

# MAT 535 Abelian Categories

## 1 Introduction

These are notes on Abelian categories and additive functors supplementing the material from our textbook. Some of the notes are cut-and-pasted from previous courses I taught. Much of the notes are exercises working through the basic results about these notions.

## 2 Categories enriched over Abelian groups

### 2.1 Review of categories

Recall that the second-order definition of a (parametrically definable Zermelo-Fraenkel) **class** is an equivalence class  $[f(s, t), a]$  of ordered pairs consisting of a predicate  $f(s, t)$  with two free variables  $s$  and  $t$  in the first-order language of Zermelo-Fraenkel set theory and a set  $a$ , where  $[f'(s', t'), a']$  equals  $[f(s, t), a]$  if (and only if)

$$\forall b (f(a, b) \Leftrightarrow f'(a', b)).$$

In particular, the von Neumann class  $V$  is the equivalence class for any tautological pair, i.e.,  $f(s, t)$  is  $s = s$  and  $a$  is the empty set. For every class  $\mathbf{C} = [f(s, t), a]$  and for every set  $b$ , the set  $b$  is a **member** of  $\mathbf{C}$  if (and only if)  $f(a, b)$  holds. Thus, by definition,  $[f(s, t), a]$  equals  $[f'(s', t'), a']$  if and only if, for every set  $b$ , the set  $b$  is a member of  $[f(s, t), a]$  if and only if  $b$  is a member of  $[f'(s', t'), a']$ , giving extensionality for classes.

For the predicate  $f(s, t)$  of  $t \in s$ , for every set  $a$ , the class  $[f(s, t), a]$  is the unique class whose members are precisely the elements of  $a$ . Typically we identify the set  $a$  with the class  $[t \in s, a]$  and we identify membership in a class with the relation of set membership for these classes of sets, thus embedding Zermelo-Fraenkel set theory in the theory of (definable Zermelo-Fraenkel) classes.

For instance, for every class  $\mathbf{A}$ , for every class  $\mathbf{B}$ , the class  $\mathbf{A}$  is a **subclass** of the class  $\mathbf{B}$  if (and only if) every member of  $\mathbf{A}$  is a member of  $\mathbf{B}$ . Also, for every class  $\mathbf{A}$ , for every class  $\mathbf{B}$ , the

**product class**  $\mathbf{A} \times \mathbf{B}$  is the unique class whose members are precisely those ordered pairs  $(a, b)$  such that  $a$  is a member of  $\mathbf{A}$  and  $b$  is a member of  $\mathbf{B}$ . For every class  $\mathbf{A}$ , for every class  $\mathbf{B}$ , the **ordered pair class**  $(\mathbf{A}, \mathbf{B})$  is the class whose members are precisely ordered pairs  $(\emptyset, a)$  such that  $a$  is a member of  $\mathbf{A}$  together with the ordered pairs  $(\{\emptyset\}, b)$  such that  $b$  is a member of  $\mathbf{B}$ . Of course we can iterate this to define ordered triples, etc.

More generally, for every class  $\mathbf{C}$ , for every class  $\mathbf{D}$ , the class  $\mathbf{D}$  is a **C-class** if (and only if) every member of  $\mathbf{D}$  is an ordered pair  $(c, d)$  such that  $c$  is a member of  $\mathbf{C}$ . For every class  $\mathbf{C}$ , for every C-class  $\mathbf{D}$ , and for every member  $c$  of  $\mathbf{C}$ , the  $c$ -**fiber**  $\mathbf{D}_c$  of  $\mathbf{D}$  is the unique class whose members are precisely those sets  $d$  such that  $(c, d)$  is a member of  $\mathbf{D}$ . For instance, every C-class  $\mathbf{D}$  for the class  $\mathbf{C}$  of the set  $\{\emptyset, \{\emptyset\}\}$  equals the ordered pair class of its fibers,  $(\mathbf{D}_\emptyset, \mathbf{D}_{\{\emptyset\}})$ . For every class  $\mathbf{A}$ , for every class  $\mathbf{B}$ , the class  $\mathbf{B}$  is a **A-function** if (and only if), every member of  $\mathbf{B}$  is an ordered pair  $(a, b)$  for precisely one member  $a$  of  $\mathbf{A}$ , i.e.,  $\mathbf{B}$  is a **A-class** such that every fiber is the class of a singleton set. This is enough to formalize the most essential notions of category theory.

**Definition 2.1.** A **category** is an ordered triple  $\mathbf{C} = (\text{Obj}_{\mathbf{C}}, \text{Hom}_{\mathbf{C}}, \text{comp}_{\mathbf{C}})$  of a class  $\text{Obj}_{\mathbf{C}}$ , of an  $\text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}}$ -class  $\text{Hom}_{\mathbf{C}}$  whose fiber class over every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$  is the class of a set, denoted  $\text{Hom}_{\mathbf{C}}(a, b) = \mathbf{C}_b^a$ , and of a  $\text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}}$ -function  $\text{comp}_{\mathbf{C}}$  from  $(\text{Hom}_{\mathbf{C}} \circ \text{pr}_{1,2}) \times (\text{Hom}_{\mathbf{C}} \circ \text{pr}_{2,3})$  to  $\text{Hom}_{\mathbf{C}} \circ \text{pr}_{1,3}$  associating to every ordered triple  $(a, b, c)$  of members of  $\text{Obj}(\mathbf{C})$  a function of sets,

$$\text{comp}_{\mathbf{C}}^{a,b,c} : \mathbf{C}_c^b \times \mathbf{C}_b^a \rightarrow \mathbf{C}_c^a,$$

such that, for every member  $a$  of  $\text{Obj}_{\mathbf{C}}$  there exists an element  $\text{Id}_a^{\mathbf{C}}$  of  $\mathbf{C}_a^a$  with both  $f \circ \text{Id}_a^{\mathbf{C}}$  equal to  $f$  and with  $\text{Id}_a^{\mathbf{C}} \circ g$  equal to  $g$  for all members  $((a, b), f)$  and  $((b, a), g)$  of  $\text{Obj}_{\mathbf{C}}$  and, for every ordered quadruple  $(a, b, c, d)$  of members of  $\text{Obj}_{\mathbf{C}}$ , the following compositions of set function are equal,

$$\text{Hom}_{\mathbf{C}}(c, d) \times \text{Hom}_{\mathbf{C}}(b, c) \times \text{Hom}_{\mathbf{C}}(a, b) \xrightarrow{\text{Id} \times \circ_{a,b,c}} \text{Hom}_{\mathbf{C}}(c, d) \times \text{Hom}_{\mathbf{C}}(a, c) \xrightarrow{\circ_{a,c,d}} \text{Hom}_{\mathbf{C}}(a, d),$$

$$\text{Hom}_{\mathbf{C}}(c, d) \times \text{Hom}_{\mathbf{C}}(b, c) \times \text{Hom}_{\mathbf{C}}(a, b) \xrightarrow{\circ_{b,c,d} \times \text{Id}} \text{Hom}_{\mathbf{C}}(b, d) \times \text{Hom}_{\mathbf{C}}(a, b) \xrightarrow{\circ_{a,b,d}} \text{Hom}_{\mathbf{C}}(a, d).$$

**Definition 2.2.** For every category  $\mathbf{C} = (\text{Obj}_{\mathbf{C}}, \text{Hom}_{\mathbf{C}}, \text{comp}_{\mathbf{C}})$ , the members of  $\text{Obj}_{\mathbf{C}}$  are **objects** of  $\mathbf{C}$ , each set  $\text{Hom}_{\mathbf{C}}(a, b)$  is a **Hom set** of  $\mathbf{C}$  whose elements are **morphisms** of  $\mathbf{C}$ , and each set function  $\text{comp}_{\mathbf{C}}^{a,b,c}$  is **composition** of morphisms of  $\mathbf{C}$ .

**Definition 2.3.** For every category  $\mathbf{C}$ , for every C-morphism  $f$  in  $\mathbf{C}_b^a$ , the morphism is **monic**, respectively **epic**, **invertible** if (and only if) for every C-object  $t$ , the postcomposition function  $f \circ -$  from  $\mathbf{C}_t^a$  to  $\mathbf{C}_t^b$  is injective, resp. the precomposition function  $- \circ f$  from  $\mathbf{C}_t^b$  to  $\mathbf{C}_t^a$  is injective, both of the postcomposition function and the precomposition function are bijective.

**Definition 2.4.** For every category  $\mathbf{C}$ , the **source class function** is the  $\text{Hom}_{\mathbf{C}}$ -function to  $\text{Obj}_{\mathbf{C}}$  associating the member  $a$  of  $\text{Obj}_{\mathbf{C}}$  to every member  $((a, b), f)$  of  $\text{Hom}_{\mathbf{C}}$ . The **target class function** is the  $\text{Hom}_{\mathbf{C}}$ -function to  $\text{Obj}_{\mathbf{C}}$  associating the member  $b$  of  $\text{Obj}_{\mathbf{C}}$  to every ordered member  $((a, b), f)$  of  $\text{Hom}_{\mathbf{C}}$ .

**Definition 2.5.** For every category  $\mathbf{C}$ , the **opposite category** is the category  $\mathbf{C}^{\text{opp}}$  whose objects are the objects of  $\mathbf{C}$ , such that for every ordered pair  $(a, b)$  of objects of  $\mathbf{C}$  the Hom set  $\text{Hom}_{\mathbf{C}^{\text{opp}}}(b, a)$  equals the Hom set  $\text{Hom}_{\mathbf{C}}(a, b)$ , and such that the composition in  $\mathbf{C}^{\text{opp}}$  of  $f$  with  $g$  equals the composition  $\text{comp}_{\mathbf{C}}^{a,b,c}(g, f)$ .

**Definition 2.6.** For every category  $\mathbf{C}$ , the category is **strictly small** if (and only if)  $\text{Obj}_{\mathbf{C}}$  equals the class of a set. A category  $\mathbf{C}$  is **small** if (and only if)  $\text{Obj}_{\mathbf{C}}$  has a subclass that equals the class of a set such that every object is isomorphic to a member of the subclass. A category  $\mathbf{C}$  is **skeletal** if (and only if), for every ordered pair  $(a, b)$  of objects of  $\mathbf{C}$ , the objects are isomorphic if and only if they are equal. A category  $\mathbf{C}$  is a **groupoid** if (and only if) every morphism is an isomorphism. A groupoid  $\mathbf{C}$  is a (large) **setoid** if (and only if), for every object  $a$  of  $\mathbf{C}$ , the Hom set  $\mathbf{C}_a^a$  equals the singleton set of  $\text{Id}_a^{\mathbf{C}}$ . A (large) setoid is a  $\mathbf{C}$  is **discrete category** if (and only if) it is skeletal.

**Definition 2.7.** For every category  $\mathbf{C} = (\text{Obj}_{\mathbf{C}}, \text{Hom}_{\mathbf{C}}, \text{comp}_{\mathbf{C}})$  and for every category  $\mathbf{D} = (\text{Obj}_{\mathbf{D}}, \text{Hom}_{\mathbf{D}}, \text{comp}_{\mathbf{D}})$ , an ordered pair  $F = (F_{\text{Obj}}, F_{\text{Hom}})$  of a  $\text{Obj}_{\mathbf{C}}$ -function  $F_{\text{Obj}}$  to  $\text{Obj}_{\mathbf{D}}$  and of a  $\text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}}$ -function  $F_{\text{Hom}}$  that associates to every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$  a set function,

$$F_b^a : \mathbf{C}_b^a \rightarrow \mathbf{D}_{F(b)}^{F(a)},$$

is a (covariant) **functor** if (and only if) both,  $F_a^a(\text{Id}_a^{\mathbf{C}})$  equals  $\text{Id}_{F(a)}^{\mathbf{D}}$  for every object  $a$  of  $\mathbf{C}$ , and  $F_c^a(\text{comp}_{\mathbf{C}}(g, f))$  equals  $\text{comp}_{\mathbf{D}}(F_c^b(g), F_b^a(f))$  for every ordered triple  $(a, b, c)$  of objects of  $\mathbf{C}$  and for every element  $(g, f)$  of  $\mathbf{C}_c^b \times \mathbf{C}_b^a$ .

**Definition 2.8.** For every category  $\mathbf{C}$ , the **identity functor** from  $\mathbf{C}$  to  $\mathbf{C}$  is the functor with  $F(a)$  equal to  $a$  for every object  $a$  of  $\mathbf{C}$  and with  $F_b^a$  equal to  $\text{Id}_{\mathbf{C}_b^a}$  for every ordered pair  $(a, b)$  of objects of  $\mathbf{C}$ .

For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , for every category  $\mathbf{E}$ , for every functor  $F$  from  $\mathbf{C}$  to  $\mathbf{D}$ , for every functor  $G$  from  $\mathbf{D}$  to  $\mathbf{E}$ , the **composite functor**  $G \circ F$  from  $\mathbf{C}$  to  $\mathbf{E}$  sends every member  $a$  of  $\text{Obj}_{\mathbf{C}}$  to  $G(F(a))$ , and sends every element  $f$  of  $\mathbf{C}_b^a$  to  $G_{F(b)}^{F(a)}(F_b^a(f))$ .

**Definition 2.9.** For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , a **contravariant functor** from  $\mathbf{C}$  to  $\mathbf{D}$  is a covariant functor from  $\mathbf{C}^{\text{opp}}$  to  $\mathbf{D}$ .

**Definition 2.10.** For every functor  $F$  from a category  $\mathbf{C}$  to a category  $\mathbf{D}$ , the functor is **faithful**, respectively **full**, **fully faithful**, if (and only if) the function  $F_b^a$  from  $\mathbf{C}_b^a$  to  $\mathbf{D}_{F(b)}^{F(a)}$  is injective, resp.

surjective, bijective, for every ordered pair  $(a, b)$  of objects of  $\mathbf{C}$ . The functor  $F$  is **essentially surjective** if (and only if) every object of  $\mathbf{D}$  is  $\mathbf{D}$ -isomorphic to an object in the image of  $F$ . A faithful functor  $F$  is **conservative** if (and only if) the only  $\mathbf{C}$ -morphisms that are mapped to  $\mathbf{D}$ -isomorphisms are  $\mathbf{C}$ -isomorphisms. A **weak equivalence** is an essentially surjective, fully faithful functor. A weak equivalence from a skeletal category is a **skeleton**.

**Definition 2.11.** For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , for every covariant functor  $F$  from  $\mathbf{C}$  to  $\mathbf{D}$ , for every covariant functor  $G$  from  $\mathbf{C}$  to  $\mathbf{D}$ , a **natural transformation** from  $F$  to  $G$  is an  $\text{Obj}_{\mathbf{C}}$ -function  $\theta$  associating to every object  $a$  of  $\mathbf{C}$  an element  $\theta_a$  of  $\mathbf{D}_{G(a)}^{F(a)}$  such that, for every member  $((a, b), f)$  of  $\text{Hom}_{\mathbf{C}}$ , the  $\mathbf{D}$ -composite  $\theta_b \circ F_b^a(f)$  equals the  $\mathbf{D}$ -composite  $G_b^a(f) \circ \theta_a$ . A natural transformation is a **natural equivalence** (or **natural isomorphism**) if (and only if) the morphism associated to each object is an isomorphism.

**Definition 2.12.** For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , and for every covariant functor  $F$  from  $\mathbf{C}$  to  $\mathbf{D}$ , the **identity natural transformation** from  $F$  to itself is the natural transformation that associates to every object  $a$  of  $\mathbf{C}$  the identity morphism  $\text{Id}_{F(a)}^{\mathbf{D}}$ ; this is denoted by  $\text{Id}_F^{\mathbf{C}, \mathbf{D}}$ .

For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , for every ordered triple  $(F, G, H)$  of covariant functors from  $\mathbf{C}$  to  $\mathbf{D}$ , for every natural transformation  $\theta$  from  $F$  to  $G$ , for every natural transformation  $\eta$  from  $G$  to  $H$ , the (vertical) **composite natural transformation**  $\eta \circ \theta$  from  $F$  to  $H$  is the natural transformation such that for every object  $a$  of  $\mathbf{C}$ , the associated  $\mathbf{D}$ -morphism from  $F(a)$  to  $H(a)$  equals the  $\mathbf{D}$ -composition of  $\eta_a$  with  $\theta_a$ .

For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , for every category  $\mathbf{E}$ , for every ordered pair  $(F, G)$  of covariant functors from  $\mathbf{C}$  to  $\mathbf{D}$ , for every ordered pair  $(H, I)$  of covariant functors from  $\mathbf{D}$  to  $\mathbf{E}$ , for every natural transformation  $\theta$  from  $F$  to  $G$ , for every natural transformation  $\eta$  from  $H$  to  $I$ , the **horizontal composition natural transformation** of  $\eta$  and  $\theta$ , or **Godement product**, is the natural transformation  $\eta * \theta : H \circ F \rightarrow I \circ G$  associating to every object  $a$  of  $\mathbf{C}$  the  $\mathbf{E}$ -morphism,

$$\eta_{G(a)} \circ_{\mathbf{C}} H_{F(a), G(a)}(\theta_a) = (\eta * \theta)_a = I_{F(a), G(a)}(\theta_a) \circ \eta_{F(a)}.$$

This is associative in both  $\theta$  and  $\eta$  separately.

For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , for every category  $\mathbf{E}$ , for every ordered pair  $(F, G)$  of covariant functors from  $\mathbf{C}$  to  $\mathbf{D}$ , for every covariant functor  $H$  from  $\mathbf{D}$  to  $\mathbf{E}$ , for every natural transformation  $\theta$  from  $F$  to  $G$ , the  **$H$ -pushforward natural transformation** is  $H_*\theta = \text{Id}_H^{\mathbf{D}, \mathbf{E}} * \theta$ , associating to every object  $a$  of  $\mathbf{A}$  the  $\mathbf{E}$ -morphism  $H_{F(a), G(a)}(\theta_a)$ . Similarly, for every category  $\mathbf{B}$ , for every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , for every covariant functor  $E$  from  $\mathbf{B}$  to  $\mathbf{C}$ , for every ordered pair  $(F, G)$  of covariant functors from  $\mathbf{C}$  to  $\mathbf{D}$ , for every natural transformation  $\theta$  from  $F$  to  $G$ , the  **$E$ -pullback natural transformation**,  $E^*\theta = \theta * \text{Id}_E^{\mathbf{B}, \mathbf{E}}$  associates to every upd

object  $b$  of  $\mathbf{B}$  the  $\mathbf{D}$ -morphism  $\theta_{E(b)}$ . Of course the Godement product can be expanded in terms of pushforward, pullback and vertical composition,

$$G^*\eta \circ H_*\theta = \eta * \theta = I_*\theta \circ F^*\eta.$$

**Definition 2.13.** For every category  $\mathbf{C}$ , for every category  $\mathbf{D}$ , an **adjoint pair** of covariant functors between  $\mathbf{C}$  and  $\mathbf{D}$  is an ordered pair  $((L, R), (\theta, \eta))$  of an ordered pair pair of covariant functors,

$$L : \mathbf{C} \rightarrow \mathbf{D},$$

$$R : \mathbf{D} \rightarrow \mathbf{C},$$

and a pair of natural transformations of covariant functors,

$$\theta : \text{Id}_{\mathbf{C}} \Rightarrow R \circ L, \quad \theta(a) : a \rightarrow R(L(a)),$$

$$\eta : L \circ R \Rightarrow \text{Id}_{\mathbf{D}}, \quad \eta(b) : L(R(b)) \rightarrow b,$$

such that the following composition of natural transformations equals  $\text{Id}_R$ , respectively equals  $\text{Id}_L$ ,

$$(*_R) : R \xRightarrow{\theta \circ R} R \circ L \circ R \xRightarrow{R \circ \eta} R,$$

$$(*_L) : L \xRightarrow{L \circ \theta} L \circ R \circ L \xRightarrow{\eta \circ L} L.$$

For every object  $a$  of  $\mathbf{C}$  and for every object  $b$  of  $\mathbf{D}$ , define set maps,

$$H_R^L(a, b) : \text{Hom}_{\mathbf{D}}(L(a), b) \rightarrow \text{Hom}_{\mathbf{C}}(a, R(b)),$$

$$(L(a) \xrightarrow{\phi} b) \mapsto \left( a \xrightarrow{\theta(a)} R(L(a)) \xrightarrow{R(\phi)} R(b) \right),$$

and

$$H_L^R(a, b) : \text{Hom}_{\mathbf{C}}(a, R(b)) \rightarrow \text{Hom}_{\mathbf{D}}(L(a), b),$$

$$(a \xrightarrow{\psi} R(b)) \mapsto \left( L(a) \xrightarrow{L(\psi)} L(R(b)) \xrightarrow{\eta(b)} b \right).$$

## 2.2 Some examples

**Definition 2.14.** The category **Set** has  $\text{Obj}_{\mathbf{Set}}$  equal to the von Neumann class  $\mathbf{V}$  of all sets, and has  $\text{Hom}_{\mathbf{Set}}(a, b)$  equal to the set of all functions from  $a$  to  $b$  for every ordered pair  $(a, b)$  of sets. Composition in **Set** is composition of functions.

**Definition 2.15.** A **concrete category** is a category **C** together with a faithful functor **F** from **C** to **Set**.

**Definition 2.16.** The objects of the concrete category **Monoid** are ordered pairs  $(H, \cdot)$  of a set  $H$  and of a function  $\cdot$  from  $H \times H$  to  $H$  that is associative and unital. For monoids  $(H, \cdot)$  and  $(H', \cdot')$ , a **monoid morphism** is a function  $f$  from  $H$  to  $H'$  that maps the identity to the identity and that respects the binary operations. The functor from **Monoid** to **Set** maps each monoid  $(H, \cdot)$  to  $H$  and is the inclusion on Hom sets. This is conservative.

**Definition 2.17.** The concrete category **Group** is the full subcategory of **Monoid** whose objects are monoids  $(H, \cdot)$  such that every element of  $H$  has a left-right inverse, i.e.,  $(H, \cdot)$  is a group.

**Definition 2.18.** The concrete category **Ab** is the full subcategory of **Group** whose objects are all groups such that the binary operation is commutative, i.e., such that the group is an Abelian group.

**Definition 2.19.** The objects of the concrete category **Ring** are ordered triples  $(R, +, \cdot)$  of a set  $R$ , of a binary operation  $+$  on  $R$  that makes  $R$  into an Abelian group with identity element  $0$ , and an associative binary operation  $\cdot$  on  $R$  that makes  $R$  into a monoid with identity element  $1$  which left-right distributes with respect to  $+$ , i.e.,  $R$  is an associative, unital ring. For rings  $(R, +, \cdot)$  and  $(R', +', \cdot')$ , a **ring homomorphism** is a function  $f$  from  $R$  to  $R'$  that is both a morphism of Abelian groups with respect to  $+$  and  $+'$  and a morphism of monoids with respect to  $\cdot$  and  $\cdot'$ . The functor from **Ring** to **Set** maps each associative, unital ring  $(R, +, \cdot)$  to the set  $R$  and is the inclusion on Hom sets. This is conservative.

**Definition 2.20.** The concrete category **CommRing** is the full subcategory of **Ring** whose objects are associative, unital rings that are commutative, i.e., the objects are commutative rings.

**Definition 2.21.** The concrete category **Field** is the full subcategory of **CommRing** whose objects are fields.

**Definition 2.22.** For every associative, unital ring  $R$ , the objects of the concrete category  $R\text{-Mod}$  are ordered triples  $(M, +, \cdot)$  of an Abelian group  $(M, +)$  with identity element  $0$  and a function  $\cdot$  from  $R \times M$  to  $M$  that makes  $M$  into a left  $R$ -module, i.e., the induced function  $R \rightarrow \text{Hom}_{\mathbb{Z}}(M, M)$  of left multiplication is a morphism of associative, unital rings. For left  $R$ -modules  $(M, +, \cdot)$  and

$(M', +', \cdot')$ , a **morphism** of left  $R$ -modules from  $(M, +, \cdot)$  to  $(M', +', \cdot')$  is a function  $f$  from  $M$  to  $M'$  that is a morphism of Abelian groups from  $(M, +)$  to  $(M', +')$  and that respects  $\cdot$  and  $\cdot'$ . The functor from  $R\text{-Mod}$  to **Set** maps each left  $R$ -module  $(M, +, \cdot)$  to the set  $M$  and is the inclusion on Hom sets. This is conservative.

**Definition 2.23.** For every associative, unital ring  $(R, +_R, \cdot_R)$ , the objects of the concrete category  $\mathbf{Mod} - R$  are ordered triples  $(M, +, \bullet)$  of an Abelian group  $(M, +)$  with identity element 0 and a function  $\bullet$  from  $M \times R$  to  $M$  that makes  $M$  into a right  $R$ -module, i.e., the induced function  $R \rightarrow \text{Hom}_{\mathbb{Z}}(M, M)$  of right multiplication is a morphism of associative, unital rings. For right  $R$ -modules  $(M, +, \bullet)$  and  $(M', +', \bullet')$ , a **morphism** of right  $R$ -modules from  $(M, +, \bullet)$  to  $(M', +', \bullet')$  is a function  $f$  from  $M$  to  $M'$  that is a morphism of Abelian groups from  $(M, +)$  to  $(M', +')$  and that respects  $\bullet$  and  $\bullet'$ . The functor from  $\mathbf{Mod} - R$  to **Set** maps each right  $R$ -module  $(M, +, \bullet)$  to the set  $M$  and is the inclusion on Hom sets. This is conservative.

**Definition 2.24.** For every ordered pair  $((R, +_R, \cdot_R), (S, +_S, \cdot_S))$  of associative, unital rings, the objects of the concrete category  $R - S - \mathbf{Mod}$  are ordered 4-tuples  $(M, +, \cdot, \bullet)$  of an Abelian group  $(M, +)$  with identity element 0, of a left  $R$ -module structure  $\cdot$  and of a right  $S$ -module structure  $\bullet$  that commute with each other, i.e.,  $(r \cdot m) \bullet s$  equals  $r \cdot (m \bullet s)$  for every element  $(r, m, s)$  of  $R \times M \times S$ . For  $R - S$ -bimodules  $(M, +, \cdot, \bullet)$  and  $(M', +', \cdot', \bullet')$ , a **morphism** of  $R - S$ -bimodules from  $(M, +, \cdot, \bullet)$  to  $(M', +', \cdot', \bullet')$  is a function  $f$  from  $M$  to  $M'$  that is simultaneously a morphism of left  $R$ -modules from  $(M, +, \cdot)$  to  $(M', +', \cdot')$  and a morphism of right  $S$ -modules from  $(M, +, \bullet)$  to  $(M', +', \bullet')$ . The functor from  $R - S - \mathbf{Mod}$  to **Set** maps each  $R - S$ -bimodule  $(M, +, \cdot, \bullet)$  to  $M$  and is the inclusion on Hom sets. This is conservative.

## 2.3 Preadditive categories

**Definition 2.25.** A **category enriched over Abelian groups** (or **preadditive category**) is an ordered pair class  $(\mathbf{C}, \text{add})$  of a category  $\mathbf{C} = (\text{Obj}_{\mathbf{C}}, \text{Hom}_{\mathbf{C}}, \text{comp}_{\mathbf{C}})$  and an  $\text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}}$ -function  $\text{add}$  that associates to every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$  an Abelian group law on  $\text{Hom}_{\mathbf{C}}(a, b)$ ,

$$\text{add}^{a,b} : \text{Hom}_{\mathbf{C}}(a, b) \times \text{Hom}_{\mathbf{C}}(a, b) \rightarrow \text{Hom}_{\mathbf{C}}(a, b),$$

such that for every ordered triple  $(a, b, c)$  of members of  $\text{Obj}_{\mathbf{C}}$ , the composition,

$$\text{comp}_{\mathbf{C}}^{a,b,c} : \text{Hom}_{\mathbf{C}}(b, c) \times \text{Hom}_{\mathbf{C}}(a, b) \rightarrow \text{Hom}_{\mathbf{C}}(a, c),$$

is biadditive for these Abelian group laws.

For categories enriched over Abelian groups,  $(\mathbf{C}, \text{add})$  and  $(\widetilde{\mathbf{C}}, \widetilde{\text{add}})$ , an **additive functor** from  $(\mathbf{C}, \text{add})$  to  $(\widetilde{\mathbf{C}}, \widetilde{\text{add}})$  is a functor  $F$  from  $\mathbf{C}$  to  $\widetilde{\mathbf{C}}$  such that, for every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$  the following function is additive for the Abelian group laws,

$$F_b^a : \text{Hom}_{\mathbf{C}}(a, b) \rightarrow \text{Hom}_{\widetilde{\mathbf{C}}}(F(a), F(b)).$$

**Example 2.26.** For every category  $\mathbf{C}$ , define  $\mathbb{Z} \cdot \mathbf{C}$  to be the category enriched over Abelian groups whose objects agree with the objects of  $\mathbf{C}$ , whose Hom set  $\text{Hom}_{\mathbb{Z} \cdot \mathbf{C}}(a, b)$  equals the free Abelian group on the set  $\text{Hom}_{\mathbf{C}}(a, b)$  for every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$ , and where  $\text{comp}_{\mathbb{Z} \cdot \mathbf{C}}^{a,b,c}$  is the unique biadditive map induced by  $\text{comp}_{\mathbf{C}}^{a,b,c}$  for every ordered triple  $(a, b, c)$  of members of  $\text{Obj}_{\mathbf{C}}$ .

**Proposition 2.27.** For every category enriched over Abelian groups,  $(\mathbf{C}, \text{add})$ , for every member  $a$  of  $\text{Obj}_{\mathbf{C}}$ , the ordered triple  $(\text{Hom}_{\mathbf{C}}(a, a), \text{add}^{a,a}, \text{comp}_{\mathbf{C}}^{a,a,a})$  is an associative, unital ring  $\text{End}_{\mathbf{C}, \text{add}}(a)$ . For every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$ , the Abelian group structure  $\text{add}^{a,b}$  and the left and right composition make  $\text{Hom}_{\mathbf{C}}(a, b)$  into a  $\text{End}_{\mathbf{C}, \text{add}}(b) \times \text{End}_{\mathbf{C}, \text{add}}(a)$ -bimodule, and this is the regular  $\text{End}_{\mathbf{C}, \text{add}}(a) - \text{End}_{\mathbf{C}, \text{add}}(a)$ -bimodule when  $b$  equals  $a$ . For every ordered triple  $(a, b, c)$  of members of  $\text{Obj}_{\mathbf{C}}$ , the biadditive pairing

$$\text{comp}_{\mathbf{C}}^{a,b,c} : \text{Hom}_{\mathbf{C}}(b, c) \times \text{Hom}_{\mathbf{C}}(a, b) \rightarrow \text{Hom}_{\mathbf{C}}(a, c),$$

is  $\text{End}_{\mathbf{C}, \text{add}}(b)$ -balanced, left  $\text{End}_{\mathbf{C}, \text{add}}(c)$ -linear and right  $\text{End}_{\mathbf{C}, \text{add}}(a)$ -linear, i.e., it is equivalent to a morphism of left  $\text{End}_{\mathbf{C}, \text{add}}(c) - \text{End}_{\mathbf{C}, \text{add}}(a)$ -bimodules,

$$\text{comp}_{\mathbf{C}}^{a,b,c} : \text{Hom}_{\mathbf{C}}(b, c) \otimes_{\text{End}_{\mathbf{C}, \text{add}}(b)} \text{Hom}_{\mathbf{C}}(a, b) \rightarrow \text{Hom}_{\mathbf{C}}(a, c).$$

Finally, for every ordered quadruple  $(a, b, c, d)$  of members of  $\text{Obj}_{\mathbf{C}}$ , via the canonical associativity isomorphism (“associator”),

$$\begin{aligned} & \text{Hom}_{\mathbf{C}}(c, d) \otimes_{\text{End}_{\mathbf{C}, \text{add}}(c)} \left( \text{Hom}_{\mathbf{C}}(b, c) \otimes_{\text{End}_{\mathbf{C}, \text{add}}(b)} \text{Hom}_{\mathbf{C}}(a, b) \right) \xrightarrow{\cong} \\ & \left( \text{Hom}_{\mathbf{C}}(c, d) \otimes_{\text{End}_{\mathbf{C}, \text{add}}(c)} \text{Hom}_{\mathbf{C}}(b, c) \right) \otimes_{\text{End}_{\mathbf{C}, \text{add}}(b)} \text{Hom}_{\mathbf{C}}(a, b), \end{aligned}$$

the two composite morphisms of  $\text{End}_{\mathbf{C}, \text{add}}(d) - \text{End}_{\mathbf{C}, \text{add}}(a)$ -bimodules to  $\text{Hom}_{\mathbf{C}}(a, d)$  are equal. Conversely, for every class  $\text{Obj}_{\mathbf{C}}$ , for every  $\text{Obj}_{\mathbf{C}}$ -function that associates an associative, unital ring  $E(a)$  to every member  $a$  of  $\text{Obj}_{\mathbf{C}}$ , for every  $\text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}}$ -function  $H(a, b)$  that associates a  $E(b) - E(a)$ -bimodule to each ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$  which equals the regular  $E(a) - E(a)$ -bimodule when  $b$  equals  $a$ , and to every  $\text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}} \times \text{Obj}_{\mathbf{C}}$ -function that associates a morphism of  $E(c) - E(a)$ -bimodules,

$$\text{comp}^{a,b,c} : H(b, c) \otimes_{E(b)} H(a, b) \rightarrow H(a, c),$$

to every ordered triple  $(a, b, c)$  of members of  $\text{Obj}_{\mathbf{C}}$ , if for every ordered quadruple  $(a, b, c, d)$  of members of  $\text{Obj}_{\mathbf{C}}$ , via the associator the two composite morphisms of  $E(d) - E(a)$ -bimodules to  $H(a, d)$  are equal, then there exists a unique structure of category enriched over Abelian groups such that, for every ordered pair  $(a, b)$  of members of  $\text{Obj}_{\mathbf{C}}$ , the Hom set equals  $H(a, b)$  and, for every ordered triple  $(a, b, c)$  of members of  $\text{Obj}_{\mathbf{C}}$ , the composition is induced by  $\text{comp}^{a,b,c}$ .

**Definition 2.28.** For every category  $\mathbf{C}$ , for every member  $z$  of  $\text{Obj}_{\mathbf{C}}$ , the member  $z$  is a **null object** if (and only if) it is both an initial object and a terminal object, i.e., for every member  $a$  of  $\text{Obj}_{\mathbf{C}}$ , both  $\text{Hom}_{\mathbf{C}}(z, a)$  and  $\text{Hom}_{\mathbf{C}}(a, z)$  are singleton sets.

**Lemma 2.29.** For every category enriched over Abelian groups,  $(\mathbf{C}, \text{add})$ , for every member  $z$  of  $\text{Obj}_{\mathbf{C}}$ , the member  $z$  is a null object if and only if  $\text{Hom}_{\mathbf{C}}(z, z)$  is a singleton set.

### 3 Additive categories

**Definition 3.1.** For every category enriched over Abelian groups  $(\mathbf{C}, \text{add})$ , this is an **additive category** if (and only if) both there exists a null object and, for every ordered pair  $(a_1, a_2)$  of members of  $\text{Obj}_{\mathbf{C}}$  there exists a product in the category  $\mathbf{C}$ ,

$$(a_1 \oplus a_2, p_1^{a_1, a_2} : a_1 \oplus a_2 \rightarrow a_1, p_2^{a_1, a_2} : a_1 \oplus a_2 \rightarrow a_2).$$

The **factor morphisms**  $q_1^{a_1, a_2}$  from  $a_1$  to  $a_1 \oplus a_2$  and  $q_2^{a_1, a_2}$  from  $a_2$  to  $a_1 \oplus a_2$  are the unique morphisms in the category such that  $p_i^{a_1, a_2} \circ q_i^{a_1, a_2}$  equals the identity on  $a_i$  and  $p_i^{a_1, a_2} \circ q_j^{a_1, a_2}$  is the zero morphism of  $i$  is different from  $j$ .

**Proposition 3.2.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every ordered pair  $(a_1, a_2)$  of objects, the ordered triple  $(a_1 \oplus a_2, q_1^{a_1, a_2} : a_1 \rightarrow a_1 \oplus a_2, q_2^{a_1, a_2} : a_2 \rightarrow a_1 \oplus a_2)$  is a coproduct in the category. In particular, for every object  $a$ , the morphism  $p_1^{a, a} + p_2^{a, a}$  from  $a \oplus a$  to  $a$  is the codiagonal.

**Exercise 3.3.** Prove this result.

**Proposition 3.4.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every ordered pair  $(a, b)$  of objects of  $\mathbf{C}$ , for every element  $(f_1, f_2)$  of  $\mathbf{C}_b^a \times \mathbf{C}_b^a$ , the sum  $\text{add}_{\mathbf{C}}^{a, b}(f_1, f_2)$  equals the composite of the diagonal morphism  $(q_1^{a, a}, q_2^{a, a}) : a \rightarrow a \oplus a$ , the product morphism  $(f_1 \circ p_1^{a, a}, f_2 \circ p_2^{a, a})$  from  $a \oplus a$  to  $b \oplus b$ , and the codiagonal morphism  $b \oplus b \rightarrow b$ .

**Exercise 3.5.** Prove this result.

**Example 3.6.** There is a unique additive category (up to weak equivalence) that has only one isomorphism class of objects. These objects must all be null objects, and the Hom sets must each be a zero Abelian group. This additive category is the **zero category**.

**Example 3.7.** For every associative, unital ring  $R$ , there is an additive category  $R\text{-FinFreeMod}$  whose objects are denoted  $R^{\oplus \ell}$  indexed by nonnegative integers  $\ell$ , such that the Hom set  $\text{Hom}(R^{\oplus \ell}, R^{\oplus \ell'})$  is the set of  $\ell' \times \ell$  matrices with entries in  $R$ , and where the composition from  $\text{Hom}(R^{\oplus \ell'}, R^{\oplus \ell''}) \times \text{Hom}(R^{\oplus \ell}, R^{\oplus \ell'})$  to  $\text{Hom}(R^{\oplus \ell}, R^{\oplus \ell''})$  is multiplication of matrices with entries in  $R$ .

**Example 3.8.** The previous additive category is a full small subcategory of the full subcategory  $R\text{-FreeMod}$  of  $R\text{-Mod}$  whose objects are all free left  $R$ -modules on sets. This additive category is not small.

**Example 3.9.** For every sextuple,

$$((R, +_R, \cdot_R), (S, +_S, \cdot_S), (M, +_M, \cdot_{R,M}, \bullet_{M,S}), (N, +_N, \bullet_{S,N}, \cdot_{N,R}), \phi, \psi),$$

of associative, unital rings  $(R, +_R, \cdot_R)$  and  $(S, +_S, \cdot_S)$ , of an  $R - S$ -bimodule  $(M, +_M, \cdot_{R,M}, \bullet_{M,S})$ , of an  $S - R$ -bimodule  $(N, +_N, \bullet_{S,N}, \cdot_{N,R})$ , of a morphism  $\phi$  of  $R - R$ -bimodules from  $M \otimes_S N$  to the regular  $R - R$ -bimodule  $R$ , and of a morphism  $\psi$  of  $S - S$ -bimodules from  $N \otimes_R M$  to the regular  $S - S$ -bimodule  $S$ , this datum is **associative** if (and only if) for every element  $(m, m')$  of  $M \times M$  and for every element  $(n, n')$  of  $N \times N$ , both the triadditive form  $\phi(m \otimes n) \cdot m'$  equals  $m \bullet \chi(n \otimes m')$  and the triadditive form  $\chi(n \otimes m) \bullet n'$  equals  $n \cdot \phi(m \otimes n')$ . In this case, there is an additive category whose objects are denoted  $R^{\oplus k} \oplus S^{\oplus \ell}$  for ordered pairs  $(k, \ell)$  of nonnegative integers, and such that the Hom Abelian group from  $R^{\oplus k} \oplus S^{\oplus \ell}$  to  $R^{\oplus k'} \oplus S^{\oplus \ell'}$  is the set of  $2 \times 2$  matrices whose  $R^{\oplus k'} \times R^{\oplus k}$  upper left entry is a  $k' \times k$  matrix with entries in  $R$ , whose  $S^{\oplus \ell'} \times S^{\oplus \ell}$  lower right entry is an  $\ell' \times \ell$  matrix with entries in  $R$  whose  $R^{\oplus k'} \times S^{\oplus \ell}$  off-diagonal entry is a  $k' \times \ell$  matrix with entries in  $M$ , and whose  $S^{\oplus \ell'} \times R^{\oplus k}$  off-diagonal entry is an  $\ell' \times k$  matrix with entries in  $N$ . Composition is obtained from matrix multiplication and the following rules: for every element  $m$  of  $M$  from  $S^{\oplus 1}$  to  $R^{\oplus 1}$ , for every element  $n$  on  $N$  from  $R^{\oplus 1}$  to  $S^{\oplus 1}$ , then the composition  $m \circ n$  equals  $\phi(m \otimes n)$ , and the composition  $n \circ m$  equals  $\psi(n \otimes m)$ . Associativity of this composition is precisely the definition above.

**Exercise 3.10.** For each object  $R^{\oplus k} \oplus S^{\oplus \ell}$  in the category above, what is the associative, unital algebra of endomorphisms of that object?

**Definition 3.11.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every  $\mathbf{C}$ -morphism  $f$  from a  $\mathbf{C}$ -object  $a$  to a  $\mathbf{C}$ -object  $b$ , a  $\mathbf{C}$ -morphism  $i$  from a  $\mathbf{C}$ -object  $k$  to  $a$  is a **kernel** of  $f$  if (and only if) both  $f \circ i$  is the zero map, and, for every  $\mathbf{C}$ -morphism  $i'$  from a  $\mathbf{C}$ -object  $k'$  to  $a$  such that  $f \circ i'$  is the zero map, there exists a unique  $\mathbf{C}$ -morphism  $g$  from  $k'$  to  $k$  such that  $i \circ g$  equals  $i'$ .

**Definition 3.12.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every  $\mathbf{C}$ -morphism  $f$  from a  $\mathbf{C}$ -object  $a$  to a  $\mathbf{C}$ -object  $b$ , a  $\mathbf{C}$ -morphism  $p$  from  $b$  to a  $\mathbf{C}$ -object  $c$  is a **cokernel** of  $f$  if (and only if) both  $p \circ f$  is the zero map, and, for every  $\mathbf{C}$ -morphism  $p'$  from  $b$  to a  $\mathbf{C}$ -object  $c'$  such that  $p' \circ f$  is the zero map, there exists a unique  $\mathbf{C}$ -morphism  $g$  from  $c$  to  $c'$  such that  $g \circ p$  equals  $p'$ .

**Exercise 3.13.** Prove that every kernel is monic.

**Exercise 3.14.** Prove that every cokernel is epic.

**Definition 3.15.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every  $\mathbf{C}$ -morphism  $f$  from a  $\mathbf{C}$ -object  $a$  to a  $\mathbf{C}$ -object  $b$ , a  $\mathbf{C}$ -morphism  $j$  from a  $\mathbf{C}$ -object  $b'$  to  $b$  is an **image** of  $f$  if (and only if)  $j$  is a kernel of a cokernel of  $f$ .

**Definition 3.16.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every  $\mathbf{C}$ -morphism  $f$  from a  $\mathbf{C}$ -object  $a$  to a  $\mathbf{C}$ -object  $b$ , a  $\mathbf{C}$ -morphism  $q$  from  $a$  to a  $\mathbf{C}$ -object  $a'$  is an **image** of  $f$  if (and only if)  $q$  is a cokernel of a kernel of  $f$ .

**Definition 3.17.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , an ordered pair  $(q, p)$  of a  $\mathbf{C}$ -morphism  $q$  from a  $\mathbf{C}$ -object  $a'$  to a  $\mathbf{C}$ -object  $a$  and a  $\mathbf{C}$ -morphism  $p$  from  $a$  to a  $\mathbf{C}$ -object  $a''$  is a  **$\mathbf{C}$ -left exact sequence**, respectively a  **$\mathbf{C}$ -right exact sequence**, a  **$\mathbf{C}$ -short exact sequence** if (and only if)  $q$  is a kernel of  $p$ , resp.  $p$  is a cokernel of  $q$ , both  $q$  is a kernel of  $p$  and  $p$  is a cokernel of  $q$ . For  $\mathbf{C}$ -short exact sequences  $\Sigma = (a' \xrightarrow{q} a, a \xrightarrow{p} a'')$  and  $\tilde{\Sigma} = (\tilde{a}' \xrightarrow{\tilde{q}} \tilde{a}, \tilde{a} \xrightarrow{\tilde{p}} \tilde{a}'')$ , a **morphism** of  $\mathbf{C}$ -short exact sequences from  $\Sigma$  to  $\tilde{\Sigma}$  is an ordered triple  $(u', u, u'')$  of a  $\mathbf{C}$ -morphism  $u'$  from  $a'$  to  $\tilde{a}'$ , of a  $\mathbf{C}$ -morphism  $u$  from  $a$  to  $\tilde{a}$ , and of a  $\mathbf{C}$ -morphism  $u''$  from  $a''$  to  $\tilde{a}''$  such that both  $\tilde{q} \circ u'$  equals  $u \circ q$  and  $\tilde{p} \circ u$  equals  $u'' \circ p$ .

For every  $\mathbf{C}$ -short exact sequence, we denote it by

$$0 \rightarrow a' \xrightarrow{q} a \xrightarrow{p} a'' \rightarrow 0.$$

**Exercise 3.18.** Prove that the category  $\mathbf{C}_{\text{ses}}$  of  $\mathbf{C}$ -short exact sequences is an additive category, and the three functors  $\pi'$ , respectively  $\pi$ ,  $\pi''$ , from this category to  $\mathbf{C}$  sending each short exact sequence to  $a'$ , resp. to  $a$ , to  $a''$ , is an additive functor.

**Exercise 3.19.** Prove that for every additive functor  $\mathbf{F}$  from the Abelian category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  to an Abelian category  $(\tilde{\mathbf{C}}, \text{add}_{\tilde{\mathbf{C}}})$ , there is a unique additive functor  $\mathbf{F}_{\text{ses}}$  from  $\mathbf{C}_{\text{ses}}$  to  $\tilde{\mathbf{C}}_{\text{ses}}$  that commutes with each of the three functors above. Similarly, for additive functors  $\mathbf{F}$  and  $\tilde{\mathbf{F}}$  from  $\mathbf{C}_{\text{ses}}$  to  $\tilde{\mathbf{C}}_{\text{ses}}$ , for every natural transformation  $\theta$  from  $\mathbf{F}$  to  $\tilde{\mathbf{F}}$ , prove that there exists a unique natural transformation  $\theta_{\text{ses}}$  from  $\mathbf{F}_{\text{ses}}$  to  $\tilde{\mathbf{F}}_{\text{ses}}$  that commutes with each of the three functors above. If you know the language of 2-categories, deduce that  $(\cdot)_{\text{ses}}$  is a 2-functor from the 2-category of strictly small Abelian categories to itself, and each of the three functors is a natural transformation from this 2-functor to the identity 2-functor.

**Definition 3.20.** A  $\mathbf{C}$ -short exact sequence  $(a' \xrightarrow{q} a, a \xrightarrow{p} a'')$  is **split** if (and only if) there exists an ordered pair  $(r, s)$  of a  $\mathbf{C}$ -morphism  $r$  from  $a$  to  $a'$  and a  $\mathbf{C}$ -morphism  $s$  from  $a''$  to  $a$  such that  $p \circ s$  equals  $\text{Id}_{a''}$ , such that  $r \circ q$  equals  $\text{Id}_{a'}$ , and  $s \circ p + q \circ r$  equals  $\text{Id}_a$ .

**Exercise 3.21.** Prove that a  $\mathbf{C}$ -short exact sequence is split if and only if it is isomorphic as a  $\mathbf{C}$ -short exact sequence to a  $\mathbf{C}$ -short exact sequence  $(a' \xrightarrow{q_1^{a', a''}} a' \oplus a'', a' \oplus a'' \xrightarrow{p_1^{a', a''}} a'')$ .

**Exercise 3.22.** For the additive category from Example 3.10, are there kernels and cokernels? If so, what are they?

## 4 Additive functors

**Definition 4.1.** For additive categories  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  and  $(\mathbf{D}, \text{add}_{\mathbf{D}})$ , an **additive functor** from  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  to  $(\mathbf{D}, \text{add}_{\mathbf{D}})$  is a functor  $\mathbf{F}$  from  $\mathbf{C}$  to  $\mathbf{D}$  such that  $\mathbf{F}$  preserves the zero object and preserves the product / coproduct structures.

By the previous proposition, for every additive functor, for every ordered pair  $(a, b)$  of objects of  $\mathbf{C}$ , the induced function  $\mathbf{F}_b^a$  from  $\mathbf{C}_b^a$  to  $\mathbf{D}_{\mathbf{F}(b)}^{\mathbf{F}(a)}$  is a homomorphism of Abelian groups. Also, by definition, every additive functor sends every split  $\mathbf{C}$ -short exact sequence to a split  $\mathbf{D}$ -short exact sequence.

**Definition 4.2.** For additive categories  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  and  $(\mathbf{D}, \text{add}_{\mathbf{D}})$ , an additive functor  $\mathbf{F}$  from  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  to  $(\mathbf{D}, \text{add}_{\mathbf{D}})$  is **left exact**, respectively **right exact**, **exact**, if (and only if)  $\mathbf{F}$  maps every  $\mathbf{C}$ -short exact sequence to a  $\mathbf{D}$ -left exact sequence, resp. to a  $\mathbf{D}$ -right exact sequence, a  $\mathbf{D}$ -short exact sequence. It is **half exact** if (and only if)  $\mathbf{F}$  maps every  $\mathbf{C}$ -short exact sequence to a pair of composable  $\mathbf{D}$ -morphisms where the kernel of the left morphism equals the image of the right morphism (but we ask neither that the left morphism is an epimorphism nor that the right morphism is a monomorphism). A fully faithful, exact, additive functor is **conservative** if (and only if) only zero objects map to zero objects (this is automatic if the domain category has a generator). Finally, a full faithful, exact, conservative, additive functor is a **Serre embedding** if (and only if) every short exact sequence in the target category whose first and third terms are in the image of the functor is equivalent to the image of a short exact sequence in the domain category.

**Example 4.3.** Every additive category has a unique functor to the zero category, and this is an exact additive functor. Similarly, there is a unique additive functor from the zero category to each additive category (mapping the null object to a null object), and this additive functor is exact.

**Example 4.4.** For every additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every  $\mathbf{C}$ -object  $a$ , for every morphism of associative, unital rings  $f$  from an associative, unital ring  $R$  to  $\mathbf{C}_a^a$ , there exists a unique additive functor from  $R\text{-FinFreeMod}$  to  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  that sends each object  $R^{\oplus \ell}$  to the direct sum  $a^{\oplus \ell}$  and that sends the Hom set  $R$  of  $R^{\oplus 1}$  to itself to the Hom set  $\mathbf{C}_a^a$  by  $f$ .

## 5 Abelian categories

**Definition 5.1.** An additive category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  is an **Abelian category** if (and only if), every  $\mathbf{C}$ -morphism has both a kernel and a cokernel, and the induced morphism from every coimage to the image is an isomorphism.

**Example 5.2.** For every associative, unital ring  $(R, +_R, \cdot_R)$ , the category  $R\text{-Mod}$  of left  $R$ -modules is an Abelian category. By the Freyd-Mitchell Embedding Theorem, every small Abelian category admits an additive, exact, fully faithful functor into the category of left  $R$ -modules for an appropriate associative, unital ring  $R$  whose cardinality can be bounded in terms of the cardinality of the small Abelian category, and such that every image  $R$ -module has cardinality bounded in terms of the cardinality of the small Abelian category. (Of course this requires the Axiom of Choice.)

**Example 5.3.** For every ordered pair  $((R, +_R, \cdot_R), (S, +_S, \cdot_S))$  of associative, unital rings, the category  $R\text{-}S\text{-Mod}$  of  $R\text{-}S$ -bimodules is an Abelian category. Of course this is naturally equivalent to the category of left modules for the ring  $R \otimes_{\mathbb{Z}} S^{\text{opp}}$ , but it is frequently useful to work with the category  $R\text{-}S\text{-Mod}$  as it is defined above.

**Example 5.4.** For every abelian category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , for every  $\mathbf{C}$ -object  $a$ , the Yoneda (covariant) functor  $h_{\mathbf{C}}^a(-) = \mathbf{C}_a^+ = \text{Hom}_{\mathbf{C}}(a, -)$  is an additive functor. By the definition of  $\mathbf{C}$ -short exact sequences, this functor is left exact. Since finite products in an additive category equal finite coproducts, also the Yoneda contravariant functor  $h_a^{\mathbf{C}}(-) = \mathbf{C}_a^- = \text{Hom}_{\mathbf{C}}(-, a)$  is an additive functor from  $\mathbf{C}^{\text{opp}}$  to  $\mathbf{C}$ . The opposite of a  $\mathbf{C}$ -short exact sequence is a  $\mathbf{C}^{\text{opp}}$ -short exact sequence, and the  $h_a^{\mathbf{C}}$  sends this to a  $\mathbf{C}$ -left exact sequence.

**Definition 5.5.** For every Abelian category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , a  $\mathbf{C}$ -object  $a$  is  **$\mathbf{C}$ -projective**, respectively  **$\mathbf{C}$ -injective**, if (and only if) we have exactness of the left exact Yoneda functor  $h_{\mathbf{C}}^a$ , resp.  $h_a^{\mathbf{C}}$ . For every Abelian category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$ , there are **enough projectives**, respectively **enough injectives**, if (and only if) every object is the target / codomain of an epic  $\mathbf{C}$ -morphism from a  $\mathbf{C}$ -projective object, resp. every object is the domain of a monic  $\mathbf{C}$ -morphism to a  $\mathbf{C}$ -injective object.

**Example 5.6.** As explained in the notes on adjoint functors, for every adjoint pair of functors between additive categories, if one of them is additive, then so is the other. Moreover, the right adjoint functor is always left exact, and the left adjoint functor is always right exact. The right adjoint of an exact additive functor preserves injective objects, and the left adjoint of an exact additive functor preserves projective objects. For appropriate choices of adjoint pairs of functors, it follows that the Abelian category  $R\text{-}S\text{-Mod}$  both has enough projective objects and enough injective objects. Moreover, the epic from a projective object can be made part of a comonad / cotriple, and the monic to an injective object can be made part of a monad / triple. This provides an alternative construction of derived functors (closely aligned with the historically earlier constructions predating Grothendieck's Tohoku approach). We will not pursue this here, but it is a key idea in extending homological algebra of an Abelian category such as  $R\text{-}S\text{-Mod}$  to a non-necessarily-Abelian category such as the category of commutative rings.

**Example 5.7.** In particular, using the adjoint pair of tensor product and Hom (the covariant Yoneda functor), it follows that tensor product is right exact. Moreover, for every morphism of associative, unital rings  $u$  from  $(R, +_R, \cdot_R)$  to  $(S, +_S, \cdot_S)$ , the functor from left  $R$ -modules to left  $S$ -modules of  $S \otimes_R -$  preserves projective objects.

The main structural theorem about Abelian categories is the following (whose proof is not covered in our course, but I am happy to point you to the proof if asked).

**Theorem 5.8** (Freyd – Mitchell Embedding Theorem). *Every small Abelian category  $\mathbf{C}$  admits a fully faithful embedding as a full, exact subcategory of the category  $R\text{-Mod}$  for an appropriate associative, unital ring  $R$  whose cardinality is bounded in terms of the cardinality of  $\mathbf{C}$  and such that the cardinality of each  $R$ -module in the image of the embedding is also bounded in terms of the cardinality of  $\mathbf{C}$ .*

## 6 Additional hypotheses on Abelian categories

Most modern treatments of higher derived functors as delta-functors follow Grothendieck’s “Tohoku” paper.

“Sur quelques points d’algèbre homologique,” *Tohoku J. Math.* **9** (1957): pp. 119-221.

In that article, Grothendieck introduces additional hypotheses on Abelian categories that are frequently useful. The first two hypotheses – (AB1) and (AB2) – are basically the definition of Abelian category above.

The next hypothesis, (AB3), is equivalent to the existence of direct sums indexed by sets. Together with the other axioms, this implies existence of direct limits (also known as colimits) indexed by a small category. Another name for this hypothesis is **cocompleteness** of the category.

The opposite hypothesis, (AB3\*), is equivalent to the existence of direct products indexed by sets. Together with the other axioms, this implies existence of inverse limits (usually just called limit) indexed by a small category. Another name for this hypothesis is **completeness** of the category.

For a fixed indexing category, there is a functor that associates to each indexed family of objects the colimit, respectively the limit. These are both additive functors. The colimit functor is a left adjoint, and thus is right exact. The limit functor is right adjoint, and thus is left exact. If the direct sum additive functor is always left exact (hence exact), then (AB4) holds. If the direct product additive functor is always right exact (hence exact), then (AB4\*) holds.

Finally, if the colimit functor for each filtered small indexing category is always left exact (hence exact), then (AB5) holds. If the limit functor for each filtered small indexing category is always right exact (hence exact), then (AB5\*) holds.

**Definition 6.1.** For every category, and object of that category is **generator** if (and only if) the Yoneda (covariant) functor of that object is faithful. An Abelian category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  is a **Grothendieck category** if (and only if) it satisfies (AB3), (AB4), (AB5), and it has a generator.

**Example 6.2.** For every associative, unital rings  $(R, +_R, \cdot_R)$ , the Abelian category  $R\text{-Mod}$  of left  $R$ -modules is a Grothendieck category, where one generator is  $R$  with its regular structure of left  $R$ -module.

**Example 6.3.** For every topological space  $(X, \tau)$ , for every sheaf  $\mathcal{R}$  of associative, unital rings on  $(X, \tau)$ , the Abelian category of sheaves of left  $\mathcal{R}$ -modules on  $(X, \tau)$  is a Grothendieck category. In fact, much more is true: the topos of Abelian sheaves on a small category with a Grothendieck topology is a Grothendieck category (and even more than this is true).

One of the main results about Grothendieck categories is the following.

**Theorem 6.4.** *Every Grothendieck category  $(\mathbf{C}, \text{add}_{\mathbf{C}})$  has a (non-additive) functor  $\text{Inj}$  to itself and a natural transformation  $\iota$  from the identity functor to  $\text{Inj}$  such that for every object  $a$  of  $\mathbf{C}$ , the morphism  $\iota_M$  from  $M$  to  $\text{Inj}(M)$  is an injective resolution. In particular,  $\mathbf{C}$  has enough injectives.*