id}_{\mathbf{Q}})_b$ equals $\mathrm{Id}_{\mathbf{Q}_b}$ for every member b of $\mathbf{B}$. Similarly, for every class $\mathbf{B}$, for every class $\mathbf{C}$, for every $(\mathbf{B}, \mathbf{C})$-span $\mathbf{M}$, the identity class isomorphism $\mathrm{Id}_{\mathbf{M}}$ is a $(\mathbf{B}, \mathbf{C})$-span isomorphism such that $(\mathrm{Id}_{\mathbf{M}})^b_c$ equals $\mathrm{Id}_{\mathbf{M}^b_c}$ for every member (b, c) of $\mathbf{B} \times \mathbf{C}$. In particular, for every class $\mathbf{O}$, the identity $\mathrm{Id}_{\mathbf{Id_O}}$ class morphism from $\mathbf{Id_O}$ to itself is an isomorphism of $\mathbf{O}$-Hom spans.

The notion of composition of functions and relations between sets extends to composition of morphisms and relations between classes, as well as composition of spans.

Definition 5.9. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every relation $\mathbf{Q}$ from $\mathbf{B}$ to $\mathbf{C}$, for every relation $\mathbf{R}$ from $\mathbf{C}$ to $\mathbf{D}$, a class the **composition** $\mathbf{R}\circ\mathbf{Q}$ of $\mathbf{R}$ and $\mathbf{Q}$ is the class whose members are all ordered pairs (b,d) such that there exists a member c of $\mathbf{C}$ with both (b,c) a member of $\mathbf{Q}$ and (c,d) a member of $\mathbf{R}$.

Definition 5.10. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, for every span $\mathbf{N}$ from $\mathbf{C}$ to $\mathbf{D}$, the **span composition** $\mathbf{N}\circ\mathbf{M}$ of $\mathbf{N}$ and $\mathbf{M}$ is the span from $\mathbf{B}$ to $\mathbf{D}$ such that for every member (b,d) of $\mathbf{B}\times\mathbf{D}$, the members of the fiber class $(\mathbf{N}\circ\mathbf{M})^b_d$ are all ordered pairs $(c,(n,m))$ of a member c of $\mathbf{C}$ and members n and m of the respective fiber categories $\mathbf{N}^c_d$ and $\mathbf{M}^b_c$.

Example 5.11. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, there is an isomorphism of $(\mathbf{B},\mathbf{C})$-spans $\mathsf{r}_{\mathbf{M}}$ from $\mathbf{M}\circ\mathbf{Id}_{\mathbf{B}}$ to $\mathbf{M}$, respectively $\mathsf{l}_{\mathbf{M}}$ from $\mathbf{Id}_{\mathbf{C}}\circ\mathbf{M}$ to $\mathbf{M}$, sending every member $((b,c),(b,(m,\mathrm{Id}_b)))$ of $\mathbf{M}\circ\mathbf{Id}_{\mathbf{B}}$ to $((b,c),m)$, respectively sending every member $((b,c),(c,(\mathrm{Id}_c,m)))$ of $\mathbf{Id}_{\mathbf{C}}\circ\mathbf{M}$ to $((b,c),m)$. The isomorphism $\mathsf{r}_{\mathbf{M}}$, respectively $\mathsf{l}_{\mathbf{M}}$, is the **right unitor** of $\mathbf{M}$, resp. the **left unitor** of $\mathbf{M}$.

Example 5.12. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every class $\mathbf{D}$, for every class $\mathbf{E}$, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, for every span $\mathbf{N}$ from $\mathbf{C}$ to $\mathbf{D}$, and for every span $\mathbf{P}$ from $\mathbf{D}$ to $\mathbf{E}$, there is an isomorphism of $(\mathbf{B},\mathbf{E})$-spans $\mathsf{a}_{\mathbf{P},\mathbf{N},\mathbf{M}}$ from $(\mathbf{P}\circ\mathbf{N})\circ\mathbf{M}$ to $\mathbf{P}\circ(\mathbf{N}\circ\mathbf{M})$ sending every member $((b,e),(c,((d,(p,n)),m)))$ of $(\mathbf{P}\circ\mathbf{N})\circ\mathbf{M}$ to the member $((b,e),(d,(p,(c,(n,m)))))$ of $\mathbf{P}\circ(\mathbf{Q}\circ\mathbf{R})$. In other words, for every member (b,e) of $\mathbf{B}\times\mathbf{E}$, the induced isomorphism of fiber classes from $((\mathbf{P}\circ\mathbf{N})\circ\mathbf{M})^b_e$ to $(\mathbf{P}\circ(\mathbf{N}\circ\mathbf{M}))^b_e$ sends $(c,((d,(p,n)),m))$ to $(d,(p,(c,(n,m))))$, i.e., it transposes c and d while leaving p, n and m in the same order. The isomorphism $\mathsf{a}_{\mathbf{P},\mathbf{N},\mathbf{M}}$ is the **associator** of $\mathbf{P}$, $\mathbf{N}$ and $\mathbf{M}$.

Example 5.13. For the von Neumann class $\mathbf{V}$ of all sets, consider the span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to $\mathbf{V}$ such that for every set b and for every set c, the members of the fiber class over (b,c) are all subsets of $b\times c$ that are (graphs of) functions from b to c. In other words, for every member (b,c) of $\mathbf{V}\times\mathbf{V}$, the fiber class is the class of the set $\mathrm{Fun}(b,c)$ of all functions from b to c. The span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself, together with the *usual composition law*, is the category $\mathbf{Set}$ of all sets.

Proposition 5.14. *Composition of relations between classes is strictly associative, and the identity relations are strict left-right identities for this composition. Composition of spans is associative up to the specified associator* a*, and the identity spans are left-right identities for this composition up to the left and right unitors* l *and* r*. The associator and unitors satisfy the triangle (coherence) identity and the pentagon (coherence) identity of monoidal categories.*

There is a notion of morphisms of spans. Together with the composition, associator and unitors, spans satisfy the axioms of (a version of) *double category*. Of course spans are classes that may not be sets, so extreme care is necessary in forming any kind of category of spans.

Exercise 5.15. Read about double categories. Formulate and verify the axioms of a double category that are satisfied by the operations above for spans.

Spans admit a more general notion of morphisms that is useful in formulating natural transformations.

Definition 5.16. For every ordered triple $(\mathbf{B}, \mathbf{C}, \mathbf{M})$ of classes $\mathbf{B}$ and $\mathbf{C}$ and a span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, for every ordered triple $(\mathbf{B}', \mathbf{C}', \mathbf{M}')$ of classes $\mathbf{B}'$ and $\mathbf{C}'$ and a span $\mathbf{M}'$ from $\mathbf{B}'$ to $\mathbf{C}'$, a **span cell** from $(\mathbf{B}, \mathbf{C}, \mathbf{M})$ to $(\mathbf{B}', \mathbf{C}', \mathbf{M}')$ is a class $\mathbf{F} = ((\mathsf{s}(\mathbf{F}), \mathsf{t}(\mathbf{F})), \mathbf{F}_{\text{mor}})$ of a morphism of classes $\mathsf{s}(\mathbf{F})$ from $\mathbf{B}$ to $\mathbf{B}'$, of a morphism of classes $\mathsf{t}(\mathbf{F})$ from $\mathbf{C}$ to $\mathbf{C}'$, and of a morphism of classes $\mathbf{F}_{\text{mor}}$ from $\mathbf{M}$ to $\mathbf{M}'$ such that for every member $((b,c),m)$ of $\mathbf{M}$, for the unique member $((b',c'),m')$ of $\mathbf{M}'$ such that $(((b,c),m),((b',c'),m'))$ is a member of $\mathbf{F}$, also (b,b') is a member of $s(\mathbf{F})$ and (c,c') is a member of $t(\mathbf{F})$.

Example 5.17. For every ordered triple $(\mathbf{B}, \mathbf{C}, \mathbf{M})$ of class $\mathbf{B}$ and $\mathbf{C}$ and a span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, the **identity span cell** is $(\text{Id}_{\mathbf{B}}, \text{Id}_{\mathbf{C}}, \text{Id}_{\mathbf{M}})$.

Exercise 5.18. Check that the identity span cell is a span cell.

Example 5.19. For every ordered triple $(\mathbf{B}, \mathbf{C}, \mathbf{M})$ of a span $\mathbf{M}$ from a class $\mathbf{B}$ to a class $\mathbf{C}$, for every ordered triple $(\mathbf{B}', \mathbf{C}', \mathbf{M}')$ of a span $\mathbf{M}'$ from a class $\mathbf{B}'$ to a class $\mathbf{C}'$, for every ordered triple $(\mathbf{B}'', \mathbf{C}'', \mathbf{M}'')$ of a span $\mathbf{M}''$ from a class $\mathbf{B}''$ to a class $\mathbf{C}''$, for every span cell $\mathbf{F} = (s(\mathbf{F}), t(\mathbf{F}), \mathbf{F}_{\text{mor}})$ from $(\mathbf{B}, \mathbf{C}, \mathbf{M})$ to $(\mathbf{B}', \mathbf{C}', \mathbf{M}')$, and for every span cell $\mathbf{F}' = (s(\mathbf{F}'), s(\mathbf{F}'), \mathbf{F}'_{\text{mor}})$ from $(\mathbf{B}', \mathbf{C}', \mathbf{M}')$ to $(\mathbf{B}'', \mathbf{C}'', \mathbf{M}'')$, the **composition span cell** is $(s(\mathbf{F}') \circ s(\mathbf{F}), t(\mathbf{F}') \circ t(\mathbf{F}), \mathbf{F}'_{\text{mor}} \circ \mathbf{F}_{\text{mor}})$ from $(\mathbf{B}, \mathbf{C}, \mathbf{M})$ to $(\mathbf{B}'', \mathbf{C}'', \mathbf{M}'')$.

Exercise 5.20. Check that the composition span cell is a span cell.

Exercise 5.21. Check that composition of span cells is strictly associative. Also check that identity span cells are strict left-right identities for composition of span cells.

One advantage of relations, and more generally of spans, over morphisms is that they have *opposites*.

Definition 5.22. For every class $\mathbf{B}$, for every class $\mathbf{C}$, for every relation $\mathbf{R}$ from $\mathbf{B}$ to $\mathbf{C}$, the **opposite relation** $\mathbf{R}^{\text{opp}}$ from $\mathbf{C}$ to $\mathbf{B}$ is the unique subclass of $\mathbf{C} \times \mathbf{B}$ whose members are all ordered pairs (c,b) such that (b,c) is a member of $\mathbf{R}$.

More generally, for every span $\mathbf{M}$ from $\mathbf{B}$ to $\mathbf{C}$, the **opposite span** $\mathbf{M}^{\text{opp}}$ from $\mathbf{C}$ to $\mathbf{B}$ is the $\mathbf{C} \times \mathbf{B}$-class such that for every member b of $\mathbf{B}$ and for every member c of $\mathbf{C}$, the fiber class $(\mathbf{M}^{\text{opp}})^c_b$ equals the fiber class $\mathbf{M}^b_c$.

Exercise 5.23. Formulate the notion of the opposite of a span cell. Check that the opposite span of a span composite is naturally span isomorphic to the span composite of the span opposites of the factors (in the opposite order). Read about *dagger categories*. Formulate and check the axioms of a dagger category that hold for spans.

6 Definition of categories

A category is a span from a class to itself whose fiber classes are required to be (classes of) sets, and equipped with a morphism of spans to itself from the span composite of the span with itself that is associative and unital.

Definition 6.1. For every class $\mathbf{O}$, an **$\mathbf{O}$-Hom span** is an $(\mathbf{O},\mathbf{O})$-set $\mathbf{M}$, i.e., a class in which every member is of the form $((a,b),f)$ for members a and b of $\mathbf{O}$ and a set f, and such that each fiber class $\mathbf{M}^a_b$ of all sets f with $((a,b),f)$ is a member of $\mathbf{M}$ is the class of a set, the **Hom set** of $\mathbf{M}$ over (a,b). For every class $\mathbf{O}$, for every $\mathbf{O}$-Hom span $\mathbf{M}'$, for every $\mathbf{O}$-Hom span $\mathbf{M}$, a **morphism** of $\mathbf{O}$-Hom spans from $\mathbf{M}'$ to $\mathbf{M}$ is a morphism of $(\mathbf{O},\mathbf{O})$-classes from $\mathbf{M}'$ to $\mathbf{M}$.

Breaking with our earlier convention, we sometimes denote the Hom set by $\mathbf{M}^a_b$. More often it is denoted $\text{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$, or just $\text{Hom}(a,b)$ when $\mathbf{O}$ and $\mathbf{M}$ are understood, i.e., the members of $\mathbf{M}$ are sets $((a,b),f)$ for members a and b of $\mathbf{O}$ and elements f of $\text{Hom}(a,b)$.

Example 6.2. For every class $\mathbf{O}$, the empty $(\mathbf{O},\mathbf{O})$-span $\mathbf{M}$ with no members is an $\mathbf{O}$-Hom span, the **initial $\mathbf{O}$-Hom span**. For every class $\mathbf{O}$, the identity $\text{Id}_{\mathbf{O}\times\mathbf{O}}$ of $\mathbf{O}\times\mathbf{O}$, considered as a span from $\mathbf{O}$ to itself, is an $(\mathbf{O},\mathbf{O})$-span, the **final $\mathbf{O}$-Hom span**. Finally, the **identity Hom span $\text{Id}_{\mathbf{O}}$** is the class whose members are all ordered pairs $((b,b),\text{Id}_b)$ for b a member of $\mathbf{O}$. This is also called the **discrete $\mathbf{O}$-Hom span**.

Example 6.3. For every set H, let $\mathbf{O}_H$ be a class with a unique member (say $\varnothing$, for definiteness), and let $\mathbf{M}_H$ be the unique $\mathbf{O}_H$-Hom span whose unique Hom set is H.

Example 6.4. Recall the earlier example, where $\mathbf{O}$ is the von Neumann class $\mathbf{V}$ of all sets, the span $\text{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself is the class of all triples $((a,b),f)$ of a set a, of a set b, and of a function f from a to b. Thus, each Hom set $\text{Hom}_{\mathbf{V},\text{mor}(\mathbf{Set})}(a,b)$ is the set $\text{Fun}(a,b)$ of all functions from a to b.

Example 6.5. For another example, again let $\mathbf{O}$ be the von Neumann class $\mathbf{V}$ of all sets, but now let the span $\text{mor}(\mathbf{Rel})$ from $\mathbf{V}$ to itself be the class of all triples $((a,b),R)$ of a set a, of a set b, and of a relation R from a to b, i.e., R is an (arbitrary) subset of $a\times b$. Thus, each Hom set $\text{Hom}_{\mathbf{V},\text{mor}(\mathbf{Rel})}(a,b)$ is the power set $\mathcal{P}(a\times b)$ of $a\times b$.

Example 6.6. For every class $\mathbf{O}$, for every $\mathbf{O}$-Hom span $\mathbf{M}$, for every $\mathbf{O}$-Hom span $\mathbf{M}'$, for every $\mathbf{O}$-Hom span $\mathbf{M}''$, for every morphism $\mathbf{F}'$ of $\mathbf{O}$-Hom spans from $\mathbf{M}'$ to $\mathbf{M}$, and for every morphism $\mathbf{F}''$ of $\mathbf{O}$-Hom spans from $\mathbf{M}''$ to $\mathbf{M}$, the fiber product $\mathbf{M}'\times_{\mathbf{F}',\mathbf{M},\mathbf{F}''}\mathbf{M}''$, or just $\mathbf{M}'\times_{\mathbf{M}}\mathbf{M}''$ when confusion is unlikely, is also an $\mathbf{O}$-Hom span whose fiber class for each ordered pair (a,b) of members of $\mathbf{O}$ equals the fiber product set $(\mathbf{M}')^a_b\times_{\mathbf{M}^a_b}(\mathbf{M}'')^a_b$. In particular, $\mathbf{M}\times_{\mathbf{O}\times\mathbf{O}}\mathbf{M}$ is the $\mathbf{O}$-Hom span whose fiber class is just the product set $\mathbf{M}^a_b\times\mathbf{M}^a_b$ for every ordered pair (a,b) of members of $\mathbf{O}$.

Of course, for a Hom span $(\mathbf{O},\mathbf{M})$, the composite span $\mathbf{M}\circ\mathbf{M}$ from $\mathbf{O}$ to $\mathbf{O}$ is typically **not** a Hom span: for all members a and c of $\mathbf{O}$, the members of $(\mathbf{M}\circ\mathbf{M})^a_c$ are all ordered triples $(b,(g,f))$ of a member b of $\mathbf{O}$, of an element f of the set $\mathrm{Hom}(a,b)$ and of an element g of $\mathrm{Hom}(b,c)$. Since b varies over members of a class (that is typically not a set), the class $(\mathbf{M}\circ\mathbf{M})^a_c$ is typically not a set. This is a Hom span if and only if $\mathbf{O}$ is the class of a set.

Definition 6.7. A Hom span $(\mathbf{O},\mathbf{M})$ is **small** if (and only if) the class $\mathbf{O}$ is the class of a set.

Example 6.8. In the example **Set**, for every set a, for every set c, the fiber class $(\mathrm{mor}(\mathbf{Set})\circ\mathrm{mor}(\mathbf{Set}))^a_c$ is the class of all triples $(b,(g,f))$ of a set b, of a function f from a to b, and of a function g from b to c. This is not the class of a set, since the class of all sets b (i.e., the von Neumann class) is not the class of a set.

Example 6.9. On the other hand, for every small Hom span $(\mathbf{O},\mathbf{M})$, for every nonnegative integer n, the n-fold composite of the $\mathbf{O}$-Hom span is again an $\mathbf{O}$-Hom span. Taking the union over all positive integers n gives a new $\mathbf{O}$-Hom span $(\mathbf{O},\mathbf{M}^*)$ where the fiber class over (a,b) is the set of **strings**, i.e., ordered pairs $(n,(a=a_0\xrightarrow{f_1}a_1,a_1\xrightarrow{f_2}a_2,\dots,a_{n-1}\xrightarrow{f_n}a_n=b))$ of a positive integer n and an ordered n-tuple of "composable" members of $\mathbf{M}$. We "complete" this by also adding a member $(0,(a=a_0,a_0=a))$ of $\mathbf{M}^*$ mapping to (a,a) in $\mathbf{O}\times\mathbf{O}$ for every member a of $\mathbf{O}$.

Definition 6.10. For every class $\mathbf{O}$, for every Hom span $\mathbf{M}$ from $\mathbf{O}$ to itself, an **$(\mathbf{O},\mathbf{M})$-composition law** is a span morphism $\circ$ from the composition $(\mathbf{O},\mathbf{O})$-span $\mathbf{M}\circ\mathbf{M}$ to $\mathbf{M}$, i.e., a morphism of $\mathbf{O}\times\mathbf{O}$-classes such that, for all members a and c of $\mathbf{O}$, the induced fiber morphism from $(\mathbf{M}\circ\mathbf{M})^a_c$ to $\mathbf{M}^a_c$ sends each member $(b,(g,f))$ of $(\mathbf{M}\circ\mathbf{M})^a_c$ to a member $g\circ f$ of $\mathbf{M}^a_c$.

A composition law is **associative** if (and only if), for all members a, b, c and d of $\mathbf{O}$, for every element (h,g,f) of $\mathrm{Hom}(d,e)\times\mathrm{Hom}(c,d),\mathrm{Hom}(b,c)$, the composition $(h\circ g)\circ f$ equals $h\circ(g\circ f)$ as elements of $\mathrm{Hom}(a,e)$.

An associative composition law is **unital** if (and only if), for every member a of $\mathbf{O}$, there exists an element $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ of $\mathrm{Hom}(a,a)$ such that, for every member b of $\mathbf{O}$, both the left composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ from $\mathrm{Hom}(b,a)$ to itself is the identity, and the right composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ from $\mathrm{Hom}(a,b)$ to itself is the identity.

A **category** is an ordered triple class $(\mathbf{O},\mathbf{M},\circ)$ of a class $\mathbf{O}$, called the **class of objects**, an $\mathbf{O}$-Hom span $\mathbf{M}$, called the **class of morphisms**, the specification of the **source** morphism, respectively **target** morphism, from $\mathbf{M}$ to $\mathbf{O}$ sending every member $((a,b),f)$ of $\mathbf{M}$ to the member a of $\mathbf{O}$, respectively to the member b of $\mathbf{O}$, and a **$(\mathbf{O},\mathbf{M})$-composition law** $\circ$ that is both associative and unital. An **isomorphism** in a category is a morphism $((a,b),f)$ such that there exists a morphism $((b,a),g)$ with both $g\circ f$ equal to Id_a and $f\circ g$ equal to Id_b; in this case we denote g by f^{-1}.

For a category $\mathbf{C}$, the class $\mathbf{O}$ is often denoted $\text{ob}(\mathbf{C})$ and its members are called **C-objects** or **objects** of $\mathbf{C}$. The class $\mathbf{M}$ is often denoted $\text{mor}(\mathbf{C})$, each set $\text{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$ is denoted $\mathbf{C}_b^a$ or $\text{Hom}_{\mathbf{C}}(a,b)$ and its elements are called **C-morphisms** from a to b. The composition law is denoted $\circ^{\mathbf{C}}$, or just $\circ$ when confusion is unlikely. For every object a of $\mathbf{C}$, the left-right identity morphism from a to itself is usually denoted $\text{Id}_a^{\mathbf{C}}$ or Id_a when confusion is unlikely (this set may or may not equal the identity function from the set a to itself, so please use caution). A category is **small** if (and only if) the class of objects is (the class of) a set.