MAT 535 Notes on Categories

1 Definition of categories

A category is a span $\mathbf{M}$ from a class $\mathbf{O}$ to itself whose fiber classes are required to be sets, and equipped with an associative, unital morphism $\circ$ of spans to $\mathbf{M}$ from the self-composite of $\mathbf{M}$.

Definition 1.1. For every class $\mathbf{O}$, an **O-Hom span** is a class $\mathbf{M}$ whose members are ordered pairs $((a,b),f)$ for members a and b of $\mathbf{O}$, such that each fiber class $\mathbf{M}_b^a$ (of all f with $((a,b),f)$ a member of $\mathbf{M}$) is the class of a set, $\mathrm{Hom}_{\mathbf{O},\mathbf{M}}(a,b)$. This is the **Hom set** of $\mathbf{M}$ over (a,b). For every class $\mathbf{O}$, and for all **O**-Hom spans $\mathbf{M}'$ and $\mathbf{M}$, a **morphism** of **O**-Hom spans from $\mathbf{M}'$ to $\mathbf{M}$ is a class whose members are ordered pairs $((a,b),(f',f))$ with $((a,b),f')$ a member of $\mathbf{M}'$ and $((a,b),f)$ a member of $\mathbf{M}$, such that for every member $((a,b),f')$ of $\mathbf{M}'$, there exists a unique member $((a,b),(f',f))$ of the morphism. Morphisms are composed in the same way as spans.

Example 1.2. For every class $\mathbf{O}$, the empty $(\mathbf{O},\mathbf{O})$-span $\mathbf{M}$ with no members is an $\mathbf{O}$-Hom span, the **initial O-Hom span**. For every class $\mathbf{O}$, the identity span $\mathbf{Id}_{\mathbf{O}\times\mathbf{O}}$ has as members all ordered pairs $((a,b),(a,b))$ for members a and b of $\mathbf{O}$. Considered as a span from $\mathbf{O}$ to itself, this is an $(\mathbf{O},\mathbf{O})$-span, the **final O-Hom span**. Finally, the **discrete O-Hom span** is the class whose members are all ordered pairs $((b,b),\mathrm{Id}_b)$ for b a member of $\mathbf{O}$.

Example 1.3. For every set H, let $\mathbf{O}_H$ be a class with a unique member (say $\varnothing$, for definiteness), and let $\mathbf{M}_H$ be the unique $\mathbf{O}_H$-Hom span whose unique Hom set is H.

Example 1.4. Let $\mathbf{O}$ be the von Neumann class $\mathbf{V}$ of all sets, and let the span $\mathrm{mor}(\mathbf{Set})$ from $\mathbf{V}$ to itself be the class of all triples $((a,b),f)$ of a set a, of a set b, and of a function f from a to b. Thus, each Hom set $\mathrm{Hom}_{\mathbf{V},\mathrm{mor}(\mathbf{Set})}(a,b)$ is the set $\mathrm{Fun}(a,b)$ of all functions from a to b.

Example 1.5. For another example, again let $\mathbf{O}$ be the von Neumann class $\mathbf{V}$ of all sets, but now let the span $\mathrm{mor}(\mathbf{Rel})$ from $\mathbf{V}$ to itself be the class of all triples $((a,b),R)$ of a set a, of a set b, and of a subset R of $a\times b$. Thus, each Hom set $\mathrm{Hom}_{\mathbf{V},\mathrm{mor}(\mathbf{Rel})}(a,b)$ is the power set $\mathcal{P}(a\times b)$ of $a\times b$.

Of course, for $\mathbf{O}$-Hom spans $\mathbf{M}$ and $\mathbf{M}'$ the composite span $\mathbf{M} \circ \mathbf{M}'$ from $\mathbf{O}$ to $\mathbf{O}$ is typically **not** a Hom span: for all members a and c of $\mathbf{O}$, the members of the fiber class $(\mathbf{M} \circ \mathbf{M})_c^a$ are all ordered triples $(b, (f, f'))$ of a member b of $\mathbf{O}$, of an element f of the set $\mathrm{Hom}_{\mathbf{O},\mathbf{M}}(b, c)$ and of an element f' of $\mathrm{Hom}_{\mathbf{O},\mathbf{M}'}(a, b)$.

Definition 1.6. For every class $\mathbf{O}$, a $\mathbf{O}$-Hom span $\mathbf{M}$ is **small** if (and only if) the class $\mathbf{O}$ is the class of a set.

Example 1.7. For every small Hom span $(\mathbf{O}, \mathbf{M})$, for every positive integer n, the n-fold composite of the $\mathbf{O}$-Hom span is again an $\mathbf{O}$-Hom span. Taking the union over all positive integers n gives a new $\mathbf{O}$-Hom span $(\mathbf{O}, \mathbf{M}^*)$, the **Kleene star** or **Kleene closure**, where the fiber class over (a, b) is the set of **strings**, i.e., ordered pairs $\left(n, \left(a = a_0 \xrightarrow{f_1} a_1, a_1 \xrightarrow{f_2} a_2, \ldots, a_{n-1} \xrightarrow{f_n} a_n = b\right)\right)$ of a positive integer n and an ordered n-tuple of "composable" members of $\mathbf{M}$. We "complete" this by also adding a member $(0, (a = a_0, a_0 = a))$ of $\mathbf{M}^*$ mapping to (a, a) in $\mathbf{O} \times \mathbf{O}$ for every member a of $\mathbf{O}$.

Definition 1.8. For every class $\mathbf{O}$, for every Hom span $\mathbf{M}$ from $\mathbf{O}$ to itself, an $(\mathbf{O}, \mathbf{M})$-**composition law** is a morphism of $\mathbf{O}$-Hom spans comp from $\mathbf{M} \circ \mathbf{M}$ to $\mathbf{M}$, i.e., an assignment to every member (c, b, a) of $\mathbf{O} \times \mathbf{O} \times \mathbf{O}$ and every member (g, f) of $\mathbf{M}_c^b \times \mathbf{M}_b^a$ of a unique member $g \circ f$ of $\mathbf{M}_c^a$.

An $(\mathbf{O}, \mathbf{M})$-composition law is **associative** if (and only if), for all members (d, c, b, a) of $\mathbf{O} \times \mathbf{O} \times \mathbf{O} \times \mathbf{O}$ and for every member (h, g, f) of $\mathbf{M}_d^c \times \mathbf{M}_c^b \times \mathbf{M}_b^a$, the composition $(h \circ g) \circ f$ equals $h \circ (g \circ f)$.

An associative composition law is **unital** if (and only if), for every member a of $\mathbf{O}$ there exists an element $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ of $\mathbf{M}_a^a$ such that, for every member b of $\mathbf{O}$, both the left composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ is the identity on $\mathbf{M}_a^b$, and the right composition with $\mathrm{Id}_a^{\mathbf{O},\mathbf{M},\circ}$ is the identity on $\mathbf{M}_b^a$.

A **category** is an ordered triple class $(\mathbf{O}, \mathbf{M}, \circ)$ of a class $\mathbf{O}$, the **class of objects**, of an $\mathbf{O}$-Hom span $\mathbf{M}$, the **class of morphisms**, and of a $(\mathbf{O}, \mathbf{M})$-**composition law** comp that is both associative and unital.

Definition 1.9. An **isomorphism** in a category is a morphism $((a, b), f)$ such that there exists a morphism $((b, a), f^{-1})$ with both $f^{-1} \circ f$ equal to Id_a and $f \circ f^{-1}$ equal to Id_b. A category is **small** if (and only if) the class of objects is (the class of) a set.

2 Examples of categories

There are many elementary examples of categories, and there are many different properties that a category can possess.

2.1   Some universal categories

Example 2.1. For every class $\mathbf{O}$, the final $\mathbf{O}$-Hom span (where every Hom set is a singleton) has a unique composition law, and this composition law is associative and unital. This is a **final category structure** on $\mathbf{O}$. Similarly, the discrete $\mathbf{O}$-Hom span (whose only morphisms are identity morphisms) also has a unique composition law, and this is associative and unital. This is the **discrete category structure** on $\mathbf{O}$. It is initial among category structures on $\mathbf{O}$.

Definition 2.2. For every small class $\mathbf{O}$, for every $\mathbf{O}$-Hom span $\mathbf{M}$, the **free category** on $(\mathbf{O}, \mathbf{M})$ is the $\mathbf{O}$-Hom span of the Kleene closure $\mathbf{M}^\star$ whose members are composable strings of morphisms from $\mathbf{M}$ with composition law given by concatenation, i.e., for every morphism $f = (m, (a_0 \xrightarrow{f_1} a_1, \ldots, a_{m-1} \xrightarrow{f_m} a_m))$ and for every morphism $g = (n, (b_0 \xrightarrow{g_1} b_1, \ldots, b_{n-1} \xrightarrow{g_n} b_n))$ such that a_m equals b_0, the composition is

$$g \circ f := (n + m, (a_0 \xrightarrow{f_1} a_1, \ldots, a_{m-1} \xrightarrow{f_m} a_m, b_0 \xrightarrow{g_1} b_1, \ldots, b_{n-1} \xrightarrow{g_n} b_n)).$$

Of course each element $(0, (a_0, a_0))$ composes as a left-right identity.

Example 2.3. The category $\mathbf{Set}$ of sets has object class equal to the von Neumann class $\mathbf{V}$ of all sets, has morphism class $\mathrm{mor}(\mathbf{Set})$ as above with fiber class $\mathbf{Set}_b^a$ equal to the (class of the) set $\mathrm{Hom}_{\mathbf{Set}}(a, b) = \mathrm{Fun}(a, b)$ of all functions f from a to b, and has the usual composition of functions.

Example 2.4. The category $\mathbf{Rel}$ of relations again has object class equal to $\mathbf{V}$, but each fiber class $\mathbf{Rel}_b^a$ equals the (class of the) power set $\mathcal{P}(a \times b)$ of $a \times b$ for all sets a and b. Composition is composition of relations.

Example 2.5. For every small category $\mathbf{B}$, for every small category $\mathbf{C}$, each object of the category $\mathbf{Span}_{\mathbf{C}}^{\mathbf{B}}$ is a (set whose associated class is a) $\mathbf{B} \times \mathbf{C}$-set, i.e., a span from $\mathbf{B}$ to $\mathbf{C}$ whose fiber classes are all (classes of) sets. The morphisms between two such spans are span cells such that the class morphism from $\mathbf{B}$ to itself is the identity and the class morphism from $\mathbf{C}$ to itself is the identity. Composition is composition of span cells.

2.2   Opposites and subcategories

There are many ways to produce new categories from given categories.

Definition 2.6. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, the **opposite category** is the category $(\mathbf{O}, \mathbf{M}^{\mathrm{opp}}, \circ^{\mathrm{opp}})$, where $\mathbf{M}^{\mathrm{opp}}$ is the opposite span of $\mathbf{M}$, and where, for every member $((a, b), f)$ of $\mathbf{M}$ and for every member $((b, c), g)$ of $\mathbf{M}$, the opposite composition is defined by

$$((b, a), f) \circ^{\mathrm{opp}} ((c, b), g) = ((c, a), g \circ f).$$

Definition 2.7. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every subclass $\mathbf{O}'$ of $\mathbf{O}$, the **full subcategory** of $\mathbf{C}$ with objects class $\mathbf{O}'$ is the category $(\mathbf{O}', \mathbf{M}|_{\mathbf{O}'}, \circ')$ where, for all members a and b of $\mathbf{O}'$, the class $(\mathbf{M}|_{\mathbf{O}'})^a_b$ equals $\mathbf{M}^a_b$, and where $\circ'$ is the restriction of $\circ$. More generally, a (not necessarily full) **subcategory** of $\mathcal{C}$ consists of a subclass $\mathbf{O}'$ of $\mathbf{O}$ and a subclass of $\mathbf{M}'$ of $\mathbf{M}|_{\mathbf{O}'}$ that contains all identity morphisms of objects of $\mathbf{O}'$ and that is stable for composition, thus defining a restriction composition on the subcategory.

Example 2.8. The category **Set** of sets is a non-full subcategory of the category **Rel** of all relations. The category of all finite sets is a full subcategory of the category of all sets.

Example 2.9. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, the discrete category on $\mathbf{O}$ is (uniquely) a (typically not full) subcategory of $\mathbf{C}$.

2.3   Preorders and partially ordered sets

Definition 2.10. A category is a **monoid** if (and only if) the object class is the class of a singleton set. A category is **thin** if (and only if) every nonempty Hom set is a singleton set. A category is a **groupoid** if (and only if) every morphism is an isomorphism. A thin groupoid is a **setoid**. A category is **skeletal** if (and only if) all isomorphic objects are equal. A skeletal setoid is a **discrete category**.

Definition 2.11. For every set A, for every relation $R \subset A \times A$, the relation R is a **preorder** if (and only if) R is both reflexive and transitive, i.e., (a, a) is in R for every a in A, and (a, c) is in R for every (a, b) and (b, c) in R. A preorder R on a set A is a (nonstrict) **partial order** if (and only if) R is antisymmetric, i.e., for every (a, b) in R, if also (b, a) is in R then b equals a. A partial order R on a set A is a (nonstrict) **total order** if (and only if), for every element (a, a') of $A \times A$, either (a, a') or (a', a) is an element of R; then we often denote this with infix notation, i.e., we write $a \leq_R a'$ if and only if (a, a') is an element of R. A total order R on a set A is a (nonstrict) **well order** if (and only if), every nonempty subset of A has a unique minimal element.

Definition 2.12. For every set A with a preorder R, the **associated thin category** has objects class equal to the (class of the) set A and has morphisms class equal to the (class of the) set R. Composition is defined in the unique way, i.e., for all elements a, b and c of R, if (a, b) is in R and (b, c) is in R, then the composition is the element (a, c) of R.

Exercise 2.13. Prove that this does define a thin category. Prove that R is a (nonstrict) partial order if and only if this thin category is skeletal. Prove that every small thin category is equivalent to the associated thin category of a set with preorder. Thus, also every small skeletal thin category is equivalent to the associated thin category of a set with (nonstrict) partial order.

Definition 2.14. For every small category $\mathbf{C}$ whose objects form a (class of a) set A, the **associated preorder** on A is the subset of $A \times A$ consisting of all ordered pairs (a, b) such that there exists a $\mathbf{C}$-morphism from a to b.

Exercise 2.15. Prove that this is a preorder.

Definition 2.16. For every set A with preorder R, for a set B with preorder S, an **order preserving function** from (A, R) to (B, S) is a function f from A to B such that for every element (a, a') of R, also $(f(a), f(a'))$ is an element of S.

Example 2.17. The objects of the category **PreOrd** are ordered pairs (A, R) of a set A and a preorder $R \subseteq A \times A$. For every preordered set (A, R), for every preordered set (B, S), the order preserving functions from (A, R) to (B, S) are the morphisms in **PreOrd** from (A, R) to (B, S). Composition is composition of functions. The full subcategory of **PreOrd** whose objects are (nonstrict) partially ordered sets is denoted **POrd**. This category has a non-full subcategory $\mathbf{POrd}_{\mathrm{inj}}$, respectively $\mathbf{POrd}_{\mathrm{surj}}$, $\mathbf{POrd}_{\mathrm{bij}}$, with the same objects and whose morphisms are the order preserving functions that are injective, resp. surjective, bijective. Similarly, the non-full subcategory with the same objects whose morphisms are order preserving, injective functions onto a lower set of the codomain is denoted $\mathbf{POrd}_{\mathrm{init}}$. The full subcategory of **POrd**, respectively of $\mathbf{POrd}_{\mathrm{inj}}$, of $\mathbf{POrd}_{\mathrm{surj}}$, of $\mathbf{POrd}_{\mathrm{bij}}$, of $\mathbf{POrd}_{\mathrm{init}}$, whose objects are totally ordered sets is denoted **Ord**, resp. $\mathbf{Ord}_{\mathrm{inj}}$, $\mathbf{Ord}_{\mathrm{surj}}$, $\mathbf{Ord}_{\mathrm{bij}}$, $\mathbf{Ord}_{\mathrm{init}}$. The full subcategory of **Ord**, respectively of $\mathbf{Ord}_{\mathrm{inj}}$, $\mathbf{Ord}_{\mathrm{surj}}$, $\mathbf{Ord}_{\mathrm{bij}}$, $\mathbf{Ord}_{\mathrm{init}}$, whose objects are totally ordered sets is denoted **TotOrd**, resp. $\mathbf{TotOrd}_{\mathrm{inj}}$, $\mathbf{TotOrd}_{\mathrm{surj}}$, $\mathbf{TotOrd}_{\mathrm{bij}}$, $\mathbf{TotOrd}_{\mathrm{init}}$.

Exercise 2.18. Check that all of these are categories. Check also that $\mathbf{TotOrd}_{\mathrm{init}}$ is a thin category (one skeletal subcategory consists of the full subcategory whose objects are von Neumann ordinals).

Exercise 2.19. One particularly important variant is the full subcategory Δ of **TotOrd** whose objects are the finite, nonempty totally ordered sets. This has a skeletal subcategory consisting of the finite ordinals. By convention, these are labeled starting from 0, i.e., $[0] = \{0\}$, $[1] = \{0, 1\}$, etc. Prove that the set of all morphisms of Δ is generated by the following two sets of morphisms. The face maps, $\delta^{n,m}$ for $n \in \mathbb{Z}_{\geq 0}$ and for $m = 0, \dots, n$, are injective, order preserving maps from $[n-1]$ to $[n]$ whose image equals $[n] \smallsetminus \{m\}$. The degeneracy maps, $\sigma^{n,m}$ for $n \in \mathbb{Z}_{\geq 0}$ and $m = 0, \dots, n$, are surjective, order preserving maps from $[n+1]$ to $[n]$ such that both m and $m+1$ map to m.

2.4   Monoids

Example 2.20. For every set H together with a binary operation $\bullet$ from $H \times H$ to H that is associative and unital (which in many places is defined to be a "monoid") there exists a monoid (category) $B(H, \bullet)$ whose unique object is, say, the set H itself (perhaps considered as a right act over itself), and whose unique Hom set is H with $\bullet$ giving the binary operation.

Example 2.21. For every category $\mathbf{C}$, for every object a of $\mathbf{C}$, prove that the restriction $\bullet$ of composition to the Hom set $H(a) = \mathrm{Hom}_{\mathbf{C}}(a, a)$ is a monoid as above. In particular, prove that every category with a unique object is strongly equivalent to the category of the monoid of the unique Hom set with its composition operation.

Example 2.22. For every category $\mathbf{C}$, for all objects a and b, prove that the Hom set $H(a, b) = \mathrm{Hom}_{\mathbf{C}}(a, b)$ has a left act (via postcomposition) by the monoid $H(b)$ and a right act (via precomposition) by the monoid $H(a)$, and these acts commute with each other. Thus, $H(a, b)$ admits a left act by the product monoid $H(b) \times H(a)^{\mathrm{opp}}$. Also, $H(a, a)$ with this left act is the regular act of $H(a) \times H(a)^{\mathrm{opp}}$ on $H(a)$.

Example 2.23. For every category $\mathbf{C}$, for all objects a, b and c, prove that the composition $\mathsf{comp}_{c,b,a}$ from $H(b, c) \times H(a, b)$ to $H(a, c)$ is a $H(b)$-balanced morphism of sets with a left act by $H(c) \times H(a)^{\mathrm{opp}}$.

Example 2.24. For every category $\mathbf{C}$, for all objects a, b, c and d, prove that the $H(b)$-balanced, $H(c)$-balanced morphism of left acts by $H(d) \times H(a)^{\mathrm{opp}}$ from $H(c, d) \times H(b, c) \times H(a, b)$ to $H(a, d)$ by $\mathsf{comp}_{d,c,b} \circ (\mathrm{Id}_{H(c,d)} \times \mathsf{comp}_{c,b,a})$ equals the morphism by $\mathsf{comp}_{d,b,a} \circ (\mathsf{comp}_{d,c,b} \times \mathrm{Id}_{H(a,b)})$.

Example 2.25. Finally, for every set O, for every function $H(-)$ from O whose value on each element a of O is a monoid $(H(a), \bullet)$, for every function $H(-, -)$ from $O \times O$ whose value on each element (a, b) is a set $H(a, b)$ with a left act by $H(b) \times H(a)^{\mathrm{opp}}$ that recovers the regular left act of $H(a) \times H(a)^{\mathrm{opp}}$ on $H(a)$ when b equals a, and for every function $\mathsf{comp}_{-,-,-}$ from $O \times O \times O$ whose value on each element (a, b, c) is a $H(b)$-balanced morphism of sets with left act by $H(c) \times H(a)^{\mathrm{opp}}$ from $H(b, c) \times H(a, b)$ to $H(a, c)$ that satisfies the associativity from the previous exercise, prove that there exists a unique small category $\mathbf{C}$ such that $\mathrm{ob}_{\mathbf{C}}$ equals (the class of) the set O, such that each Hom set equals $\mathrm{Hom}_{\mathbf{C}}(a, b) = H(a, b)$, and such that composition is $\mathsf{comp}_{-,-,-}$.

Example 2.26. In particular, for every set S, for a category $\mathbf{O}$ with a unique object $*$, for the $\mathbf{O}$-Hom span $\mathbf{M}_S$ whose unique Hom set is S, for the free category $(\mathbf{O}, \mathbf{M}_S^*, \circ)$, the associated monoid is the **free monoid** on the set S. The unique Hom set S^* is also called the free monoid on S, and it is also the **Kleene star** of S, i.e.,

$$S^* = (\{0\} \times \{\mathrm{Id}_*\}) \sqcup (\{1\} \times S) \sqcup (\{2\} \times (S \times S)) \sqcup (\{3\} \times (S \times S \times S)) \sqcup \ldots.$$

Example 2.27. For every monoid $(H, \bullet)$, the opposite category of $B(H, \bullet)$ is (canonically equivalent to) the category of the **opposite monoid** $(H, \bullet^{\mathrm{opp}})$ where $a \bullet^{\mathrm{opp}} b$ is defined to equal $b \bullet a$ for all elements a and b of H.

Example 2.28. For every monoid $(H, \bullet)$, the monoid is a group if and only if every element of H is invertible. In this case, the category of the group is a skeletal groupoid. In this case, the nerve of

the small category $B(H, \bullet)$ is the **classifying simplicial set** of the (discrete) group $(H, \bullet)$. The geometric realization of this simplicial set is the **classifying space** of $(H, \bullet)$. For every groupoid, for every object $*$ of that groupoid, the restriction of composition to the Hom set from $*$ to itself is a group $(H, \bullet)$, and the full subcategory whose unique object is $*$ is strongly equivalent to $B(H, \bullet)$. In particular, every groupoid with a unique object is strongly equivalent to $B(H, \bullet)$ for the unique Hom set $(H, \bullet)$ with its composition operation.

Example 2.29. We could "deskeletonize" the previous example by considering the category whose objects are all right acts over the monoid $(H, \bullet)$ that are principal homogeneous spaces, and whose morphisms are all morphisms of right $(H, \bullet)$-acts.

Example 2.30. The category of right principal homogenous spaces for $(H, \bullet)$, as above, is a full subcategory of the category of all right $(H, \bullet)$-acts. Another full subcategory is the category of all right $(H, \bullet)$-acts that are trivial in the sense that every element of $(H, \bullet)$ acts identically on the set. This full subcategory is strongly equivalent to the category **Set** of all sets. Of course if $(H, \bullet)$ is itself a singleton monoid, then this full subcategory equals the entire category of right $(H, \bullet)$-acts, so that this category is strongly equivalent to **Set**.

Example 2.31. The category **Monoid** has as objects all ordered pairs $(H, \bullet)$ of a set H together with a unital, associative binary operation $\bullet$ from H to itself, and has morphisms from $(H, \bullet)$ to $(H', \bullet')$ being all functions f from H to H' that preserve the identity and preserve the binary operation (i.e., usual morphisms of monoids): $f(e_H)$ equals $e_{H'}$ and $f(h \bullet k)$ equals $f(h) \bullet' f(k)$ for all elements h, k of H. The category **Monoid** has a full subcategory **Grp** whose objects are groups. The category **Grp** has a full subcategory **Ab** whose objects are Abelian groups. The category **Ab** has a full subcategory $\mathbb{Q} - mbfMod$ whose objects are Abelian groups such that multiplication by n is a bijection of the group to itself for every nonzero integer n, i.e., the Abelian group is a $\mathbb{Q}$-vector space. The category $\mathbb{Q} - \mathbf{Mod}$ has a full subcategory whose objects are finite-dimensional $\mathbb{Q}$-vector spaces, etc.

Example 2.32. A hybrid of the previous two examples is the category whose objects are all ordered pairs $((H, \bullet), (S, \rho))$ of a monoid $(H, \bullet)$ together with a right H-act $\rho : S \times H \to S$. The morphisms from $((H, \bullet), (S, \rho))$ to $((H', \bullet'), (S', \rho'))$ are all ordered pairs (f, g) of a morphism f of monoids from $(H, \bullet)$ to $(H', \bullet')$ together with a function g from S to S' such that, for every element h of H and every element s of S, the image $g(\rho(s, h))$ equals $\rho'(g(s), f(h))$, i.e., g is a morphism of right H-acts for the induced right H-act on S' obtained from ρ' and f. This hybrid category is an example of the "Grothendieck construction" for fibered categories (one of the basic notions in extending from schemes to stacks).

2.5   Categories of modules

For Abelian monoids, respectively for Abelian groups, there is an enrichment of the Hom sets to Abelian monoids, resp. to Abelian groups.

Definition 2.33. For every Abelian monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, the **addition law** on $\text{Hom}_{\mathbf{Monoid}}((H', \bullet'), (H, \bullet))$ is the binary operation that associates to every pair (f, g) of monoid homomorphisms from $(H', \bullet')$ to $(H, \bullet)$ the monoid homomorphism $f \bullet g$ that sends every element h' of H' to $f(h') \bullet g(h')$.

Example 2.34. Check that the set function $f \bullet g$ is a monoid homomorphism. Check that the addition law is both associative and commutative, and it has a left-right identity consisting of the constant set function from H' with image the singleton of the monoid identity in H. Thus $\text{Hom}_{\mathbf{Monoid}}((H', \bullet'), (H, \bullet))$ with this addition law is itself an Abelian monoid. If $(H, \bullet)$ is an Abelian group, check that also $\text{Hom}_{\mathbf{Monoid}}((H', \bullet'), (H, \bullet))$ is an Abelian group.

Example 2.35. For the full subcategory **Ab** of all Abelian groups, check that the addition laws makes composition into a biadditive map of Abelian groups

$$\text{Hom}_{\mathbf{Ab}}((H', \bullet'), (H, \bullet)) \times \text{Hom}_{\mathbf{Ab}}((H'', \bullet''), (H', \bullet')) \to \text{Hom}_{\mathbf{Ab}}((H'', \bullet''), (H, \bullet)).$$

In particular, check that the addition law together with composition makes $\text{Hom}_{\mathbf{Ab}}((H, \bullet), (H, \bullet))$ into an *associative, unital ring*, i.e., composition is a monoid structure that distributes with respect to addition both on the left and right.

This allows a concise definition of associative, unital rings. Moreover, for each associative, unital rings, there are (Abelian) categories of modules over that ring.

Definition 2.36. An **associative, unital ring** $(R, +, \cdot)$ is an Abelian group $(R, +)$ together with an (injective) homomorphism of Abelian groups

$$L_\bullet : (R, +) \to \text{Hom}_{\mathbf{Ab}}((R, +), (R, +)), \quad r \mapsto (L_r : (R, +) \to (R, +))$$

whose image is a submonoid under composition that is right unital, i.e., there exists a (unique) element 1 in R with $L_1 = \text{Id}_R$ (so 1 is a left multiplicative identity) and also with $L_r(1) = r$ for every element r of R (so 1 is also a right right multiplicative identity), and, for every (r, s) in $R \times R$, there exists a (unique) element $r \cdot s$ of R such that $L_r \circ L_s$ equals $L_{r \cdot s}$ (notice that $L_r(s) = L_r(L_s(1)) = L_{r \cdot s}(1) = r \cdot s$). For every associative, unital ring $(R, +, \cdot)$, for every associative, unital ring $(R', +', \cdot')$, a **morphism** of associative, unital rings from $(R, +, \cdot)$ to $(R', +', \cdot')$ is a set function f from R to R' that is simultaneously a homomorphism from the Abelian group $(R, +)$ to $(R', +')$ and a homomorphism from the monoid $(R, \cdot)$ to the monoid $(R', \cdot')$.

Exercise 2.37. Check that, for every associative, unital ring $(R, +, \cdot)$, the identity function Id_R is a morphism of associative, unital rings from $(R, +, \cdot)$ to itself. Also check that the composition function of morphisms of associative, unital rings is again a morphism of associative, unital rings.

Definition 2.38. The **category of associative, unital rings**, denoted **Ring**, has as objects all associative, unital rings, has as morphisms the morphisms of associative, unital rings, and has the composition from the previous exercise.

Definition 2.39. For every associative, unital ring $(R, +, \cdot)$, the **opposite product** $\cdot^{\text{opp}}$ is the binary operation on R defined by $s \cdot^{\text{opp}} r = r \cdot s$ for every element (r, s) of $R \times R$.

Exercise 2.40. Check that for every associative, unital ring $(R, +, \cdot)$, also $(R, +, \cdot^{\text{opp}})$ is an associative, unital ring.

Definition 2.41. A **commutative, associative, unital ring** is an associative, unital ring $(R, +, \cdot)$ such that $\cdot^{\text{opp}}$ equals $\cdot$, i.e., $r \cdot s$ equals $s \cdot r$ for every element (r, s) of $R \times R$. For every commutative, associative, unital ring $(R, +, \cdot)$, for every commutative, associative, unital ring $(R', +', \cdot')$, a **morphism** of commutative, associative, unital rings from $(R, +, \cdot)$ to $(R', +', \cdot')$ is a morphism of associative, unital rings from $(R, +, \cdot)$ to $(R', +', \cdot')$. The category **CRing** is the full subcategory of **Ring** whose objects are all commutative, associative, unital rings.

Definition 2.42. For every associative, unital ring $(R, +, \cdot)$, for every Abelian group $(M, +)$, a **left R-module** structure on $(M, +)$ is a morphism of associative, unital rings λ from $(R, +, \cdot)$ to the associative, unital ring $\text{Hom}_{\textbf{Ab}}((M, +), (M, +))$, i.e., for every element r of R, the function λ_r from M to itself is a group homomorphism, λ_1 equals Id_M, and, for every element (r, s) of $R \times R$, the image λ_{r+s} equals $\lambda_r + \lambda_s$ and $\lambda_{r \cdot s}$ equals $\lambda_r \circ \lambda_s$. Stated differently, this is a biadditive map $*$ from $R \times M$ to M that is also a monoid homomorphism for $\cdot$ on R and for composition of Abelian group homomorphisms of M, i.e., $1 * m$ equals m and $(r \cdot s) * m$ equals $r * (s * m)$ for every element m of M and for every element (r, s) of $R \times R$. For every left R-module $(M, +, \lambda)$, for every left R-module $(M', +', \lambda')$, a **morphism** of left R-modules from $(M, +, \lambda)$ to $(M', +', \lambda')$ is a set function f from M to M' that is a homomorphism of Abelian groups from $(M, +)$ to $(M', +')$ and that commutes with λ and λ', i.e., $f \circ \lambda_r$ equals $\lambda'_r \circ f$ for every element r of R.

Exercise 2.43. Check that the identity function from M to itself is a morphism of left R-modules from $(M, +, \lambda)$ to itself. Check that the composition function of two morphisms of left R-modules is again a morphism of left R-modules.

Definition 2.44. For every associative, unital ring $(R, +, \cdot)$, the **category of left R-modules**, denoted $R - \textbf{Mod}$, has objects that are all left R-modules, has morphisms that are all morphisms of left R-modules, and has composition as defined above.

Exercise 2.45. For every left R-module $(M, +, \lambda)$, for every Abelian group $(M', +')$, define a left R-module structure on $\text{Hom}_{\mathbf{Ab}}((M', +'), (M, +))$ by $\lambda_{M,r}^{M'}(f)(m') := \lambda_r(f(m'))$ for every element r of R, for every element m' of M', and for every Abelian group homomorphism f from $(M', +')$ to $(M, +)$. Check that this is a structure of left R-module. For every ordered pair of left R-modules $(M, +, \lambda)$ and $(M', +', \lambda')$, for every Abelian group homomorphism f from $(M', +')$ to $(M, +)$, check that f is a morphism of left R-modules from $(M', +', \lambda')$ to $(M, +, \lambda)$ if and only if $\lambda_{M,r}^{M'}(f)$ equals $f \circ \lambda_r'$ for every element r of R.

Definition 2.46. For every associative, unital ring $(R, +, \cdot)$, for every Abelian group $(M, +)$, a **right R-module** structure on $(M, +)$ is a morphism of associative, unital rings ρ from $(R, +, \cdot^{\text{opp}})$ to the associative, unital ring $\text{Hom}_{\mathbf{Ab}}((M, +), (M, +))$. Stated differently, this is a biadditive map $*$ from $M \times R$ to M that is also a monoid homomorphism for $\cdot$ on R and for composition of Abelian group homomorphisms of M, i.e., $m \cdot 1$ equals m and $m * (r \cdot s)$ equals $(m * r) * s$ for every element m of M and for every element (r, s) of $R \times R$. For every right R-module $(M, +, \rho)$, for every right R-module $(M', +', \rho')$, a **morphism** of right R-modules from $(M, +, \rho)$ to $(M', +', \rho')$ is a set function f from M to M' that is a homomorphism of Abelian groups from $(M, +)$ to $(M', +')$ and that commutes with ρ and ρ', i.e., $f \circ \rho_r$ equals $\rho_r' \circ f$ for every element r of R.

Exercise 2.47. Check that the identity function from M to itself is a morphism of right R-modules from $(M, +, \rho)$ to itself. Check that the composition function of two morphisms of right R-modules is again a morphism of right R-modules.

Definition 2.48. For every associative, unital ring $(R, +, \cdot)$, the **category of right R-modules**, denoted $\mathbf{Mod} - R$, has objects that are all right R-modules, has morphisms that are all morphisms of right R-modules, and has composition as defined above.

Exercise 2.49. For every left R-module $(M, +, \lambda)$, for every Abelian group $(M', +')$, define a right R-module structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$ by $\rho_{M'}^{M,r}(f)(m) := f(\lambda_r(m))$ for every element r of R, for every element m of M, and for every Abelian group homomorphism f from $(M, +)$ to $(M', +')$. Check that this is a structure of right R-module. For every ordered pair of left R-modules $(M, +, \lambda)$ and $(M', +', \lambda')$, for every Abelian group homomorphism f from $(M, +)$ to $(M', +')$, check that f is a morphism of left R-modules from $(M, +, \lambda)$ to $(M', +', \lambda')$ if and only if $\rho_{M'}^{M,r}(f)$ equals $\lambda_r \circ f$ (which also equals $\lambda_{M',r}^M(f)$, by definition) for every element r of R.

Exercise 2.50. Formulate and prove the analogous results for a right R-module structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$ associated to a right R-module structure on $(M', +')$ and for a left R-module structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$ associated to a right R-module structure on $(M, +)$.

Definition 2.51. For every associative, unital ring $(R, +_R, \cdot_R)$, for every associative, unital ring $(S, +_S, \cdot_S)$, an $R - S$-**bimodule** is a quadruple $(M, +, \lambda, \rho)$ of an Abelian group $(M, +)$ with a left

R-module structure λ and a right S-module structure ρ such that, for every element (r, r') of $R \times R'$, the Abelian group homomorphism $\lambda_r \circ \rho_{r'}$ equals $\rho_{r'} \circ \lambda_r$, i.e., the images of λ and ρ in $\text{Hom}_{\mathbf{Ab}}((M, +), (M, +))$ centralize one another. For every $R - S$-bimodule $(M, +, \lambda, \rho)$, for every R–S-bimodule $(M', +', \lambda', \rho')$, a **morphism** of R–S-bimodules from $(M, +, \lambda, \rho)$ to $(M', +', \lambda', \rho')$ is a set function f from M to M' that is simultaneously a morphism of left R-modules from $(M, +, \lambda)$ to $(M', +', \lambda')$ and a morphism of right S-modules from $(M, +, \rho)$ to $(M', +', \rho')$.

Exercise 2.52. Check that the identity set function Id_M is a morphism of $R - S$-bimodules from $(M, +, \lambda, \rho)$ to itself. Also check that the composition function of morphisms of $R - S$-bimodules is again an $R - S$-bimodule.

Definition 2.53. For every associative, unital ring $(R, +_R, \cdot_R)$, for every associative, unital ring $(S, +_S, \cdot_S)$, the **category of $R - S$-bimodules**, denoted $R - S - \mathbf{Mod}$, has objects that are all $R - S$-bimodules, has morphisms that are all morphisms of $R - S$-bimodules, and has composition as defined above.

Exercise 2.54. For every R–S-bimodule $(M, +, \lambda, \rho)$, for every Abelian group $(M', +')$, check that the operations $\lambda_{M,r}^{M'}$ and $\rho_{M,s}^{M'}$ make $\text{Hom}_{\mathbf{Ab}}((M', +'), (M, +))$ into an $R - S$-bimodule. Similarly, define an $S - R$-bimodule structure on $\text{Hom}_{\mathbf{Ab}}((M, +), (M', +'))$. For every Abelian group homomorphism f from an R–S-bimodule $(M, +, \lambda, \rho)$ to an R–S-bimodule $(M', +', \lambda', \rho')$, check that f is a morphism of $R - S$-bimodules if and only if both $\rho_{M'}^{M,r}(f)$ equals $\lambda_{M',r}^{M}(f)$ and $\rho_{M',s}^{M}(f)$ equals $\lambda_{M'}^{M,s}(f)$ for every element r of R and for every element s of S.

2.6   Topological spaces

Of course there are also many categories arising from topological spaces and geometric objects.

Definition 2.55. For every set X, a **topology** (of open subsets of X) is a subset τ of the power set $\mathcal{P}(X)$ of X satisfying all of the following.

(i) Both $\varnothing$ and X are elements of τ.

(ii) For every ordered pair (U, V) of elements of τ, also $U \cap V$ is an element of τ.

(iii) For every subset I of τ, the union over all elements of I (considered as a subset of X) is an element of τ.

A **topological space** is an ordered pair (X, τ) of a set X and a topology τ on X.

For every ordered pair $((X, \tau), (X', \tau'))$ of topological spaces, a **continuous map** from (X, τ) to (X', τ') is a function f from X to X' such that for every element U' of τ', the preimage $f^{\text{pre}}(U')$ is an element of τ.

Exercise 2.56. For every topological space (X, τ), check that Id_X is a continuous map from (X, τ) to itself. Check that the composition function of continuous maps is again a continuous map. Thus, topological spaces with continuous maps form a category, **Top**.

Definition 2.57. For every set X, a **topological basis** (of open subsets of X) is a subset $\mathcal{B}$ of the power set $\mathcal{P}(X)$ of X satisfying all of the following.

(i) The set X is the union over all elements of $\mathcal{B}$.

(ii) For every ordered pair (U, V) of elements of $\mathcal{B}$, the set $U \cap V$ equals the union over all elements of $\mathcal{B}$ that are subset of $U \cap V$.

Occasionally, a function to $\mathcal{P}(X)$ whose image is a topological basis is also called a topological basis. The **topology generated** by a topological basis is the subset $\tau(\mathcal{B})$ of $\mathcal{P}(X)$ of all subsets U of X that equal the union over all elements of $\mathcal{B}$ that are a subset of U (thus, also $\varnothing$ is tautologically an element of $\mathcal{B}$).

Exercise 2.58. Check that $\tau(\mathcal{B})$ is a topology for $\mathcal{X}$. For every topological space (X', τ') and for every function f from X' to X, check that f is a continuous map from (X', τ') to $(X, \tau(\mathcal{B})$ if and only if, for every element U of $\mathcal{B}$, the preimage subset $f^{\mathrm{pre}}(U)$ is an element of τ'.

2.7 Manifolds

Definition 2.59. For every set X, denote by $\mathrm{Arr}(\mathcal{P}(X))$ the subset of $\mathcal{P}(X) \times \mathcal{P}(X)$ of all ordered pairs (U, V) of subsets of X with U a subset of V. For every set R, for every subset U of X, the **set of R-valued discontinuous functions** on U is the set $\mathrm{Hom}_{\mathbf{Set}}(U, R)$ of all R-valued functions on U. By convention, when U is the empty set, this set is a singleton. For every element (U, V) of $\mathrm{Arr}(\mathcal{P}(X))$, the **restriction homomorphism** rest_U^V is the function of precomposition by the inclusion incl_V^U from $\mathrm{Hom}_{\mathbf{Set}}(V, R)$ to $\mathrm{Hom}_{\mathbf{Set}}(U, R)$. Altogether, the ordered pair $(\mathrm{Hom}_{\mathbf{Set}}(\bullet, R), \mathsf{rest}_\bullet^\bullet)$ of the function $\mathrm{Hom}_{\mathbf{Set}}(\bullet, R)$ on $\mathcal{P}(X)$ and the function $\mathsf{rest}_\bullet^\bullet$ on $\mathrm{Arr}(\mathcal{P}(X))$ is the **sheaf of R-valued discontinuous functions**.

Exercise 2.60. For each element (U, U) of $\mathrm{Arr}(\mathcal{P}(X))$, prove that rest_U^U is the identity function. Also, for every pair (U, V) and (V, W) of elements of $\mathrm{Arr}(\mathcal{P}(X))$, prove that the composition $\mathsf{rest}_U^V \circ \mathsf{rest}_V^W$ equals rest_U^W. Thus, restriction is contravariant for the category of open sets with inclusions as morphisms.

Exercise 2.61. For every element V of $\mathcal{P}(X)$, for every subset $\mathfrak{U}$ of $\mathcal{P}(V)$ such that V equals the union of all elements U of $\mathfrak{U}$, for every tuple $(f_U)_{U \in \mathfrak{U}}$ of elements f_U of $\mathrm{Hom}_{\mathbf{Set}}(U, R)$, prove that $\mathsf{rest}_{U \cap U'}^U(f_U)$ equals $\mathsf{rest}_{U' \cap U}^{U'}(f_{U'})$ for every element (U, U') of $\mathfrak{U} \times \mathfrak{U}$ if and only if there exists an

element f of $\mathrm{Hom}_{\mathbf{Set}}(V,R)$ such that $\mathsf{rest}^V_U(f)$ equals f_U for every element U of $\mathfrak{U}$. Moreover, prove such f is unique. This is the sheaf axiom for the sheaf of R-valued discontinuous functions (for the discrete topology on X, hence for every other topology on X as well).

Definition 2.62. For every function g from a set Y to a set X, for every element V of $\mathcal{P}(X)$, pre-composition by g gives the **pushforward homomorphism** $g^{\#}_V$ from $\mathrm{Hom}_{\mathbf{Set}}(V,R)$ to $\mathrm{Hom}_{\mathbf{Set}}(g^{\mathrm{pre}}(V),R)$.

Exercise 2.63. Prove that for every element (U,V) of $\mathrm{Arr}(\mathcal{P}(X))$, also $(g^{\mathrm{pre}}(U),g^{\mathrm{pre}}(V))$ is an element of $\mathrm{Arr}(\mathcal{P}(Y))$, and the composition $g^{\#}_U \circ \mathsf{rest}^V_U$ equals $\mathsf{rest}^{g^{\mathrm{pre}}(V)}_{g^{\mathrm{pre}}(U)} \circ g^{\#}_V$. Thus, the collection $(g^{\#}_V)_{V\in\mathcal{P}(X)}$ is a morphism of sheaves on X (for the discrete topology) from the sheaf of R-valued discontinuous functions on X to the pushforward by g of the sheaf of R-valued discontinuous functions on Y.

Exercise 2.64. For every set X, for every set Y, for every set Z, for every function g from Y to X, for every function h from Z to Y, and for every element V of $\mathcal{P}(X)$, prove that $h^{\#}_{g^{\mathrm{pre}}(V)} \circ g^{\#}_V$ equals $(g \circ h)^{\#}_V$. Also, for every set X, for every element V of $\mathcal{P}(X)$, prove that $(\mathrm{Id}_X)^{\#}_V$ is the identity function. Thus $g^{\#}$ is contravariant in g.

Definition 2.65. For every topological space (X,τ), let $\mathrm{Arr}(\tau)$ denote the subset of $\tau \times \tau$ of all ordered pairs (U,V) with U a subset of V. For every element U of τ, the $\mathbb{R}$**-algebra of** $\mathbb{R}$**-valued continuous functions** is the commutative $\mathbb{R}$-subalgebra $C^0_{(X,\tau),\mathbb{R}}(U)$ of $\mathrm{Hom}_{\mathbf{Set}}(U,\mathbb{R})$ whose elements are all continuous functions from $(U,\tau|_U)$ to $\mathbb{R}$ (with the Euclidean topology on $\mathbb{R}$). Similarly the $\mathbb{C}$**-algebra of continuous functions** is the commutative $\mathbb{C}$-subalgebra $C^0_{(X,\tau),\mathbb{C}}(U)$ of $\mathrm{Hom}_{\mathbf{Set}}(U,\mathbb{C})$ whose elements are continuous functions from $(U,\tau|_U)$ to $\mathbb{C}$ (with the Euclidean topology on $\mathbb{C}$). For every element (U,V) of $\mathrm{Arr}(\tau)$, the **restriction homomorphism** res^V_U from $C^0_{(X,\tau),\mathbb{R}}(V)$ to $C^0_{(X,\tau),\mathbb{R}}(U)$, respectively fron $C^0_{(X,\tau),\mathbb{C}}(V)$ to $C^0_{(X,\tau),\mathbb{C}}(U)$, algebra homomorphism induced by the restriction homomorphism from the previous definition, i.e., precomposition with the continuous inclusion $U \hookrightarrow V$. Altogether, the ordered pair $(C^0_{(X,\tau),\mathbb{R}}(\bullet),\mathsf{res}^{\bullet}_{\bullet})$ of the function on τ associating $C^0_{(X,\tau),\mathbb{R}}(U)$ to each element U of τ, and the function on the subset $\mathrm{Arr}(\tau)$ associating res^V_U to each element (U,V) of $\mathrm{Arr}(\tau)$ is the **sheaf of continuous functions** to $\mathbb{R}$, usually denoted by $C^0_{(X,\tau),\mathbb{R}}$, or just $C^0_{X,\mathbb{R}}$ when τ is understood. Similarly, $(C^0_{(X,\tau),\mathbb{C}}(\bullet),\mathsf{res}^{\bullet}_{\bullet})$ is the **sheaf of continuous functions** to $\mathbb{C}$.

Exercise 2.66. For every element V of τ, for every subset $\mathfrak{U}$ of $\tau|_V$ such that V equals the union of all elements U of $\mathfrak{U}$, for every tuple $(f_U)_{U\in\mathfrak{U}}$ of elements f_U of $C^0_{(X,\tau),\mathbb{R}}(U)$, respectively of $C^0_{(X,\tau),\mathbb{C}}(U)$, prove that $\mathsf{rest}^U_{U\cap U'}(f_U)$ equals $\mathsf{rest}^{U'}_{U'\cap U}(f_{U'})$ for every element (U,U') of $\mathfrak{U}\times\mathfrak{U}$ if and only if there exists an element f of $C^0_{(X,\tau),\mathbb{R}}(V)$, resp. of $C^0_{(X,\tau),\mathbb{C}}(V)$, such that $\mathsf{rest}^V_U(f)$ equals f_U for every element U of $\mathfrak{U}$. Moreover, prove such f is unique. This is the sheaf axiom for $C^0_{(X,\tau),\mathbb{R}}$, resp. for $C^0_{(X,\tau),\mathbb{C}}$.

Definition 2.67. For every continuous function g from a topological space (Y, σ) to a topological space (X, τ), for every element V of τ, the **pushforward homomorphism** $g_V^\#$ from $C_{(X,\tau),\mathbb{R}}^0(V)$ to $C_{(Y,\sigma),\mathbb{R}}^0(g^{\mathrm{pre}}(V))$, respectively from $C_{(X,\tau),\mathbb{C}}^0(V)$ to $C_{(Y,\sigma),\mathbb{C}}^0(g^{\mathrm{pre}}(V))$, is the restriction of the pushforward homomorphism $g_\tau^\#$ for the sheaves of discontinuous functions, i.e., precomposition by the continuous function g.

Exercise 2.68. For every continuous function g from a topological space (Y, σ) to a topological space (X, τ), for every element V of τ, prove that $g_V^\#$ maps each element of $C_{(X,\tau),\mathbb{R}}^0(V)$ to an element of $C_{(Y,\sigma),\mathbb{R}}^0(g^{\mathrm{pre}}(V))$, respectively maps each element of $C_{(X,\tau),\mathbb{C}}^0(V)$ to an element of $C_{(Y,\sigma),\mathbb{C}}^0(g^{\mathrm{pre}}(V))$. Thus the morphism of sheaves $g^\#$ defined above restricts to a morphism of sheaves from $C_{(X,tau),\mathbb{R}}^0$ to the pushforward $g_* C_{(Y,\sigma),\mathbb{R}}^0$.

Exercise 2.69. For every continuous function g from a topological space (Y, σ) to a topological space (X, τ), for every continuous function h from a topological space (Z, ρ) to (Y, σ), deduce that $h_{g^{\mathrm{pre}}(V)}^\# \circ g_V^\#$ equals $(g \circ h)_V^\#$ as morphisms of sheaves on (X, τ) from $C_{(X,\tau),\mathbb{R}}^0$ to $(g \circ h)_* C_{(Z,\rho),\mathbb{R}}^0$, respectively from $C_{(X,\tau),\mathbb{C}}^0$ to $(g \circ h)_* C_{(Z,\rho),\mathbb{C}}^0$, Also, for every set X, for every element V of $\mathcal{P}(X)$, prove that $(\mathrm{Id}_X)_V^\#$ is the identity function. Thus, the pushforward homomorphisms $g^\#$ are contravariant in g.

Definition 2.70. For every topological space (X, τ), for every element V of τ, a **chart** for V is a homeomorphism ϕ from $(V, \tau|_V)$ to an open subset $\phi(V)$ of some Euclidean topological space $\mathbb{R}^n$. An element V of τ is **chartable** if (and only if) there exists a chart for V. A topological space (X, τ) is a **topological manifold** if (and only if) the chartable open subsets cover X. The full subcategory of **Top** whose objects are topological manifolds is denoted **TopMan**.

Exercise 2.71. For every topological manifold (X, τ), for every topological space (Y, σ), for every function g from Y to X, prove that g is continuous if and only if $g_V^\#$ maps $C_{(X,\tau),\mathbb{R}}^0(V)$ to the subset $C_{(Y,\sigma),\mathbb{R}}^0(g^{\mathrm{pre}}(V))$ of $\mathrm{Hom}_{\mathbf{Set}}(V, \mathbb{R})$ for every element V of τ, respectively g is continuous if and only if $g_V^\#$ maps $C_{(X,\tau),\mathbb{C}}^0(V)$ to the subset $C_{(Y,\sigma),\mathbb{C}}^0(g^{\mathrm{pre}}(V))$ of $\mathrm{Hom}_{\mathbf{Set}}(V, \mathbb{C})$ for every element V of τ.

Definition 2.72. For every integer k, respectively for ∞, for every topological manifold (X, τ), a collection $\mathfrak{A}$ of ordered pairs (V, ϕ) of a chartable open V and a chart ϕ for V is a C^k**-atlas**, respectively a C^∞**-atlas**, if (and only if) both X equals the union of all the open subsets V of elements (V, ϕ) in $\mathfrak{A}$, and, for all elements (V, ϕ) and (V', ϕ') of $\mathfrak{A}$, the unique homeomorphism $\mathsf{trans}_{(V',\phi')}^{(V,\phi)}$ from $\phi(V \cap V')$ to $\phi'(V \cap V)$ satisfying $\mathsf{trans}_{(V',\phi')}^{(V,\phi)} \circ \phi|_{V \cap V'} = \phi'|_{V \cap V'}$ is a C^k function, resp. is a C^∞ function. A C^k**-manifold structure** on (X, τ), resp. a C^∞**-manifold structure** on (X, τ), is a maximal C^k-atlas, resp. a maximal C^∞-atlas, i.e., one that cannot be made bigger

by adding more pairs (V, ϕ). For every continuous function g from a C^k-manifold, resp. a C^∞-manifold, $(Y, \sigma, \mathfrak{B})$ to a C^k-manifold $(X, \tau, \mathfrak{A})$, the continuous function g is a C^k **map** if (and only if), for every element (V, ϕ) of $\mathfrak{A}$, for every element (U, ψ) of $\mathfrak{B}$ such that U is a subset of $g^{\mathrm{pre}}(V)$, the unique continuous map $\mathsf{trans}_{g,(V,\phi)}^{(U,\psi)}$ from $\psi(U)$ to $\phi(V)$ satisfying $\phi \circ g|_U = \mathsf{trans}_{g,(V,\phi)}^{(U,\psi)} \circ \phi$ is a C^k map.

Definition 2.73. For every nonnegative integer k and for $k = \infty$, for every continuous function g from a C^k-manifold $(Y, \sigma, \mathfrak{B})$ to a C^k-manifold $(X, \tau, \mathfrak{A})$, the continuous function g is a C^k **function** if (and only if), for every element (V, ϕ) of $\mathfrak{A}$, for every element (U, ψ) of $\mathfrak{B}$ such that U is a subset of $g^{\mathrm{pre}}(V)$, the unique continuous map $\mathsf{trans}_{g,(V,\phi)}^{(U,\psi)}$ from $\psi(U)$ to $\phi(V)$ satisfying $\phi \circ g|_U = \mathsf{trans}_{g,(V,\phi)}^{(U,\psi)} \circ \phi$ is a C^k function of open subsets of Euclidean space. In particular, for every element V of τ, the $\mathbb{R}$**-algebra of** C^k **functions** is the commutative $\mathbb{R}$-subalgebra $C_{(X,\tau),\mathbb{R}}^k(V)$ of $C_{(X,\tau),\mathbb{R}}^0(V)$ whose elements are all C^k maps from $(V, \tau|_V, \mathfrak{A}|_V)$ to $\mathbb{R}$ (with its usual C^k-manifold structure). Similarly the $\mathbb{C}$**-algebra of** C^k **functions** is the commutative $\mathbb{C}$-subalgebra $C_{(X,\tau),\mathbb{C}}^k(V)$ of $C_{(X,\tau),\mathbb{C}}^0(V)$ whose elements are all C^k maps from $(V, \tau|_V, \mathfrak{A}|_V)$ to $\mathbb{C}$ (with its usual C^k-manifold structure). For every element (U, V) of $\mathrm{Arr}(\tau)$, the **restriction homomorphism** res_U^V from $C_{(X,\tau),\mathbb{R}}^k(V)$ to $C_{(X,\tau),\mathbb{R}}^k(U)$, respectively fron $C_{(X,\tau),\mathbb{C}}^k(V)$ to $C_{(X,\tau),\mathbb{C}}^k(U)$, algebra homomorphism induced by the restriction homomorphism from the previous definition, i.e., precomposition with the continuous inclusion $U \hookrightarrow V$. Altogether, the ordered pair $(C_{(X,\tau),\mathbb{R}}^k(\bullet), \mathsf{res}_\bullet^\bullet)$ of the function on τ associating $C_{(X,\tau),\mathbb{R}}^k(V)$ to each element V of τ, and the function on the subset $\mathrm{Arr}(\tau)$ associating res_U^V to each element (U, V) of $\mathrm{Arr}(\tau)$ is the **sheaf of** C^k **functions** to $\mathbb{R}$, usually denoted by $C_{(X,\tau,\mathfrak{A}),\mathbb{R}}^k$, or just $C_{X,\mathbb{R}}^k$ when τ and $\mathfrak{A}$ are understood. Similarly, $(C_{(X,\tau,\mathfrak{A}),\mathbb{C}}^k(\bullet), \mathsf{res}_\bullet^\bullet)$ is the **sheaf of** C^k **functions** to $\mathbb{C}$.

Exercise 2.74. For every nonnegative integer k and for $k = \infty$, for every C^k-manifold $(X, \tau, \mathfrak{A})$, for every element V of τ, for every subset $\mathfrak{U}$ of $\tau|_V$ such that V equals the union of all elements U of $\mathfrak{U}$, for every tuple $(f_U)_{U \in \mathfrak{U}}$ of elements f_U of $C_{(X,\tau,\mathfrak{A}),\mathbb{R}}^k(U)$, respectively of $C_{(X,\tau,\mathfrak{A}),\mathbb{C}}^k(U)$, prove that $\mathsf{rest}_{U \cap U'}^U(f_U)$ equals $\mathsf{rest}_{U' \cap U}^{U'}(f_{U'})$ for every element (U, U') of $\mathfrak{U} \times \mathfrak{U}$ if and only if there exists an element f of $C_{(X,\tau,\mathfrak{A}),\mathbb{R}}^k(V)$, resp. of $C_{(X,\tau,\mathfrak{A}),\mathbb{C}}^k(V)$, such that $\mathsf{rest}_U^V(f)$ equals f_U for every element U of $\mathfrak{U}$. Moreover, prove such f is unique. This is the sheaf axiom for $C_{(X,\tau,\mathfrak{A}),\mathbb{R}}^k$, resp. for $C_{(X,\tau,\mathfrak{A}),\mathbb{C}}^k$.

Exercise 2.75. For every nonnegative integer k and for $k = \infty$, for every C^k-manifold $(X, \tau, \mathfrak{A})$, for every C^k-manifold $(Y, \sigma, \mathfrak{B})$, for every continuous function g from (Y, σ) to (X, τ), prove that g is a C^k-function if and only if $g_V^\#$ maps $C_{(X,\tau),\mathbb{R}}^k(V)$ to the subset $C_{(Y,\sigma),\mathbb{R}}^k(g^{\mathrm{pre}}(V))$ of $\mathrm{Hom}_{\mathbf{Set}}(V, \mathbb{R})$ for every element V of τ, respectively g is a C^k-function if and only if $g_V^\#$ maps $C_{(X,\tau),\mathbb{C}}^k(V)$ to the subset $C_{(Y,\sigma),\mathbb{C}}^k(g^{\mathrm{pre}}(V))$ of $\mathrm{Hom}_{\mathbf{Set}}(V, \mathbb{C})$ for every element V of τ. Since this also holds for open subsets of C^k-manifolds with their induced structure of C^k-manifold, this uniquely determines the C^k-charts of open subsets of a C^k-manifold in terms of the sheaf of $\mathbb{R}$-subalgebras $C_{(X,\tau,\mathfrak{A}),\mathbb{R}}^k$ of $C_{(X,\tau),\mathbb{R}}^0$,

respectively in terms of the sheaf of $\mathbb{C}$-subalgebras $C^k_{(X,\tau,\mathfrak{A}),\mathbb{C}}$ of $C^0_{(X,\tau),\mathbb{C}}$. Thus, the C^k-manifold structure $\mathfrak{A}$ on the topological manifold (X,τ) is uniquely determined by the sheaf of $\mathbb{R}$-subalgebras $C^k_{(X,\tau,\mathfrak{A}),\mathbb{R}}$ of $C^0_{(X,\tau),\mathbb{R}}$, respectively is uniquely determined by the sheaf of $\mathbb{C}$-subalgebras $C^k_{(X,\tau,\mathfrak{A}),\mathbb{C}}$ of $C^0_{(X,\tau),\mathbb{C}}$.

Exercise 2.76. For every nonnegative integer k and for $k = \infty$, for every C^k-function g from a C^k-manifold $(Y,\sigma,\mathfrak{B})$ to a C^k-manifold $(X,\tau,\mathfrak{A})$, for every C^k-function h from a C^k-manifold $(Z,\rho,\mathfrak{C})$ to $(Y,\sigma,\mathfrak{B})$, prove that the composition $g \circ h$ is a C^k-function from $(Z,\rho,\mathfrak{C})$ to $(X,\tau,\mathfrak{A})$. Also prove that Id_X is a C^k-function from $(X,\tau,\mathfrak{A})$ to itself. Deduce that $h^{\#}_{g^{\mathrm{pre}}(V)} \circ g^{\#}_V$ equals $(g \circ h)^{\#}_V$ as morphisms of sheaves on (X,τ) from $C^k_{(X,\tau),\mathbb{R}}$ to $(g \circ h)_* C^k_{(Z,\rho),\mathbb{R}}$, respectively from $C^k_{(X,\tau),\mathbb{C}}$ to $(g \circ h)_* C^k_{(Z,\rho),\mathbb{C}}$. Also, for every C^k-manifold $(X,\tau,\mathfrak{A})$, for every element V of τ, prove that $(\mathrm{Id}_X)^{\#}_V$ is the identity function. Thus, the pushforward homomorphisms $g^{\#}$ of sheaves of C^k-functions are contravariant in the C^k-map g.

Definition 2.77. For every nonnegative integer k and for $k = \infty$, the category $C^k - \mathbf{Man}$ has as objects all C^k-manifolds $(X,\tau,\mathfrak{A})$ and has as morphisms all C^k-functions between C^k-manifolds.

Definition 2.78. For every C^∞-manifold $(X,\tau,\mathfrak{A})$, a C^∞-atlas $\mathfrak{A}'$ of ordered pairs (V,ϕ) of a chartable open V and a diffeomorphism ϕ for V to an open subset of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is a **holomorphic atlas** if (and only if) for all elements (V,ϕ) and (V',ϕ') of $\mathfrak{A}$, the unique diffeomorphism $\mathsf{trans}^{(V,\phi)}_{(V',\phi')}$ from $\phi(V \cap V')$ to $\phi'(V \cap V)$ satisfying $\mathsf{trans}^{(V,\phi)}_{(V',\phi')} \circ \phi|_{V \cap V'} = \phi'|_{V \cap V'}$ is a holomorphic function between open subsets of $\mathbb{C}^n$. A **complex manifold structure** on $(X,\tau,\mathfrak{A})$, is a maximal holomorphic atlas, i.e., one that cannot be made bigger by adding more pairs (V,ϕ). For every C^∞-function g from a complex manifold $(Y,\sigma,\mathfrak{B}')$ to a complex manifold $(X,\tau,\mathfrak{A}')$, the C^∞-function g is a **holomorphic map** if (and only if), for every element (V,ϕ) of $\mathfrak{A}'$, for every element (U,ψ) of $\mathfrak{B}'$ such that U is a subset of $g^{\mathrm{pre}}(V)$, the unique C^∞-map $\mathsf{trans}^{(U,\psi)}_{g,(V,\phi)}$ from $\psi(U)$ to $\phi(V)$ satisfying $\phi \circ g|_U = \mathsf{trans}^{(U,\psi)}_{g,(V,\phi)} \circ \phi$ is a holomorphic function between open subsets of complex manifolds $\mathbb{C}^m$ and $\mathbb{C}^n$.

Definition 2.79. For every C^∞-function g from a complex manifold $(Y,\sigma,\mathfrak{B}')$ to a complex manifold $(X,\tau,\mathfrak{A}')$, the C^∞-function g is a **holomorphic function** if (and only if), for every element (V,ϕ) of $\mathfrak{A}'$, for every element (U,ψ) of $\mathfrak{B}'$ such that U is a subset of $g^{\mathrm{pre}}(V)$, the unique continuous map $\mathsf{trans}^{(U,\psi)}_{g,(V,\phi)}$ from $\psi(U)$ to $\phi(V)$ satisfying $\phi \circ g|_U = \mathsf{trans}^{(U,\psi)}_{g,(V,\phi)} \circ \phi$ is a holomorphic function between open subsets of complex spaces $\mathbb{C}^m$ and $\mathbb{C}^n$. In particular, for every element V of τ, the **$\mathbb{C}$-algebra of holomorphic functions** is the commutative $\mathbb{C}$-subalgebra $\mathcal{O}_{X,\tau,\mathfrak{A}'}(V)$ of $C^\infty_{(X,\tau,\mathfrak{A}),\mathbb{R}}(V)$ whose elements are all holomorphic functions from $(U,\tau|_U,\mathfrak{A}'|_U)$ to $\mathbb{C}$ (with its usual complex manifold structure). For every element (U,V) of $\mathrm{Arr}(\tau)$, the **restriction homomorphism** res^V_U from $\mathcal{O}_{X,\tau,\mathfrak{A}'}(V)$ to $\mathcal{O}_{X,\tau,\mathfrak{A}'}(U)$ is the unique $\mathbb{C}$-algebra homomorphism induced by the

restriction homomorphism above, i.e., precomposition with the holomorphic $U \hookrightarrow V$. Altogether, the ordered pair $(\mathcal{O}_{X,\tau,\mathfrak{A}'}(\bullet), \mathsf{res}^{\bullet}_{\bullet})$ of the function on τ associating $\mathcal{O}_{X,\tau,\mathfrak{A}'}(V)$ to each element V of τ, and the function on the subset $\mathrm{Arr}(\tau)$ associating res^V_U to each element (U,V) of $\mathrm{Arr}(\tau)$ is the **sheaf of holomorphic functions** to $\mathbb{C}$, usually denoted by $\mathcal{O}_{X,\tau,\mathfrak{A}'}$, or just $\mathcal{O}_X$ when τ and $\mathfrak{A}'$ are understood.

Exercise 2.80. For every complex manifold $(X, \tau, \mathfrak{A}')$, for every element V of τ, for every subset $\mathfrak{U}$ of $\tau|_V$ such that V equals the union of all elements U of $\mathfrak{U}$, for every tuple $(f_U)_{U \in \mathfrak{U}}$ of elements f_U of $\mathcal{O}_{X,\tau,\mathfrak{A}'}(U)$, prove that $\mathsf{rest}^U_{U \cap U'}(f_U)$ equals $\mathsf{rest}^{U'}_{U' \cap U}(f_{U'})$ for every element (U, U') of $\mathfrak{U} \times \mathfrak{U}$ if and only if there exists an element f of $\mathcal{O}_{X,\tau,\mathfrak{A}'}(V)$ such that $\mathsf{rest}^V_U(f)$ equals f_U for every element U of $\mathfrak{U}$. Moreover, prove such f is unique. This is the sheaf axiom for $\mathcal{O}_{X,\tau,\mathfrak{A}'}$.

Exercise 2.81. For every complex manifold $(X, \tau, \mathfrak{A}')$, for every complex manifold $(Y, \sigma, \mathfrak{B}')$, for every C^∞-function g from $(Y, \sigma, \mathfrak{B}')$ to $(X, \tau, \mathfrak{A}')$, prove that g is a holomorphic function if and only if $g^\#_V$ maps $\mathcal{O}_{X,\tau,\mathfrak{A}'}(V)$ to the subset $\mathcal{O}_{Y,\sigma,\mathfrak{B}'}(g^{\mathrm{pre}}(V))$ of $C^\infty_{(Y,\sigma,\mathfrak{B}),\mathbb{C}}(V)$ for every element V of τ. Since this also holds for open subsets of complex manifolds with their induced structure of complex manifold, this uniquely determines the holomorphic charts of open subsets of a complex manifold in terms of the sheaf of $\mathbb{C}$-subalgebras $\mathcal{O}_{X,\tau,\mathfrak{A}'}$ of $C^\infty_{(X,\tau,\mathfrak{A}),\mathbb{C}}$. Thus, the complex manifold structure $\mathfrak{A}'$ on the C^∞-manifold $(X, \tau, \mathfrak{A})$ is uniquely determined by the sheaf of $\mathbb{C}$-subalgebras $\mathcal{O}_{X,\tau,\mathfrak{A}'}$ of $C^\infty_{(X,\tau,\mathfrak{A}),\mathbb{C}}$.

Exercise 2.82. For every holomorphic function g from a complex manifold $(Y, \sigma, \mathfrak{B}')$ to a complex manifold $(X, \tau, \mathfrak{A}')$, for every holomorphic function h from a complex manifold $(Z, \rho, \mathfrak{C}')$ to $(Y, \sigma, \mathfrak{B}')$, prove that the composition $g \circ h$ is a holomorphic function from $(Z, \rho, \mathfrak{C}')$ to $(X, \tau, \mathfrak{A}')$. Also prove that Id_X is a holomorphic function from $(X, \tau, \mathfrak{A}')$ to itself. Deduce that $h^\#_{g^{\mathrm{pre}}(V)} \circ g^\#_V$ equals $(g \circ h)^\#_V$ as morphisms of sheaves on (X, τ) from $\mathcal{O}_{X,\tau,\mathfrak{A}'}$ to $(g \circ h)_* \mathcal{O}_{Z,\rho,\mathfrak{C}'}$. Thus, the pushforward homomorphisms $g^\#$ of sheaves of holomorphic functions are contravariant in the holomorphic map g.

Definition 2.83. The category **CpxMan** has as objects all complex manifolds and has as morphisms all holomorphic functions between complex manifolds.

Remark 2.84. Because there are more algebraic operations one can perform on them, and thanks to the exercises above showing that the two approaches are equivalent, many mathematicians prefer to define C^k-manifolds and complex manifolds using the sheaves defined above rather than atlases.

2.8   Cores and slice categories

Associated to every category there is a maximal subcategory that is a groupoid.

Definition 2.85. For every category $\mathbf{C}$, the **core** of $\mathbf{C}$ is the (usually non-full) groupoid sub-category with the same objects, but whose Hom set is the subset of invertible elements in the corresponding Hom set of $\mathbf{C}$.

Example 2.86. For every monoid $(H, \bullet)$, the core of the category of $(H, \bullet)$ is $B(H, \bullet)^\times$, where $(H, \bullet)^\times$ is the submonoid (in fact, group) of $(H, \bullet)$ whose elements are all invertible elements of H. The core of **Set** is the the groupoid of sets whose morphisms are bijections of sets. This also equals the core of **Rel**. The core of a product of categories is (canonically equivalent to) the product of the cores of the categories.

Example 2.87. In particular, in the core of the hybrid category, for every object $((H, \bullet), (H, r_H))$ where $(H, \bullet)$ is a group and where (H, r_H) is the right regular H-action on itself, the group of automorphisms of this object is the classical notion of *holomorph* of the group, i.e., the semidirect product of the group with its automorphism group.

Example 2.88. For every set S together with a relation R from S to itself, consider the class of S as a class of objects, and consider R as a span from this class to itself. An associative, unital composition law extending this to a category is unique if it exists. In fact, this span extends to a category if and only if R is a **preorder**, i.e., if and only if R is both transitive and reflexive. In this case, the corresponding category is small and thin. Every small, thin category is strongly equivalent to the category of a preordered set. The core of the category of a preordered set (S, R) is (canonically equivalent to) the category of the associated **Bishop set**, i.e., the set S together with an equivalence relation $\sim_R$, where $a \sim_R b$ if and only if both (a, b) and (b, a) are elements of R. The category of a preordered set is skeletal if and only if $\sim_R$ is equality, i.e., if and only if the preorder is a partial order: a transitive, reflexive relation that is also asymmetric. Similarly, the category of a preordered set is a groupoid if and only if the relation R is already an equivalence relation, i.e., if and only if the transitive, reflexive relation is also symmetric. Every preordered set is the pullback of a partial order under a surjection whose associated equivalence relation is $\sim_R$ (and this surjection to a partially ordered set is unique up to unique isomorphism). If we accept the Axiom of Choice, there exists a subset of the original set that surjects isomorphically to the partially ordered set, and this defines a full subcategory of the category of the preordered set that is a skeleton.

Definition 2.89. For every category $\mathbf{C}$, the objects of the **arrow category** $\mathrm{Arr}(\mathbf{C})$ are objects of $\mathrm{mor}(\mathbf{C})$, i.e., tuples $((s, t), f)$ of an ordered pair (s, t) of objects of $\mathbf{C}$ and a $\mathbf{C}$-morphism f from s to t. For every ordered pair of $\mathbf{C}$-morphisms, say $((s, t), f)$ and $((s', t'), f')$, the morphisms of $\mathrm{Arr}(\mathbf{C})$ from $((s, t), f)$ to $((s', t'), f')$ are ordered pairs (σ, τ) of a $\mathbf{C}$-morphism σ from s to s' and a $\mathbf{C}$-morphism τ from t to t' such that $f' \circ \sigma$ equals $\tau \circ f$. Composition of morphisms is componentwise.

Example 2.90. For every monoid $(H, \bullet)$, for the associated monoid category, the objects of the arrow category are elements h of H, and for every ordered pair (h, h') of elements of H, the morphisms from h to h' in the arrow category are ordered pairs (σ, τ) of elements of H such that $\tau \bullet h$ equals $h' \bullet \sigma$. In particular, if $(H, \bullet)$ is a group, this is the same as the set of pairs $(\sigma, h' \bullet \sigma \sigma h^{-1})$, which projects under pr_1 as a bijection to H (where composition corresponds to $\bullet$). Thus, for every group $(H, \bullet)$, the arrow category of $B(H, \bullet)$ is weakly equivalent to $B(H, \bullet)$.

Example 2.91. For each partially ordered set (S, R), denote by $R^{(2)}$ the partial order on $S \times S$ whose elements are all elements $((s, t), (s', t'))$ of $(S \times S) \times (S \times S)$ such that both (s, s') and (t, t') are elements of R. Denote by $R^{(2)}|_R$ the restriction of this partial order to the subset R of $S \times S$. Then for the category of the partially ordered set (S, R), the arrow category is strongly equivalent to the category of the partially ordered set $(R, R^{(2)}|_R)$.

Definition 2.92. For every category $\mathbf{C}$ and for every object b, the **under category** of $\mathbf{C}$ under the object b, denoted $\mathbf{C}_b$ or $b/\mathbf{C}$, is the subcategory of the arrow category whose objects are arrows $((s, t), f)$ such that t equals b, and whose morphisms from $((s, b), f)$ to $((s', b), f')$ are all ordered pairs (σ, Id_b) of a morphism σ from s to s' such that $f' \circ \sigma$ equals $\mathrm{Id}_b \circ f$, i.e., equals f.

Definition 2.93. For every category $\mathbf{C}$ and for every object a, the **over category** of $\mathbf{C}$ over the object a, denoted $\mathbf{C}^a$ or $\mathbf{C}/a$, is the subcategory of the arrow category whose objects are arrows $((s, t), f)$ such that s equals a, and whose morphisms from $((a, t), f)$ to $((a, t'), f')$ are all ordered pairs (Id_a, τ) of a morphism τ from t to t' such that $\tau \circ f$ equals $f' \circ \mathrm{Id}_a$, i.e., equals f'.

Example 2.94. For every monoid $(H, \bullet)$, for the associated monoid category, for the unique object, both for the under category and the over category, the objects are the elements h of H. For every ordered pair (h, h') of elements of H, the morphisms from h to h' in the under category are ordered pairs (σ, Id_H) of elements of H such that h equals $h' \bullet \sigma$, and the morphisms from h to h' in the over category are ordered pairs (Id_H, τ) such that $\tau \bullet h$ equals h'. In particular, if $(H, \bullet)$ is a group, both the over category and the under category are weakly equivalent to the discrete category with only one object and only one morphism (the identity morphism).

Example 2.95. For each partially ordered set (S, R), for each element b of S, denote by S_b the **lower subset** of b in (S, R), i.e., the subset of all elements a of S such that (a, b) is an element of R. Denote by R_b the restriction of R to this subset. Then the under category of the category of (S, R) under the object b is strongly equivalent to the category of the partially ordered set (S_b, R_b). Similarly, the over category over an element a is strongly equivalent to the category of the partially ordered set (S^a, R^a), where S^a is the **upper subset** of a in (S, R), i.e., the subset of all elements b of S such that (a, b) is an element of R.

3 Functors

The usual notion of morphisms between categories, called *functors*, are morphisms of spans that respect both composition and identities.

Definition 3.1. For every ordered pair class $(\mathbf{O}, \mathbf{M})$ of a class $\mathbf{O}$ and a $\mathbf{O}$-Hom span $\mathbf{M}$, for every ordered pair class $(\mathbf{O}', \mathbf{M}')$ of a class $\mathbf{O}'$ and a $\mathbf{O}'$-Hom span $\mathbf{M}'$, a **morphism** of Hom spans from $(\mathbf{O}, \mathbf{M})$ to $(\mathbf{O}', \mathbf{M}')$ is an ordered pair class $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ such that $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ is a span cell from $(\mathbf{O}, \mathbf{O}, \mathbf{M})$ to $(\mathbf{O}', \mathbf{O}', \mathbf{M}')$, i.e., for every member a of $\mathbf{O}$ there exists a unique member a' of $\mathbf{O}'$ such that (a, a') is a member of $\mathbf{F}_{\mathrm{obj}}$ (and all members are of this form), and for every member $((a, a'), (b, b'))$ of $\mathbf{F}_{\mathrm{obj}} \times \mathbf{F}_{\mathrm{obj}}$, for every member f of $\mathbf{M}_b^a$, there is a unique member f' of $(\mathbf{M}')_{b'}^{a'}$ such that $(((a, b), f), ((a', b'), f'))$ is a member of $\mathbf{F}_{\mathrm{mor}}$. We denote a' by $\mathbf{F}(a)$, and we denote by $\mathbf{F}_b^a$ the function from $\mathbf{M}_b^a$ to $(\mathbf{M}')_{b'}^{a'}$ that sends each f to f'.

Example 3.2. For every ordered pair class $(\mathbf{O}, \mathbf{M})$ with $\mathbf{M}$ a $\mathbf{O}$-Hom span, for every ordered pair class $(\mathbf{O}', \mathbf{M}')$ with $\mathbf{M}'$ a $\mathbf{O}'$-Hom span such that $\mathbf{O}'$ is a subclass of $\mathbf{O}$ and such that $\mathbf{M}'$ is a subclass of $\mathbf{M}$, the inclusion from the class $\mathbf{O}'$ to $\mathbf{O}$ and the inclusion from $\mathbf{M}'$ to $\mathbf{M}$ together define a morphism of Hom spans, $\mathrm{incl}_{\mathbf{O}, \mathbf{M}}^{\mathbf{O}', \mathbf{M}'}$ from $(\mathbf{O}', \mathbf{M}')$ to $(\mathbf{O}, \mathbf{M})$.

Exercise 3.3. For every ordered pair class $(\mathbf{O}, \mathbf{M})$ as above, prove that the identity span cell from $(\mathbf{O}, \mathbf{O}, \mathbf{M})$ to itself is a morphism of Hom spans from $(\mathbf{O}, \mathbf{M})$ to itself. Also, for all ordered pair classes as above, say $(\mathbf{O}, \mathbf{M})$, $(\mathbf{O}', \mathbf{M}')$ and $(\mathbf{O}'', \mathbf{M}'')$, for every morphism of Hom spans $\mathbf{F} = (\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ from $(\mathbf{O}, \mathbf{O}, \mathbf{M})$ to $(\mathbf{O}', \mathbf{O}', \mathbf{M}')$ and for every morphism of Hom spans $\mathbf{F}' = (\mathbf{F}'_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{mor}})$ from $(\mathbf{O}', \mathbf{O}', \mathbf{M}')$ to $(\mathbf{O}'', \mathbf{O}'', \mathbf{M}'')$, prove that the composition $(\mathbf{F}'_{\mathrm{obj}} \circ \mathbf{F}_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{mor}} \circ \mathbf{F}_{\mathrm{mor}})$ is a morphism of Hom spans from $(\mathbf{O}, \mathbf{O}, \mathbf{M})$ to $(\mathbf{O}'', \mathbf{O}'', \mathbf{M}'')$. Prove that composition of morphisms of Hom spans is associative, and the identity morphisms of Hom spans are left-right identities for this composition.

Definition 3.4. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every category $\mathbf{C}' = (\mathbf{O}', \mathbf{M}', \circ')$, a (covariant) **functor** from $\mathbf{C}$ to $\mathbf{C}'$ is a morphism $\mathbf{F}$ of Hom spans from $(\mathbf{O}, \mathbf{M})$ to $(\mathbf{O}', \mathbf{M}')$ that maps identities to identities and that is compatible with composition laws: for every object a of $\mathbf{C}$, the morphism $\mathbf{F}_{\mathrm{mor}}$ maps $(a, a, \mathrm{Id}_a^{\mathbf{C}})$ to $(a', a', \mathrm{Id}_{a'}^{\mathbf{C}'})$, and for every ordered pair $((a, b), f), ((b, c), g)$ of members of $\mathbf{M}$ with images $((a', b'), f')$ and $((b', c'), g')$ under $\mathbf{F}_{\mathrm{mor}}$, also $((a, c), g \circ f)$ has image $((a', c'), g' \circ' f')$.

Definition 3.5. For every category, the **identity functor** from the category to itself maps every object to itself and maps every morphism to itself. For every functor $\mathbf{F} = (\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every functor $\mathbf{F}' = (\mathbf{F}'_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{mor}})$ from the category $\mathbf{C}'$ to a category $\mathbf{C}''$, the **composite functor** is the composite of span cells, $\mathbf{F}' \circ \mathbf{F} = (\mathbf{F}'_{\mathrm{obj}} \circ \mathbf{F}_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{obj}} \circ \mathbf{F}_{\mathrm{obj}}, \mathbf{F}'_{\mathrm{mor}} \circ \mathbf{F}_{\mathrm{mor}})$ that sends every $\mathbf{C}$-object a to $\mathbf{F}'_{\mathrm{obj}}(\mathbf{F}_{\mathrm{obj}}(a))$ and that sends every $\mathbf{C}$-morphism $((a, b), f)$ in $\mathbf{C}_b^a$ to $\mathbf{F}'_{\mathrm{mor}}(\mathbf{F}_{\mathrm{mor}}((a, b), f))$.

Proposition 3.6. *Composition of functors is associative, and it is unital for the identity functors. A composition of functors is faithful, respectively full, fully faithful, essentially surjective, if each of the component functors is of this type.*

Exercise 3.7. Prove this proposition.

Definition 3.8. For every functor $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ from $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$ to $\mathbf{C}' = (\mathbf{O}', \mathbf{M}', \circ')$, the functor is **full**, respectively **faithful**, **fully faithful**, if for all members a and b of $\mathbf{O}$ with values $a' = \mathbf{F}_{\mathrm{obj}}(a)$ and $b' = \mathbf{F}_{\mathrm{obj}}(b)$, the function $\mathbf{F}^a_{\mathrm{mor},b}$ from $\mathrm{Hom}_{\mathbf{C}}(a,b)$ to $\mathrm{Hom}_{\mathbf{C}'}(a',b')$ is surjective, resp. injective, bijective. A functor is **essentially surjective** if every object of $\mathbf{C}'$ is isomorphic to an object of the form $\mathbf{F}_{\mathrm{obj}}(a)$ for some member a of $\mathbf{C}$. A faithful functor is **conservative** if (and only if) every morphism that is mapped to an isomorphism under the functor is already an isomorphism. A functor that is essentially surjective and fully faithful is a **weak equivalence** of categories. An essentially surjective inclusion functor of a full, skeletal subcategory in a category is a **skeleton** of the category. (If we assume a strong version of the Axiom of Choice then every category has a skeleton).

Definition 3.9. For every functor $\mathbf{G}$ from a category $\mathbf{E}$ to a category $\mathbf{B}$, the **G-variant** $\mathbf{E_G}$ of $\mathbf{E}$ is the category whose objects are ordered pairs (s,r) of a $\mathbf{E}$-object s and a $\mathbf{B}$-isomorphism r from $\mathbf{G}(s)$ to an $\mathbf{B}$-object U. For all $\mathbf{E_G}$-objects (s,r) and (s',r'), the Hom set $\mathrm{Hom}_{\mathbf{E_G}}((s,r),(s',r'))$ equals $\mathrm{Hom}_{\mathbf{E}}(s,s')$, and composition in $\mathbf{E_G}$ is composition in $\mathbf{E}$. The **weak equivalence** between $\mathbf{E}$ and $\mathbf{E_G}$ associates the ordered pair $(s, \mathrm{Id}_{\mathbf{G}(s)})$ to every $\mathbf{E}$-object s and associates the $\mathbf{E}$-object s to every $\mathbf{E_G}$-object $(s, \mathrm{Id}_{\mathbf{G}(s)})$. The weak equivalence is the identity on Hom sets. The **induced functor** $\widetilde{\mathbf{G}}$ from $\mathbf{E_G}$ to $\mathbf{B}$ associates the object U to every $\mathbf{E_G}$-object (s,r) as above, and it associates the morphism $r' \circ \mathbf{G}(u) \circ r^{-1}$ to every $\mathbf{E_G}$-morphism u from (s,r) to (s',r').

For every object U of $\mathbf{B}$, the **fiber category** $\mathbf{E}_{\mathbf{G},U}$ of $\mathbf{G}$ over U is the (non-full) subcategory of $\mathbf{E_G}$ whose objects are all ordered pairs (s,r) of an object s of $\mathbf{E}$ and a $\mathbf{B}$-isomorphism r from $\mathbf{G}(s)$ to U. For all objects (s,r) and (s',r') of $\mathbf{E}_{\mathbf{G},U}$, an $\mathbf{E}_{\mathbf{G},U}$-morphism from (s,r) to (s',r') is a $\mathbf{E}$-morphism u from s to s' such that $r' \circ \mathbf{G}(w)$ equals r. Composition of $\mathbf{E}_{\mathbf{G},U}$-morphisms is composition in $\mathbf{E}$. The **weak inclusion** from $\mathbf{E}_{\mathbf{G},U}$ to $\mathbf{E}$ sends every $\mathbf{E}_{\mathbf{G},U}$-object (s,r) to the $\mathbf{E}$-object s and sends every $\mathbf{E}_{\mathbf{G},U}$-morphism u to itself.

Exercise 3.10. Prove that $\mathbf{E_G}$ is a category, and prove that $\widetilde{\mathbf{G}}$ is a functor whose image equals the essential image of $\mathbf{G}$. Prove that if $\mathbf{G}$ is faithful, then every fiber category is a thin category.

4   Examples of functors

4.1   Some universal functors

Example 4.1. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$ for every (not necessarily full) subcategory $\mathbf{C'} = (\mathbf{O'}, \mathbf{M'}, \circ)$ the inclusion is a functor $\mathrm{incl}_{\mathbf{C}}^{\mathbf{C'}}$ from $\mathbf{C'}$ to $\mathbf{C}$, the **inclusion functor**.

Exercise 4.2. Prove that the inclusion functor of a subcategory into a category is a faithful functor that is full if and only if the subcategory is a full subcategory.

Example 4.3. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every class morphism $\mathbf{F}_{\mathrm{obj}}$ from a class $\mathbf{O'}$ to $\mathbf{O}$, this extends uniquely to a functor $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ from the discrete category structure on $\mathbf{O'}$ to $(\mathbf{O}, \mathbf{M}, \circ)$. Similarly for every class morphism $\mathbf{F}_{\mathrm{obj}}$ from $\mathbf{O}$ to a class $\mathbf{O'}$, this extends uniquely to a functor $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ from $(\mathbf{O}, \mathbf{M}, \circ)$ to the final category structure on $\mathbf{O'}$.

Definition 4.4. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, a **C-Hom equivalence relation** is a subclass $\mathbf{R}$ of $\mathbf{M} \times_{\mathbf{O} \times \mathbf{O}} \mathbf{M}$ that is stable for (component-wise) composition and such that each fiber class is an equivalence relation on the associated fiber class $\mathbf{M}_b^a$.

Example 4.5. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every $\mathbf{C}$-Hom equivalence relation $\mathbf{R}$, there exists a unique $\mathbf{O}$-Hom span $\mathbf{M}/\mathbf{R}$ and a unique composition law $\circ$ making $(\mathbf{O}, \mathbf{M}/\mathbf{R}, \circ)$ into a category such that for every ordered pair (a, b) of members of $\mathbf{O}$, the Hom set $(\mathbf{M}/\mathbf{R})_b^a$ is the set of $\mathbf{R}_b^a$-equivalence classes in $\mathbf{M}_b^a$, and such that the identity class morphism on $\mathbf{O}$ together with the quotient class morphism $\mathbf{M} \to \mathbf{M}/\mathbf{R}$ defines a full, strictly surjective functor from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O}, \mathbf{M}/\mathbf{R}, \circ)$.

Definition 4.6. For every category $\mathbf{C}$, for every $\mathbf{C}$-Hom equivalence relation $\mathbf{R}$, the functor of the previous example is the **quotient functor** of $\mathbf{C}$ by the $\mathbf{C}$-Hom equivalence relation $\mathbf{R}$.

Example 4.7. For every monoid $(H, \bullet)$, for the associated monoid category, a Hom-equivalence relation is equivalent to an equivalence relation R on H such that, for every (h, h') in R and for every k in H, also $(k \bullet h, k \bullet h')$ and $(h \bullet k, h' \bullet k)$ are elements of R. In particular, if $(H, \bullet)$ is a group, Hom-equivalence relations are precisely the equivalence relations of (left or right) congruence modulo *normal* subgroups of $(H, \bullet)$, and the quotient functor corresponds to the quotient group homomorphism by the normal subgroup.

Definition 4.8. For every functor $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}})$ from $(\mathbf{O}, \mathbf{M}, \circ)$ to $(\mathbf{O'}, \mathbf{M'}, \circ')$, the span cell of opposites spans is a functor $(\mathbf{F}_{\mathrm{obj}}, \mathbf{F}_{\mathrm{mor}}^{\mathrm{opp}})$ from $(\mathbf{O}, \mathbf{M}^{\mathrm{opp}}, \circ^{\mathbf{opp}})$ to $(\mathbf{O'}, (\mathbf{M'})^{\mathrm{opp}}, (\circ')^{\mathrm{opp}})$. This is the **opposite functor**. The opposite functor of the opposite functor equals the original functor.

For every category $\mathbf{C}$, for every category $\mathbf{C'}$, a functor from $\mathbf{C}^{\mathrm{opp}}$ to $\mathbf{C'}$ is equivalent (up to taking opposites) to a functor from $\mathbf{C}$ to $(\mathbf{C'})^{\mathrm{opp}}$, and these are both called **contravariant functors** from $\mathbf{C}$ to $\mathbf{C'}$.

Example 4.9. For all associative, unital rings R, S and T, for every $R - T$-bimodule N, the rule sending every $R - S$-bimodule M to the $S - T$-bimodule $\mathrm{Hom}_{R-\mathbf{Mod}}(M, N)$ and sending every morphisms of $R - S$-bimodules, say f from M to M', to the morphism of $S - T$-bimodules of precomposition by f, $\mathrm{Hom}_{R-\mathbf{Mod}}(M', N) \to \mathrm{Hom}_{R-\mathbf{Mod}}(M, N)$, is a contravariant functor from $R - S - \mathbf{Mod}$ to $S - T - \mathbf{Mod}$. In particular, if R equals S equals T is a commutative ring, and if N is the regular $R - R$-bimodule R, then $\mathrm{Hom}_{R-\mathbf{Mod}}(-, R)$ is the **dual module** contravariant functor from $R - \mathbf{Mod}$ to itself.

Definition 4.10. For every category $\mathbf{C}$, for every object a of $\mathbf{C}$, the set-valued **covariant Yoneda functor** $\mathbf{C}^a$ of a from $\mathbf{C}$ maps every $\mathbf{C}$-object b to the set $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a, b)$. This is also denoted $h^a_{\mathbf{C}}(b)$, or just $h^a(b)$ when confusion is unlikely. Also, for every $\mathbf{C}$-morphism v from b to b', the functor maps u to $\mathbf{C}^a_v$, the postcomposition with v from $\mathbf{C}^a_b$ to $\mathbf{C}^a_{b'}$. This is also denoted $h^a(v)$. Similarly, for every set S, the set-valued functor $S \times h^a$ maps every $\mathbf{C}$-object b to $S \times h^a(b)$ and maps every $\mathbf{C}$-morphism v to $\mathrm{Id}^{\mathbf{Set}}_S \times h^a(v)$ from $S \times h^a(b)$ to $S \times h^a(b')$.

Similarly, for every object b of $\mathbf{C}$, the set-valued **contravariant Yoneda functor** $\mathbf{C}_b$ of b is the covariant functor from $\mathbf{C}^{\mathrm{opp}}$ that maps every $\mathbf{C}$-object a to the set $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a, b)$. This is also denoted $h_{\mathbf{C},b}(a)$, or just $h_b(a)$ when confusion is unlikely. Also, for every $\mathbf{C}$-morphism u from a to a', the functor map u to $\mathbf{C}^u_b$, the precomposition with u from $\mathbf{C}^{a'}_b$ to $\mathbf{C}^a_b$ (note this is contravariant). This is denoted $h_b(u)$. Similarly, for every set S, the set-valued functor $S \times h_b$ maps every $\mathbf{C}$-object a to $S \times h_b(a)$ and maps every $\mathbf{C}$-morphism u to $\mathrm{Id}^{\mathbf{Set}}_S \times h_b(u)$ from $S \times h_b(a')$ to $S \times h_b(a)$.

Exercise 4.11. Check that each of these does preserve identities and composition, so that it is a functor.

Example 4.12. Let $(S, \leq)$ be a partially ordered set. For every element a of S, for every element b of S, the Yoneda functor $h^a(b)$ is a singleton set if and only if $a \leq b$, and otherwise it is empty, i.e., the image in $\mathbf{Set}$ is either an initial object or a final object. If we define the **support** of such a function to be the subset of S where the image is not the empty set, then the support of h^a is the subset $S_{\geq a}$ of all elements b with $a \leq b$. Similarly, the support of h_b is the subset $S_{\leq b}$ of all elements of b with $a \leq b$.

4.2   Forgetful functors

Definition 4.13. A faithful functor from a category $\mathbf{C}$ to $\mathbf{Set}$ is a **concrete functor**, and this functor makes $\mathbf{C}$ into a **concrete category**.

Most of the categories that arise in analysis, algebra, geometry, etc. are concrete, and typically the concrete functor is a "forgetful functor" that "forgets" some of the structure of the objects of $\mathbf{C}$.

Example 4.14. The forgetful functor from **Monoid** to **Set** that forgets the binary operation is a faithful functor; in fact, it is conservative. Thus, we also get concrete (and conservative) functors by restricting to the full subcategories **Grp**, **Ab** and $\mathbb{Q} - \mathbf{Mod}$.

Example 4.15. Similarly, the forgetful functors on $R - \mathbf{Mod}$, on $\mathbf{Mod} - S$ and on $R - S - \mathbf{Mod}$ are each concrete (and conservative) functors.

Example 4.16. The forgetful functor on **Ring** is a concrete (and conservative) functor, hence so is its restriction to the full subcategory **CRing**.

Example 4.17. Similarly, the forgetful functor from **Top** to **Set** is faithful, but it is not conservative (because there can be many different topologies on the same underlying set).

Example 4.18. Also the forgetful functors from $C^k - \mathbf{Man}$ to **TopMan** are faithful, but not conservative.

Example 4.19. Finally, the forgetful functor from **CpxMan** to $C^\infty - \mathbf{Man}$ is both faithful and conservative.

Exercise 4.20. The inclusion of **Set** as a (non-full) subcategory of **Rel** is faithful. Prove that the following defines a faithful (non-full) functor $\mathcal{P}$ from **Rel** to **Set**: map every object a to the power set $\mathcal{P}(a)$ of a. For every ordered pair (a,b) of sets, map every element R of $\mathbf{Rel}_b^a = \mathcal{P}(a \times b)$ to the set function $\mathcal{P}_R$ from $\mathcal{P}(a)$ to $\mathcal{P}(b)$ sending every subset a' of a to the subset $\mathcal{P}_R(a') = \mathrm{pr}_{(a,b),2}(\mathrm{pr}_{(a,b),1}^{\mathrm{pre}}(a') \cap R)$ of b, where $\mathrm{pr}_{(a,b),1}$, respectively $\mathrm{pr}_{(a,b),2}$, is the usual projection function from the Cartesian product $a \times b$ to a, resp. to b.

4.3 Functors of monoids and partially ordered sets

Example 4.21. For every monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, for every monoid homomorphism f from $(H, \bullet)$ to $(H', \bullet')$, there is a unique functor from the category of $(H, \bullet)$ to the category of $(H', \bullet')$ that maps the unique object to the unique object, and that maps Hom sets via f. Every functor between these categories is of this form for a unique monoid homomorphism f. More generally, for every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every object a of $\mathbf{C}$ with image $a' = \mathbf{F}(a)$, the function $\mathbf{F}_a^a$ from $\mathbf{C}_a^a$ to $(\mathbf{C}')_{a'}^{a'}$ is a monoid homomorphism. Moreover, for every ordered pair (a, b) of objects of $\mathbf{C}$, for the set $\mathbf{C}_b^a$ with its natural left $\mathbf{C}_b^b$-act and its natural right $\mathbf{C}_a^a$-act, for the set $(\mathbf{C}')_{b'}^{a'}$ with the induced left $\mathbf{C}_b^b$-act and right $\mathbf{C}_a^a$-act arising from the monoid homomorphisms $\mathbf{F}_b^b$ and $\mathbf{F}_a^a$, the function $\mathbf{F}_b^a$ from $\mathbf{C}_b^a$ to $(\mathbf{C}')_{b'}^{a'}$ is compatible with the left and right acts.

Example 4.22. Specializing the previous example to the case when $(H, \bullet)$ and $(H', \bullet')$ are groups, the functors from BH to BH' are equivalent to group homomorphisms from $(H, \bullet)$ to $(H', \bullet')$. More generally, every functor between groupoids induces group homomorphisms between automorphism groups of objects and the induced functioms between general Hom sets are compatible with both the left and right actions by these automorphism groups.

Example 4.23. The **opposite monoid functor** from **Monoid** to itself maps every monoid $(H, \bullet)$ to the opposite monoid $(H, \bullet^{\mathrm{opp}})$ and maps every morphism of monoids f from $(H_1, \bullet_1)$ to $(H_2, \bullet_2)$ to the same function f, now as a morphism of monoids from $(H_1, \bullet_1^{\mathrm{opp}})$ to $(H_2, \bullet_2^{\mathrm{opp}})$. This is an equivalence of categories, and the composition of this functor with itself is the identity functor.

Example 4.24. The restriction of the opposite monoid functor to the full subcategory **Group** is again an equivalence of **Group** to itself whose composition with itself equals the identity functor. Similarly when restricted to the full subcategories of finite groups, of finitely presented groups, of finitely generated groups, etc. The restriction to the full subcategory **Ab** is the identity functor.

Example 4.25. The **submonoid lattice covariant functor** from **Monoid** to the category of partially ordered sets **POrder** (in fact to its full subcategory of lattices) sends every monoid $(H, \bullet)$ to the subset of $\mathcal{P}(H)$ whose elements are all submonoids of $(H, \bullet)$, where the partial order is set inclusion. For every morphism f from a monoid $(H, \bullet)$ to a monoid $(H', \bullet')$, the image of f under the functor sends every submonoid K of $(H, \bullet)$ to the image submonoid $f(K)$ of $(H', \bullet')$.

Example 4.26. The **submonoid lattice contravariant functor** from **Monoid** to the category of partially ordered sets **POrder** is the same as the previous functor on objects. However, for every morphism f from a monoid $(H, \bullet)$ to a monoid $(H', \bullet')$, the image of f under the contravariant functor sends every submonoid K' of $(H', \bullet')$ to the preimage submonoid $f^{\mathrm{pre}}(K')$ of $(H, \bullet)$.

Example 4.27. For every category $\mathbf{C}$, for every preordered set (S', R'), every functor from $\mathbf{C}$ to the category of (S', R') is equivalent to a morphism $\mathbf{F}_{\mathrm{obj}}$ from $\mathrm{obj}(\mathbf{C})$ to (the class of) S' that is *nondecreasing*, i.e., for every ordered pair (a, b) of objects of $\mathbf{C}$ such that $\mathbf{C}_b^a$ is nonempty, then $(f(a), f(b))$ is an element of R'. There is a similar result when we replace (S', R') by any thin category.

Definition 4.28. For every catebory $\mathbf{C}$, a **simplicial object** of $\mathbf{C}$ is a functor to $\mathbf{C}$ from the opposite category Δ^{opp} of the category of nonempty, finite, totally ordered sets with order preserving maps as morphisms. A **cosimplicial object** of $\mathbf{C}$ is a functor to $\mathbf{C}$ from Δ.

Example 4.29. For the category **Set**, respectively **Monoid**, **Group**, **Ring**, **CRing**, etc., for every object X, the simplicial object $\mathsf{cosk}_0(X)$ maps each object $[n]$ of Δ^{opp} to the $(n+1)$-fold fiber product X^{n+1}, it maps each face map $\delta^{n,m}$ to the projection morphism from X^{n+1} to X^n that forgets the $(m+1)^{\mathrm{st}}$ coordinate, and it maps each degeneracy map $\sigma^{n,m}$ to the diagonal morphism from X^{n+1} to X^{n+1} that uses the $(m+1)^{\mathrm{st}}$ coordinate in X^{n+1} for both the $(m+1)^{\mathrm{st}}$ and the $(m+2)^{\mathrm{nd}}$ coordinate in X^{n+1}. This is the 0-**coskeleton** simplicial object associated to X.

4.4 Functors between categories of modules

Definition 4.30. For every ordered pair $((R, +_R, \cdot_R), (S, +_S, \cdot_S))$ of associative, unital rings, for every ordered pair $((R', +_{R'}, \cdot_{R'}), (S', +_{S'}, \cdot_{S'}))$ of associative, unital rings, a functor $\mathbf{F}$ from $R - S - \mathbf{Mod}$ to $R' - S' - \mathbf{Mod}$ is an **additive functor** if (and only if) both F preserves finite products and, for every ordered pair (M, N) of $R - S$-bimodules with image ordered pairs (M', N') of $R' - S'$-bimodules, the function $\mathbf{F}_N^M$ from $\mathrm{Hom}_{R-S-\mathbf{Mod}}(M, N)$ to $\mathrm{Hom}_{R'-S'-\mathbf{Mod}}(M', N')$ is additive, i.e., a homomorphism of Abelian groups. An additive functor is **right exact**, respectively **left exact**, **exact**, if (and only if) it maps every short exact sequence to a right exact sequence, resp. left exact sequence, short exact sequence. Similarly, a functor from the opposite category $(R - S - \mathbf{Mod})^{\mathrm{opp}}$ to $R' - S' - \mathbf{Mod}$ is an **additive functor** if it satisfies the conditions above, and it is **right exact**, resp. **left exact**, **exact**, if it satisfies the conditions above (here we use that the opposite of a short exact sequence is still a short exact sequence).

Example 4.31. For every $S - S'$-bimodule L, the functor $- \otimes_S L$ from $R - S - \mathbf{Mod}$ to $R - S' - \mathbf{Mod}$ maps every $R - S$-bimodule M to the $R - S'$-bimodule $M \otimes_S L$ and maps every morphism f of $R - S$-bimodules from M to N to $f \otimes_S \mathrm{Id}_L$ from $M \otimes_S L$ to $N \otimes_S L$. This is a right exact, additive functor.

Example 4.32. For every $R' - R$-bimodule H, the functor $H \otimes_R -$ from $R - S - \mathbf{Mod}$ to $R' - S - \mathbf{Mod}$ maps every $R - S$-bimodule M to the $R' - S$-bimodule $H \otimes_R M$ and maps every morphism f of $R - S$-bimodules from M to N to $\mathrm{Id}_H \otimes_R f$ from $H \otimes_S M$ to $H \otimes_S N$. This is also a right exact, additive functor.

Example 4.33. For every $R - R'$-bimodule K, the functor $\mathrm{Hom}_{R-\mathbf{Mod}}(K, -)$ from $R - S - \mathbf{Mod}$ to $R' - S - \mathbf{Mod}$ maps every $R - S$-bimodule M to the $R' - S$-bimodule $\mathrm{Hom}_{R-\mathbf{Mod}}(K, M)$ and maps every morphism f of $R - S$-bimodules from M to N to composition $f \circ -$ from $\mathrm{Hom}_{R-\mathbf{Mod}}(K, M)$ to $\mathrm{Hom}_{R-\mathbf{Mod}}(K, N)$. This is a left exact, additive functor.

Example 4.34. For every $S' - S$-bimodule P, the functor $\mathrm{Hom}_{\mathbf{Mod}-S}(P, -)$ from $R - S - \mathbf{Mod}$ to $R - S' - \mathbf{Mod}$ maps every $R - S$-bimodule M to the $R - S'$-bimodule $\mathrm{Hom}_{\mathbf{Mod}-S}(P, M)$ and maps every morphism f of $R - S$-bimodules from M to N to composition $f \circ -$ from $\mathrm{Hom}_{\mathbf{Mod}-S}(P, M)$ to $\mathrm{Hom}_{\mathbf{Mod}-S}(P, N)$. This is a left exact, additive functor.

Example 4.35. For every $R - S'$-bimodule J, the functor $\mathrm{Hom}_{R-\mathbf{Mod}}(-, J)$ from $(R - S - \mathbf{Mod})^{\mathrm{opp}}$ to $S - S' - \mathbf{Mod}$ maps every $R - S$-bimodule M to the $S - S'$-bimodule $\mathrm{Hom}_{R-\mathbf{Mod}}(M, J)$ and maps every morphism f of $R - S$-bimodules from M to N to precomposition $- \circ f$ from $\mathrm{Hom}_{R-\mathbf{Mod}}(N, J)$ to $\mathrm{Hom}_{R-\mathbf{Mod}}(M, J)$. This is a left exact, additive functor.

Example 4.36. For every $R' - S$-bimodule Q, the functor $\mathrm{Hom}_{\mathbf{Mod}-S}(-, Q)$ from $(R - S - \mathbf{Mod})^{\mathrm{opp}}$ to $R' - R - \mathbf{Mod}$ maps every $R - S$-bimodule M to the $R' - R$-bimodule $\mathrm{Hom}_{\mathbf{Mod}-S}(M, Q)$ and maps

every morphism f of $R-S$-bimodules from M to N to precomposition $-\circ f$ from $\mathrm{Hom}_{\mathbf{Mod}-S}(N, Q)$ to $\mathrm{Hom}_{\mathbf{Mod}-S}(M, Q)$. This is a left exact, additive functor.

Example 4.37. In particular, for every morphism u of associative, unital rings from R to R', for every morphism v of associative, unital rings from S to S', there is a functor $R' \otimes_R - \otimes_S S'$ from $R-S-\mathbf{Mod}$ to $R'-S'-\mathbf{Mod}$ that sends every $R-S$-bimodule M to the $R'-S'$-bimodule $R' \otimes_R M \otimes_S S'$. This is a right exact, additive functor.

Example 4.38. Similarly, for every morphism u of associative, unital rings from R to R', for every morphism v of associative, unital rings from S to S', there is a functor $^{u,v}(-)$ from $R'-S'-\mathbf{Mod}$ to $R-S-\mathbf{Mod}$ that makes every $R'-S'$-bimodule M into an $R-S$-bimodule via $r\cdot m\cdot s := u(r)\cdot m\cdot v(s)$. This is an exact, additive functor.

4.5   Functors of topologies

Example 4.39. For every continuous function f from a topological space (Y, σ) to a topological space (X, τ), the preimage function f^{pre} from τ to σ is a functor when we view the partially ordered sets σ and τ (with the subset order) as thin categories. If (X, τ) is a T_1 topological space, then the function f is uniquely determined by f^{pre}.

Definition 4.40. For every topological space (X, τ), for every category $\mathbf{C}$, a functor $\mathbf{F}$ from the category of τ to $\mathbf{C}$ is a $\mathbf{C}$-**presheaf** on (X, τ). For a category $\mathbf{C}$ that has a final object and that has products, a $\mathbf{C}$-presheaf $\mathbf{F}$ on (X, τ) is a $\mathbf{C}$-**sheaf** on (X, τ) if (and only if) for every element V of τ and for every subset $\mathfrak{U}$ of $\tau|_V$ such that V equals the union of all elements U of $\mathfrak{U}$, the restriction map from $\mathbf{F}(V)$ to $\prod_{U \in \mathfrak{U}} \mathbf{F}(U)$ equals the limit (i.e., coequalizer, fiber product) of the two maps from $\prod_{U \in \mathfrak{U}} \mathbf{F}(U)$ to $\prod_{(U_1, U_2) \in \mathfrak{U}^2} \mathbf{F}(U_1 \cap U_2)$ coming from restriction of U_1 to $U_1 \cap U_2$, respectively coming from restriction of U_2 to $U_1 \cap U_2$.

Definition 4.41. For every functor p from a category $\mathbf{E}$ to a category $\mathbf{B}$ for every $\mathbf{E}$-morphism $s' \xrightarrow{v'} s''$, this morphism is p-**Cartesian** if (and only if), for every $\mathbf{E}$-morphism $s \xrightarrow{v''} s''$, and for every $\mathbf{B}$-morphism $p(s) \xrightarrow{i} p(s')$ such that $p(v')\circ i$ equals $p(v'')$, there exists a unique $\mathbf{E}$-morphism $s \xrightarrow{v} s'$ such that both $v'\circ v$ equals v'' and $p(v)$ equals i. The functor p is a **Grothendieck fibration** if (and only if), for every object (s'', r'') of the fiber category $\mathbf{B}_{p,U''}$ over each $\mathbf{B}$-object U'', and for every $\mathbf{B}$-morphism $U' \xrightarrow{i'} U''$, there exists an object (s', r') of the fiber category $\mathbf{B}_{p,U'}$ and a p-Cartesian $\mathbf{E}$-morphism $s' \xrightarrow{v'} s''$ with $r'' \circ p(v')$ equal to $i' \circ r'$. Note that by the uniqueness in the definition of p-Cartesian morphisms, the p-Cartesian morphism v' is unique up to unique isomorphism. A **clivage** of a Grothendieck fibration is a class morphism assigning to each $\mathbf{B}$-morphism $U' \xrightarrow{i'} U''$ and each object (s'', r'') of the fiber category $\mathbf{E}_{p,U''}$ an object (s', r') of the fiber category $\mathbf{E}_{p,U'}$ and

a p-Cartesian morphism $s' \xrightarrow{v'} s''$ with $r'' \circ p(v')$ equal to $i' \circ r'$. Using some variant of the Axiom of Choice, there always exists a clivage. A clivage is **normalized** if it assigns an identity **E**-morphism to each identity **B**-morphism and object of the fiber category.

Exercise 4.42. Prove that every clivage determines a functor of **pullback by** i' from the fiber category $\mathbf{E}_{p,U''}$ to the fiber category $\mathbf{E}_{p,U'}$ for each **B**-morphism $U' \xrightarrow{i'} U''$.

Example 4.43. For every presheaf of sets **F** over a topological space (X,τ), the **Grothendieck fibration** of **F** is the category $\int \mathbf{F}$ whose objects are ordered pairs (U,f_U) of an element U of τ and of an element f_U of the set $\mathbf{F}(U)$, whose Hom set from an ordered pair (U,f_U) to (V,f_V) is empty unless U is a subset of V and f_U is the restriction to U of f_V, in which case it equals the Hom set of U to V in τ, i.e., the singleton of the inclusion $U \hookrightarrow V$, and where the composition is the usual composition of inclusions. The functor $p_\mathbf{F}$ from $\int \mathbf{F}$ to τ maps each ordered pair (U,f_U) to the element U of τ and maps each nonempty Hom set identically to itself. The restriction maps of **F** define a clivage of this Grothendieck fibration; again this is unique, in this case, since nonempty Hom sets are singletons. The presheaf **F** is uniquely recovered from the functor $p_\mathbf{F}$ from the category $\int \mathbf{F}$ to τ, which is one instance of the *Grothendieck construction*.

Example 4.44. For presheaves of sets **F** and **G** over (X,τ), for the corresponding Grothendieck fibrations $p_\mathbf{F}$ from $\int \mathbf{F}$ to τ and $p_\mathbf{G}$ from $\int \mathbf{G}$ to τ, a **morphism of Grothendieck fibrations** over τ is a functor ϕ from $\int \mathbf{F}$ to $\int \mathbf{G}$ such that $p_\mathbf{G} \circ \phi$ equals $p_\mathbf{F}$ (in other contexts, we would only ask that they are 2-equivalent, but since nonempty Hom sets are singletons here, this is strict equality). Of course this is equivalent to an assignment to each element U of τ of a set function ϕ_U from $\mathbf{F}(U)$ to $\mathbf{G}(U)$ such that for every inclusion $U \hookrightarrow V$ of elements of τ, we have $\phi_U \circ \mathsf{rest}^V_{\mathbf{F},U}$ equals $\mathsf{rest}^V_{\mathbf{G},U} \circ \phi_V$. Such a functor is also called a **morphism** of presheaves of sets on (X,τ). Composition of morphisms of presheaves of sets is composition of functors. Every identity functor of each presheaf of sets is an identity for this composition, and composition is associative. Altogether this defines a category $\mathbf{Presh}^{X,\tau}_{\mathbf{Set}}$ of presheaves of sets on (X,τ). In case **F** and **G** are both sheaves of sets, the morphisms of sheaves of sets from **F** to **G** are defined to be the same as the morphisms of presheaves of sets from **F** to **G**. This defines a full subcategory $\mathbf{Sh}^{X,\tau}_{\mathbf{Set}}$ of $\mathbf{Presh}^{X,\tau}_{\mathbf{Set}}$.

Example 4.45. For every presheaf of sets **F** over a topological space (X,τ), for every element x of X, an x-**section** of **F** is an object (U,f_U) of $\int \mathbf{F}$ such that x is an element of U. For x-sections (U,f_U) and (V,f_V), these are **equivalent** if (and only if) there exists an element W of $\tau|_{U \cap V}$ such that x is an element of W and the restrictions of f_U and f_V to W are equal. An x-**germ** of **F** is an equivalence class $f_p = [U,f_u]$ of x-sections. The **stalk** $\mathbf{F}_x$ of **F** at x is the set of all x-germs. The rule that associates to each presheaf of sets, **F**, the stalk $\mathbf{F}_x$ extends to a functor from the category $\mathbf{Presh}^{X,\tau}_{\mathbf{Set}}$ to **Set** for each element x of X.

Example 4.46. For every presheaf of sets $\mathbf{F}$ over a topological space (X, τ), the **éspace étalè**, denoted $\mathrm{Esp}(\mathbf{F})$, is the Cartesian product set $\prod_{x \in X} \mathbf{F}_x$ of all ordered pairs (x, f_x) of an element x of X and an x-germ f_x. This has a natural projection function $q_{\mathbf{F}}$ to X. There is a topology with a basis indexed by all objects (U, f_U) of $\int \mathbf{F}$ whose associated basis set $\mathcal{B}_{U, f_U}$ is all ordered pairs (x, f_x) of an element x of U and the corresponding x-germ $f_x = [U, f_u]$. For this topology, the function $q_{\mathbf{F}}$ from $\mathrm{Esp}(\mathbf{F})$ to (X, τ) is continuous, hence this is an object of the slice category of **Top** over (X, τ). The rule that associates to every presheaf of sets $\mathbf{F}$ over (X, τ) the object $\mathrm{Esp}(\mathbf{F}) \xrightarrow{q_{\mathbf{F}}} (X, \tau)$ of the slice category extends to a functor Esp from the category $\mathbf{Presh}_{\mathbf{Set}}^{X, \tau}$ to the slice category $\mathbf{Top}_{X, \tau}$.

Example 4.47. Conversely, for every object of the slice category $\mathbf{Top}_{X, \tau}$, i.e., for every continuous function p from a topological space (Y, σ) to (X, τ), and for every element V of τ, a continuous function f_U from $(U, \tau|_U)$ to $(p^{\mathrm{pre}}(U), \sigma|_{p^{\mathrm{pre}}(V)})$ such that $p \circ f$ equals the inclusion is a **continuous section** of p over U. The set of all continuous sections of p over U is denoted by $\mathsf{Sec}_{(Y, \sigma), (X, \tau), p}(U)$, or just $\mathsf{Sec}_{Y, \sigma}(U)$ when confusion is unlikely. For every element U of $\tau|_V$, the restriction map res_U^V from $\mathsf{Sec}_{Y, \sigma}(V)$ to $\mathsf{Sec}_{Y, \sigma}(U)$ is the usual restriction of continuous functions to an open subset. Altogether this defines a presheaf of sets $\mathsf{Sec}_{Y, \sigma}$ on (X, τ) that is even a sheaf, the **sheaf of sections** of the continuous function p from (Y, σ) to (X, τ).

Example 4.48. Continuing the previous example, for every continuous function p' from a topological space (Y', σ') to (X, τ), for every continuous function g from a topological space (Y, σ) to (Y', σ'), for the composite continuous function p from (Y, σ) to (X, τ), there is an induced morphism $\mathsf{Sec}_{(Y, \sigma), (Y', \sigma'), (X, \tau), (g, p')}$ of sheaves of sets from $\mathsf{Sec}_{(Y, \sigma), (X, \tau), p}$ to $\mathsf{Sec}_{(Y', \sigma'), (X, \tau), p'}$ by postcomposition with g. This morphism is denoted Sec_g when confusion is unlikely. This rule sends Id_Y to the identity morphism of $\mathrm{Sec}(Y, \tau)$. Also, it sends composition of morphisms in the slice category $\mathbf{Top}_{X, \tau}$ to composition in $\mathbf{Sh}_{\mathbf{Set}}^{X, \tau}$. Altogether, this defines a functor from the slice category $\mathbf{Top}_{X, \tau}$ to the category $\mathbf{Sh}_{\mathbf{Set}}^{X, \tau}$ of sheaves of sets on (X, τ).

Example 4.49. As a special case of the previous examples, for every topological space (X, τ), for every topological space (Z, ρ), for the product topological space (Y, σ) of (X, τ) and (Z, ρ) with its continous first projection p to (X, τ), the sheaf of sections of p is the same as the sheaf $\mathrm{Hom}_{\mathbf{Top}}(-, (Z, \rho))$ on (X, τ) of continuous functions to (Z, ρ). This is functorial in (Z, ρ).

4.6   Functors between categories of manifolds

Definition 4.50. For every topological space (X, τ), for every field k, the sheaf of k**-valued cofinitely continuous functions** is the subsheaf $\mathrm{Hom}_{\mathbf{Top}}(-, (k, \mathsf{Cof}))$ of the sheaf $\mathrm{Hom}_{\mathbf{Set}}(-, k)$ of k-valued discontinuous functions such that for every element U of τ, the subset $\mathrm{Hom}_{\mathbf{Top}}((U, \tau|_U), (k, \mathsf{Cof}))$ of $\mathrm{Hom}_{\mathbf{Set}}(U, k)$ is the set of all continuous functions f from $(U, \tau|_U)$ to k with its cofinite topology.

This is a sheaf of k-subalgebras on (X,τ) of the sheaf of k-algebras $\mathrm{Hom}_{\mathbf{Set}}(-,k)$s A sheaf $\mathcal{A}$ of k-subalgebras of $\mathrm{Hom}_{\mathbf{Top}}(-,(k,\mathsf{Cof}))$ is **locally ringed** if (and only if), for every element U of τ, for every element f of $\mathcal{A}(U)$ such that $f(U)$ is contained in $k^{\times} = k \smallsetminus \{0\}$, so that $1/f$ is an element of $\mathrm{Hom}_{\mathbf{Top}}((U,\tau|_U),(k,\mathsf{Cof}))$, then $1/f$ is also an element of $\mathcal{A}(U)$. In this case, the ordered pair $((X,\tau),\mathcal{A})$ is a **space with functions** to k.

Example 4.51. For every field k, for every topological space (X,τ), the sheaf $\mathrm{Hom}_{\mathbf{Top}}(-,(k,\mathsf{Cof}))$ on (X,τ) makes (X,τ) into a space with functions to k. For metrizable topological spaces, far more important is the subsheaf $C^0_{(X,\tau),\mathbb{R}}$, i.e., $\mathrm{Hom}_{\mathbf{Top}}(-,(\mathbb{R},\mathrm{Eucl}))$, of the sheaf $\mathrm{Hom}_{\mathbf{Top}}(-,(\mathbb{R},\mathsf{Cof}))$. The sheaf $C^0_{(X,\tau),\mathbb{R}}$ also makes (X,τ) into a space with functions to $\mathbb{R}$. Similarly, for every nonnegative integer k, and for $k = \infty$, for every C^k-manifold structure on a topological manifold (X,τ), the corresponding subsheaf $C^k_{(X,\tau),\mathbb{R}}$ makes (X,τ) into a space with functions to $\mathbb{R}$. By the earlier discussion, the C^k-manifold structure can be uniquely recovered from this space with functions structure. Similarly, for every complex manifold structure, the sheaf $\mathcal{O}_X$ of holomorphic functions makes (X,τ) into a space with functions to $\mathbb{C}$, from which the complex manifold structure can be uniquely recovered. More is true: the category of C^k-manifolds with C^k-functions as morphisms is a full subcategory of the category of spaces with functions to $\mathbb{R}$, and the category of complex manifolds with holomorphic functions as morphisms is a full subcategory of the category of spaces with functions to $\mathbb{C}$. Since the categories of spaces with functions also contain other important full subcategories, e.g., categories of (reduced and locally separated) analytic spaces and categories of (reduced and locally separated) algebraic spaces, this gives a convenient context for relating properties of manifolds and properties of analytic spaces, respectively algebraic spaces (what are usually called "GAGA theorems").

Example 4.52. There are many functors induced by the previous notion, e.g., the global sections functor of the locally ringed sheaf of k-algebras. In particular, $C^0_{\mathbb{R}}(-)$ defines a contravariant functor from the category of topological spaces (X,τ) to the category of $\mathbb{R}$-algebras. For C^k-manifolds, also $C^k_{\mathbb{R}}(-)$ is such a contravariant functor. For complex manifolds, also $\mathcal{O}(-)$ is a contravariant functor to the category of $\mathbb{C}$-manifolds. These functors are not faithful (much less fully faithful), whereas the functor associating a space with functions to a C^k-manifold, respectively to a complex manifold, is a fully faithful functor.

4.7 Constant functors

Definition 4.53. For every every category $\mathbf{C}$, for every category $\mathbf{D}$, for every object a of the category $\mathbf{D}$, the **constant functor** $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},a}$ from $\mathbf{C}$ to $\mathbf{D}$ with value a assigns the object a to every object of $\mathbf{C}$ and assigns the identity morphism $\mathrm{Id}^{\mathbf{D}}_a$ to every $\mathbf{C}$-morphism. In other words, $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},a}$ is the composition of the unique functor from $\mathbf{C}$ to the trivial monoid $B\{e\}$ with the unique functor from $B\{e\}$ to $\mathbf{D}$ sending the unique object of $B\{e\}$ to the object a of $\mathbf{D}$.

Example 4.54. In particular, for the category **Set**, the functor $\mathbf{L} = \mathrm{const}^{\mathbf{Set}}_{\mathbf{Set},\varnothing}$ from **Set** to itself has the special property that $\mathrm{Hom}_{\mathbf{Set}}(\mathbf{L}(a), b)$ is always a singleton set.

Definition 4.55. For every category **C**, an object 0 of **C** is an **initial object** if (and only if), for every object a of **C**, there exists a unique **C**-morphism from 0 to a.

Example 4.56. Similarly, for the category **Set**, for every singleton set, say $\mathbf{1} := \{\varnothing\}$, the functor $\mathbf{R} = \mathrm{const}^{\mathbf{Set}}_{\mathbf{Set},\mathbf{1}}$ from **Set** to itself has the special property that $\mathrm{Hom}_{\mathbf{Set}}(a, \mathbf{R}(b))$ is always a singleton set.

Definition 4.57. For every category **C**, an object 1 of **C** is a **final object** if (and only if), for every object a of **C**, there exists a unique **C**-morphism from a to 1. An object that is both initial and final is a **zero object**.

Exercise 4.58. For every category **C** that has an initial object, prove that the initial object is unique up to unique isomorphism. Similarly, for every category **C** that has a final object, prove that the final object is unique up to unique isomorphism (you can use opposites to reduce to the previous assertion). Conclude that for every category **C** that has a zero object, the zero object is unique up to unique isomorphism.

Exercise 4.59. Prove that **Set** has an initial object and a final object, but these are not isomorphic, hence **Set** does not have a zero object. Prove the same for **Top**, and the concrete forgetful functor maps the initial object of **Top**, respectively each final object of **Top**, to the initial object of **Set**, resp. to a final object of **Set**. On the other hand, prove that the empty set is the unique zero object of **Rel**. Similarly, prove that $\{e\}$ is the unique zero object in **Monoid**, in the full subcategory **Grp**, in the full subcategory **Ab**, etc.

Exercise 4.60. Prove that the (standard) ring of integers $\mathbb{Z}$ is an initial object in the category **Ring** of associative, unital rings, and also in the full subcategory **CRing** of commutative, associative, unital rings. Prove that the zero ring is a final object in each of these categories.

5   Natural transformations

Morphisms between functors are *natural transformations*. This is very analogous to the operation on group homomorphisms of postcomposition by a conjugation, i.e., an inner automorphism.

Definition 5.1. For every category **C**, for every category **C**′, for every covariant functor **F** from **C** to **C**′, and for every covariant functor **G** from **C** to **C**′, a **natural transformation** from **F** to **G** is a morphism of classes θ from $\mathrm{ob}_{\mathbf{C}}$ associating to every object a of **C** an element θ_a

of $\mathrm{Hom}_{\mathbf{C}'}(\mathbf{F}(a), \mathbf{G}(a))$ such that, for every ordered pair (a, b) of objects of $\mathbf{C}$ and for every $\mathbf{C}$-morphism u from a to b, the $\mathbf{C}'$-composite $\theta_b \circ F_b^a(u)$ equals the $\mathbf{C}'$-composite $G_b^a(u) \circ \theta_a$. A natural transformation is a **natural equivalence** (or **natural isomorphism**) if (and only if) the morphism associated to each object is an isomorphism.

Example 5.2. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, and for every covariant functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{C}'$, the **identity natural equivalence** from $\mathbf{F}$ to itself is the natural transformation that associates to every object a of $\mathbf{C}$ the identity morphism $\mathrm{Id}_{\mathbf{F}(a)}^{\mathbf{C}'}$. This is denoted by $\mathrm{Id}_{\mathbf{C}',\mathbf{F}}^{\mathbf{C}}$, or just $\mathrm{Id}_{\mathbf{F}}$ when confusion is unlikely.

Definition 5.3. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every ordered triple $(\mathbf{F}, \mathbf{G}, \mathbf{H})$ of covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, for every natural transformation η from $\mathbf{G}$ to $\mathbf{H}$, the (vertical) **composition natural transformation** $\eta \circ \theta$ from $\mathbf{F}$ to $\mathbf{H}$ is the natural transformation that associates to every object a of $\mathbf{C}$ the composite morphism $\eta_a \circ \theta_a$ from $\mathbf{F}(a)$ to $\mathbf{H}(a)$.

Exercise 5.4. Check that the composition natural transformation is, indeed, a natural transformation. Also check that composition is (strictly) associative for natural transformations. Finally check that identity natural transformations are (strict) left-right identities for composition.

There is another notion of composition for functors.

Definition 5.5. For every category $\mathbf{C} = (\mathbf{O}, \mathbf{M}, \circ)$, for every category $\mathbf{C}' = (\mathbf{O}', \mathbf{M}', \circ')$, for every category $\mathbf{C}'' = (\mathbf{O}'', \mathbf{M}'', \circ'')$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, for every ordered pair $(\mathbf{F}', \mathbf{G}')$ of covariant functors from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, for every natural transformation θ' from $\mathbf{F}'$ to $\mathbf{G}'$, the **horizontal composition natural transformation** of θ' and θ, sometimes called the **Godement product**, is the natural transformation $\theta' \star \theta$ from $\mathbf{F}' \circ \mathbf{F}$ to $\mathbf{G}' \circ \mathbf{G}$ associating to every object a of $\mathbf{C}$ the $\mathbf{C}''$-morphism,

$$\theta'_{\mathbf{G}(a)} \circ'' (\mathbf{F}')_{\mathbf{G}(a)}^{\mathbf{F}(a)}(\theta_a) = (\theta' \star \theta)_a = (\mathbf{G}')_{\mathbf{G}(a)}^{\mathbf{F}(a)}(\theta_a) \circ'' \theta'_{\mathbf{F}(a)}.$$

Exercise 5.6. Check that the Godement product is a natural transformation from $\mathbf{F}' \circ \mathbf{F}$ to $\mathbf{G}' \circ \mathbf{G}$. Also check that the Godement product is associative in both θ and θ' separately for the (vertical) composition of natural transformations.

There are some important special cases of the Godement product.

Definition 5.7. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every category $\mathbf{C}''$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of covariant functors from $\mathbf{C}$ to $\mathbf{C}'$, for every covariant functor $\mathbf{H}'$ from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **$\mathbf{H}'$-pushforward natural transformation** is $\mathbf{H}'_* \theta = \mathrm{Id}_{\mathbf{C}'',\mathbf{H}'}^{\mathbf{C}'} \star \theta$, associating to every object a of $\mathbf{A}$ the $\mathbf{C}''$-morphism $(\mathbf{H}')_{\mathbf{G}(a)}^{\mathbf{F}(a)}(\theta_a)$.

Definition 5.8. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every category $\mathbf{C}''$, for every covariant functor $\mathbf{E}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every ordered pair $(\mathbf{F}', \mathbf{G}')$ of covariant functors from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation θ from $\mathbf{F}'$ to $\mathbf{G}'$, the **E-pullback natural transformation**, $\mathbf{E}^*\theta = \theta * \mathrm{Id}^{\mathcal{C}}_{\mathcal{C}',\mathbf{E}}$ associates to every object a of $\mathbf{C}$ the $\mathbf{C}''$-morphism $\theta_{\mathbf{E}(a)}$.

Exercise 5.9. Check that the Godement product can be expanded in terms of pushforward, pullback and vertical composition as follows,

$$\mathbf{G}^*\eta \circ \mathbf{F}'_*\theta = \eta * \theta = \mathbf{G}'_*\theta \circ \mathbf{F}^*\eta.$$

6 Examples of natural transformations

6.1 Some universal natural transformations

Definition 6.1. For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism u from a $\mathbf{C}$-object b to a $\mathbf{C}$-object b', the **Yoneda natural transformation** $\mathbf{C}_u$ from the functor $\mathbf{C}_b$ to the functor $\mathbf{C}_{b'}$ (as functors from $\mathbf{C}^{\mathrm{opp}}$ to the category of sets) maps every $\mathbf{C}$-object a to the set function $\mathbf{C}^a_u$ of postcomposition by u from $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a,b)$ to $\mathbf{C}^a_{b'} = \mathrm{Hom}_{\mathbf{C}}(a,b)$. This is also denoted by h_u. Similarly, for every $\mathbf{C}$-morphism v from a $\mathbf{C}$-object a to a $\mathbf{C}$-object a', the **contravariant Yoneda natural transformation** $\mathbf{C}^v$ from the functor $\mathbf{C}^{a'}$ to $\mathbf{C}^a$ maps every $\mathbf{C}$-object b to the set function $\mathbf{C}^v_b$ of precomposition by v from $\mathbf{C}^{a'}_b = \mathrm{Hom}_{\mathbf{C}}(a',b)$ to $\mathbf{C}^a_b = \mathrm{Hom}_{\mathbf{C}}(a,b)$. This is also denoted by h^v.

Exercise 6.2. Check that each of these is a natural transformation. Also check that h_{Id_b} is the identity natural equivalence. Finally, check that this is compatible with composition, i.e., for composable $\mathbf{C}$-morphisms u and u', respectively v and v', check that the natural transformation $h_{u' \circ u}$ equals the composition $h_{u'} \circ h_u$, resp. check that the natural transformation $h^{v' \circ v}$ equals the composition $h^v \circ h^{v'}$.

Exercise 6.3. For the inclusion functor $\mathrm{incl}^{\mathbf{Set}}_{\mathbf{Rel}}$ from $\mathbf{Set}$ to $\mathbf{Rel}$, for the power set functor $\mathcal{P}$ from $\mathbf{Rel}$ to $\mathbf{Set}$, check that the following defines a natural transformation θ from the identity functor $\mathrm{Id}^{\mathbf{Set}}$ to the composite functor $\mathcal{P} \circ \mathrm{incl}^{\mathbf{Set}}_{\mathbf{Rel}}$. For every set a, the set function θ_a from a to $\mathcal{P}(a)$ sends every element y of a to the singleton set $\{y\}$ considered as an element of $\mathcal{P}(a)$.

Exercise 6.4. Continuing the previous exercise, check that the following defines a natural transformation η from $\mathrm{incl}^{\mathbf{Set}}_{\mathbf{Rel}} \circ \mathcal{P}$ to the identity functor $\mathrm{Id}^{\mathbf{Rel}}$. For every set a, the relation η_a from $\mathcal{P}(a)$ to a is the subset η_a of $\mathcal{P}(a) \times a$ of all ordered pairs (x,y) of a subset x of a and an element y of a, the ordered pair (x,y) is an element of η_a if and only if y is an element of x (i.e., η_a is the opposite relation of the relation ϵ_a of being an element of a set).

6.2 Natural transformations of constant functors

Example 6.5. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every $\mathbf{C}'$-morphism f' from an object a' to an object b', there is an associated natural transformation $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',f'}$ from the constant functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',a'}$ to the constant functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',b'}$ associating to every object a of $\mathbf{C}$ the morphism f'.

Exercise 6.6. For every category $\mathbf{C}$, for every category $\mathbf{C}'$, for every object a' of $\mathbf{C}'$, prove that $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',\mathrm{Id}_{a'}}$ is the identity natural transformation from $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',a'}$ to itself. Also, for every ordered triple (a',b',c') of $\mathbf{C}'$-objects, for every $\mathbf{C}'$-morphism f' from a' to b' and for every $\mathbf{C}'$-morphism g' from b' to c', prove that $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',g'\circ f'}$ equals the composition of natural transformations $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',g'} \circ \mathrm{const}^{\mathbf{C}}_{\mathbf{C}',f'}$.

Example 6.7. For every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, if there exists an initial object $0'$ of $\mathbf{C}'$, then there is a unique natural transformation from the constant functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',0'}$ to $\mathbf{F}$ that associates to every object a of $\mathbf{C}$ the unique $\mathbf{C}'$-morphism from $0'$ to $\mathbf{F}(a)$.

Example 6.8. For every functor $\mathbf{F}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, if there exists a final object $1'$ of $\mathbf{C}'$, then there is a unique natural transformation from $\mathbf{F}$ to the constant functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{C}',1'}$ that associates to every object a of $\mathbf{C}$ the unique $\mathbf{C}'$-morphism from $\mathbf{F}(a)$ to $1'$.

6.3 Natural transformations and monoids

Example 6.9. For every monoid $(H,\bullet)$, for every monoid $(H',\bullet')$, for monoid homomorphisms f and g from $(H,\bullet)$ to $(H',\bullet')$, for the associated functors from the category of $(H,\bullet)$ to the category of $(H',\bullet')$, a natural transformation between these functors is an element h' of H' such that for every element h of H, the composite $h' \bullet' f(h)$ equals $g(h) \bullet' h'$. In particular, if g equals f, then the natural self-transformations of $g = f$ are equivalent to elements of H' that centralize the image of f. So the center of $(H',\bullet')$ is equivalent to the set of natural self-transformations of the identity functor of the category of $(H',\bullet')$.

Example 6.10. Continuing the previous example, if the monoid $(H',\bullet')$ is a group (i.e., if every morphism is an isomorphism), then a natural transformation from f to g, monoid homomorphisms from $(H,\bullet)$ to $(H',\bullet')$, are equivalent to elements h' of H' such that g equals the composite $\mathrm{inner}_{h'} \circ f$, where $\mathrm{inner}_{h'}$ is the conjugation (inner) automorphism of $(H',\bullet')$ associated to h'.

Example 6.11. Similarly, for every natural transformation θ between functors $\mathbf{F}$ and $\mathbf{G}$ from a category $\mathbf{C}$ to a category $\mathbf{C}'$, for every object a of $\mathbf{C}$ that maps under both $\mathbf{F}$ and $\mathbf{G}$ to a common object a', the monoid homomorphisms $\mathbf{F}^a_a$ and $\mathbf{G}^a_a$ from $\mathbf{C}^a_a$ to $(\mathbf{C}')^{a'}_{a'}$ are *intertwined* by the element θ_a of $(\mathbf{C}')^{a'}_{a'}$ in the sense that $\theta_a \circ \mathbf{F}^a_a(u)$ equals $\mathbf{G}^a_a(u) \circ \theta_a$ for every element u of $\mathbf{C}^a_a$.

6.4 Natural transformations and partially ordered sets

Example 6.12. For every category $\mathbf{C}$, for every preordered set (S, R), for functors $\mathbf{F}$ and $\mathbf{G}$ from $\mathbf{C}$ to the thin category of (S, R), i.e., nondecreasing morphisms from $\mathrm{obj}(\mathbf{C})$ to (the class of) S, there exists a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ if and only if $(\mathbf{F}(a), \mathbf{G}(a))$ is an element of R for every object a of $\mathbf{C}$, and then the natural transformation is unique. In particular, for every preordered set (S', R'), and for all order preserving functions $\mathbf{F}$ and $\mathbf{G}$ from S' to S, considered as functors between the corresponding thin categories, there is a natural transformation from $\mathbf{F}$ to $\mathbf{G}$ if and only if $\mathbf{G}$ is valuewise greater than or equal to $\mathbf{F}$, in which case the natural transformation is unique. Thus, there is a natural equivalence from $\mathbf{F}$ to $\mathbf{G}$ if and only if $\mathbf{F}$ is valuewise equivalent to $\mathbf{G}$ for the equivalence relation on S induced by the preorder R.

Example 6.13. Continuing the previous example, for functors $\mathbf{F}$ and $\mathbf{G}$ from the thin category of (S, R) to a category $\mathbf{C}$, a natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$ is a morphism from (the class of) S to the Hom class of $\mathbf{C}$ such that for every element s of S, the $\mathbf{C}$-morphism θ_s is from $\mathbf{F}(s)$ to $\mathbf{G}(s)$, and for every element (s, s') of R, we have $\theta_{s'} \circ \mathbf{F}_{s,s'}$ equals $\mathbf{G}_{s,s'} \circ \theta_s$.

6.5 Natural transformations and modules

Example 6.14. For all associative, unital rings R, S and S', for every morphism u of $S - S'$-bimodules from L to $\widetilde{L}$, the **associated natural transformation** of u from the functor $- \otimes_S L$ to the functor $- \otimes_S \widetilde{L}$, as right exact, additive functors from $R - S - \mathbf{Mod}$ to $R - S' - \mathbf{Mod}$, is the assignment to every $R - S$-bimodule M of the unique morphism $\mathrm{Id}_M \otimes_S u$ from $M \otimes_S L$ to $M \otimes_S L'$.

Example 6.15. For all associative, unital rings R, R' and S, for every morphism v of $R' - R$-bimodules from H to $\widetilde{H}$, the **associated natural transformation** of v from the functor $H \otimes_R -$ to the functor $\widetilde{H} \otimes_R -$, as right exact, additive functors from $R - S - \mathbf{Mod}$ to $R' - S - \mathbf{Mod}$, is the assignment to every $R - S$-bimodule M of the unique morphism $v \otimes_R \mathrm{Id}_M$ from $H \otimes_R M$ to $\widetilde{H} \otimes_R M$.

Example 6.16. For every $R - R'$-bimodule morphism w from K to $\widetilde{K}$, the **associated natural transformation** of w from the functor $\mathrm{Hom}_{R-\mathbf{Mod}}(\widetilde{K}, -)$ to $\mathrm{Hom}_{R-\mathbf{Mod}}(K, -)$, as left exact, additive functors from $R - S - \mathbf{Mod}$ to $R' - S - \mathbf{Mod}$, assigns to every $R - S$-bimodule M the function $\mathrm{Hom}_{R-\mathbf{Mod}}(w, M)$ of precomposition by w from $\mathrm{Hom}_{R-\mathbf{Mod}}(\widetilde{K}, M)$ to $\mathrm{Hom}_{R-\mathbf{Mod}}(K, M)$. In other words, this is the Yoneda natural transformation h^w.

Example 6.17. For every $R - S'$-bimodule morphism t from J to $\widetilde{J}$, the **associated natural transformation** of t from the functor $\mathrm{Hom}_{R-\mathbf{Mod}}(-, J)$ to $\mathrm{Hom}_{R-\mathbf{Mod}}(-, \widetilde{J})$, as left exact, additive functors from $R - S - \mathbf{Mod}^{\mathrm{opp}}$ to $S - S' - \mathbf{Mod}$, assigns to every $R - S$-bimodule M the function $\mathrm{Hom}_{R-\mathbf{Mod}}(M, t)$ of postcomposition by t from $\mathrm{Hom}_{R-\mathbf{Mod}}(M, J)$ to $\mathrm{Hom}_{R-\mathbf{Mod}}(M, \widetilde{J})$. In other words, this is the Yoneda natural transformation h_t.

Example 6.18. For every $R' - R$-bimodule H, for every $R - S$-bimodule M, for every element m of M, define $\theta^H_{M,m}$ to be the R'-module morphism from H to $H \otimes_R M$ by $\theta^H_{M,m}(h) := h \otimes m$. As a function of m, this is an $R - S$-bimodule morphism θ^H_M from M to $\mathrm{Hom}_{R'-\mathbf{Mod}}(H, H \otimes_R M)$. Define θ^H to be the natural transformation from the identity functor on $R - S - \mathbf{Mod}$ to the composite functor $\mathrm{Hom}_{R'-\mathbf{Mod}}(H, -) \circ (H \otimes_R -)$ of the functor $H \otimes_R -$ from $R - S - \mathbf{Mod}$ to $R' - S - \mathbf{Mod}$ and the functor $\mathrm{Hom}_{R'-\mathbf{Mod}}(H, -)$ from $R' - S - \mathbf{Mod}$ to $R - S - \mathbf{Mod}$.

Example 6.19. Continuing the previous example, for every $R' - S$-bimodule N, the function from $H \times \mathrm{Hom}_{R'-\mathbf{Mod}}(H, N)$ to N by $(h, u) \mapsto u(h)$ is biadditive and R-balanced, hence defines a function η^H_N from $H \otimes_R \mathrm{Hom}_{R'-\mathbf{Mod}}(H, N)$ to N. This is an $R' - S$-bimodule morphism that is natural in N. Altogether η^H is a natural transformation from the composite functor $(H \otimes_R -) \circ \mathrm{Hom}_{R'-\mathbf{mod}}(H, -)$ to the identity functor on $R' - S - \mathbf{Mod}$.

Example 6.20. For every morphism u of associative, unital rings from R to R', for every morphism v of associative, unital rings from S to S', there is a natural transformation θ from the identity functor on $R - S - \mathbf{Mod}$ to the composite functor $^{u,v}(-) \circ (R' \otimes_R - \otimes_S S')$ from $R - S - \mathbf{Mod}$ to $R - S - mbfMod$ that associates to every $R - S$-bimodule M the $R - S$-bimodule morphism θ_M from M to $^{u,v}(R' \otimes_R M \otimes_S S')$ by $\theta_M(m) := 1 \otimes m \otimes 1$.

Example 6.21. For every morphism u of associative, unital rings from R to R', for every morphism v of associative, unital rings from S to S', there is a natural transformation η from the composite functor $(R' \otimes_R - \otimes_S S') \circ {}^{u,v}(-)$ to the identity functor on $R' - S' - \mathbf{Mod}$ that associates to every $R' - S'$-bimodule N the $R' - S'$-bimodule morphism η_N from $R' \otimes_R {}^{u,v}N \otimes_S S'$ to N by $\eta_N(r' \otimes n \otimes s') := r' \cdot n \cdot s'$.

6.6   Natural transformations and topologies

Exercise 6.22. For every T_1 topological space (X, τ), for continuous functions f and g from a topological space (Y, σ) to (X, τ), prove that f equals g if and only if there exists a natural transformation between the associated functors f^{pre} and g^{pre} from the thin category of τ to the thin category of σ, in which case this natural transformation is unique (the identity natural transformation).

Definition 6.23. For every topological space (X, τ), for every category $\mathbf{C}$, for $\mathbf{C}$-presheaves $\mathbf{F}$ and $\mathbf{G}$ on (X, τ), a $\mathbf{C}$-**presheaf morphism** from $\mathbf{F}$ to $\mathbf{G}$ is a natural transformation ϕ from $\mathbf{F}$ to $\mathbf{G}$. If $\mathbf{C}$ has a final object and products, so that the notion of $\mathbf{C}$-sheaf is defined, then $\mathbf{C}$-**sheaf morphisms** are defined to be the same as $\mathbf{C}$-presheaf morphisms between the associated $\mathbf{C}$-presheaves. In particular, for each $\mathbf{C}$-presheaf $\mathbf{F}$, a $\mathbf{C}$-presheaf morphism $\mathsf{assoc}_{\mathbf{F}}$ from $\mathbf{F}$ to a $\mathbf{C}$-sheaf $\mathbf{F}^{\mathrm{sh}}$ is an **associated $\mathbf{C}$-sheaf** of $\mathbf{F}$ if (and only if), for every $\mathbf{C}$-sheaf $\mathbf{G}$ and for every

natural transformation ϕ from $\mathbf{F}$ to $\mathbf{G}$, there exists a unique $\mathbf{C}$-sheaf morphism ϕ^{sh} from $\mathbf{F}^{\mathrm{sh}}$ to $\mathbf{G}$ such that $\phi^{\mathrm{sh}} \circ \mathsf{assoc}_{\mathbf{F}}$ equals ϕ.

Exercise 6.24. For presheaves of sets, prove that this definition of morphism is equivalent to the earlier definition in terms of functors between the associated Grothendieck fibrations.

Exercise 6.25. For every presheaf of sets $\mathbf{F}$ on a topological space (X, τ), for every object (U, f_U) of $\int \mathbf{F}$, prove that the function f_U from (U, τ_U) to the topological space $\mathrm{Esp}(\mathbf{F})$ that sends every element x of U to the x-germ f_x of f_U is continuous. Altogether, this defines a morphism $\theta_{\mathbf{F}}$ of presheaves of sets from $\mathbf{F}$ to $\mathrm{Sec}(\mathrm{Esp}(\mathbf{F}))$. Prove that this is a natural transformation θ of functors of $\mathbf{Presh}_{\mathbf{Set}}^{X,\tau}$ to itself from the identity functor to the composite functor $\mathrm{Sec} \circ \mathrm{Esp}$.

Exercise 6.26. For every object $(Y, \sigma) \xrightarrow{p} (X, \tau)$ of the slice category $\mathbf{Top}_{X,\tau}$, for every element x of X, for every object (U, f_U) of $\int \mathrm{Sec}_{Y,\sigma}$, the element $f_U(x)$ of Y depends only on the x-germ $f_x = [U, f_u]$. Thus, there is a well-defined function $\eta_{Y,\sigma}$ from $\mathrm{Esp}(\mathrm{Sec}_{Y,\sigma})$ to Y. This is a continuous functor for the given topologies, and this function commutes with the projections to (X, τ), i.e., this is a morphism of the slice category $\mathbf{Top}_{X,\tau}$. Prove that this is a natural transformation η of functors of $\mathbf{Top}_{X,\tau}$ to itself from the composite functor $\mathrm{Esp} \circ \mathrm{Sec}$ to the identity functor.

7   Adjoint pairs

Definition 7.1. For every category $\mathbf{C}$, for every category $\mathbf{D}$, an **adjoint pair** of covariant functors between $\mathbf{C}$ and $\mathbf{D}$ is $((\mathbf{L}, \mathbf{R}), (\theta, \eta))$ consisting of an ordered pair of covariant functors,

$$\mathbf{L} : \mathbf{C} \to \mathbf{D},$$

$$\mathbf{R} : \mathbf{D} \to \mathbf{C},$$

and an ordered pair of natural transformations of covariant functors,

$$\theta : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L}, \ \theta(a) : a \to \mathbf{R}(\mathbf{L}(a)),$$

$$\eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathbf{D}}, \ \eta(b) : \mathbf{L}(\mathbf{R}(b)) \to b,$$

such that the following composition of natural transformations equals $\mathrm{Id}_{\mathbf{R}}$, respectively equals $\mathrm{Id}_{\mathbf{L}}$,

$$(*_{\mathbf{R}}) : \mathbf{R} \overset{\theta \circ \mathbf{R}}{\Rightarrow} \mathbf{R} \circ \mathbf{L} \circ \mathbf{R} \overset{\mathbf{R} \circ \eta}{\Rightarrow} \mathbf{R},$$

$$(*_{\mathbf{L}}) : \mathbf{L} \overset{\mathbf{L} \circ \theta}{\Rightarrow} \mathbf{L} \circ \mathbf{R} \circ \mathbf{L} \overset{\eta \circ \mathbf{L}}{\Rightarrow} \mathbf{L}.$$

For every object a of $\mathbf{C}$ and for every object b of $\mathbf{D}$, define set maps,

$$H_{\mathbf{R}}^{\mathbf{L}}(a,b) : \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(a), b) \to \mathrm{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)),$$

$$(\mathbf{L}(a) \xrightarrow{\phi} b) \mapsto \left(a \xrightarrow{\theta(a)} \mathbf{R}(\mathbf{L}(a)) \xrightarrow{\mathbf{R}(\phi)} \mathbf{R}(b) \right),$$

and

$$H_{\mathbf{L}}^{\mathbf{R}}(a,b) : \mathrm{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)) \to \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(a), b),$$

$$(a \xrightarrow{\psi} \mathbf{R}(b)) \mapsto \left(\mathbf{L}(a) \xrightarrow{\mathbf{L}(\psi)} \mathbf{L}(\mathbf{R}(b)) \xrightarrow{\eta(b)} b \right).$$

Exercise 7.2. For $\mathbf{L}$, $\mathbf{R}$, θ and η as above, prove that the conditions $(*_{\mathbf{R}})$ and $(*_{\mathbf{L}})$ hold if and only if, for every object a of $\mathbf{C}$ and for every object b of $\mathbf{D}$, the morphisms $H_{\mathbf{R}}^{\mathbf{L}}(a,b)$ and $H_{\mathbf{L}}^{\mathbf{R}}(a,b)$ are inverse bijections.

Exercise 7.3. Prove that both $H_{\mathbf{R}}^{\mathbf{L}}(a,b)$ and $H_{\mathbf{L}}^{\mathbf{R}}(a,b)$ are binatural in a and b.

Exercise 7.4. For functors $\mathbf{L}$ and $\mathbf{R}$, and for binatural inverse bijections $H_{\mathbf{R}}^{\mathbf{L}}(a,b)$ and $H_{\mathbf{L}}^{\mathbf{R}}(a,b)$ between the bifunctors

$$\mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(a), b), \mathrm{Hom}_{\mathbf{C}}(a, \mathbf{R}(b)) : \mathbf{C} \times \mathbf{D} \to \mathbf{Set},$$

prove that there exist unique θ and η extending $\mathbf{L}$ and $\mathbf{R}$ to an adjoint pair such that $H_{\mathbf{R}}^{\mathbf{L}}$ and $H_{\mathbf{L}}^{\mathbf{R}}$ agree with the binatural inverse bijections defined above.

Exercise 7.5. Let $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ be an adjoint pair as above. For every covariant functor,

$$\widetilde{\mathbf{R}} : \mathbf{D} \to \mathbf{C},$$

for every natural transformation η from $\mathbf{L} \circ \widetilde{\mathbf{R}}$ to $\mathrm{Id}_{\mathbf{D}}$, prove that $\widetilde{\eta}' := R_* \widetilde{\eta} \circ \widetilde{\mathbf{R}}^* \theta$ is the unique natural transformation from $\widetilde{\mathbf{R}}$ to $\mathbf{R}$ such that $\widetilde{\eta}$ equals $\eta \circ L_* \widetilde{\eta}'$. Conversely, for every natural transformation $\widetilde{\eta}'$ from $\widetilde{\mathbf{R}}$ to $\mathbf{R}$, prove that $\widetilde{\eta} := \eta \circ L_* \widetilde{\eta}'$ is the unique natural transformation from $\mathbf{L} \circ \widetilde{\mathbf{R}}$ such that $\widetilde{\eta}'$ equals $R_* \widetilde{\eta} \circ \widetilde{\mathbf{R}}^* \theta$. Formulate and prove the analogous correspondence between natural transformations $\widetilde{\theta}$ from $\mathrm{Id}_{\mathbf{C}}$ to $\mathbf{R} \circ \widetilde{\mathbf{L}}$ and natural transformations $\widetilde{\theta}'$ from $\mathbf{L}$ to a functor $\widetilde{\mathbf{L}}$.

Exercise 7.6. Let $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ be an adjoint pair as above. Let a covariant functor

$$\widetilde{\mathbf{R}} : \mathbf{D} \to \mathbf{C},$$

and natural transformations,

$$\widetilde{\theta} : \mathrm{Id}_{\mathbf{C}} \Rightarrow \widetilde{\mathbf{R}} \circ \mathbf{L}, \widetilde{\eta} : \mathbf{L} \circ \widetilde{\mathbf{R}} \Rightarrow \mathrm{Id}_{\mathbf{D}},$$

be natural transformations such that $(\mathbf{L}, \widetilde{\mathbf{R}}, \widetilde{\theta}, \widetilde{\eta})$ is also an adjoint pair. For every object b of $\mathbf{D}$, define $\iota(b)$ in $\mathrm{Hom}_{\mathbf{D}}(\mathbf{R}(b), \widetilde{\mathbf{R}}(b))$ to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathbf{D}}(b, b) \xrightarrow{\mathrm{Hom}_{\mathbf{D}}(\theta(b), b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(\mathbf{R}(b)), b) \xrightarrow{H_{\mathbf{L}}^{\widetilde{\mathbf{R}}}(\mathbf{R}(b), b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{R}(b), \widetilde{\mathbf{R}}(b)).$$

Similarly, define $\kappa(b)$ in $\mathrm{Hom}_{\mathbf{D}}(\widetilde{\mathbf{R}}(b), \mathbf{R}(b))$, to be the image of Id_b under the composition,

$$\mathrm{Hom}_{\mathbf{D}}(b, b) \xrightarrow{\mathrm{Hom}_{\mathbf{D}}(\widetilde{\theta}(b), b)} \mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(\widetilde{\mathbf{R}}(b)), b) \xrightarrow{H_{\mathbf{L}}^{\mathbf{R}}(\widetilde{\mathbf{R}}(b), b)} \mathrm{Hom}_{\mathbf{D}}(\widetilde{\mathbf{R}}(b), \mathbf{R}(b)).$$

Prove that ι and κ are the unique natural transformations of functors,

$$\iota : \mathbf{R} \Rightarrow \widetilde{\mathbf{R}}, \ \ \kappa : \widetilde{\mathbf{R}} \Rightarrow \mathbf{R},$$

such that $\widetilde{\theta}$ equals $(\iota \circ \mathbf{L}) \circ \theta$, θ equals $(\kappa \circ \mathbf{L}) \circ \widetilde{\theta}$, $\widetilde{\eta}$ equals $\eta \circ (\mathbf{L} \circ \iota)$, and η equals $\widetilde{\eta} \circ (\mathbf{L} \circ \kappa)$. Moreover, prove that ι and κ are inverse natural equivalences. In this sense, every extension of a functor $\mathbf{L}$ to an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ is unique up to unique natural isomorphisms (ι, κ). Formulate and prove the symmetric statement for all extensions of a functor $\mathbf{R}$ to an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ (you could use opposite categories to simplify this).

Exercise 7.7. For every adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$, prove that also $(\mathbf{R}^{\mathrm{opp}}, \mathbf{L}^{\mathrm{opp}}, \eta^{\mathrm{opp}}, \theta^{\mathrm{opp}})$ is an adjoint pair.

Exercise 7.8. Formulate the corresponding notions of adjoint pairs when $\mathbf{L}$ and $\mathbf{R}$ are contravariant functors (just replace one of the categories by its opposite category).

Exercise 7.9. For every ordered triple of categories, $(\mathbf{C}, \mathbf{D}, \mathbf{E})$ for all covariant functors,

$$\mathbf{L}' : \mathbf{C} \to \mathbf{D}$$

$$\mathbf{R}' : \mathbf{D} \to \mathbf{C},$$

for all natural transformations that form an adjoint pair,

$$\theta' : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R}'\mathbf{L}',$$

$$\eta' : \mathbf{L}'\mathbf{R}' \Rightarrow \mathrm{Id}_{\mathbf{D}},$$

for all covariant functors,

$$\mathbf{L}'' : \mathbf{D} \to \mathbf{E},$$
$$\mathbf{R}'' : \mathbf{E} \to \mathbf{D},$$

and for all natural transformations that form an adjoint pair,

$$\theta'' : \mathrm{Id}_{\mathbf{D}} \Rightarrow \mathbf{R}''\mathbf{L}'',$$

$$\eta'' : \mathbf{L}''\mathbf{R}'' \Rightarrow \mathrm{Id}_{\mathbf{E}},$$

define covariant functors

$$\mathbf{L} : \mathbf{C} \to \mathbf{E}, \ \mathbf{R} : \mathbf{E} \to \mathbf{C}$$

by $\mathbf{L} = \mathbf{L}'' \circ \mathbf{L}'$, $\mathbf{R} = \mathbf{R}' \circ \mathbf{R}''$, define the natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L},$$

to be the composition of natural transformations,

$$\mathrm{Id}_{\mathbf{C}} \overset{\theta'}{\Rightarrow} \mathbf{R}' \circ \mathbf{L}' \overset{\mathbf{R}' \circ \theta'' \circ \mathbf{L}'}{\Rightarrow} \mathbf{R}' \circ \mathbf{R}'' \circ \mathbf{L}'' \circ \mathbf{L}',$$

and define the natural transformation,

$$\eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathbf{E}},$$

to be the composition of natural transformations,

$$\mathbf{L}'' \circ \mathbf{L}' \circ \mathbf{R}' \circ \mathbf{R}'' \overset{\mathbf{L}'' \circ \eta' \circ \mathbf{R}''}{\Rightarrow} \mathbf{L}'' \circ \mathbf{R}'' \overset{\eta''}{\Rightarrow} \mathrm{Id}_{\mathbf{E}}.$$

Prove that $\mathbf{L}$, $\mathbf{R}$, θ and η form an adjoint pair of functors. This is the **composition** of $(\mathbf{L}', \mathbf{R}', \theta', \eta')$ and $(\mathbf{L}'', \mathbf{R}'', \theta'', \eta'')$.

Exercise 7.10. If $\mathbf{C}$ equals $\mathbf{D}$, if $\mathbf{L}'$ and $\mathbf{R}'$ are the identity functors, and if θ' and η' are the identity natural transformations, prove that $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ equals $(\mathbf{L}'', \mathbf{R}'', \theta'', \eta'')$. Similarly, if $\mathbf{D}$ equals $\mathbf{E}$, if $\mathbf{L}''$ and $\mathbf{R}''$ are the identity functors, and if θ'' and η'' are the identity natural transformations, prove that $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ equals $(\mathbf{L}', \mathbf{R}', \theta', \eta')$. Finally, prove that composition of three adjoint pairs is associative.

Example 7.11. Let $\mathbf{C}$ be a category that has a final object f, and let $\mathbf{D}$ be a category that has an initial object e. Let $\mathbf{L}$ be $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},e}$, and let $\mathbf{R}$ be $\mathrm{const}^{\mathbf{D}}_{\mathbf{C},f}$. Thus, $\mathbf{R} \circ \mathbf{L}$ equals $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$, and $\mathbf{L} \circ \mathbf{R}$ equals $\mathrm{const}^{\mathbf{D}}_{\mathbf{D},e}$. Since f is a final object of $\mathbf{C}$, there is a unique natural transformation from every endofunctor of $\mathbf{C}$ to $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$. In particular, there exists a unique natural transformation θ from the identity functor to $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$. Since e is an initial object of $\mathbf{C}$, there is a unique natural transformation from $\mathrm{const}^{\mathbf{D}}_{\mathbf{D},e}$ to every endofunctor of $\mathbf{D}$. In particular, there exists a unique natural transformation η from $\mathrm{const}^{\mathbf{C}}_{\mathbf{C},f}$ to the identity functor. Together, these define an adjoint pair giving binatural bijections for every object a of $\mathbf{C}$ and every object b of $\mathbf{D}$,

$$\mathrm{Hom}_{\mathbf{D}}(\mathrm{const}^{\mathbf{C}}_{\mathbf{D},e}(a), b) \cong \mathrm{Hom}_{\mathbf{C}}(a, \mathrm{const}^{\mathbf{D}}_{\mathbf{C},f}(b)).$$

Example 7.12. Let $(S, \leq)$ and $(S', \leq')$ be partially ordered sets. Let $\mathbf{L}$ be a nondecreasing function from $(S, \leq)$ to $(S', \leq')$ considered as a functor between the associated categories. Let $\mathbf{R}$ be a nondecreasing function from $(S', \leq')$ to $(S, \leq)$ considered as a functor between the associated categories. There exist natural transformations completing this to an adjoint pair if and only if, for every element a of S, for every element a' of S', we have $L(a) \leq' a'$ if and only $a \leq R(a')$. In this case, the natural transformations extending to an adjoint pair are unique.

Definition 7.13. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every adjoint pair

$$(\mathbf{L} : \mathbf{C} \to \mathbf{D}, \mathbf{R} : \mathbf{D} \to \mathbf{C}, \theta : \mathrm{Id}_{\mathbf{C}} \Rightarrow \mathbf{R} \circ \mathbf{L}, \eta : \mathbf{L} \circ \mathbf{R} \Rightarrow \mathrm{Id}_{\mathbf{D}}),$$

the adjoint pair is a **strict equivalence** from $\mathbf{C}$ to $\mathbf{D}$ if (and only if) both θ is a natural equivalence and η is a natural equivalence.

Exercise 7.14. Prove that identity adjoint pairs are strict equivalences. Prove that the composition adjoint pair of strict equivalences is a strict equivalence. For every strict equivalence from $\mathbf{C}$ to $\mathbf{D}$ as above, prove that also $(\mathbf{R}, \mathbf{L}, \eta^{-1}, \theta^{-1})$ is a strict equivalence from $\mathbf{D}$ to $\mathbf{C}$ that is a left-right inverse of the original strict equivalence.

Exercise 7.15. Prove that each of the functors in a strict equivalence is a weak equivalence. Prove that every composition of weak equivalences is a weak equivalence.

Exercise 7.16. Let $\mathbf{C}$ and $\mathbf{D}$ be strictly small categories. Prove that for every weak equivalence L from $\mathbf{C}$ to $\mathbf{D}$ there exists a strict equivalence (L, R, θ, η) from $\mathbf{C}$ to $\mathbf{D}$, and this strict equivalence is unique up to isomorphism (which is not necessarily unique). Thus, using a strong variant of the Axiom of Choice, every weak equivalence should arise (non-uniquely) from a strict equivalence.

8   Products and Coproducts

Definition 8.1. For every category $\mathbf{C}$, for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$, an **arrow over** (b_1, b_2) is an ordered pair (p_1, p_2) of a $\mathbf{C}$-morphism p_1 from an object a to b_1 and a $\mathbf{C}$-morphism p_2 from a to b_2. For every arrow $((p_1 : a \to b_1, p_2 : a \to b_2)$ over (b_1, b_2), for every arrow $(p_1' : a' \to b_1, p_2' : a' \to b_2))$ over (b_1, b_2), a **morphism** of arrows over (b_1, b_2) from (p_1, p_2) to (p_1', p_2') is a $\mathbf{C}$-morphism f from a to a' such that both $p_1' \circ f$ equals p_1 and $p_2' \circ f$ equals p_2. A **product** of (b_1, b_2) in $\mathbf{C}$ is an arrow over (b_1, b_2), say

$$(\mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),1} : b_1 \times b_2 \to b_1, \mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),2} : b_1 \times b_2 \to b_2),$$

such that for every arrow (p_1, p_2) over (b_1, b_2), there exists a unique morphism of arrows over (b_1, b_2) from (p_1, p_2) to $(\mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),1}, \mathrm{pr}^{\mathbf{C}}_{(b_1,b_2),2})$. More generally, for every object c of $\mathbf{C}$, for every ordered

pair $(g_1 : b_1 \to c, g_2 : b_2 \to c)$ of objects of the under category $\mathbf{C}_c$, a **fiber product** of (g_1, g_2) is a product of (g_1, g_2) in the under category $\mathbf{C}_c$, i.e., an arrow over (b_1, b_2),

$$(\mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),1} : b_1 \times_{g_1,c,g_2} b_2 \to b_1, \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),2} : b_1 \times_{g_1,c,g_2} b_2 \to b_2),$$

such that $g_1 \circ \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),1}$ equals $g_2 \circ \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),2}$, and such that, for every arrow over (b_1, b_2), say $(p_1 : a \to b_1, p_2 : a \to b_2)$ that satisfies $g_1 \circ p_1 = g_2 \circ p_2$, there exists a unique morphism of arrows over (b_1, b_2) from (p_1, p_2) to $(\mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),1}, \mathrm{pr}^{\mathbf{C}}_{(g_1,g_2),2})$.

Definition 8.2. For every category $\mathbf{C}$, for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$, an **arrow under** (b_1, b_2) is an ordered pair (i_1, i_2) of a $\mathbf{C}$-morphism i_1 from b_1 to an object c and a $\mathbf{C}$-morphism p_2 from b_2 to c. For every arrow $((i_1 : b_1 \to c, i_2 : b_2 \to c)$ under (b_1, b_2), for every arrow $(i_1' : b_1 \to c', i_2' : b_2 \to c'))$ under (b_1, b_2), a **morphism** of arrows under (b_1, b_2) from (i_1, i_2) to (i_1', i_2') is a $\mathbf{C}$-morphism h from c to c' such that both $h \circ i_1$ equals i_1' and $h \circ i_2$ equals i_2'. A **coproduct** of (b_1, b_2) in $\mathbf{C}$ is an arrow under (b_1, b_2), say

$$(\mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),1} : b_1 \to b_1 \sqcup b_2, \mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),2} : b_2 \to b_1 \sqcup b_2),$$

such that for every arrow (i_1', i_2') under (b_1, b_2), there exists a unique morphism of arrows under (b_1, b_2) from $(\mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),1}, \mathrm{incl}^{\mathbf{C}}_{(b_1,b_2),2})$ to (i_1', i_2'). More generally, for every object a of $\mathbf{C}$, for every ordered pair $(f_1 : a \to b_1 \to b, f_2 : a \to b_2)$ of objects of the over category $\mathbf{C}^a$, a **cofiber coproduct** of (f_1, f_2) is a coproduct of (f_1, f_2) in the over category $\mathbf{C}^a$, i.e., an arrow under (b_1, b_2),

$$(\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),1} : b_1 \to b_1 \sqcup^{f_1,a,f_2} b_2, \mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),2} : b_2 \to b_1 \sqcup^{f_1,a,f_2} b_2),$$

such that $\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),1} \circ f_1$ equals $\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),2} \circ f_2$, and such that for every arrow under (b_1, b_2), say $(i_1' : b_1 \to c', i_2' : b_2 \to c')$ that satisfies $f_1 \circ i_1' = f_2 \circ i_2'$, there exists a unique morphism of arrows under (b_1, b_2) from $(\mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),1}, \mathrm{incl}^{\mathbf{C}}_{(f_1,f_2),2})$ to (i_1', i_2').

Lemma 8.3. *When products exist, respectively when fiber products exist, when coproducts exist, when cofiber coproducts exist, they are unique up to unique isomorphism. Products and fiber products in the opposite category are coproducts and cofiber coproducts in the original category. Coproducts and cofiber coproducts in the opposite category are products and fiber products in the original category.*

Definition 8.4. A category $\mathbf{C}$ has **all finite products**, respectively has **all finite coproducts**, if (and only if) for every ordered pair (b_1, b_2) of objects of $\mathbf{C}$ there exists a product of (b_1, b_2) in $\mathbf{C}$, resp. there exists a coproduct of (b_1, b_2) in $\mathbf{C}$. A category has **all finite limits** if (and only if), for every object c of $\mathbf{C}$, the under category $\mathbf{C}_c$ has all finite products. A category has **all finite colimits** if (and only if), for every object a of $\mathbf{C}$, the over category $\mathbf{C}^a$ has all finite coproducts.

Example 8.5. In the category **Set**, Cartesian products with the usual projection functions are products, and disjoint unions with the usual inclusion functions are coproducts. Thus, **Set** has all finite products, and it has all finite coproducts. Similarly, for fiber products, the equalizer subset in the Cartesian product of the pair of morphisms is a fiber product in the category of sets, and the coequalizer quotient set of the disjoint union for the pair of morphisms is a cofiber coproduct in the category of sets. Thus, **Set** has all finite limits, and it has all finite colimits. In the category **Rel**, again disjoint union with inclusion functions (considered as relations) are coproducts. The opposite relations of the inclusions functions make disjoint unions into products in the category of **Rel**. Thus, **Rel** has all finite products, and it has all finite coproducts. However, the category **Rel** does not have all fiber products, nor does it have all cofiber coproducts.

Definition 8.6. For every category **C**, for every category **C**′, for every functor **F** from **C** to **C**′, the functor **preserves finite products** if (and only if), for every ordered pair (b_1, b_2) of objects of **C** and for every ordered pair $(\text{pr}^{\mathbf{C}}_{(b_1,b_2),1}, \text{pr}^{\mathbf{C}}_{(b_1,b_2),2})$ of **C**-morphisms that is a product of (b_1, b_2), for the **C**′-objects $b_i' = \mathbf{F}(b_i)$, and for the **C**′-morphisms $\text{pr}^{\mathbf{C}'}_{(b_1',b_2'),i} = \mathbf{F}(\text{pr}^{\mathbf{C}}_{(b_1,b_2),i})$, the ordered pair $(\text{pr}^{\mathbf{C}'}_{(b_1',b_2'),1}, \text{pr}^{\mathbf{C}'}_{(b_1',b_2'),2})$ is a product of (b_1', b_2') in **C**′.

Similarly, the functor **preserves finite limits** if (and only if), for every object c of **C** with image object $c' = \mathbf{F}(c)$ of **C**′, the associated functor $\mathbf{F}_c$ from the under category $\mathbf{C}_c$ to the under category $\mathbf{C}'_{c'}$ preserves finite products, i.e., **F** preserves (finite) fiber products.

Definition 8.7. For every category **C**, for every category **C**′, for every functor **F** from **C** to **C**′, the functor **preserves finite coproducts** if (and only if), for every ordered pair (b_1, b_2) of objects of **C** and for every ordered pair $(\text{incl}^{\mathbf{C}}_{(b_1,b_2),1}, \text{incl}^{\mathbf{C}}_{(b_1,b_2),2})$ of **C**-morphisms that is a coproduct of (b_1, b_2), for the **C**′-objects $b_i' = \mathbf{F}(b_i)$, and for the **C**′-morphisms $\text{incl}^{\mathbf{C}'}_{(b_1',b_2'),i} = \mathbf{F}(\text{incl}^{\mathbf{C}}_{(b_1,b_2),i})$, the ordered pair $(\text{incl}^{\mathbf{C}'}_{(b_1',b_2'),1}, \text{incl}^{\mathbf{C}'}_{(b_1',b_2'),2})$ is a coproduct of (b_1', b_2') in **C**′.

Similarly, the functor **preserves finite colimits** if (and only if), for every object a of **C** with image object $a' = \mathbf{F}(c)$ of **C**′, the associated functor $\mathbf{F}_a$ from the over category $\mathbf{C}^a$ to the over category $(\mathbf{C}')^{a'}$ preserves finite coproducts, i.e., **F** preserves (finite) cofiber coproducts.

Exercise 8.8. For every monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, define a binary operation on the Cartesian product $H \times H'$ by $(h_1, h_1') * (h_2, h_2') := (h_1 \bullet h_2, h_1' \bullet' h_2')$. Check that the projection function $\text{pr}_{(H,H'),1}$, respectively $\text{pr}_{(H,H'),2}$, is a monoid homomorphism from $(H \times H', *)$ to $(H, \bullet)$, resp. to $(H', \bullet')$. Check that this operation makes $(H \times H', *)$ into a product of $(H, \bullet)$ and $(H', \bullet')$ in the category of monoids. Conclude that **Monoid** has all finite products, and the forgetful concrete functor from **Monoid** to **Set** preserves finite products. Similarly, check that **Monoid** has all finite limits, and the forgetful concrete functor preserves finite limits.

Exercise 8.9. Prove that a full subcategory of a category that has all finite products, respectively that has all finite coproducts, both has all finite products, resp. all finite coproducts, and the inclusion functor preserves all finite products, resp. all finite coproducts, if and only if every product in the ambient category, resp. every coproduct in the ambient category, of objects of the full subcategory is isomorphic to an object in the full subcategory. Formulate and prove the analogous result for finite limits, resp. for finite colimits.

Exercise 8.10. Prove that the full subcategory **Grp** of **Monoid** has all finite limits and the inclusion functor preserves all finite limits. Similarly, prove that the full subcategory **Ab** of **Grp** has all finite limits and the inclusion functor preserves all finite limits. Similarly, prove that the full subcategory $\mathbb{Q} - \textbf{Mod}$ of **Ab** has all finite limits and the inclusion functor preserves all finite limits. More generally, for all associative, unital rings R and S, for the forgetful functor to **Ab** from $R - \textbf{Mod}$, respectively from $\textbf{Mod} - S$, from $R - S - \textbf{Mod}$, mapping each module to its underlying additive group, prove that each of these categories has all finite limits and the forgetful functor preserves all finite limits.

Exercise 8.11. For the forgetful functor from **Ring** to **Ab** that maps each associative, unital rings to its underling additive group, prove that **Ring** has all finite limits and the forgetful functor preserves all finite limits. Prove that the full subcategory **CRing** of **Ring** has all finite limits and the inclusion functor preserves all finite limits.

Exercise 8.12. For every ordered pair $((X_1, \tau_1), (X_2, \tau_2))$ of topological spaces, prove that there exists a coarsest topology $\tau_1 \otimes \tau_2$ on the product set $X_1 \times X_2$ such that for both $i = 1$ and $i = 2$, the projection function $\text{pr}_{(X_1, X_2), i}$ is a continuous map from $(X_1 \times X_2, \tau_1 \otimes \tau_2)$ to (X_i, τ_i), namely the topology generated by the topological basis $\mathcal{B}$ of all subsets $\text{pr}^{\text{pre}}_{(X_1, X_2), 1}(U_1) \cap \text{pr}^{\text{pre}}_{(X_1, X_2), 2}(U_2)$ with U_1 an element of τ_1 and with U_2 an element of τ_2. This is the **product topology** on $X_1 \times X_2$ of τ_1 and τ_2. Prove that the pair of continuous maps $((\text{pr}_{(X_1, X_2), 1}, \text{pr}_{(X_1, X_2), 2})$ is a product of (X_1, τ_1) and (X_2, τ_2) in the category of topological spaces. Conclude that the category of topological spaces has all finite products.

Exercise 8.13. For every topological space (X, τ), and for every subset X' of X, prove that the subset $\tau|_{X'} := \{U \cap X' | U \in \tau\}$ of $\mathcal{P}(X')$ is the coarsest topology on X' such that the inclusion function $\text{incl}_X^{X'}$ is a continuous map from $(X', \tau|_{X'})$ to (X, τ). Show also that for every topological space (X'', τ''), for every continuous map f from (X'', τ'') to (X, τ), the image of f is contained in the subset X' if and only if there exists a continuous map f' from (X'', τ'') to $(X', \tau|_{X'})$ such that f equals $\text{incl}_X^{X'} \circ f'$, and then f' is unique. The topology $\tau|_{X'}$ is the **subspace topology**.

Exercise 8.14. For every ordered triple of topological spaces, say (X_1, τ_1), (X_2, τ_2) and (X, τ), for every ordered pair of continuous maps g_1 from (X_1, τ_1) to (X, τ) and g_2 from (X_2, τ_2) to (X, τ), prove that the subspace topology on the subset $X_1 \times_{g_1, X, g_2} X_2$ of $(X_1 \times X_2, \tau_1 \otimes \tau_2)$ gives a fiber

product of g_1 and g_2 in the category of topological spaces. Conclude that the category of topological spaces has all finite limits, and the forgetful functor from **Top** to **Set** preserves all finite limits.

The description of coproduct in each of these concrete categories is different. The notion of left adjoint functors to each concrete forgetful functor gives a uniform construction of the coproducts.

9 Product categories

Definition 9.1. For every category $\mathbf{C}_1$ and for every category $\mathbf{C}_2$, the **product category** $\mathbf{C}_1 \times \mathbf{C}_2$ of $\mathbf{C}_1$ and $\mathbf{C}_2$ is the category whose objects are ordered pairs (a_1, a_2) of a $\mathbf{C}_1$-object a_1 and a $\mathbf{C}_2$-object a_2. For every ordered pair $((a_1, a_2), (b_1, b_2))$ of objects a_1 and b_1 of $\mathbf{C}_1$ and objects a_2 and b_2 of $\mathbf{C}_2$, the Hom set in $\mathbf{C}_1 \times \mathbf{C}_2$ is the product set

$$\mathrm{Hom}_{\mathbf{C}_1 \times \mathbf{C}_2}((a_1, a_2), (b_1, b_2)) = \mathrm{Hom}_{\mathbf{C}_1}(a_1, b_1) \times \mathrm{Hom}_{\mathbf{C}_2}(a_2, b_2).$$

Finally, composition is defined componentwise: for every ordered pair $((g_1, g_2), (f_1, f_2))$ of $\mathbf{C}_1$-morphisms f_1 from a_1 to b_1 and g_1 from b_1 to c_1 and $\mathbf{C}_2$-morphisms f_2 from a_2 to b_2 and g_2 from b_2 to c_2, the composition $(g_1, g_2) \circ (f_1, f_2)$ is defined to equal $(g_1 \circ_1 f_1, g_2 \circ_2 f_2)$.

Example 9.2. For every monoid $(H, \bullet)$, for every monoid $(H', \bullet')$, the product of the category of $(H, \bullet)$ and the category of $(H', \bullet')$ is (canonically equivalent to) the category of the direct product monoid $(H \times H', *)$ where $(a, a') * (b, b')$ equals $(a \bullet a', b \bullet b')$ for all elements a and b of H and for all elements a' and b' of H'. Note, this is (usually) quite different from the free product of the two monoids (which is the coproduct in the category of monoids), i.e., the quotient of the free monoid on the set $H \sqcup H'$ by the equivalence relation arising from the identities and group operations on H and on H'. The direct product is a further quotient by the equivalence relation identifying each product $(e, h') * (h, e')$ with the product $(h, e') * (e, h')$, for identity elements e and e' of H and H'.

Definition 9.3. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, for the product category $\mathbf{C}_1 \times \mathbf{C}_2$, the **first projection functor** $\mathrm{pr}_{\mathbf{C}_1,1}^{\mathbf{C}_1,\mathbf{C}_2}$ from $\mathbf{C}_1 \times \mathbf{C}_2$ to $\mathbf{C}_1$ maps every object (a_1, a_2) of $\mathbf{C}_1 \times \mathbf{C}_2$ to the object a_1 of $\mathbf{C}_1$ and maps (f_1, f_2) to f_1 for every ordered pair (f_1, f_2) of a $\mathbf{C}_1$-morphism f_1 from a_1 to b_1 and a $\mathbf{C}_2$-morphism f_2 from a_2 to b_2. This functor is denoted by pr_1 when confusion is unlikely.

Similarly, the **second projection functor** $\mathrm{pr}_{\mathbf{C}_2,2}^{\mathbf{C}_1,\mathbf{C}_2}$ from $\mathbf{C}_1 \times \mathbf{C}_2$ to $\mathbf{C}_2$ maps every object (a_1, a_2) to a_2 and maps every $\mathbf{C}_1 \times \mathbf{C}_2$-morphism (f_1, f_2) to f_2. This functor is denoted by pr_2 when confusion is unlikely.

Example 9.4. For every monoid $(H_1, \bullet_1)$, for every monoid $(H_2, \bullet_2)$ the projection functors from the product category correspond to the projection monoid homomorphisms from the product monoid $(H_1 \times H_2, \bullet_1 \times \bullet_2)$ to the factors $(H_1, \bullet_1)$ and $(H_2, \bullet_2)$.

Proposition 9.5. *For every category* $\mathbf{B}$*, for every category* $\mathbf{C}_1$*, for every category* $\mathbf{C}_2$*, for every functor* $\mathbf{F}_1$ *from* $\mathbf{B}$ *to* $\mathbf{C}_1$*, for every functor* $\mathbf{F}_2$ *from* $\mathbf{B}$ *to* $\mathbf{C}_2$*, there exists a unique functor* $(\mathbf{F}_1, \mathbf{F}_2)$ *from* $\mathbf{B}$ *to the product category* $\mathbf{C}_1 \times \mathbf{C}_2$ *such that the composite functor* $pr_1 \circ (\mathbf{F}_1, \mathbf{F}_2)$ *equals* $\mathbf{F}_1$ *and the composite functor* $pr_2 \circ (\mathbf{F}_1, \mathbf{F}_2)$ *equals* $\mathbf{F}_2$*.*

Proposition 9.6. *For every category* $\mathbf{B}$*, for every category* $\mathbf{C}_1$*, for every category* $\mathbf{C}_2$*, for every ordered pair* $(\mathbf{F}_1, \mathbf{G}_1)$ *of functors from* $\mathbf{B}$ *to* $\mathbf{C}_1$*, for every ordered pair* $(\mathbf{F}_2, \mathbf{G}_2)$ *of functors from* $\mathbf{B}$ *to* $\mathbf{C}_2$*, for every natural transformation* θ_1 *from* $\mathbf{F}_1$ *to* $\mathbf{G}_1$*, for every natural transformation* θ_2 *from* $\mathbf{F}_2$ *to* $\mathbf{G}_2$*, there exists a unique natural transformation* (θ_1, θ_2) *from* $(\mathbf{F}_1, \mathbf{F}_2)$ *to* $(\mathbf{G}_1, \mathbf{G}_2)$ *such that the pushforward of* (θ_1, θ_2) *by* pr_1 *equals* θ_1 *and the pushforward of* (θ_1, θ_2) *by* pr_2 *equals* θ_2*.*

Corollary 9.7. *For every functor of categories,* $\mathbf{F}_1$ *from* $\mathbf{C}_1$ *to* $\mathbf{D}_1$*, for every functor of categories,* $\mathbf{F}_2$ *from* $\mathbf{C}_2$ *to* $\mathbf{D}_2$*, there is a unique functor* $(\mathbf{F}_1 \circ pr_1, \mathbf{F}_2 \circ pr_2)$ *from the product category* $\mathbf{C}_1 \times \mathbf{C}_2$ *to the product category* $\mathbf{D}_1 \times \mathbf{D}_2$*, such that the composite functor* $pr_1 \circ (\mathbf{F}_1 \circ pr_1, \mathbf{F}_2 \circ pr_2)$ *equals* $\mathbf{F}_1 \circ pr_1$ *and the composite functor* $pr_2 \circ (\mathbf{F}_1 \circ pr_1, \mathbf{F}_2 \circ pr_2)$ *equals* $\mathbf{F}_2 \circ pr_2$*.*

Corollary 9.8. *For every ordered pair* $(\mathbf{F}_1, \mathbf{G}_1)$ *of functors from* $\mathbf{C}_1$ *to* $\mathbf{D}_1$*, for every ordered pair* $(\mathbf{F}_2, \mathbf{G}_2)$ *of functors from* $\mathbf{C}_2$ *to* $\mathbf{D}_2$*, for every natural transformation* θ_1 *from* $\mathbf{F}_1$ *to* $\mathbf{G}_1$*, for every natural transformation* θ_2 *from* $\mathbf{F}_2$ *to* $\mathbf{G}_2$*, there is a unique natural transformation* $(pr_1^* \theta_1, pr_2^* \theta_2)$ *from* $(\mathbf{F}_1 \circ pr_1, \mathbf{F}_2 \circ pr_2)$ *to* $(\mathbf{G}_1 \circ pr_1, \mathbf{G}_2 \circ pr_2)$ *whose pushforward by* pr_1 *equals the pullback* $pr_1^* \theta_1$ *and whose pushforward by* pr_2 *equals the pullback* $pr_2^* \theta_2$*.*

Definition 9.9. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, for every category $\mathbf{D}$, a **bifunctor** (or **strict** 2-**functor**) $\mathbf{F}$ to $\mathbf{D}$ from $\mathbf{C}_1$ and $\mathbf{C}_2$ is an ordered triple class $(\mathbf{F}_{\text{obj,obj}}, (\mathbf{F}_{\text{mor,obj}}, \mathbf{F}_{\text{obj,mor}}))$ of a class morphism $\mathbf{F}_{\text{obj,obj}}$ from $\text{obj}(\mathbf{C}_1) \times \text{obj}(\mathbf{C}_2)$ to $\text{obj}(\mathbf{D})$, of a class morphism $\mathbf{F}_{\text{mor,obj}}$ from $\text{mor}(\mathbf{C}_1) \times \text{obj}(\mathbf{C}_2)$ to $\text{mor}(\mathbf{D})$, and of a class morphism $\mathbf{F}_{\text{obj,mor}}$ from $\text{obj}(\mathbf{C}_1) \times \text{mor}(\mathbf{C}_2)$ to $\text{mor}(\mathbf{D})$ such that, for every member (a_1, a_2) of $\text{obj}(\mathbf{C}_1) \times \text{obj}(\mathbf{C}_2)$, the ordered pair class $(\mathbf{F}_{\text{obj,obj}}(\bullet, a_2), \mathbf{F}_{\text{mor,obj}}(\bullet, a_2))$ is a functor from $\mathbf{C}_1$ to $\mathbf{D}$, the ordered pair class $(\mathbf{F}_{\text{obj,obj}}(a_1, \bullet), \mathbf{F}_{\text{obj,mor}}(a_1, \bullet))$ is a functor from $\mathbf{C}_2$ to $\mathbf{D}$, and we have

$$\mathbf{F}_{\text{obj,mor}}(b_1, f_1) \circ \mathbf{F}_{\text{mor,obj}}(f_1, a_2) = \mathbf{F}_{\text{mor,obj}}(f_1, b_2) \circ \mathbf{F}_{\text{obj,mor}}(a_1, f_2)$$

for every $\mathbf{C}_1$-morphism f_1 from a_1 to an object b_1 and for every $\mathbf{C}_2$-morphism f_2 from a_2 to an object b_2.

Example 9.10. For every category $\mathbf{C}$, the **Hom bifunctor** $\text{Hom}_{\mathbf{C}}$, or just Hom when confusion is unlikely, is the bifunctor to **Set** from $\mathbf{C}^{\text{opp}}$ and $\mathbf{C}$ that maps every ordered pair (a, b) of objects of $\mathbf{C}$ to the set $\text{Hom}_{\mathbf{C}}(a, b)$, that maps every ordered pair $(u : a' \to a, b)$ of a $\mathbf{C}$-morphism u from a' to a and an object b of $\mathbf{C}$ to the set function $\text{Hom}_{\mathbf{C}}(u, b)$ from $\text{Hom}_{\mathbf{C}}(a, b)$ to $\text{Hom}_{\mathbf{C}}(a', b)$ of precomposition by u, and that maps every ordered pair $(a, v : b \to b')$ of an object a of $\mathbf{C}$ and a $\mathbf{C}$-morphism v from b to b' to the set function $\text{Hom}_{\mathbf{C}}(a, v)$ from $\text{Hom}_{\mathbf{C}}(a, b)$ to $\text{Hom}_{\mathbf{C}}(a, b')$ of postcomposition by v. This satisfies the bifunctor identities because of associativity of composition.

Exercise 9.11. Check that this is a bifunctor.

Example 9.12. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, the **braiding bifunctor** $B_{\mathbf{C}_1,\mathbf{C}_2}$ is the bifunctor to $\mathbf{C}_2 \times \mathbf{C}_1$ from $\mathbf{C}_1$ and $\mathbf{C}_2$ that maps every ordered pair (a_1, a_2) of an object a_1 of $\mathbf{C}_1$ and an object a_2 of $\mathbf{C}_2$ to the object (a_2, a_1) of $\mathbf{C}_2 \times \mathbf{C}_1$, that maps every ordered pair $(u_1 : a_1 \to a_1', a_2)$ of a $\mathbf{C}_1$-morphism u_1 from a_1 to a_1' and an object a_2 of $\mathbf{C}_2$ to the morphism (Id_{a_2}, u_1) from (a_2, a_1) to (a_2, a_1') in $\mathbf{C}_2 \times \mathbf{C}_1$, and that maps every ordered pair $(a_1, u_2 : a_2 \to a_2')$ of a $\mathbf{C}_2$-morphism u_2 from a_2 to a_2' and an object a_1 of $\mathbf{C}_1$ to the morphism (u_2, Id_{a_1}) from (a_2, a_1) to (a_2', a_1) in $\mathbf{C}_2 \times \mathbf{C}_1$.

Proposition 9.13. *For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, and for every category $\mathbf{D}$, every bifunctor to $\mathbf{D}$ from $\mathbf{C}_1$ and $\mathbf{C}_2$ extends uniquely to a functor from the product category $\mathbf{C}_1 \times \mathbf{C}_2$ to $\mathbf{D}$.*

Proposition 9.14. *For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, and for every category $\mathbf{D}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of bifunctor to $\mathbf{D}$ from $\mathbf{C}_1$ and $\mathbf{C}_2$, for every class morphism θ from $obj(\mathbf{C}_1) \times obj(\mathbf{C}_2)$ to $mor(\mathbf{D})$, this is a natural transformation from the functor of $\mathbf{F}$ to the functor of $\mathbf{G}$ if and only if, for every member (a_1, a_2) of $obj(\mathbf{C}_1) \times obj(\mathbf{C}_2)$, both $\theta_{\bullet, a_2}$ is a natural transformation from $\mathbf{F}(\bullet, a_2)$ to $\mathbf{G}(\bullet, a_2)$ and $\theta_{a_1, \bullet}$ is a natural transformation from $\mathbf{F}(a_1, \bullet)$ to $\mathbf{G}(a_1, \bullet)$.*

Example 9.15. For every category $\mathbf{C}_1$, for every category $\mathbf{C}_2$, there is a **projection bifunctor** $\mathrm{pr}_{\mathbf{C}_1,1}^{\mathbf{C}_1,\mathbf{C}_2}$, respectively $\mathrm{pr}_{\mathbf{C}_2,2}^{\mathbf{C}_1,\mathbf{C}_2}$, from $\mathbf{C}_1$ and $\mathbf{C}_2$ to $\mathbf{C}_1$, resp. to $\mathbf{C}_2$, that sends every ordered pair (a_1, a_2) of an object a_1 of $\mathbf{C}_1$ and an object a_2 of $\mathbf{C}_2$ to the object a_1 of $\mathbf{C}_1$, resp. to the object a_2 of $\mathbf{C}_2$. For every $\mathbf{C}_1$-morphism f_1 from a_1 to b_1, the associated morphism from $\mathrm{pr}_1(a_1, a_2) = a_1$ to $\mathrm{pr}_1(b_1, a_2) = b_1$, resp. from $\mathrm{pr}_2(a_1, a_2) = a_2$ to $\mathrm{pr}_1(b_1, a_2) = a_2$, is f, resp. is Id_{a_2}. For every $\mathbf{C}_2$-morphism f_2 from a_s to b_2, the associated morphism from $\mathrm{pr}_1(a_1, a_2) = a_1$ to $\mathrm{pr}_1(a_1, b_2) = a_1$, resp. from $\mathrm{pr}_2(a_1, a_2) = a_2$ to $\mathrm{pr}_1(a_1, b_2) = b_2$, is Id_{a_1}, resp. is f_2.

Example 9.16. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, the **associated bifunctor $\mathbf{F}^{\mathrm{opp}} \times \mathbf{F}$** of $\mathbf{F}$ from $\mathbf{C}^{\mathrm{opp}} \times \mathbf{C}$ to $\mathbf{D}^{\mathrm{opp}} \times \mathbf{D}$ is the unique functor such that both $\mathrm{pr}_1 \circ (\mathbf{F}^{\mathrm{opp}} \times \mathbf{F})$ equals $\mathbf{F}^{\mathrm{opp}} \circ \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ (\mathbf{F}^{\mathrm{opp}} \times \mathbf{F})$ equals $\mathbf{F} \circ \mathrm{pr}_2$. The **associated natural transformation of Hom bifunctors $\mathbf{F}_\bullet^\bullet$** of $\mathbf{F}$ from $\mathrm{Hom}_{\mathbf{C}}$ to $\mathrm{Hom}_{\mathbf{D}} \circ (\mathbf{F}^{\mathrm{opp}} \times \mathbf{F})$ maps every ordered pair (a, b) of objects of $\mathbf{C}$ to the set function $\mathbf{F}_b^a$ from $\mathrm{Hom}_{\mathbf{C}}(a, b)$ to $\mathrm{Hom}_{\mathbf{D}}(\mathbf{F}(a), \mathbf{F}(b))$.

Exercise 9.17. Check that $\mathbf{F}_\bullet^\bullet$ is, indeed, a natural transformation of bifunctors. For every functor $\mathbf{G}$ from $\mathbf{D}$ to a category $\mathbf{E}$, check that $(\mathbf{G}^{\mathrm{opp}} \times \mathbf{G}) \circ (\mathbf{F}^{\mathrm{opp}} \times \mathbf{F})$ equals $(\mathbf{G} \circ \mathbf{F})^{\mathrm{opp}} \times (\mathbf{G} \circ \mathbf{F})$. Also check that the composition natural transformation $(\mathbf{F}^{\mathrm{opp}} \times \mathbf{F})_* \mathbf{G}_\bullet^\bullet \circ \mathbf{F}_\bullet^\bullet$ equals $(\mathbf{G} \circ \mathbf{F})_\bullet^\bullet$.

Exercise 9.18. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for functors $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ from $\mathbf{C}$ to $\mathbf{D}$, for every natural equivalence θ from $\mathbf{F}$ to $\widetilde{\mathbf{F}}$ with inverse natural equivalence $\widetilde{\theta}$, prove that $\widetilde{\theta}^{\mathrm{opp}} \times \theta$ from $\mathbf{F}^{\mathrm{opp}} \times \mathbf{F}$ to $\widetilde{\mathbf{F}}^{\mathrm{opp}} \times \widetilde{\mathbf{F}}$ is a natural equivalence. Formulate and prove the compatibility of this natural transformation with the natural transformations $\mathbf{F}_\bullet^\bullet$ and $\widetilde{\mathbf{F}}_\bullet^\bullet$.

Definition 9.19. For every category $\mathbf{C}$, a **product bifunctor** is an ordered triple $(-\times-, \mathrm{pr}_1^{\mathbf{C}}, \mathrm{pr}_2^{\mathbf{C}})$ of a bifunctor $-\times-$ to $\mathbf{C}$ from $\mathbf{C}$ and $\mathbf{C}$, a natural transformation $\mathrm{pr}_1^{\mathbf{C}}$ from the bifunctor $-\circ-$ to the bifunctor $\mathrm{pr}_{\mathbf{C},1}^{\mathcal{C},\mathcal{C}}$, and a natural transformation $\mathrm{pr}_2^{\mathbf{C}}$ from the bifunctor $-\circ-$ to the bifunctor $\mathrm{pr}_{\mathbf{C},2}^{\mathcal{C},\mathcal{C}}$ such that, for every ordered pair (a_1, a_2) of objects of $\mathbf{C}$, the following ordered pair is a product of a_1 and a_2 in $\mathbf{C}$,
$$(\mathrm{pr}_{(a_1,a_2),1}^{\mathbf{C}} : a_1 \times a_2 \to a_1, \mathrm{pr}_{(a_1,a_2),2}^{\mathbf{C}} : a_1 \times a_2 \to a_2).$$

Exercise 9.20. For every category $\mathbf{C}$, if a product bifunctor exists, prove that it is unique up to unique natural equivalence.

Exercise 9.21. Let $\mathbf{C}$ be a small category such that for every ordered pair (a_1, a_2) of objects of $\mathbf{C}$, there exists a product. Using the Axiom of Choice, prove that there is a product bifunctor. Up to some much stronger Axiom of Choice, every category that admits finite products has a product bifunctor.

Example 9.22. For every product bifunctor on a category $\mathbf{C}$, a product bifunctor on the arrow category $\mathrm{Arr}(\mathbf{C})$ maps every ordered pair $(((s_1,t_1),f_1),((s_2,t_2),f_2))$ to $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$, maps every morphism (σ_1, τ_1) from $((s_1,t_1),f_1)$ to $((s_1',t_1'),f_1')$ to the morphism $(\sigma_1 \times \mathrm{Id}_{s_2}, \tau_1 \times \mathrm{Id}_{t_2})$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_1' \times s_2, t_1' \times t_2), f_1' \times f_2)$, and maps every morphism (σ_2, τ_2) from $((s_2,t_2),f_2)$ to $((s_2',t_2'),f_2')$ to the morphism $(\mathrm{Id}_{s_1} \times \sigma_2, \mathrm{Id}_{t_1} \times \tau_2)$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_1 \times s_2', t_1 \times t_2'), f_1 \times f_2')$. The projection natural transformation $\mathrm{pr}_1^{\mathrm{Arr}(\mathbf{C})}$ maps every ordered pair $(((s_1,t_1),f_1),((s_2,t_2),f_2))$ to the projection morphism $(\mathrm{pr}_{(s_1,s_2),1}^{\mathbf{C}}, \mathrm{pr}_{(t_1,t_2),1}^{\mathbf{C}})$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_1,t_1),f_1)$. The projection natural transformation $\mathrm{pr}_2^{\mathrm{Arr}(\mathbf{C})}$ maps every ordered pair $(((s_1,t_1),f_1),((s_2,t_2),f_2))$ to the projection morphism $(\mathrm{pr}_{(s_1,s_2),2}^{\mathbf{C}}, \mathrm{pr}_{(t_1,t_2),2}^{\mathbf{C}})$ from $((s_1 \times s_2, t_1 \times t_2), f_1 \times f_2)$ to $((s_2,t_2),f_2)$.

Example 9.23. For the category $\mathbf{Set}$, the bifunctor $-\times-$ associates to every ordered pair (a_1, a_2) of sets the Cartesian product set $a_1 \times a_2$, associates to every function f_1 from a set a_1 to a set a_1' the function $f_1 \times \mathrm{Id}_{a_2}$ from $a_1 \times a_2$ to $a_1' \times a_2$, and associates to every function f_2 from a_2 to a_2' the function $\mathrm{Id}_{a_1} \times f_2$ from $a_1 \times a_2$ to $a_1 \times a_2'$. The natural transformation $\mathrm{pr}_1^{\mathbf{Set}}$ associates to every ordered pair (a_1, a_2) of sets the first projection function $\mathrm{pr}_{(a_1,a_2),1}^{\mathbf{Set}}$ from $a_1 \times a_2$ to a_1. The natural transformation $\mathrm{pr}_2^{\mathbf{Set}}$ associates to every ordered pair (a_1, a_2) of sets the second projection function $\mathrm{pr}_{(a_1,a_2),1}^{\mathbf{Set}}$ from $a_1 \times a_2$ to a_2.

Exercise 9.24. Check that this defines a product bifunctor on the category $\mathbf{Set}$.

Exercise 9.25. Each of the categories $\mathbf{Monoid}$, $\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Ring}$, $\mathbf{CRing}$, $R-\mathbf{Mod}$, $\mathbf{Mod}-S$ and $R-S-\mathbf{Mod}$, and $\mathbf{Top}$ has all finite products, and the (standard) concrete forgetful functor from each to $\mathbf{Set}$ preserves all finite products. Use this "lift" to each of these categories the product bifunctor for $\mathbf{Set}$, thus proving that each of these categories has a product bifunctor.

Exercise 9.26. Formulate and prove analogues of each of the general theorems about for a *coproduct functor* (e.g., by applying the theorems above to the opposite category). However, the standard concrete forgetful functors in the previous exercise do not preserve all coproducts, except for the concrete functor on **Top**. Adjoint pairs give coproducts in the other cases.

Exercise 9.27. For every monoid $(H, \bullet)$, there is a bifunctor $\sqcup$ to the category $H - \mathbf{Act}$ of left H-acts from the category $H - \mathbf{Act}$ and $H - \mathbf{Act}$ that sends every ordered pair $((S, \rho), (S', \rho'))$ of left H-acts to the left H-act $\rho \sqcup \rho'$ on the disjoint union set $S \sqcup S'$. Deduce that $H - \mathbf{Act}$ has all finite coproducts, and the concrete forgetful functor to **Set** preserves all finite coproducts. If $(H, \bullet)$ is a group, prove that the left H-actions that are indecomposable with respect to $\sqcup$ are precisely the left regular action of H on the right coset space H/K of a subgroup K of H.

Exercise 9.28. For every monoid $(H, \bullet)$, use the same technique as earlier to construct a product bifunctor $\times$ to $H - \mathbf{Act}$ from the category $H - \mathbf{Act}$ and $H - \mathbf{Act}$ sending every ordered pair $((S, \rho), (S', \rho'))$ of left H-acts to the left H-act $\rho \times \rho'$ on the Cartesian product set $S \times S'$. Deduce that the category $H - \mathbf{Act}$ has all finite products, and the concrete forgetful functor to **Set** preserves finite products. In particular, if $(H, \bullet)$ is a group, then for $\sqcup$-indecomposible left H-actions H/K and H/K' for subgroups K and K' of H, the $\sqcup$-components of $(H/K) \times (H/K')$ are of the form H/K'' for K'' a subgroup of the form $(hKh^{-1}) \cap (h'K'(h')^{-1})$. Thus, the $\sqcup$-components are all isomorphic (so that $(H/K) \times (H/K')$ is "isotypic") if at least two of K, K' and $K \cap K'$ are normal.

Exercise 9.29. For associative, unital rings $(R, +_R, \cdot_R)$, $(S, +_S, \cdot_S)$ and $(T, +_T, \cdot_T)$, there is a bifunctor $\otimes_S$ to the category $R - T - \mathbf{Mod}$ of $R - T$-bimodules from the category $R - S - \mathbf{Mod}$ and $S - T - \mathbf{Mod}$ that sends every ordered pair $((M, +, (\rho, \sigma)), (M', +', (\sigma', \tau')))$ of an $R - S$-bimodule and a $S - T$-bimodule to the associated tensor product $R - T$-bimodule $M \otimes_S M'$, where the set function from $M \times M'$ to the Abelian group $M \otimes_S M'$ is initial among all biadditive maps from $M \times M'$ to an Abelian group that are S-**balanced**: for every element (m, m') of $M \times M'$ and for every element s of S, both $(m \cdot s, m')$ and $(m, s \cdot m')$ have the same image. Formulate and prove existence of associator isomorphisms $(M \otimes_S M') \otimes_T M'' \cong M \otimes_S (M' \otimes_T M'')$ for every T-module M''. Formulate and prove existence of left / right unitor isomorphisms of $S \otimes_S M' \cong M'$ and $M \otimes_S S \cong M$. Formulate and prove the triangle (coherence) identity and the pentagon (coherence) identity for the unitors and associators.

Example 9.30. For every Abelian monoid $(H, \bullet)$, there is a bifunctor $\mathrm{sum}_{H, \bullet}$ to $B(H, \bullet)$ from $B(H, \bullet)$ and $B(H, \bullet)$ that maps the unique object $(*, *)$ to the unique object $*$, and, for every element h of H, maps both $(h, *)$ and $(*, h)$ to h. The bifunctor axiom is precisely the Abelian hypothesis on the monoid.

10   Comma categories

Definition 10.1. For every ordered triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of categories, for every ordered pair (F, G) of a covariant functors, F from $\mathbf{A}$ to $\mathbf{C}$ and a covariant functor G from $\mathbf{B}$ to $\mathbf{C}$, the **comma category**, $F \downarrow G$, has as objects ordered triples (a, b, u) of an object a of $\mathbf{A}$, an object b of $\mathbf{B}$, and a $\mathbf{C}$-morphism $u : F(a) \to G(b)$. For an ordered pair of such objects, $((a, b, u), (a', b', u'))$, a morphism in the comma category is an ordered pair (q, r) of $q \in \mathrm{Hom}_{\mathbf{A}}(a, a')$ and $r \in \mathrm{Hom}_{\mathbf{B}}(b, b')$ such that $u' \circ F(q)$ equals $G(r) \circ u$ in $\mathrm{Hom}_{\mathbf{C}}(F(a), G(b'))$. Composition is defined in the evident way.

In particular, the arrow category of $\mathbf{C}$ is the comma category where $\mathbf{A}$ equals $\mathbf{B}$ equals $\mathbf{C}$ and each of F and G is the identity functor on $\mathbf{C}$.

Definition 10.2. For every ordered triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of categories, for every ordered pair (F, G) of a covariant functors, F from $\mathbf{A}$ to $\mathbf{C}$ and a covariant functor G from $\mathbf{B}$ to $\mathbf{C}$, the **domain functor**, or **source functor**, from $F \downarrow G$ to $\mathbf{A}$, associates to every object (a, b, u) of $F \downarrow G$ the $\mathbf{A}$-object a and associates to every morphism (q, r) of $F \downarrow G$ the $\mathcal{A}$-morphism q. Also the **codomain functor**, or **target functor**, from $F \downarrow G$ to $\mathbf{B}$, associates to every object (a, b, u) the $\mathbf{B}$-object b and associates to every morphism (q, r) the $\mathbf{B}$-morphism r. Finally, the **arrow functor** from $F \downarrow G$ to $\mathbf{C}^{\to}$ associates to every object (a, b, u) the $\mathbf{C}^{\to}$-object $(F(a), G(b), u)$ and associates to every morphism (q, r) the $\mathbf{C}^{\to}$-morphism $(F(q), G(r))$.

11   Functor categories

Functors give a formalism for working with labelled collections of objects in some fixed category $\mathbf{D}$, where the labels or indices are themselves objects of some small category $\mathbf{C}$ (such as a partially ordered set). The indexed collections then form objects of a new category, called a functor category.

Definition 11.1. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **functor category object class** $\mathrm{obj}(\mathbf{D}^{\mathbf{C}})$, sometimes also denoted $\mathrm{obj}([\mathbf{C}, \mathbf{D}])$ or $\mathrm{obj}(\mathrm{Fun}(\mathbf{C}, \mathbf{D}))$, is the unique class whose members are precisely the sets whose classes give functors from $\mathbf{C}$ to $\mathbf{D}$.

For every ordered pair $(\mathbf{F}, \mathbf{G})$ of functors from $\mathbf{C}$ to $\mathbf{D}$, again because $\mathbf{C}$ is small, every natural transformation from $\mathbf{F}$ to $\mathbf{G}$ is the class of a set, and the class of all sets whose classes are natural transformations from $\mathbf{F}$ to $\mathbf{G}$ is itself a set.

Definition 11.2. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **functor category morphisms class** $\mathrm{mor}(\mathbf{D}^{\mathbf{C}})$, sometimes also denoted $\mathrm{mor}([\mathbf{C}, \mathbf{D}])$ or $\mathrm{mor}(\mathrm{Fun}(\mathbf{C}, \mathbf{D})$, is the span from

obj($\mathbf{D^C}$) whose fiber class over each ordered pair $(\mathbf{F}, \mathbf{G})$ of sets whose classes are functors from $\mathbf{C}$ to $\mathbf{D}$ is the class whose members are precisely the sets whose classes give natural transformation from $\mathbf{F}$ to $\mathbf{G}$.

Together this defines a category.

Definition 11.3. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **functor category $\mathbf{D^C}$** from $\mathbf{C}$ to $\mathbf{D}$, also denoted $\text{Fun}(\mathbf{C}, \mathbf{D})$ or $[\mathbf{C}, \mathbf{D}]$, is the category with objects class obj($\mathbf{D^C}$) and with morphisms class mor($\mathbf{D^C}$). So the objects of the class are equivalent to functors from $\mathbf{C}$ to $\mathbf{D}$, and the morphisms of the class are equivalent to natural transformations. The composition law of this category is composition of natural transformations.

Please note, the way we formalize (parametrically definable) classes there is a distinction between sets and the associated classes. Thus the objects of the functor category are sets whose classes are functors from $\mathbf{C}$ to $\mathbf{D}$, and the morphisms are sets whose classes are natural transformations between such functors. Nonetheless, we shall treat this category as if the objects are functors and as if the morphisms are natural transformations.

Definition 11.4. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, the **H-composition functor $\mathbf{H^C}$** from $\mathbf{D^C}$ to $(\mathbf{D}')^{\mathbf{C}}$ maps every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$ to the composite functor $\mathbf{H} \circ \mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}'$, and maps every natural transformation θ from a functor $\mathbf{F}$ to a functor $\mathbf{G}$ to the **H**-pushforward natural transformation $\mathbf{H}_*\theta$.

Exercise 11.5. Prove that the **H**-composition functor is a functor. Prove that the $\text{Id}_{\mathbf{D}}$-composition functor is the identity functor from $\mathbf{D^C}$ to itself. Prove that for every ordered pair $(\mathbf{I}, \mathbf{H})$ of a functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$ and a functor $\mathbf{I}$ from $\mathbf{D}'$ to $\mathbf{D}''$, the $\mathbf{I} \circ \mathbf{H}$-composition functor $(\mathbf{I} \circ \mathbf{H})^{\mathbf{C}}$ equals the composition of functors $\mathbf{I^C} \circ \mathbf{H^C}$.

Definition 11.6. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every functor $\mathbf{J}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every category $\mathbf{D}$, the **J-precomposition functor $\mathbf{D^J}$** from $\mathbf{D}^{\mathbf{C}'}$ to $\mathbf{D^C}$ maps every functor $\mathbf{F}'$ from $\mathbf{C}'$ to $\mathbf{D}$ to the composite functor $\mathbf{F}' \circ \mathbf{J}$ from $\mathbf{C}$ to $\mathbf{D}$, and maps every natural transformation θ' from a functor $\mathbf{F}'$ to a functor $\mathbf{G}'$ to the **J**-pullback natural transformation $\mathbf{J}_*\theta'$.

Exercise 11.7. Prove that the **J**-precomposition functor is a functor. Prove that the $\text{Id}_{\mathbf{C}}$-precomposition functor is the identity functor from $\mathbf{D^C}$ to itself. Prove that for every ordered pair $(\mathbf{K}, \mathbf{J})$ of a functor $\mathbf{J}$ from $\mathbf{C}$ to $\mathbf{C}'$ and a functor $\mathbf{K}$ from $\mathbf{C}'$ to $\mathbf{C}''$, the $\mathbf{K} \circ \mathbf{I}$-precomposition functor $\mathbf{D^{K \circ J}}$ equals the composition of functors $\mathbf{D^J} \circ \mathbf{D^K}$.

Exercise 11.8. Prove that for every functor $\mathbf{J}$ from a small category $\mathbf{C}$ to a small category $\mathbf{C}'$ and for every functor $\mathbf{H}$ from a category $\mathbf{D}$ to a category $\mathbf{D}'$, the composite functor $(\mathbf{D}')^{\mathbf{J}} \circ \mathbf{H}^{\mathbf{C}'}$ equals the composite functor $\mathbf{H}^{\mathbf{C}} \circ \mathbf{D}^{\mathbf{J}}$.

Definition 11.9. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, the θ-**composition natural transformation** $\theta^{\mathbf{C}}$ from the functor $\mathbf{H}_1^{\mathbf{C}}$ to the functor $\mathbf{H}_2^{\mathbf{C}}$ maps every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$ to the $\mathbf{F}$-pullback natural transformation $\mathbf{F}^*\theta$ from $\mathbf{H}_1 \circ \mathbf{F}$ to $\mathbf{H}_2 \circ \mathbf{F}$.

Exercise 11.10. Prove that $\theta^{\mathbf{C}}$ is a natural transformation. For every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, for the identity natural transformation $\mathrm{Id}_{\mathbf{H}}$ from $\mathbf{H}$ to itself, prove that $(\mathrm{Id}_{\mathbf{H}})^{\mathbf{C}}$ is the identity natural transformation from $\mathbf{H}^{\mathbf{C}}$ to itself. For every ordered pair (θ_2, θ_1) of a natural transformation θ_1 of functors from $\mathbf{H}_1$ to $\mathbf{H}_2$ and of a natural transformation θ_2 of functors from $\mathbf{H}_2$ to $\mathbf{H}_3$, prove that $(\theta_2 \circ \theta_1)^{\mathbf{C}}$ equals the composite natural transformation $\theta_2^{\mathbf{C}} \circ \theta_1^{\mathbf{C}}$.

Exercise 11.11. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every category $\mathbf{D}''$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, for every ordered pair $(\mathbf{H}_1', \mathbf{H}_2')$ of functors from $\mathbf{D}'$ to $\mathbf{D}''$, for every natural transformation θ' from $\mathbf{H}_1'$ to $\mathbf{H}_2'$, prove that for the Godement product $\theta' * \theta$ natural transformation from $\mathbf{H}_1' \circ \mathbf{H}_1$ to $\mathbf{H}_2' \circ \mathbf{H}_2$, also $(\theta' * \theta)^{\mathbf{C}}$ equals the Godement product $(\theta')^{\mathbf{C}} * \theta^{\mathbf{C}}$. Deduce special cases of compatibility of $(-)^{\mathbf{C}}$ with pushforward and pullback by functors of natural transformations.

Definition 11.12. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every ordered pair $(\mathbf{I}_1, \mathbf{I}_2)$ of functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation η from $\mathbf{I}_1$ to $\mathbf{I}_2$, for every category $\mathbf{D}$, the η-**precomposition natural transformation** $\mathbf{D}^{\eta}$ from the functor $\mathbf{D}^{\mathbf{I}_1}$ to the functor $\mathbf{D}^{\mathbf{I}_2}$ maps every functor $\mathbf{F}'$ from $\mathbf{C}'$ to $\mathbf{D}$ to the $\mathbf{F}'$-pushforward natural transformation $(\mathbf{F}')_*\eta$ from $\mathbf{F}' \circ \mathbf{I}_1$ to $\mathbf{F}' \circ \mathbf{I}_2$.

Exercise 11.13. Prove that $\mathbf{D}^{\eta}$ is a natural transformation. For every functor $\mathbf{I}$ from $\mathbf{C}$ to $\mathbf{C}'$, for the identity natural transformation $\mathrm{Id}_{\mathbf{I}}$ from $\mathbf{I}$ to itself, prove that $\mathbf{D}^{\mathrm{Id}_{\mathbf{I}}}$ is the identity natural transformation from $\mathbf{D}^{\mathbf{I}}$ to itself. For every ordered pair (η_2, η_1) of a natural transformation η_1 of functors from $\mathbf{I}_1$ to $\mathbf{I}_2$ and of a natural transformation η_2 of functors from $\mathbf{I}_2$ to $\mathbf{I}_3$, prove that $\mathbf{D}^{\eta_2 \circ \eta_1}$ equals the composite natural transformation $\mathbf{D}^{\eta_2} \circ \mathbf{D}^{\eta_1}$. Also prove that $\mathbf{D}^{\bullet}$ is compatible with Godement products.

Exercise 11.14. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every small category $\mathbf{C}''$, for every ordered pair $(\mathbf{I}_1, \mathbf{I}_2)$ of functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation η from $\mathbf{I}_1$ to $\mathbf{I}_2$, for every ordered pair $(\mathbf{I}_1', \mathbf{I}_2')$ of functors from $\mathbf{C}'$ to $\mathbf{C}''$, for every natural transformation

η' from $\mathbf{I}'_1$ to $\mathbf{I}'_2$, for every category $\mathbf{D}$, prove that for the Godement product $\eta' * \eta$ natural transformation from $\mathbf{I}'_1 \circ \mathbf{I}_1$ to $\mathbf{I}'_2 \circ \mathbf{I}_2$, also $\mathbf{D}^{\eta' * \eta}$ equals the Godement product $\mathbf{D}^\eta * \mathbf{D}^{\eta'}$ (in the opposite order). Deduce special cases of compatibility of $\mathbf{D}^\bullet$ with pushforward and pullback by functors of natural transformations.

Exercise 11.15. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every ordered pair $(\mathbf{I}_1, \mathbf{I}_2)$ of functors from $\mathbf{C}$ to $\mathbf{C}'$, for every natural transformation η from $\mathbf{I}_1$ to $\mathbf{I}_2$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, prove that the Godement product $\theta^{\mathbf{C}} * \mathbf{D}^\eta$ equals the Godement product $(\mathbf{D}')^\eta * \theta^{\mathbf{C}'}$. Deduce special cases for pushforward and pullback by functors of natural transformations.

12   Constant functors

Definition 12.1. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **constant functor** $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, \bullet}$ from $\mathbf{D}$ to $\mathbf{D}^{\mathbf{C}}$ maps every object a of $\mathbf{D}$ to the object $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, a}$ of $\mathbf{D}^{\mathbf{C}}$, and maps every $\mathbf{D}$-morphism f from a to b to the natural transformation $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, f}$ from $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, a}$ to $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, b}$.

Exercise 12.2. Prove that this is a functor.

Exercise 12.3. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, prove that the composite functor $\mathbf{H}^{\mathbf{C}} \circ \mathrm{const}^{\mathbf{C}}_{\mathbf{D}, \bullet}$ equals the composite functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}', \bullet} \circ \mathbf{H}$ as functors from $\mathbf{D}$ to $(\mathbf{D}')^{\mathbf{C}}$.

Exercise 12.4. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every functor $\mathbf{I}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every category $\mathbf{D}$, prove that the composite functor $\mathbf{D}^{\mathbf{I}} \circ \mathrm{const}^{\mathbf{C}'}_{\mathbf{D}, \bullet}$ equals $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, \bullet}$.

Exercise 12.5. For every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for every functor $\mathbf{I}$ from $\mathbf{C}$ to $\mathbf{C}'$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every functor $\mathbf{H}$ from $\mathbf{D}$ to $\mathbf{D}'$, use the compatibilities above to deduce the compatibilities between the functors $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, \bullet}$, $\mathrm{const}^{\mathbf{C}'}_{\mathbf{D}, \bullet}$, $\mathrm{const}^{\mathbf{C}}_{\mathbf{D}', \bullet}$, $\mathrm{const}^{\mathbf{C}'}_{\mathbf{D}', \bullet}$, $\mathbf{H}^{\mathbf{C}}$, $\mathbf{H}^{\mathbf{C}'}$, $\mathbf{D}^{\mathbf{I}}$ and $(\mathbf{D}')^{\mathbf{I}}$, e.g., the composite functor $\mathbf{H}^{\mathbf{C}} \circ \mathbf{D}^{\mathbf{I}} \circ \mathrm{const}^{\mathbf{C}'}_{\mathbf{D}, \bullet}$ equals the composite functor $(\mathbf{D}')^{\mathbf{I}} \circ \mathrm{const}^{\mathbf{C}'}_{\mathbf{D}', \bullet} \circ \mathbf{H}$ as functors from $\mathbf{D}$ to $(\mathbf{D}')^{\mathbf{C}}$.

Exercise 12.6. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every category $\mathbf{D}'$, for every ordered pair $(\mathbf{H}_1, \mathbf{H}_2)$ of functors from $\mathbf{D}$ to $\mathbf{D}'$, for every natural transformation θ from $\mathbf{H}_1$ to $\mathbf{H}_2$, prove that the pullback natural transformation $(\mathrm{const}^{\mathbf{C}}_{\mathbf{D}, \bullet})^* \theta^{\mathbf{C}}$ equals the pushforward natural transformation $(\mathrm{const}^{\mathbf{C}}_{\mathbf{D}', \bullet})_* \theta$ as natural transformation between functors from $\mathbf{D}$ to $(\mathbf{D}')^{\mathbf{C}}$.

13 Category of small categories

Definition 13.1. The class of **small categories** is the class obj(**Cat**) whose members are sets whose associated class is a small category. The class of **functors of small categories** is the span mor(**Cat**) from obj(**Cat**) to itself whose fiber class over each pair $(\mathbf{C}, \mathbf{D})$ has for members those sets whose associated class is a functor from $\mathbf{C}$ to $\mathbf{D}$. Composition of functors defines a composition law that completes these classes to a category **Cat**, the **category of small categories**.

Technically we distinguish each set from its associated class, and thus the objects of **Cat** are sets whose associated class is a small category, rather than the small category itself (since we do not allow classes to be members of other classes). Similarly, the morphisms of **Cat** are sets whose associated class is a functor between small categories, rather than the functor itself.

The standard usage is different: most authors identify each set with the associated class (this is built in to the axioms of von Neumann – Bernays – Gödel class theory). At any rate, even though it is technically incorrect, we will refer to small categories as objects of **Cat**, and we will refer to functors between small categories as morphisms of **Cat**.

Exercise 13.2. Read about (strict) 2-categories. Formulate and prove the assertion that the natural transformations between functors make **Cat** into a 2-category.

Definition 13.3. The **opposite functor** from **Cat** to **Cat** is the functor that maps every small category $\mathbf{C}$ to its opposite category $\mathbf{C}^{\mathrm{opp}}$, that maps every functor $\mathbf{F}$ from a small category $\mathbf{C}$ to a small category $\mathbf{D}$ to the functor $\mathbf{F}^{\mathrm{opp}}$ from $\mathbf{C}^{\mathrm{opp}}$ to $\mathbf{D}^{\mathrm{opp}}$. The 2-**cell dual** of the 2-category **Cat** is the 2-category $\mathbf{Cat}^{\mathrm{co}}$ with the same objects as **Cat**, respectively the same and 1-morphisms as **Cat**, namely small categories, resp. functors between small categories, yet with opposite 2-morphism sets. The **opposite 2-functor** is the strict 2-functor from $\mathbf{Cat}^{\mathrm{co}}$ to **Cat** extending the opposite functor by mapping every natural transformation θ from a functor $\mathbf{F}$ to a functor $\mathbf{G}$ (both from a small category $\mathbf{C}$ to a small category $\mathbf{D}$) to the natural transformation θ^{opp} from $\mathbf{G}^{\mathrm{opp}}$ to $\mathbf{F}^{\mathrm{opp}}$.

Exercise 13.4. Prove that this is a strict 2-functor from $\mathbf{Cat}^{\mathrm{co}}$ to **Cat**.

14 Evaluation bifunctor

Definition 14.1. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **evaluation bifunctor** $\mathrm{Hom}_{\mathbf{D}}^{\mathbf{C}}$, or just Hom when confusion is unlikely, is the bifunctor to $\mathbf{D}$ from $\mathbf{D}^{\mathbf{C}}$ and $\mathbf{C}$ that maps every ordered pair $(\mathbf{F}, a)$ of an object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ and an object a of $\mathbf{C}$ to the object $\mathbf{F}_{\mathrm{obj}}(a)$ of $\mathbf{D}$, that maps every ordered pair (θ, a) of a $\mathbf{D}^{\mathbf{C}}$-morphism θ from $\mathbf{F}$ to $\mathbf{G}$ and of an object a of $\mathbf{C}$ to the $\mathbf{D}$-morphism θ_a from $\mathbf{F}_{\mathrm{obj}}(a)$ to $\mathbf{G}_{\mathrm{obj}}(a)$, and that maps every ordered pair $(\mathbf{F}, u)$ of an object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ and of a $\mathbf{C}$-morphism u from a to b to the $\mathbf{D}$-morphism $\mathbf{F}_{\mathrm{mor}}(u)$ from $\mathbf{F}_{\mathrm{obj}}(a)$ to $\mathbf{F}_{\mathrm{obj}}(b)$.

Exercise 14.2. Prove that this is a bifunctor.

Definition 14.3. For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every bifunctor $\mathbf{F}$ to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, the **classifying functor** $S_{\mathbf{D}}^{\mathbf{B},\mathbf{C}}\mathbf{F}$, or just $S\mathbf{F}$ when confusion is unlikely, from $\mathbf{B}$ to $\mathbf{D}^{\mathbf{C}}$ maps every object b of $\mathbf{B}$ to the functor $\mathbf{F}(b, \bullet)$ from $\mathbf{C}$ to $\mathbf{D}$ and maps every $\mathbf{B}$-morphism u from b to b' to the natural transformation $\mathbf{F}(u, \bullet)$ from $\mathbf{F}(b, \bullet)$ to $\mathbf{F}(b', \bullet)$.

Exercise 14.4. Prove that $S\mathbf{F}$ is a functor.

Proposition 14.5. *For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every bifunctor $\mathbf{F}$ to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, the functor $S\mathbf{F}$ from $\mathbf{B}$ to $\mathbf{D}^{\mathbf{C}}$ is the unique functor such that the pullback of the bifunctor Hom by the functor $S\mathbf{F} \times Id_{\mathbf{C}}$ equals F.*

Exercise 14.6. Formulate and prove functoriality of the construction $S_{\mathbf{D}}^{\mathbf{B},\mathbf{C}}$ in $\mathbf{B}$, in $\mathbf{C}$, and in $\mathbf{D}$.

Definition 14.7. For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of bifunctors to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the **classifying natural transformation** $S_{\mathbf{D}}^{\mathbf{B},\mathbf{C}}\theta$, or just $S\theta$ when confusion is unlikely, from $S\mathbf{F}$ to $S\mathbf{G}$ maps every object b of $\mathbf{B}$ to the natural transformation $\theta_{b,\bullet}$ from $\mathbf{F}(b, \bullet)$ to $\mathbf{G}(b, \bullet)$.

Exercise 14.8. Prove that $S\theta$ is a natural transformation.

Proposition 14.9. *For every category $\mathbf{B}$, for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every ordered pair $(\mathbf{F}, \mathbf{G})$ of bifunctors to $\mathbf{D}$ from $\mathbf{B}$ and $\mathbf{C}$, for every natural transformation from $S\mathbf{F}$ to $S\mathbf{G}$ there exists a unique natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$ such that the natural transformation equals $S\theta$.*

Exercise 14.10. Use this universal property (or any other argument) to formulate and prove compatibility of the operations S with Godement products of natural transformations. Specialize this to formulate and prove compatibility of S with pushforwards and pullbacks by functors of natural transformations.

15   Yoneda embedding

Recall that for a category $\mathbf{C}$, for every object a, the covariant Yoneda functor $\mathbf{C}^a$ is the **Set**-valued functor on $\mathbf{C}$ mapping every object b to $\mathbf{C}_b^a = \mathrm{Hom}_{\mathbf{C}}(a, b)$. When confusion is unlikely, this functor is often denoted by h^a (by long convention). Similarly, the contravariant Yoneda functor $\mathbf{C}_a$ is the **Set**-valued functor on $\mathbf{C}^{\mathrm{opp}}$ mapping every object b to $\mathbf{C}_a^b = \mathrm{Hom}_{\mathbf{C}}(b, a)$. When confusion is unlikely, this functor is often denoted by h_a.

Example 15.1. For every monoid $(H, \bullet)$, for the unique object (which, recall, is chosen to be H itself considered as a set), the Yoneda functor h^H associates to the unique object (i.e., H) the set H, and associates to each element a of H, considered as a morphism from the unique object to itself, the associated bijection of H of left-multiplication by a, i.e., h^H is the left regular representation of $(H, \bullet)$. Similarly h_H is the right regular representation of $(H, \bullet)$.

Definition 15.2. For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism u from a to a', the **Yoneda natural transformation of covariant functors** from $h^{a'}$ to h^a associates to every object b the set function of right-composition with u from $h^{a'}(b) = \mathbf{C}_b^{a'}$ to $h^a(b) = \mathbf{C}_b^a$. This natural transformation is denoted by h^u. Similarly, for every set S, the natural transformation $\mathrm{Id}_S^{\mathbf{Set}} \times h^u$ maps every set $S \times h^{a'}(b)$ to $S \times h^a(b)$ by $\mathrm{Id}_S^{\mathbf{Set}} \times h^u(b)$.

For every category $\mathbf{C}$, for every $\mathbf{C}$-morphism v from b to b', the **Yoneda natural transformation of contravariant functors** from h_b to $h_{b'}$ associates to every object a of the set function of left-composition with v from $h_b(a)$ to $h_{b'}(a)$. This natural transformation is denoted by h_v. Similarly, for every set S, the natural transformation $\mathrm{Id}_S^{\mathbf{Set}} \times h_v$ maps every set $S \times h_b(a)$ to $S \times h_{b'}(a)$ by $\mathrm{Id}_S^{\mathbf{Set}} \times h_v(a)$.

Exercise 15.3. Check that each of these is a natural transformation of set-valued functors from $\mathbf{C}$.

Exercise 15.4. For every $\mathbf{C}$-morphism u from a to a', for every $\mathbf{C}$-morphism u' from a' to a'', check that $h^u \circ h^{u'}$ equals $h^{u' \circ u}$; thus, also, $(\mathrm{Id}_S^{\mathbf{Set}} \times h^u) \circ (\mathrm{Id}_s^{\mathbf{Set}} \times h^{u'})$ equals $\mathrm{Id}_S^{\mathbf{Set}} \times h^{u' \circ u}$. Conclude *contravariance* of the assignment to every $\mathbf{C}$-object a of the covariant Yoneda functor h^a and to every $\mathbf{C}$-object u of the Yoneda natural transformation h^u.

Exercise 15.5. For every $\mathbf{C}$-morphism v from b to b', for every $\mathbf{C}$-morphism v' from b' to b'', check that $h_{v'} \circ h_v$ equals $h_{v' \circ v}$; thus, also, $(\mathrm{Id}_S^{\mathbf{Set}} \times h_{v'}) \circ (\mathrm{Id}_S^{\mathbf{Set}} \times h_v)$ equals $\mathrm{Id}_S^{\mathbf{Set}} \times h_{v' \circ v}$. Conclude *covariance* of the assignment to every $\mathbf{C}$-object a of the contravariant Yoneda functor h_a and to every $\mathbf{C}$-object v of the Yoneda natural transformation h_v.

Exercise 15.6. For every set-valued functor $\mathbf{F}$ from $\mathbf{C}$, respectively from $\mathbf{C}^{\mathrm{opp}}$, for every set S, for the set-valued functor $S \times \mathbf{F}$ from $\mathbf{C}$, resp. from $\mathbf{C}^{\mathrm{opp}}$, check covariance in S.

Definition 15.7. For every category $\mathbf{B}$, for every set-valued covariant functor $\mathbf{F}$ from $\mathbf{B}^{\mathrm{opp}}$, for every $\mathbf{C}$-object b, for every element γ of the set $\mathbf{F}(b)$, the **Yoneda evaluation natural transformation** from h_b to $\mathbf{F}$ associates to every $\mathbf{C}$-object a the set-function from $h_b(a) = \mathrm{Hom}_{\mathbf{C}}(a, b)$ to $\mathbf{F}(a)$ sending each element w of $\mathrm{Hom}_{\mathbf{C}}(a, b)$ to the image of γ under the set function $\mathbf{F}(w)$ from $\mathbf{F}(b)$ to $\mathbf{F}(a)$. This natural transformation is denoted by $\eta_b^{\gamma, \bullet}(\mathbf{F})$, so that w maps to $\eta_b^{\gamma, \bullet}(\mathbf{F})(w)$. Similarly, $\eta_b(\mathbf{F})$ is the natural transformation from $\mathbf{F}(b) \times h_b$ to $\mathbf{F}$ that associates to every $\mathbf{C}$-object a the set-function from $\mathbf{F}(b) \times h_b(a)$ to $\mathbf{F}(a)$ sending every element (γ, w) to $\eta_b^{\gamma, \bullet}(\mathbf{F})(w)$.

For every category $\mathbf{B}$, for every set-valued covariant functor $\mathbf{F}$ from $\mathbf{B}$, for every $\mathbf{C}$-object a, for every element δ of the set $\mathbf{F}(a)$, the **Yoneda evaluation natural transformation** from h^a to $\mathbf{F}$ associates to every $\mathbf{C}$-object b the set-function from $h^a(b) = \text{Hom}_{\mathbf{C}}(a,b)$ to $\mathbf{F}(b)$ sending each element w of $\text{Hom}_{\mathbf{C}}(a,b)$ to the image of δ under the set function $\mathbf{F}(w)$ from $\mathbf{F}(a)$ to $\mathbf{F}(b)$. This natural transformation is denoted by $\eta^b_{\delta,\bullet}(\mathbf{F})$, so that w maps to $\eta^a_{\gamma,\bullet}(\mathbf{F})(w)$. Similarly, $\eta^a(\mathbf{F})$ is the natural transformation from $\mathbf{F}(a) \times h^a$ to $\mathbf{F}$ that associates to every $\mathbf{C}$-object b the set-function from $\mathbf{F}(a) \times h^a(b)$ to $\mathbf{F}(b)$ sending every element (δ, w) to $\eta^a_{\gamma,\bullet}(\mathbf{F})(w)$.

Exercise 15.8. Check that $\eta_b(\mathbf{F})$ and $\eta^a(\mathbf{F})$ are natural transformations.

Exercise 15.9. For every natural transformation α from $\mathbf{F}$ to $\mathbf{G}$ of set-valued covariant functors from $\mathbf{C}$, check that $\alpha \circ \eta_b(\mathbf{F})$ equals the composition of $\eta_b(\mathbf{G})$ with the natural transformation of functors $\alpha(b) \times \text{Id}_{h_b}$ from $\mathbf{F}(b) \times h_b$ to $\mathbf{G}(b) \times h_b$ induced by the set function $\alpha(b)$ from $\mathbf{F}(b)$ to $\mathbf{G}(b)$. Thus, $\eta_b(\mathbf{F})$ is "covariant" in $\mathbf{F}$.

Lemma 15.10 (Yoneda Lemma). *For every category $\mathbf{C}$, for every covariant set-valued functor $\mathbf{F}$ from $\mathbf{C}^{opp}$, for every $\mathbf{C}$-object b, every natural transformation Γ from h_b to $\mathbf{F}$ is of the form $\eta^{\gamma,\bullet}_b(\mathbf{F})$ for a unique element γ of $\mathbf{F}(b)$, namely the image under Γ of the element $Id^{\mathbf{C}}_b$ of $h_b(b) = \text{Hom}_{\mathbf{C}}(b,b)$.*

Exercise 15.11. Formulate and prove the analogous result for covariant set-valued functors from $\mathbf{C}$ and the Yoneda functors h^a.

Definition 15.12. For every set S, the **identity section** is the set function from S to $S \times h_b(b) = S \times \text{Hom}_{\mathbf{C}}(b,b)$ that pairs each element of S with $\text{Id}^{\mathbf{C}}_b$.

Exercise 15.13. Check that the identity section is covariant in S.

Definition 15.14. For every small category $\mathbf{C}$, for every $\mathbf{C}$-object b, the set-valued **left Yoneda functor $\mathbf{L}_b$** from the functor category $\mathbf{Set}^{(\mathbf{C}^{\text{opp}})}$ associates to every set-valued covariant functor $\mathbf{F}$ from $\mathbf{C}^{\text{opp}}$ the set $\mathbf{F}(b)$ and associates to every natural transformation α from $\mathbf{F}$ to $\mathbf{G}$ the set function $\alpha(b)$ from $\mathbf{F}(b)$ to $\mathbf{G}(b)$.

Similarly, the **right Yoneda functor $\mathbf{R}_b$** from $\mathbf{Set}$ to $\mathbf{Set}^{\mathbf{C}^{\text{opp}}}$ associates to every set S the covariant set-valued functor $S \times h_b$ from $\mathbf{C}^{\text{opp}}$, and associates to every set function f from S to S' the natural transformation $f \times \text{Id}_{h_b}$ from $S \times h_b$ to $S' \times h_b$.

Exercise 15.15. Check that each of these is a functor. Check that the identity section is a natural transformation from the identity functor of $\mathbf{Set}$ to the composite functor $\mathbf{R}_b \circ \mathbf{L}_b$.

Lemma 15.16 (Yoneda Lemma II). *For every small category $\mathbf{C}$, for every $\mathbf{C}$-object b, the left Yoneda functor and the right Yoneda functor extend to an adjoint pair of functors using the natural transformation η_b above and the identity section natural transformation.*

Exercise 15.17. For every small category $\mathbf{C}$, conclude that the Yoneda functor from $\mathbf{C}$ to $\mathbf{Set}^{(\mathbf{C}^{\mathrm{opp}}}$ sending every $\mathbf{C}$-object b to h_b is a fully faithful embedding of categories.

In the sense of the previous lemma, the Yoneda functors give examples of adjoint pairs. Conversely, extension of a functor to an adjoint pair is an example of a *representability problem*.

Definition 15.18. For every category $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}^{\mathrm{opp}}$ to $\mathbf{Set}$, a **representation** of $\mathbf{F}$ is an ordered pair (a, x) of an object a of $\mathbf{C}$ and an element x of the set $\mathbf{F}(a)$ such that the induced natural transformation $h_a \Rightarrow \mathbf{F}$ is a natural equivalence. A functor from $\mathbf{C}^{\mathrm{opp}}$ is **representable** if (and only if) there exists a representation.

Exercise 15.19. Formulate the opposite notion of representable for functors from $\mathbf{C}$ to $\mathbf{Set}$.

Exercise 15.20. For every category $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}^{\mathrm{opp}}$ to $\mathbf{Set}$, for every representation (a, x) of $\mathbf{F}$, for every representation (a', x') of $\mathbf{F}$, prove that there exists a unique $\mathbf{C}$-isomorphism f from a to a' that pulls x' back to x. Conclude that a representation of a representable functor is unique up to unique isomorphism. Formulate and prove the opposite result for covariant functors from $\mathbf{C}$ to $\mathbf{Set}$.

Exercise 15.21. For every category $\mathbf{C}$, for every small category $\mathbf{D}$, for every covariant functor $\mathbf{L}$ from $\mathbf{C}$ to $\mathbf{D}$ such that the set-valued functor $\mathrm{Hom}_{\mathbf{D}}(\mathbf{L}(\bullet), b)$ on $\mathbf{C}^{\mathrm{opp}}$ is representable for every object b of $\mathbf{D}$, prove that there exists an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$ (which is unique up to unique natural equivalences by an earlier exercise). Thus, show that extension of a functor to an adjoint pair is a special case of representability of functors.

Exercise 15.22. Prove the variant of the previous result for opposite categories: for every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every covariant functor $\mathbf{R}$ from $\mathbf{D}$ to $\mathbf{C}$, if the set-valued functor $\mathrm{Hom}_{\mathbf{C}}(a, \mathbf{R}(\bullet))$ on $\mathbf{D}$ is representable for every object $\mathbf{a}$ of $\mathbf{C}$, prove there exists an adjoint pair $(\mathbf{L}, \mathbf{R}, \theta, \eta)$.

16   Fiber categories

Recall that for a functor $\mathbf{G}$ from a category $\mathbf{B}$ to a category $\mathbf{C}$, for every object U of $\mathbf{C}$, the fiber category of $\mathbf{G}$ over U is

Lemma 16.1. *For every functor $\mathbf{G}$ from a category $\mathbf{B}$ to a category $\mathbf{C}$, for every object U of $\mathbf{C}$, for every functor $\mathbf{I}$ from a category $\mathbf{A}$ to $\mathbf{B}$, and for every natural isomorphism α from $\mathbf{G} \circ \mathbf{I}$ to the constant functor $\mathrm{const}_{\mathbf{C}, U}$, there exists a unique functor $\widehat{\mathbf{I}}$ from $\mathbf{A}$ to $\mathbf{B}_{\mathbf{G}, U}$ such that both $\mathsf{incl}_{\mathbf{B}} \circ \widehat{\mathbf{I}}$ equals $\mathbf{I}$ and α equals the pullback of θ by $\widehat{\mathbf{I}}$.*

Corollary 16.2. *[HERE]*

17   Limits and colimits

Definition 17.1. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every object U of $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, the U-**sections object** of $\mathbf{F}$ over U is the object $\Gamma_{\mathbf{D}}^{\mathbf{C}}(U, \mathbf{F}) := \mathbf{F}(U)$ of $\mathbf{D}$, denoted also $\Gamma(U, \mathbf{F})$ when confusion is unlikely. For every ordered pair $(\mathbf{F}, \mathbf{G})$ of functors from $\mathbf{C}$ to $\mathbf{D}$, for every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$, the U-**sections morphism** of $\mathbf{F}$ over U is the $\mathbf{D}$-morphism $\Gamma(U, \theta) := \theta_U$ from $\mathbf{F}(U)$ to $\mathbf{G}(U)$.

Exercise 17.2. Prove that these rules preserve identities and composition.

Definition 17.3. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every object U of $\mathbf{C}$, the U-**sections functor** $\Gamma(U, -)$ from $\mathbf{D}^{\mathbf{C}}$ to $\mathbf{D}$ maps every object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ to $\Gamma(U, \mathbf{F}) := \mathbf{F}(U)$ and maps every $\mathbf{D}^{\mathbf{C}}$-morphism θ from $\mathbf{F}$ to $\mathbf{G}$ to the $\mathbf{D}$-morphism $\Gamma(U, \theta) := \theta_U$.

For every ordered pair (V, U) of objects of $\mathbf{C}$, for every $\mathbf{C}$-morphism r from V to U, the r-**sections natural transformation** from $\Gamma(V, \bullet)$ to $\Gamma(U, \bullet)$ maps every object $\mathbf{F}$ of $\mathbf{D}^{\mathbf{C}}$ to the $\mathbf{D}$-morphism $\Gamma(r, \mathbf{F}) := \mathbf{F}_U^V(r)$ from $\mathbf{F}(V)$ to $\mathbf{F}(U)$.

Exercise 17.4. Prove that $\Gamma(r, \bullet)$ is a natural transformation. For every object U of $\mathbf{C}$, prove that $\Gamma(\mathrm{Id}_U, \bullet)$ is the identity natural transformation from $\Gamma(U, \bullet)$ to itself. For every triple (W, V, U) of objects of $\mathbf{C}$, for every $\mathbf{C}$-morphism r from W to V, for every $\mathbf{C}$-morphism s from V to U, prove that $\Gamma(s \circ r, \bullet)$ equals the composition of natural transformations $\Gamma(s, \bullet) \circ \Gamma(r, \bullet)$.

Exercise 17.5. For every small category $\mathbf{C}$, for every small category $\mathbf{D}$, prove that $\mathbf{D}^{\mathbf{C}}$ is a small category.

Definition 17.6. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, the **sections bifunctor** is the functor $\Gamma_{\mathbf{D}}^{\mathbf{C}}(-, \bullet)$, or just $\Gamma(-, \bullet)$ when confusion is unlikely, from the product category $\mathbf{C} \times \mathbf{D}^{\mathbf{C}}$ to the category $\mathbf{D}$ that sends every object $(U, \mathbf{F})$ of $\mathbf{C} \times \mathbf{D}^{\mathbf{C}}$ to $\Gamma(U, \mathbf{F}) := \mathbf{F}(U)$, that sends every $\mathbf{C}$-morphism r from a to b to the $\mathbf{D}$-morphism $\Gamma(r, \mathbf{F})$, and that sends every natural transformation θ from $\mathbf{F}$ to $\mathbf{G}$ to the $\mathbf{D}$-morphism $\Gamma(U, \theta) := \theta_U$.

Exercise 17.7. Prove that the sections bifunctor is a bifunctor.

Exercise 17.8. Formulate and prove the statement that formation of $\mathbf{D}^{\mathbf{C}}$ is covariant in the category $\mathbf{D}$ and is contravariant in the small category $\mathbf{C}$. In particular, for every small category $\mathbf{C}$, prove that the covariant Yoneda functor of $\mathbf{C}$ in $\mathbf{Cat}$ enriches to a functor from $\mathbf{Cat}$ to itself. Similarly, for every small category $\mathbf{D}$, prove that the contravariant Yoneda functor of $\mathbf{D}$ in $\mathbf{Cat}$ enriches to a functor from $\mathbf{Cat}^{\mathrm{opp}}$ to $\mathbf{Cat}$.

Notation 17.9. For every category $\mathbf{C}$, for every category $\mathbf{D}$, for every object a of $\mathbf{D}$, denote by

$$\underline{a}^{\mathbf{C}} : \mathbf{C} \to \mathbf{D}$$

the constant functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},a}$ that sends every object to a and that sends every morphism to Id_a. For every morphism in $\mathbf{D}$, say $p : a \to b$, denote by

$$\underline{p}^{\mathbf{C}} : \underline{a}^{\mathbf{C}} \Rightarrow \underline{b}^{\mathbf{C}}$$

the natural transformation that assigns p to every object of $\mathbf{C}$.

Exercise 17.10. For every small category $\mathbf{C}$ with at least one object, for every category $\mathbf{D}$, prove that the functor $\mathrm{const}^{\mathbf{C}}_{\mathbf{D},\bullet}$ from $\mathbf{D}$ to $\mathbf{D}^{\mathbf{C}}$ is a full embedding.

Exercise 17.11. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, if $\mathbf{C}$ has an initial object X, prove that $(\mathrm{const}^{\mathbf{C}}_{\mathbf{D},\bullet}, \Gamma(X, -))$ extends to an adjoint pair of functors.

Definition 17.12. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, a **limit** of $\mathbf{F}$ is a natural transformation $\eta : \underline{a}^{\mathbf{C}} \Rightarrow \mathbf{F}$ that is final among all such natural transformations, i.e., for every natural transformation $\theta : \underline{b}^{\mathbf{C}} \Rightarrow \mathbf{F}$, there exists a unique morphism $t : b \to a$ in $\mathbf{D}$ such that θ equals $\eta \circ \underline{t}^{\mathbf{C}}$.

Exercise 17.13. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for all functors $\mathbf{F}$ and $\mathbf{G}$ from $\mathbf{C}$ to $\mathbf{D}$, for every natural transformation ϕ from $\mathbf{F}$ to $\mathbf{G}$, for all limits $\eta : \underline{a}^{\mathbf{C}} \Rightarrow \mathbf{F}$ and $\theta : \underline{b}^{\mathbf{C}} \Rightarrow \mathbf{G}$, prove that there exists a unique morphism $f : a \to b$ in $\mathbf{D}$ such that $\theta \circ \underline{f}^{\mathbf{C}}$ equals $\phi \circ \eta$. Thus, that if a limit of $\mathbf{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object a of $\mathbf{D}$, the identity transformation $\mathrm{Id}_{\underline{a}^{\mathbf{C}}} : \underline{a}^{\mathbf{C}} \to \underline{a}^{\mathbf{C}}$ is a limit of $\underline{a}^{\mathbf{C}}$.

Definition 17.14. A category $\mathbf{D}$ is **complete** if (and only if), for every small category $\mathbf{C}$, every functor from $\mathbf{C}$ to $\mathbf{D}$ has a limit (which is then unique up to unique isomorphism by the previous exercise).

For every complete category $\mathbf{D}$, for every small category $\mathbf{C}$, some version of the Axiom of Choice (e.g., Hilbert's epsilon operator) produces a rule $\varprojlim$ that assigns to every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$ an object $\Gamma = \varprojlim \mathbf{F}$ and a natural transformation $\eta_{\mathbf{F}} : \underline{\Gamma}^{\mathbf{C}} \to \mathbf{F}$ that is a limit. (In many concrete categories, there is an explicit "construction" of such a rule.)

Exercise 17.15. For every small category $\mathbf{C}$, for every complete category $\mathbf{C}$, and for every rule $\varprojlim$ as above, using the existence of the unique morphism f in the previous exercise, prove that there

is a unique extension to a functor compatible with the natural transformations η and θ (defined next),

$$\varprojlim : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}.$$

The morphisms $\eta_{\mathbf{F}}$ define a natural transformation of functors,

$$\eta : \mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}} \circ \varprojlim \Rightarrow \mathrm{Id}_{\mathbf{D}^{\mathbf{C}}}.$$

Moreover, the rule sending every object a of $\mathbf{D}$ to the identity natural transformation $\mathrm{Id}_{\underline{a}^{\mathbf{C}}}$ defines a natural transformation of functors, $\theta : \mathrm{Id}_{\mathbf{D}} \Rightarrow \varprojlim \circ \mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}}$. Prove that the quadruple $(\mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}}, \varprojlim, \theta, \eta)$ is an adjoint pair of functors. In particular, the limit functor $\varprojlim$ preserves monomorphisms and sends injective objects of $\mathbf{D}^{\mathbf{C}}$ to injective objects of $\mathbf{D}$.

Exercise 17.16. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, if $\mathbf{C}$ has a final object O, prove that $(\Gamma(O, -), \mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}})$ extends to an adjoint pair of functors.

Definition 17.17. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, a **colimit** of $\mathbf{F}$ is a natural transformation $\theta : \mathbf{F} \Rightarrow \underline{a}^{\mathbf{C}}$ that is final among all such natural transformations, i.e., for every natural transformation $\eta : \mathbf{F} \Rightarrow \underline{b}^{\mathbf{C}}$, there exists a unique morphism $h : a \to b$ in $\mathbf{D}$ such that $\underline{h}^{\mathbf{C}} \circ \theta$ equals η. A category $\mathbf{D}$ is **cocomplete** if (and only if) for every small category $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, there exists a colimit of $\mathbf{F}$.

Exercise 17.18. For every small category $\mathbf{C}$, for every category $\mathbf{D}$, for all functors $\mathbf{F}$ and $\mathbf{G}$ from $\mathbf{C}$ to $\mathbf{D}$, for every natural transformation ϕ from $\mathbf{F}$ to $\mathbf{G}$, for all colimits $\theta : \mathbf{F} \Rightarrow \underline{a}^{\mathbf{C}}$ and $\eta : \mathbf{G} \Rightarrow \underline{b}^{\mathbf{C}}$, prove that there exists a unique morphism $f : a \to b$ in $\mathbf{D}$ such that $\underline{f}^{\mathbf{C}} \circ \theta$ equals $\eta \circ \phi$. Thus, if a colimit of $\mathbf{F}$ exists, then it is unique up to unique isomorphism. In particular, for every object a of $\mathbf{D}$, the identity transformation $\mathrm{Id}_{\underline{a}^{\mathbf{C}}} : \underline{a}^{\mathbf{C}} \to \underline{a}^{\mathbf{C}}$ is a colimit of $\underline{a}^{\mathbf{C}}$. Finally, repeat the previous results for each cocomplete category $\mathbf{D}$ with colimits in place of limits to obtain a functor $\varinjlim$ from $\mathbf{D}^{\mathbf{C}}$ to $\mathbf{D}$ that is left adjoint to $\mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}}$, whereas $\varprojlim$ is right adjoint to $\mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}}$. Deduce that colimits (if they exist) preserve epimorphisms and projective objects. (You can use opposite categories to reduce most of this to the case of limits.) Altogether, for every category $\mathbf{D}$ that is both complete and cocomplete, the ordered triple of functors, $(\varinjlim, \mathrm{const}_{\mathbf{D},\bullet}^{\mathbf{C}}, \varprojlim)$, is an *adjoint triple*.

Definition 17.19. For every complete category $\mathbf{D}$, for every functor x from a small category $\mathbf{C}'$ to a small category $\mathbf{C}$, for every functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$, the x-**pullback** $\mathbf{F}_x$ of $\mathbf{F}$ is the composite functor $\mathcal{F} \circ x$, which is a functor from $\mathbf{C}'$ to $\mathbf{D}$. For every natural transformation, say $\phi : \mathbf{F} \Rightarrow \mathbf{G}$, of functors form $\mathbf{C}$ to $\mathbf{D}$, the x-**pullback** ϕ_x from the σ-family $\mathbf{F}_x$ to $\mathbf{G}_x$ is $\phi \circ x$, which is a natural transformation of functors from $\mathbf{C}'$ to $\mathbf{D}$.

Exercise 17.20. For every complete category $\mathbf{D}$, for every functor x from a small category $\mathbf{C}'$ to a small category $\mathbf{C}$, prove that the x-pullback defines a functor

$$*_x : \mathbf{D}^{\mathbf{C}} \to \mathbf{D}^{\mathbf{C}'}$$

For the identity functor $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$, prove that $\mathrm{Id}_{\mathbf{C}}$-pullback is the identity functor from $\mathbf{D}^{\mathbf{C}}$ to itself. For every functor y from a small category $\mathbf{C}''$ to $\mathbf{C}'$, prove that the $x \circ y$-pullback equals the composite $*_y \circ *_x$. In this sense, deduce that pullback is contravariant in x.

Definition 17.21. For every complete category $\mathbf{D}$, for every small category $\mathbf{C}$, for every small category $\mathbf{C}'$, for all functors x and $\overline{x}$ from $\mathbf{C}'$ to $\mathbf{C}$, and for every natural transformation n from x to $\overline{x}$, the n-**pullback** of each functor $\mathbf{F}$ from $\mathbf{C}$ to $\mathbf{D}$ is the natural transformation $\mathbf{F}_n$ from the x-pullback $\mathbf{F}_x$ to the $\overline{x}$-pullback $\mathbf{F}_{\overline{x}}$ that sends every $\mathbf{C}'$-object V to the morphism $\mathbf{F}(n(V))$ from $\mathbf{F}(x(V))$ to $\mathbf{F}(\overline{x}(V))$.

Exercise 17.22. Prove that $\mathbf{F}_n$ is a natural transformation. Also, for every natural transformation, say $\phi : \mathbf{F} \Rightarrow \mathbf{G}$, of functors from $\mathbf{C}$ to $\mathbf{D}$, prove that $\phi_{\overline{x}} \circ \mathbf{F}_n$ equals $\mathbf{G}_n \circ \phi_x$. Thus, the operation $*_n$ is a natural transformation from the functor $*_x$ to $*_{\overline{x}}$. For the identity natural transformation Id_x from x to itself, also $*_{\mathrm{Id}_x}$ is the identity natural transformation of $*_x$. Finally, for every functor $\overline{\overline{x}}$ from $\mathbf{C}'$ to $\mathbf{C}$, and for every natural transformation $\overline{n}$ from $\overline{x}$ to $\overline{\overline{x}}$, the natural transformation $\mathbf{F}_{\overline{n} \circ n}$ equals $\mathbf{F}_{\overline{n}} \circ \mathbf{F}_n$. In this sense, the operation $*_x$ is also compatible with natural transformations. In particular, if (x, y, θ, η) is an adjoint pair of functors, then also $(*_y, *_x, *_\theta, *_\eta)$ is an adjoint pair of functors.

Many categories are complete, many categories are cocomplete, and many categories are both. Most often this is proved by reducing to an explicit construction of limits and colimits. There are a few categorical notions that help.

Definition 17.23. For every functor x from a category $\mathbf{C}'$ to a category $\mathbf{C}$, for every object U of $\mathbf{C}$, a $\mathbf{C}'_{x,U}$-**object** is a pair $(V, r : x(V) \to U)$ of an object V of $\mathbf{C}'$ and a $\mathbf{C}$-isomorphism $r : x(V) \to U$. For $\mathbf{C}'_{x,U}$-objects (V, r) and (V', r'), a $\mathbf{C}'_{x,U}$-**morphism** from (V, r) to (V', r') is a $\mathbf{C}'$-morphism, $s : V \to V'$, such that $r' \circ x(s)$ equals r.

Exercise 17.24. For every $\mathbf{C}'_{x,U}$-object (V, r), prove that Id_V is a $\mathbf{C}'_{x,U}$-morphism from (V, r) to itself. More generally, the $\mathbf{C}'_{x,U}$-morphisms from (V, r) to (V, r) are precisely the $\mathbf{C}'$-morphisms s from V to V such that $x(s)$ equals $\mathrm{Id}_{x(V)}$. For every pair of $\mathbf{C}'_{x,U}$-morphisms, say $s : (V, r) \to (V', r')$ and $s' : (V', r') \to (V'', r'')$, prove that $s' \circ s$ is a $\mathbf{C}'_{x,U}$-morphism from (V, r) to (V'', r''). Conclude that these rules define a category, denoted $\mathbf{C}'_{x,U}$. Prove that the rules $(V, r) \mapsto V$ and $s \mapsto s$ defines a faithful functor,

$$\Phi_{x,U} : \mathbf{C}'_{x,U} \to \mathbf{C}',$$

and $r : x(V) \to U$ defines a natural isomorphism,

$$\theta_{x,U} : x \circ \Phi_{x,U} \Rightarrow \underline{U}^{\mathbf{C}'_{x,U}}$$

Finally, for every category $\mathbf{C}''$, for every functor $\Phi' : \mathbf{C}'' \to \mathbf{C}'$, and for every natural isomorphism $\theta' : x \circ \Phi' \Rightarrow \underline{U}^{\mathbf{C}''}$, prove that there exists a unique functor $F : \mathbf{C}'' \to \mathbf{C}'_{x,U}$ such that Φ' equals $\Phi_{x,U} \circ F$ and θ' equals $\theta_{x,U} \circ F$. In this sense, $(\Phi_{x,U}, \theta_{x,U})$ is final among pairs (Φ', θ') as above.

For every pair of functors $x, x_1 : \mathbf{C}' \to \mathbf{C}$, and for every natural isomorphism $n : x \Rightarrow x_1$, for every $\mathbf{C}'_{x_1,U}$-object $(V, r_1 : x_1(V) \to U)$, prove that $(V, r_1 \circ n_V : x(V) \to U)$ is an object of $\mathbf{C}_{x,U}$. For every morphism in $\mathbf{C}_{x_1,U}$, say $s : (V, r_1) \to (V', r'_1)$, prove that s is also a morphism in $\mathbf{C}'_{x,U}$, namely $(V, r_1 \circ n_V) \to (V', r'_1 \circ n_{V'})$. Conclude that these rules define a functor,

$$\mathbf{C}'_{n,U} : \mathbf{C}'_{x_1,U} \to \mathbf{C}'_{x,U}.$$

Prove that this functor is a *strict equivalence* of categories: it is a bijection on Hom sets (as for all equivalences), but it is also a bijection on objects (rather than merely being essentially surjective). Prove that $\mathbf{C}'_{n,U}$ is functorial in n, i.e., for a second natural isomorphism $m : x_1 \Rightarrow x_2$, prove that $\sigma_{m \circ n, U}$ equals $\sigma_{n,U} \circ \sigma_{m,U}$.

For every pair of functors, $x : \sigma \to \tau$ and $y : \rho \to \tau$, and for every functor $z : \sigma \to \rho$ such that x equals $y \circ z$ equals x, for every $\sigma_{x,U}$-object (V, r), **prove** that $(z(V), r)$ is a $\rho_{y,U}$-object. For every $\sigma_{x,U}$-morphism $s : (V, r) \to (V', r')$, **prove** that $z(s)$ is a $\rho_{y,U}$-morphism $(z(V), r) \to (z(V'), r')$. **Prove** that $z(\mathrm{Id}_V)$ equals $\mathrm{Id}_{z(V)}$, and **prove** that z preserves composition. Conclude that these rules define a functor,

$$z_U : \sigma_{x,U} \to \rho_{y,U}.$$

Prove that this is functorial in z: $(\mathrm{Id}_\sigma)_U$ equals $\mathrm{Id}_{\sigma_{x,U}}$, and for a third functor $w : \pi \to \tau$ and functor $z' : \rho \to \pi$ such that y equals $w \circ z'$, then $(z' \circ z)_U$ equals $z'_U \circ z_U$. For an object (W, r_W) of $\rho_{y,U}$, for each object $((V, r_V), q : Z(V) \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, define the *associated* object of $\sigma_{z,W}$ to be (V, q). For an object $((V', r_{V'}), q' : Z(V') \to W)$ of $(\sigma_{x,U})_{z,(W,r_W)}$, for every morphism $s : (V, r_V) \to (V', r_{V'})$ such that q equals $q' \circ z(s)$, define the *associated* morphism of $\sigma_{z,W}$ to be s. **Prove** that this defines a functor

$$\widetilde{z}_{U,(W,r_W)} : (\sigma_{x,U})_{z_U,(W,r_W)} \to \sigma_{z,W}.$$

Prove that this functor is a strict equivalence of categories. **Prove** that this equivalence is functorial in z. Finally, for two functors $z, z_1 : \sigma \to \rho$ such that x equals both $y \circ z$ and $y \circ z_1$, and for a natural transformation $m : z \Rightarrow z_1$, for every object $(V, r : x(V) \to U)$ of $\sigma_{x,U}$, **prove** that m_V is a morphism in $\rho_{y,U}$ from $(z(V), r)$ to $(z_1(V), r)$. Moreover, for every morphism in $\sigma_{x,U}$,

$s : (V, r) \to (V', r')$, **prove** that $m_{V'} \circ z(s)$ equals $z_1(s) \circ m_V$. Conclude that this rule is a natural transformation $m_U : z_U \Rightarrow (z_1)_U$. **Prove** that this is functorial in m. If m is a natural isomorphism, **prove** that also m_U is a natural isomorphism, and the strict equivalence $(m_U)_{(W,r_W)}$ is compatible with the strict equivalence m_W. Finally, **prove** that $m \mapsto m_U$ is compatible with precomposition and postcomposition of m with functors of categories over τ.

(vii)(Colimits and Limits along an Essentially Surjective Functor) Let $x : \sigma \to \tau$ be a functor of small categories. **Prove** that every fiber category $\sigma_{x,U}$ is small. Next, assume that x is *essentially surjective*, i.e., for every object U of τ, there exists a $\sigma_{x,U}$-object (V, r). Let $y : \tau \to \sigma$ be a functor, and let $\alpha : \mathrm{Id}_\sigma \Rightarrow y \circ x$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\alpha_V) : x(V) \to x(y(x(V)))$ is an isomorphism and $(y(x(V)), x(\alpha_V)^{-1})$ is a final object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and α extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has a final object.) For every adjoint pair (x, y, α, β), also $(*_y, *_x, *_\alpha, *_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and α, yet let L_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $L_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\theta_\mathcal{F} : \mathcal{F} \to *_x \circ L_x(\mathcal{F}),$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\theta_{\mathcal{F}, x, U} : \mathcal{F} \circ \Phi_{x,U} \Rightarrow L_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(L_x(\mathcal{F})(U), \theta_{\mathcal{F}, x, U})$ is a colimit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$L_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation

$$\theta_x : \mathrm{Id}_{\mathbf{Fun}(\sigma, \mathcal{C})} \Rightarrow *_x \circ L_x.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_\mathcal{G} : \mathcal{G} \circ x \circ \Phi_{x,U} \to \mathcal{G} \circ \underline{U}_{\sigma_{x,U}},$$

factors uniquely through a $\mathcal{C}$-morphism $L_x(\mathcal{G} \circ x)(U) \to \mathcal{G}(U)$. **Prove** that this defines a morphism $\eta_\mathcal{G} : L_x(\mathcal{G} \circ x) \to \mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that is a natural transformation,

$$\eta : L_x \circ *_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\tau, \mathcal{C})}.$$

Prove that $(L_x, *_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists a Γ^x and θ as above.)

Next, as above, let $x : \sigma \to \tau$ be a functor of small catgories that is essentially surjective. Let $y : \tau \to sigma$ be a functor, and let $\beta : y \circ x \Rightarrow \mathrm{Id}_\sigma$ be a natural transformation. **Prove** that this extends to an adjoint pair of functors (x, y, α, β) if and only if for every object V of σ, the morphism $x(\beta_v) : x(y(x(V))) \to x(V)$ is an isomorphism and $(y(x(V)), x(\beta_v))$ is an initial object of the fiber category $\sigma_{x,x(V)}$. (Conversely, up to some form of the Axiom of Choice, there exists y and β extending to an adjoint pair if and only if every fiber category $\sigma_{x,U}$ has an initial object.) For every adjoint pair $(y, x, \alpha, beta)$ also $(*_x, *_y, *_\alpha, *_\beta)$ is an adjoint pair. More generally, no longer assume that there exists y and β, yet let R_x be a rule that assigns to every object $\mathcal{F}$ of $\mathbf{Fun}(\sigma, \mathcal{C})$ an object $R_x(\mathcal{F})$ of $\mathbf{Fun}(\tau, \mathcal{C})$ and a natural transformation,

$$\eta_\mathcal{F} : *_x \circ R_x(\mathcal{F}) \to \mathcal{F},$$

of objects in $\mathbf{Fun}(\sigma, \mathcal{C})$. For every object U of τ, this defines a natural transformation

$$\eta_{\mathcal{F},x,U} : R_x(\mathcal{F}) \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{F} \circ \Phi_{x,U},$$

of objects in $\mathbf{Fun}(\sigma_{x,U}, \mathcal{C})$. Assume that each $(R_x(\mathcal{F})(U), \eta_{\mathcal{F},x,U})$ is a limit of $\mathcal{F} \circ \Phi_{x,U}$. **Prove** that this extends uniquely to a functor,

$$R_x : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}),$$

and a natural transformation,

$$\eta : *_x \circ R_x \Rightarrow \mathrm{Id}_{\mathbf{Fun}(\sigma, \mathcal{C})}.$$

Moreover, for every $\mathcal{G}$ in $\mathbf{Fun}(\tau, \mathcal{C})$, the identity morphism,

$$\mathrm{Id}_\mathcal{G} : \mathcal{G} \circ \underline{U}_{\sigma_{x,U}} \Rightarrow \mathcal{G} \circ x \circ \Phi_{x,U},$$

factors uniquely through a $\mathcal{G}(U) \to \mathcal{C}$-morphism $R_x(\mathcal{G} \circ x)(U)$. **Prove** that this defines a morphism $\theta_\mathcal{G} : \mathcal{G} \to R_x(\mathcal{G} \circ x)$ in $\mathbf{Fun}(\tau, \mathcal{C})$. **Prove** that this is a natural transformation,

$$\theta : \mathrm{Id}_{\mathbf{Fun}(\tau, \mathcal{C})} \Rightarrow R_x \circ *_x.$$

Prove that $(*_x, R_x, \theta, \eta)$ is an adjoint pair of functors. (Using some version of the Axiom of Choice, if every $\mathcal{F} \circ \Phi_{x,U}$ admits a colimit, then there exists R_x and η as above.)

(viii)(Adjoints Relative to a Full, Upper Subcategory) In a complementary direction to the previous case, let $x : \sigma \to \tau$ be an embedding of a full subcategory (thus, x is essentially surjective if and only if x is an equivalence of categories). In this case, the functor

$$*_x : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\sigma, \mathcal{C})$$

is called *restriction*. Assume further that σ is *upper* (a la the theory of partially ordered sets) in the sense that every morphism of τ whose source is an object of σ also has target an object of σ. Assume that $\mathcal{C}$ has an initial object, $\odot$. Let $\mathcal{G}$ be a σ-family of objects of $\mathcal{C}$. Also, let $\phi : \mathcal{G} \to \mathcal{H}$ be a morphism of σ-families. For every object U of τ, if U is an object of σ, then define $_x\mathcal{G}(U)$ to be $\mathcal{G}(U)$, and define $_x\phi(U)$ to be $\phi(U)$. For every object U of τ that is not an object of σ, define $_x\mathcal{G}(U)$ to be $\odot$, and define $_x\phi(U)$ to be $\mathrm{Id}_\odot$. For every morphism $r : U \to V$, if U is an object of σ, then r is a morphism of σ. In this case, define $_x\mathcal{G}(r)$ to be $\mathcal{G}(r)$. On the other hand, if U is not an object of σ, then $\mathcal{G}(U)$ is the initial object $\odot$. In this case, define $_x\mathcal{G}(r)$ to be the unique morphism $_x\mathcal{G}(U) \to {_x\mathcal{G}(V)}$. **Prove** that $_x\mathcal{G}$ is a τ-family of objects, i.e., the definitions above are compatible with composition of morphisms in τ and with identity morphisms. Also **prove** that $_x\phi$ is a morphism of τ-families. **Prove** that $_x\mathrm{Id}_\mathcal{G}$ equals $\mathrm{Id}_{_x\mathcal{G}}$. Also, for a second morphism of σ-families, $\psi : \mathcal{H} \to \mathcal{I}$, **prove** that $_x(\psi \circ \phi)$ equals $_x\psi \circ {_x\phi}$. Conclude that these rules form a functor,

$$_x* : \mathbf{Fun}(\sigma, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}).$$

Prove that $(_x*, *_x)$ extends to an adjoint pair of functors. In particular, conclude that $*_x$ preserves epimorphisms and $_x*$ preserves monomorphisms.

Next assume that $\mathcal{C}$ is an Abelian category that satisfies (AB3). For every τ-family $\mathcal{F}$, for every object U of τ, define $\theta_\mathcal{F}(U) : \mathcal{F}(U) \to {^x\mathcal{F}(U)}$ to be the cokernel of $\mathcal{F}(U)$ by the direct sum of the images of

$$\mathcal{F}(s) : \mathcal{F}(T) \to \mathcal{F}(U),$$

for all morphisms $s : T \to U$ with V *not* in σ (possibly empty, in which case $\theta_\mathcal{F}(U)$ is the identity on $\mathcal{F}(U)$). In particular, if U is not in σ, then $^x\mathcal{F}(U)$ is zero. For every morphism $r : U \to V$ in τ, **prove** that the composition $\theta_\mathcal{F}(V) \circ \mathcal{F}(r)$ equals $^x\mathcal{F}(r) \circ \theta_\mathcal{F}(U)$ for a unique morphism

$$^x\mathcal{F}(r) : {^x\mathcal{F}(U)} \to {^x\mathcal{F}(V)}.$$

Prove that $^x\mathcal{F}(\mathrm{Id}_U)$ is the identity morphism of $^x\mathcal{F}(U)$. **Prove** that $r \mapsto {^x\mathcal{F}(r)}$ is compatible with composition in τ. Conclude that $^x\mathcal{F}$ is a τ-family, and $\theta_\mathcal{F}$ is a morphism of τ-families. For every morphism $\phi : \mathcal{F} \to \mathcal{E}$ of τ-families, for every object U of τ, **prove** that $\theta_\mathcal{E}(U) \circ \phi(U)$ equals $^x\phi(U) \circ \theta_\mathcal{F}(U)$ for a unique morphism

$$^x\phi(U) : {^x\mathcal{F}(U)} \to {^x\mathcal{E}(U)}.$$

Prove that the rule $U \mapsto {^x\phi(U)}$ is a morphism of τ-families. **Prove** that $^x\mathrm{Id}_\mathcal{F}$ is the identity on $^x\mathcal{F}$. Also **prove** that $\phi \mapsto {^x\phi}$ is compatible with composition. Conclude that these rules define a functor

$$^x* : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{C}).$$

Prove that the rule $\mathcal{F} \mapsto \theta_{\mathcal{F}}$ is a natural transformation $\mathrm{Id}_{\mathbf{Fun}(\tau,\mathcal{C})} \Rightarrow {}^x *$. **Prove** that the natural morphism of τ-families,

$$ {}^x\mathcal{F} \to {}_x(({}^x\mathcal{F})_x), $$

is an isomorphism. Conclude that there exists a unique functor,

$$ *^x : \mathbf{Fun}(\tau,\mathcal{C}) \to \mathbf{Fun}(\sigma,\mathcal{C}), $$

and a natural isomorphism ${}^x* \Rightarrow {}_x(*^x)$. **Prove** that $(*^x, {}_x*, \theta)$ extends to an adjoint pair of functors. In particular, conclude that ${}_x*$ preserves epimorphisms and $*^x$ preserves monomorphisms.

Finally, drop the assumption that $\mathcal{C}$ has an initial object, but assume that σ is upper, assume that σ has an initial object, W_σ, and assume that there is a functor

$$ y : \tau \to \sigma $$

and a natural transformation $\theta : \mathrm{Id}_\tau \Rightarrow x \circ y$, such that for every object U of τ, the unique morphism $W_\sigma \to y(U)$ and the morphism $\theta_U : U \to y(U)$ make $y(U)$ into a coproduct of W_σ and U in τ. For simplicity, for every object U of σ, assume that $\theta_U : U \to y(U)$ is the identity Id_U (rather than merely being an isomorphism), and for every morphism $r : U \to V$ in σ, assume that $y(r)$ equals r. Thus, for every object V of σ, the identity morphism $y(V) \to V$ defines a natural transformation $\eta : y \circ x \Rightarrow \mathrm{Id}_\sigma$. **Prove** that (y, x, θ, η) is an adjoint pair of functors. Conclude that $(*_x, *_y, *_\theta, *_\eta)$ is an adjoint pair of functors. In particular, conclude that $*_x$ preserves monomorphisms and $*_y$ preserves epimorphisms.

(ix)(Compatibility of Limits and Colimits with Functors) Denote by 0 the "singleton category" 0 with a single object and a single morphism. **Prove** that $\Gamma(0,-)$ is an equivalence of categories. For an arbitrary category τ, for the unique natural transformation $\hat{\tau} : \tau \to 0$, **prove** that $*_{\hat{\tau}}$ equals the composite $\underline{*}_\tau \circ \Gamma(0,-)$ so that $\underline{*}_\tau$ is an example of this construction. In particular, for every functor $x : \sigma \to \tau$, **prove** that $(\underline{a}_\tau)_x$ equals $\underline{a}_\sigma$. If $\eta : \underline{a}_\tau \Rightarrow \mathcal{F}$ is a limit of a τ-family $\mathcal{F}$, and if $\theta : \underline{b}_\sigma \Rightarrow \mathcal{F}_x$ is a limit of the associated σ-family $\mathcal{F}_x$, then **prove** that there is a unique morphism $h : a \to b$ in $\mathcal{C}$ such that η_x equals $\theta \circ \underline{p}_\sigma$. If there are right adjoints Γ_τ of $\underline{*}_\tau$ and Γ_σ of $\underline{*}_\sigma$, conclude that there exists a unique natural transformation

$$ \Gamma_x : \Gamma_\tau \Rightarrow \Gamma_\sigma \circ *_x $$

so that $\eta_{\mathcal{F}_x} \circ \underline{\Gamma_x(\mathcal{F})}_\sigma$ equals $(\eta_{\mathcal{F}})_x$. **Repeat** this construction for colimits.

(x)(Limits / Colimits of a Concrete Category) Let σ be a small category in which the only morphisms are identity morphisms: identify σ with the underlying set of objects. Let $\mathcal{C}$ be the category **Sets**. For every σ-family $\mathcal{F}$, **prove** that the rule

$$ \Gamma_\sigma(\mathcal{F}) := \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) $$

together with the morphism
$$\eta_{\mathcal{F}} : \underline{\Gamma_{\sigma}(\mathcal{F})}_{\sigma} \Rightarrow \mathcal{F},$$
$$\eta_{\mathcal{F}}(V) = \mathrm{pr}_V : \prod_{U \in \Sigma} \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F}),$$

is a limit of $\mathcal{F}$. Next, for every small category τ, define σ to be the category with the same objects as τ, but with the only morphisms being identity morphisms. Define $x : \sigma \to \tau$ to be the unique functor that sends every object to itself. Define $\Gamma_{\tau}(\mathcal{F})$ to be the subobject of $\Gamma_{\sigma}(\mathcal{F}_x)$ of data $(f_U)_{U \in \Sigma}$ such that for every morphism $r : U \to V$, $\mathcal{F}(r)$ maps f_U to f_V. **Prove** that with this definition, there exists a unique natural transformation $\eta_{\mathcal{F}} : \underline{\Gamma_{\tau}(\mathcal{F})}_{\tau} \Rightarrow \mathcal{F}$ such that the natural transformation $\underline{\Gamma_{\tau}(\mathcal{F})}_{\sigma} \Rightarrow \underline{\Gamma_{\sigma}(\mathcal{F}_x)} \Rightarrow \mathcal{F}_x$ equals $(\eta_{\mathcal{F}})_x$. **Prove** that $\eta_{\mathcal{F}}$ is a limit of $\mathcal{F}$. Conclude that **Sets** has all small limits. Similarly, for associative, unital rings R and S, **prove** that the forgetful functor

$$\Phi : R - S - \mathrm{mod} \to \mathbf{Sets}$$

sends products to products. Let $\mathcal{F}$ be a τ-family of $R - S$-modules. **Prove** that the defining relations for $\Gamma_{\tau}(\Phi \circ \mathcal{F})$ as a subset of $\Gamma_{\sigma}(\Phi \circ \mathcal{F})$ are the simultaneous kernels of $R - S$-module homomorphisms. Conclude that there is a natural $R - S$-module structure on $\Gamma_{\tau}(\Phi \circ \mathcal{F})$, and use this to **prove** that $R - S$-mod has all limits.

(xi)(Functoriality in the Target) For every functor of categories,

$$H : \mathcal{C} \to \mathcal{D},$$

for every τ-family $\mathcal{F}$ in $\mathcal{C}$, **prove** that $H \circ \mathcal{F}$ is a τ-family in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \Rightarrow \mathcal{G}$, **prove** that $H \circ \phi$ is a morphism of τ-families in $\mathcal{D}$. **Prove** that this defines a functor

$$H_{\tau} : \mathbf{Fun}(\tau, \mathcal{C}) \to \mathbf{Fun}(\tau, \mathcal{D}).$$

For the identity functor $\mathrm{Id}_{\mathcal{C}}$, **prove** that $(\mathrm{Id}_{\mathcal{C}})_{\tau}$ is the identity functor. For $I : \mathcal{D} \to \mathcal{E}$ a functor of categories, **prove** that $(I \circ H)_{\tau}$ is the composite $I_{\tau} \circ H_{\tau}$. In this sense, deduce that H_{τ} is functorial in H.

For two functors, $H, I : \mathcal{C} \to \mathcal{D}$, and for a natural transformation $N : H \Rightarrow I$, for every τ-family $\mathcal{F}$ in $\mathcal{C}$, define $N_{\tau}(\mathcal{F})$ to be

$$N \circ \mathcal{F} : H \circ \mathcal{F} \Rightarrow I \circ \mathcal{F}.$$

Prove that $N_{\tau}(\mathcal{F})$ is a morphism of τ-families in $\mathcal{D}$. For every morphism of τ-families in $\mathcal{C}$, $\phi : \mathcal{F} \to \mathcal{G}$, **prove** that $N_{\tau}(\mathcal{G}) \circ H_{\tau}(\phi)$ equals $I_{\tau}(\phi) \circ N_{\tau}(\mathcal{F})$. In this sense, conclude that N_{τ} is a natural transformation $H_{\tau} \Rightarrow I_{\tau}$. For the identity natural transformation $\mathrm{Id}_H : H \Rightarrow H$, **prove** that $(\mathrm{Id}_H)_{\tau}$ is the identity natural transformation of H_{τ}. For a second natural transformation

$M : I \Rightarrow J$, **prove** that $(M \circ N)_\tau$ equals $M_\tau \circ N_\tau$. In this sense, deduce that $(-)_\tau$ is also compatible with natural transformations.

(xii) (Reductions of Limits to Finite Systems for Concrete Categories) A category is *cofiltering* if for every pair of objects U and V there exists a pair of morphisms, $r : W \to U$ and $s : W \to V$, and for every pair of morphisms, $r, s : V \to U$, there exists a morphism $t : W \to V$ such that $r \circ t$ equals $s \circ t$ (both of these are automatic if the category has an initial object X). Assume that the category $\mathcal{C}$ has limits for all categories τ with finitely many objects, and also for all small cofiltering categories. For an arbitrary small category τ, define $\widehat{\tau}$ to be the small category whose objects are finite full subcategories σ of τ, and whose morphisms are inclusions of subcategories, $\rho \subset \sigma$, of τ. **Prove** that $\widehat{\tau}$ is cofiltering. Let $\mathcal{F}$ be a τ-family in $\mathcal{C}$. For every finite full subcategory $\sigma \subset \tau$, denote by $\mathcal{F}_\sigma$ the restriction as in (f) above. By hypothesis, there is a limit $\eta_\sigma : \underline{\widehat{\mathcal{F}}(\sigma)}_\sigma \Rightarrow \mathcal{F}_\sigma$. Moreover, by (g), for every inclusion of full subcategories $\rho \subset \sigma$, there is a natural morphism in $\mathcal{C}$, $\widehat{\mathcal{F}}(\rho) \to \widehat{\mathcal{F}}(\sigma)$, and this is functorial. Conclude that $\widehat{\mathcal{F}}$ is a $\widehat{\tau}$-family in $\mathcal{C}$. Since $\widehat{\tau}$ is filtering, there is a limit

$$\eta_{\widehat{\mathcal{F}}} : \underline{a}_{\widehat{\tau}} \Rightarrow \widehat{\mathcal{F}}.$$

Prove that this defines a limit $\eta_\mathcal{F} \underline{a}_\tau \Rightarrow \mathcal{F}$.

Finally, use this to **prove** that limits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R - S$-bimodules (where R and S are associative, unital rings).

(xiii) (bis, Colimits) Repeat the steps above for colimits in place of limits. Use this to **prove** that colimits exist in each of the following categories: the category of (not necessarily Abelian) groups, the category of Abelian groups, the category of associative, unital (not necessarily commutative) rings, the category of commutative rings, and the category of $R - S$-bimodules (where R and S are associative, unital rings).

Practice with Limits and Colimits Exercise. In each of the following cases, say whether the given category (a) has an initial object, (b) has a final object, (c) has a zero object, (d) has finite products, (e) has finite coproducts, (f) has arbitrary products, (g) has arbitrary coproducts, (h) has arbitrary limits (sometimes called *inverse limits*), (i) has arbitrary colimits (sometimes called *direct limits*), (j) coproducts / filtering colimits preserve monomorphisms, (k) products / cofiltering limits preserve epimorphisms.

(i) The category **Sets** whose objects are sets, whose morphisms are set maps, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(ii) The opposite category **Sets**$^{\mathrm{opp}}$.

(iii) For a given set S, the category whose objects are elements of the set, and where the only morphisms are the identity morphisms from an element to that same element. What if the set is the empty set? What if the set is a singleton set?

(iv) For a partially ordered set $(S, \preceq)$, the category whose objects are elements of S, and where the Hom set between two elements x, y of S is a singleton set if $x \preceq y$ and empty otherwise. What if the partially ordered set $(S, \preceq)$ is a **lattice**, i.e., every finite subset (resp. arbitrary subset) has a least upper bound and has a greatest lower bound?

(v) For a monoid $(M, \cdot, 1)$, the category with only one object whose Hom set, with its natural composition and identity, is $(M, \cdot, 1)$. What is M equals $\{1\}$?

(vi) For a monoid $(M, \cdot, 1)$ and an action of that monoid on a set, $\rho : M \times S \to S$, the category whose objects are the elements of S, and where the Hom set from x to y is the subset $M_{x,y} = \{m \in M | m \cdot x = y\}$. What if the action is both transitive and faithful, i.e., S equals M with its left regular representation?

(vii) The category **PtdSets** whose objects are pairs (S, s_0) of a set S and a specified element s_0 of S, i.e., *pointed sets*, whose morphisms are set maps that send the specified point of the domain to the specified point of the target, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(viii) The category **Monoids** whose objects are monoids, whose morphisms are homomorphisms of monoids, whose composition is sual composition, and whose identity morphisms are usual identity maps.

(ix) For a specified monoid $(M, \cdot, 1)$, the category whose objects are pairs (S, ρ) of a set S and an action $\rho : M \times S \to S$ of M on S, whose morphisms are set maps compatible with the action, whose composition is usual composition, and whose identity morphisms are usual identity maps.

(x) The full subcategory **Groups** of **Monoids** whose objects are groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xi) The full subcategory $\mathbb{Z}-\text{mod}$ of **Groups** whose objects are Abelian groups. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xii) The full subcategory **FiniteGroups** of **Groups** whose objects are finite groups. Are coproducts, resp. products, in the subcategory also coproducts, resp. products, in the larger category **Groups**? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xiii) The full subcategory $\mathbb{Z} - \text{mod}_{\text{tor}}$ of $\mathbb{Z} - \text{mod}$ consisting of torsion Abelian groups, i.e., every element has finite order (allowed to vary from element to element). Are coproducts, resp. products, preserved by the inclusion functor? Are monomorphisms, resp. epimorphisms preserved?

(xiv) The category **Rings** whose objects are associative, unital rings, whose morphisms are homomorphisms of rings (preserving the multiplicative identity), whose composition is the usual composition, and whose identity morphisms are the usual identity maps. **Hint.** For the coproduct of two associative, unital rings $(R', +, 0, \cdot', 1')$ and $(R'', +, 0, \cdot'', 1'')$, first form the coproduct $R' \oplus R''$ of $(R', +, 0)$ and $(R'', +, 0)$ as a $\mathbb{Z}$-module, then form the total tensor product ring $T_{\mathbb{Z}}^{\bullet}(R' \oplus R'')$ as in the previous problem set. For the two natural maps $q' : R' \hookrightarrow T_{\mathbb{Z}}^1(R' \oplus R'')$ and $q'' : R'' \hookrightarrow T_{\mathbb{Z}}^1(R' \oplus R'')$ form the left-right ideal $I \subset T_{\mathbb{Z}}^{\bullet}(R' \oplus R'')$ generated by $q'(1') - 1$, $q''(1'') - 1$, $q'(r' \cdot' s') - q'(r') \cdot q'(s')$, and $q''(r'' \cdot'' s'') - q''(r'') \cdot q''(s'')$ for all elements $r', s' \in R'$ and $r'', s'' \in R''$. Define

$$p : T_{\mathbb{Z}}^1(R' \oplus R'') \to R,$$

to be the quotient by I. Prove that $p \circ q' : R' \to R$ and $p \circ q'' : R'' \to R$ are ring homomorphisms that make R into a coproduct of R' and R''.

(xv) The full subcategory **CommRings** of **Rings** whose objects are commutative, unital rings. Does the inclusion functor preserve coproducts, resp. products? Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvi) The full subcategory **NilCommRings** of **CommRings** whose objects are commutative, unital rings such that every noninvertible element is nilpotent. Does the inclusion functor preserve coproducts, resp. products? (Be careful about products!) Does the inclusion functor preserve monomorphisms, resp. epimorphisms?

(xvii) Let R and S be associative, unital rings. Let $R - \text{mod}$, resp. $\text{mod} - S$, $R - S - \text{mod}$, be the category of left R-modules, resp. right S-modules, $R - S$-bimodules. Does the inclusion functor from $R - S - \text{mod}$ to $R - \text{mod}$, resp. to $\text{mod} - S$, preserve coproduct, products, monomorphisms and epimorphisms?

(xviii) Let $(I, \leq)$ be a partially ordered set. Let $\mathcal{C}$ be a category. An $(I, \leq)$-*system* in $\mathcal{C}$ is a datum

$$c = \left((c_i)_{i \in I}, (f_{i,j})_{(i,j) \in I \times I, i \leq j} \right)$$

where every c_i is an object of $\mathcal{C}$, where for every pair $(i, j) \in I \times I$ with $i \leq j$, $c_{i,j}$ is an element of $\text{Hom}_{\mathcal{C}}(c_i, c_j)$, and satisfying the following conditions: (a) for every $i \in I$, $c_{i,i}$ equals Id_{c_i}, and (b) for every triple $(i, j, k) \in I$ with $i \leq j$ and $j \leq k$, $c_{j,k} \circ c_{i,j}$ equals $c_{i,k}$. For every pair of $(I, \leq)$-systems in $\mathcal{C}$, $c = ((c_i)_{i \in I}, (c_{i,j})_{i \leq j})$ and $c' = ((c_i')_{i \in I}, (c_{i,j}')_{i \leq j})$, a morphism $g : c \to c'$ is defined to be a datum $(g_i)_{i \in I}$ of morphisms $g_i \in \text{Hom}_{\mathcal{C}}(c_i, c_i')$ such that for every $(i, j) \in I \times I$ with $i \leq j$, $g_j \circ c_{i,j}$ equals $c_{i,j}' \circ g_i$. Composition of morphisms g and g' is componentwise $g_i' \circ g_i$, and identities are $\text{Id}_c = (\text{Id}_{c_i})_{i \in I}$. This category is $\text{Fun}((I, \leq), \mathcal{C})$, and is sometimes referred to as the category of $(I, \leq)$-*presheaves*. Assuming $\mathcal{C}$ has finite coproducts, resp. finite products, arbitrary coproducts, arbitrary products, a zero object, kernels, cokernels, etc., what can you say about $\text{Fun}((I, \leq), \mathcal{C})$?

(xix) Let $\mathcal{C}$ be a category that has arbitrary products. Let $(I, \leq)$ be a partially ordered set whose associated category as in (iv) has finite coproducts and has arbitrary products. The main example is when $I = \mathfrak{U}$ is the collection of all open subsets U of a topology on a set X, and where $U \leq V$ if $U \supseteq V$. Then coproduct is intersection and product is union. Motivated by this case, an *covering* of an element i of I is a collection $\underline{j} = (j_\alpha)_{\alpha \in A}$ of elements j_α of I such that for every α, $i \leq j_\alpha$, and such that i is the product of $(j_\alpha)_{\alpha \in A}$ in the sense of (iv). In this case, for every $(\alpha, \beta) \in A \times A$, define $j_{\alpha,\beta}$ to be the element of I such that $j_\alpha \leq j_{\alpha,\beta}$, such that $j_\beta \leq j_{\alpha,\beta}$, and such that $j_{\alpha,\beta}$ is a coprodcut of (j_α, j_β). An $(I, \leq)$-presheaf $c = ((c_i)_{i \in I}, (c_{i,j})_{i \leq j})$ is an $(I, \leq)$-*sheaf* if for every element i of I and for every covering $\underline{j} = (j_\alpha)_{\alpha \in A}$, the following diagram in $\mathcal{C}$ is *exact* in a sense to be made precise,

$$c_i \xrightarrow{q} \prod_{\alpha \in A} c_{j_\alpha} \overset{p'}{\underset{p''}{\rightrightarrows}} \prod_{(\alpha,\beta) \in A \times A} c_{j_{\alpha,\beta}}.$$

For every $\alpha \in A$, the factor of q,

$$\mathrm{pr}_\alpha \circ q : c_i \to c_{j_\alpha},$$

is defined to be c_{i,j_α}. For every $(\alpha, \beta) \in A \times A$, the factor of p',

$$\mathrm{pr}_{\alpha,\beta} \circ p' : \prod_{\gamma \in A} c_{j_\gamma} \to c_{j_{\alpha,\beta}},$$

is defined to be $c_{j_\alpha, j_{\alpha,\beta}} \circ \mathrm{pr}_\alpha$. Similarly, $\mathrm{pr}_{\alpha,\beta} \circ p''$ is defined to be $c_{j_\beta, j_{\alpha,\beta}} \circ \mathrm{pr}_\beta$. The diagram above is *exact* in the sense that q is a monomorphism in $\mathcal{C}$ and q is a fiber product in $\mathcal{C}$ of the pair of morphisms (p', p''). The category of $(I, \leq)$ is the full subcategoryof the category of $(I, \leq)$-presheaves whose objects are $(I, \leq)$-sheaves. Does this subcategory have coproducts, products, etc.? Does the inclusion functor preserve coproducts, resp. products, monomorphisms, epimorphisms? Before considering the general case, it is probably best to first consider the case that $\mathcal{C}$ is $\mathbb{Z} - \mathrm{mod}$, and then consider the case that $\mathcal{C}$ is **Sets**.