

MAT 535 Review Sheet for Midterm 1

Remark. Some of these practice problems are harder than questions that would be asked on the midterm. If you are comfortable with these questions, then you will be well prepared for the final exam. This review sheet will be periodically updated as the semester continues.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Review Topics.

Definitions. Please know all of the following definitions.

Category. Hom set. Composition. Small category. Functor. Natural transformation. Adjoint pair. Left module. Right module. Bimodule. Tensor product. Short exact sequence. Additive functor. Half exact functor. Left exact functor. Right exact functor. Exact functor. Delta-functor. Universal delta-functor. Effaceable

delta-functor. Injective object. Projective object. Cochain complex. Differential. Cochain map. Cochain homotopy. Homotopy equivalence. Cohomology modules. Quasi-isomorphism. Translation. Mapping cone. Exact complex. Injective resolution. Projective resolution. Right derived functors. Left derived functors. Ext. Tor. Flat module. Primary module. Cyclic module. Companion matrix. Frobenius decomposition. Eigenvalue. Eigenspace. Generalized eigenspace. Semisimple part. Nilpotent part. Jordan-Chevalley decomposition.

Results. Please know all of the following lemmas, propositions, theorems and corollaries.

Yoneda embedding. Every natural transformation from the contravariant Yoneda functor of h_b of an object b to a set-valued functor \mathbf{F} is the natural transformation of a unique element of the set $\mathbf{F}(b)$. In particular, the natural transformations from h_b to $h_{\tilde{b}}$ are the same as the morphisms from b to \tilde{b} .

Existence of tensor products. Tensor products exist and form a left adjoint to Hom. In particular, tensor product is right exact, preserves all direct sums and preserves all colimits (i.e., direct limits). Since Hom is right adjoint to tensor product, it is left exact, preserves all direct products and preserves all (inverse) limits.

Grothendieck's criterion. Every left exact δ -functor concentrated in cohomological degrees ≥ 0 that is effaceable is universal. Every right exact δ -functor concentrated in cohomological degrees ≤ 0 (thus in homological degrees ≥ 0) that is coeffaceable is universal.

Injective resolutions. For every Abelian category that has enough injective objects, injective resolutions exist, unique up to homotopy equivalence, and are functorial, up to null homotopy. Moreover, short exact sequences correspond to mapping cones of injective resolutions.

Existence of universal δ -functors. Every Abelian category that has enough injective objects has a right derived functor of each left exact, additive functor \mathbf{F} , and this is constructed by applying \mathbf{F} to the injective resolutions and then taking the cohomology modules.

Existence of Ext. For every associative, unital ring R , for every left R -module M , the left exact, additive functor $\text{Hom}_{R\text{-Mod}}(M, -)$ from $R\text{-Mod}$ to $\mathbb{Z}\text{-Mod}$ has an extension to a universal δ -functor $\text{Ext}_{R\text{-Mod}}^\bullet(M, -)$ in degrees ≥ 0 . A left R -module N is acyclic for $\text{Ext}_{R\text{-Mod}}^\bullet(M, -)$ for all left R -modules M if and only if N is an injective left R -module. These functors commute with limits (i.e., inverse limits). Similarly, for every left R -module N , the left exact, additive functor $\text{Hom}_{R\text{-Mod}}(-, N)$ from $(R\text{-Mod})^{\text{opp}}$ to $\mathbb{Z}\text{-Mod}$ has an extension to a universal δ -functor $\text{Ext}_{R\text{-Mod}}^\bullet(-, N)$ in degrees ≥ 0 . A left R -module M is acyclic for $\text{Ext}_{R\text{-Mod}}^\bullet(-, N)$ for all left R -modules N if and only if M is a projective left R -module. These contravariant functors convert colimits (i.e., direct limits) into limits (i.e., inverse limits). Finally, for each integer n , both right derived functors for M and N are naturally isomorphic to the Yoneda-Ext group of equivalence

classes of exact complexes of left R -modules with $n + 1$ terms whose left-most term is N and whose right-most term is M . In particular, both right derived functors are naturally isomorphic to each other, which justifies the common notation $\text{Ext}_{R\text{-Mod}}^n(M, N)$.

Existence of Tor. For every associative, unital ring R , for every right R -module M , the right exact, additive functor $M \otimes_R -$ from $R\text{-Mod}$ to $\mathbb{Z}\text{-Mod}$ has an extension to a universal δ -functor $\text{Tor}_\bullet^R(M, -)$ in homological degrees ≥ 0 (thus in cohomological degrees ≤ 0). A left R -module N is acyclic for $\text{Tor}_\bullet^R(M, -)$ for all right R -modules M if and only if N is a flat left R -module. Similarly, for every left R -module N , the left exact, additive functor $- \otimes_R N$ from $\text{Mod} - R$ to $\mathbb{Z}\text{-Mod}$ has an extension to a universal δ -functor $\text{Tor}_\bullet^R(-, N)$ in homological degrees ≥ 0 . A right R -module M is acyclic for $\text{Tor}_\bullet^R(-, N)$ for all left R -modules N if and only if M is a flat right R -module. Finally, for each integer n , both derived functors for M and N are naturally isomorphic to each other, which justifies the common notation $\text{Tor}_n^R(M, N)$.

Theorem of Govorov-Lazard. Every projective left R -module is a flat left R -module, and every projective right R -module is a flat right R -module. The Tor functors commute with colimits (i.e., direct limits). Thus, every colimit of projective left R -modules is a flat left R -module, and every colimit of projective right R -modules is a flat right R -module. In fact, every flat left R -module is a filtering colimit of finitely generated, free left R -modules, and every flat right R -module is a filtering colimit of finitely generated, free right R -modules.

Cayley-Hamilton Theorem. For every commutative ring R , for every positive integer d , for every $d \times d$ matrix $A = [a_{i,j}]_{1 \leq i,j \leq d}$ with entries $a_{i,j}$ in R , the characteristic polynomial $\det_{d \times d}(t\text{Id}_{d \times d} - A)$ is a monic polynomial $\text{char}_A^R(t)$ of degree d with coefficients in R . Moreover, the $d \times d$ matrix $\text{char}_A^R(A)$ is the zero matrix.

Rational Canonical Form / Frobenius Decomposition. For every field F , for every positive integer d , every $d \times d$ matrix A with entries in F has a minimal polynomial $f(t)$ that is a monic polynomial of degree $\leq d$ that factors as a product of (primary) powers of distinct irreducible, monic polynomials, i.e., $f = f_1^{e_1} \cdots f_n^{e_n}$ for pairwise distinct, irreducible, monic polynomials f_i and for integers $e_i \geq 1$ (the **geometric multiplicity** of f_i for A). There is an ordered F -vector space basis of $F^{\oplus n}$ with respect to which the matrix representative is in block form with blocks $C_{i,1}, \dots, C_{i,\ell_i}$ for each $i = 1, \dots, n$ that are companion matrices $C_{i,j}$ of (primary) powers $f_i^{d_{i,j}}$ for integers $e_i = d_{i,1} \geq \dots \geq d_{i,\ell_i} \geq 1$. The characteristic polynomial of A equals $f_1^{d_1} \cdots f_n^{d_n}$ for the positive integers $d_i = d_{i,1} + \dots + d_{i,\ell_i}$, the **algebraic multiplicities**, that are $\geq e_i$.

Jordan Canonical Form / Jordan-Chevalley Decomposition. For every field F , for every positive integer d , for every $d \times d$ matrix A with entries in F whose minimal polynomial factors as a product of (primary) powers of distinct linear monic polynomials, i.e., $f = (t - \lambda_1)^{e_1} \cdots (t - \lambda_n)^{e_n}$ for pairwise distinct elements λ_i of F and for integers $e_i \geq 1$, there is an ordered F -vector space basis of $F^{\oplus n}$ with respect to which the matrix representative is in block form with blocks $J_{i,1}, \dots, J_{i,\ell_i}$

for each $i = 1, \dots, n$ that are Jordan blocks of size $d_{i,j} \times d_{i,j}$, i.e., every diagonal entry equals λ_i , every entry directly above the diagonal equals 1, and all other entries equals 0, for integers $e_i = d_{i,1} \geq \dots \geq d_{i,\ell_i} \geq 1$. The characteristic polynomial of A equals $(t - \lambda_1)^{d_1} \dots (t - \lambda_n)^{d_n}$ for the positive integers $d_i = d_{i,1} + \dots + d_{i,\ell_i}$. The matrix A equals $S + N$ for $d \times d$ matrices S and N with entries in F such that SN equals NS , such that N is nilpotent, i.e., N^d equals the zero matrix, and such that S is a diagonalizable matrix, diagonalized by the same ordered basis putting T into Jordan canonical form, where the $d_{i,j} \times d_{i,j}$ block for S corresponding to $C_{i,j}$ is just $\lambda_i \text{Id}_{d_{i,j} \times d_{i,j}}$.

Practice Problems.

(1) For every monoid, prove that there exists a category with a unique object whose associated endomorphism monoid is the given monoid. Prove that this category is a groupoid if and only if the monoid is a group.

(2) For every small category, assuming the Axiom of Choice, prove that there exists a full subcategory that is skeletal, i.e., every object of the original category is isomorphic to precisely one object of the full subcategory.

(3) For every category \mathbf{C} , for every full, skeletal subcategory \mathbf{S} , define the objects of $\mathbf{C}_{\mathbf{S}}$ to be ordered pairs (a, u) of an object a of \mathbf{C} and a \mathbf{C} -isomorphism u from a to an object $u(a)$ of \mathbf{S} . For objects (a, u) and (\tilde{a}, \tilde{u}) , a $\mathbf{C}_{\mathbf{S}}$ -morphism from (a, u) to (\tilde{a}, \tilde{u}) is a \mathbf{C} -morphism f from a to \tilde{a} . Prove that the forgetful functor \mathbf{F} from $\mathbf{C}_{\mathbf{S}}$ to \mathbf{C} sending every object (a, u) to a is a strictly surjective equivalence of categories. Also prove that the functor \mathbf{G} from $\mathbf{C}_{\mathbf{S}}$ sending every object (a, u) to $u(a)$ and sending each $\mathbf{C}_{\mathbf{S}}$ -morphism f to $\tilde{u} \circ f \circ u^{-1}$ is a strictly surjective equivalence of categories. Prove that there exists a natural isomorphism from \mathbf{F} to \mathbf{G} that sends every object (a, u) to the \mathbf{C} -isomorphism u from a to $u(a)$.

(4) A small category is **specializing** if every pair of objects that admit morphisms from each to the other are isomorphic, e.g., the category of finitely generated left R -modules with morphisms being surjective morphisms of left R -modules. For every skeletal, specializing, small category, prove that the relation defined on the set of objects by nonemptiness of the Hom set is a partial order, the **specialization order**.

(5) For every functor between small, specializing, skeletal categories, prove that the induced map between sets of objects sends specializing pairs to specializing pairs.

(6) For functors \mathbf{F} and $\tilde{\mathbf{F}}$ from a small, specializing, skeletal category \mathbf{C} to a small, specializing, skeletal category $\tilde{\mathbf{C}}$, if there exists a natural transformation θ from \mathbf{F} to $\tilde{\mathbf{F}}$, prove that $\tilde{\mathbf{F}}$ is valuewise greater than or equal to \mathbf{F} for the specialization orders.

(7) A **monoid-poset** is a partially ordered set together with an assignment of a monoid to each element of the set. A **morphism** of monoid-posets is an ordered pair of a morphism of partially ordered sets and, for each element of the domain poset, a morphism of monoids from the monoid

of the domain monoid-poset to the monoid of the target monoid-poset. For each monoid-poset, prove that there exists a small, specializing, skeletal category whose associated monoid-poset is the given monoid-poset. For each morphism of monoid-posets, prove that there exists a functor of small, skeletal categories whose associated morphism of monoid-posets is the given morphism. Prove that this extends to an adjoint pair of functors between the category of small, specializing, skeletal categories and the category of monoid-posets.

(8) For every field k , for every associative, unital k -algebra A (with the image of k in the center of A), for every right A -module R that has finite k -dimension, for every left A -module L that has finite k -dimension, prove that the tensor product $R \otimes_A L$ is a quotient k -vector space of the k -vector space $R \otimes_k L$. Thus, the k -vector space dimension of $R \otimes_A L$ is finite, bounded by $\dim_k(R) \cdot \dim_k(L)$.

(9) For every field k , for every k -vector space $(V, +_V, \cdot_{k,V})$, prove that the k -algebra $\text{Hom}_{k\text{-Mod}}(V, V)$ has finite k -dimension if and only if V has finite k -dimension. When V does have finite k -dimension, describe all left ideals and describe all right ideals in $\text{Hom}_{k\text{-Mod}}(V, V)$. Prove that the only two-sided ideals are the zero ideal and the entire ring.

(10) For every field k , for every k -vector space $(V, +_V, \cdot_{k,V})$ that has finite k -dimension, describe all finitely generated left $\text{Hom}_{k\text{-Mod}}(V, V)$ -modules, and describe all finitely generated right $\text{Hom}_{k\text{-Mod}}(V, V)$ -modules. In particular, describe the simple modules.

(11) For every field k , for every k -vector space $(V, +_V, \cdot_{k,V})$ that has finite k -dimension, for every nonzero simple left $\text{Hom}_{k\text{-Mod}}(V, V)$ -module, say L , for every nonzero simple right $\text{Hom}_{k\text{-Mod}}(V, V)$ -module, say R , describe the tensor product $L \otimes_{\text{Hom}_{k\text{-Mod}}(V, V)} R$ as a k -vector space. What is the k -dimension of this k -vector space?

(12) Let \mathbb{H} be the Hamilton quaternions as an \mathbb{R} -algebra. What are the simple left \mathbb{H} -modules, and what are the simple right \mathbb{H} -modules. When you tensor a right module with a left module over \mathbb{H} , what is the dimension of this \mathbb{R} -vector space?

(13) Part of adjointness of Hom and tensor product is the following Abelian group homomorphism every left R -module M and for every Abelian group L ,

$$\epsilon_L : \text{Hom}_{\mathbb{Z}\text{-Mod}}(M, L) \otimes_R M \rightarrow L.$$

Give examples where this homomorphism is an isomorphism, where this homomorphism is surjective but not injective, and where this homomorphism is injective but not surjective.

(14) Part of adjointness of Hom and tensor product is the following Abelian group homomorphism every left R -module M and for every right R -module N ,

$$\eta_N : N \rightarrow \text{Hom}_{\mathbb{Z}\text{-Mod}}(M, N \otimes_R M).$$

Give examples where this homomorphism is an isomorphism, where this homomorphism is surjective but not injective, and where this homomorphism is injective but not surjective.

(15) Let R be a commutative ring with a unique maximal ideal \mathfrak{m} (the set of all non-units), and denote the quotient field R/\mathfrak{m} by k . Let M be an R -module (and thus an $R - R$ -bimodule) admitting a projective resolution $P^\bullet \xrightarrow{p} M[0]$ where every projective R -module P^{-n} is finitely generated. Prove that there exists such a projective resolution where for each nonnegative integer n , the nonnegative integer $\dim_k(P^{-n}/\mathfrak{m}P^{-n})$ is minimal among all projective resolutions $\tilde{P}^\bullet \xrightarrow{\tilde{p}} M[0]$ such that, for every integer $r = 0, \dots, n - 1$, the nonnegative integer $\dim_k(\tilde{P}^{-r}/\mathfrak{m}\tilde{P}^{-r})$ equals $\dim_k(P^{-r}/\mathfrak{m}P^{-r})$. Such a projective resolution is called *minimal*.

(16) Continuing the previous problem, for every minimal projective resolution, prove that it is unique up to (typically non-unique) isomorphism of complexes (not just homotopy equivalence). The sequence of integers $b_n(M) := \dim_k(P^{-n}/\mathfrak{m}P^{-n})$ are the **Betti numbers** of the R -module M .

(17) Continuing the previous problems, for every R -module M with a minimal projective resolution, prove that each $b_n(M)$ equals both $\dim_k \text{Tor}_n^R(M, R/\mathfrak{m}R)$ and $\dim_k \text{Ext}_R^n(M, R/\mathfrak{m}R)$.

(18) Let R be a commutative ring. A cochain complex $((A^{-n})_{n \in \mathbb{Z}_{\geq 0}}, (d^{-(n+1)})_{n \in \mathbb{Z}_{\geq 0}})$ of R -modules (hence $R - R$ -bimodules, since R is commutative) together with a sequence $(- * - : A^{-\ell} \otimes_R A^{-n} \rightarrow A^{-(n+\ell)})_{(\ell, n) \in \mathbb{Z}_{\geq 0}^2}$ of morphisms of R -modules is a **differential graded R -algebra** in cohomological degrees ≤ 0 if (and only if) $*$ makes $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} A^{-n}$ into an associative, unital R -algebra that is **graded-commutative**, i.e., $a_\ell * a_n$ equals $(-1)^{\ell n} a_n * a_\ell$ for all (ℓ, n) in $\mathbb{Z}_{\geq 0}^2$, and it satisfies the **graded Leibniz rule**, i.e., $d^{-(\ell+n)}(a_\ell * a_n)$ equals $d^{-\ell}(a_\ell) * a_n + (-1)^\ell a_\ell * d^{-n}(a_n)$ for every $(\ell, n) \in \mathbb{Z}_{\geq 0}$. For every differential graded R -algebra, prove that the graded R -module $(H^{-n}(A))_{n \in \mathbb{Z}_{\geq 0}}$ has an induced structure of associative, unital and graded-commutative R -algebra.

(19) Let R be a commutative ring. For every R -module M , the **exterior R -algebra** on M is the unique sequence $(- * - : \bigwedge_R^\ell(M) \otimes_R \bigwedge_R^n(M) \rightarrow \bigwedge_R^{\ell+n}(M))$ making $\bigwedge_R^\bullet(M) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \bigwedge_R^n(M)$ into an associative, unital graded R -algebra concentrated in degrees ≤ 0 that is a quotient of the associative, unital, graded R -algebra (very much not graded-commutative) $T_R^\bullet(M) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M^{\otimes n}$ by the two-sided ideal generated by all elements of the form $a \otimes m \otimes b \otimes m \otimes c$ for m an element of $M = M^{\otimes 1}$, and for elements a, b and c of $T_R^\bullet(M)$. Prove that this is graded-commutative.

(20) Continuing the previous exercise, for every R -module homomorphism u from $M = \bigwedge_R^1(M)$ to $R = \bigwedge_R^0(M)$, prove that there exists a unique sequence $(d_u^{-(n+1)})_{n \in \mathbb{Z}_{\geq 0}}$ of differentials that make $\bigwedge_R^\bullet(M)$ into a differential graded R -algebra such that d^{-1} equals u . This is the **Koszul R -algebra** $K_R^\bullet(M, u)$ determined by $M \xrightarrow{u} R$.

(21) Let R be a local commutative ring with unique maximal ideal \mathfrak{m} and with residue field R/\mathfrak{m} denoted by k . For every finite sequence (u_1, \dots, u_s) of elements of \mathfrak{m} , denote by $u : R^{\oplus s} \rightarrow R$

the corresponding R -module homomorphism. There is a morphism of R -cochain complexes from the Koszul algebra $K_R^\bullet(R^{\oplus s}, u)$ to $R/\text{Image}(u)[0]$ that is an isomorphism of H^0 -modules. The sequence (u_1, \dots, u_s) is a **m-regular sequence** if (and only if) this is a quasi-isomorphism, i.e., the Koszul algebra is a projective resolution of $R/\text{Image}(u)$. In this case, prove that each Betti number $b_n(R/\langle u_1, \dots, u_s \rangle)$ equals the binomial coefficient $\binom{s}{n}$. By famous work of Auslander-Buchsbaum and Serre, a Noetherian local commutative ring has a regular sequence that generates \mathfrak{m} (i.e., it is a regular local ring) if and only if every finitely generated R -module has a finite projective resolution (finitely many of the terms are nonzero, and each of these is finitely generated).

(22) Find an example of a Noetherian local commutative ring R and a finitely generated R -module M where $b_n(M)$ is nonzero for infinitely many nonnegative integers n . By the theorem of Auslander-Buchsbaum and Serre, you will need to choose R to be a local ring that is not regular.

(23) Let R be a local commutative ring with unique maximal ideal \mathfrak{m} and with residue field R/\mathfrak{m} denoted by k . Assume that R is a principal ideal domain (so that actually R is a discrete valuation ring). Prove that an element u_1 of \mathfrak{m} defines a **m-regular sequence** with s equal to 1 if and only if u_1 is nonzero. Thus, every discrete valuation ring is a regular local ring (and the discrete valuation rings are precisely the regular local rings whose maximal ideal is principal).

(24) For each field k , for each positive integer s , for each localized commutative ring $R = k[t_1, \dots, t_s]_{\langle t_1, \dots, t_s \rangle}$ of the polynomial k -algebra $k[t_1, \dots, t_s]$, prove that (t_1, \dots, t_s) is an **m-regular sequence**.

(25) For every finitely generated Abelian group M considered as a \mathbb{Z} -module, prove that M is projective if and only if M is free if and only if M is flat, i.e., $\text{Tor}_1^{\mathbb{Z}}(M, N)$ is a zero group for every Abelian group N .

(26) For every associative, unital ring R , for every left R -module M , prove that M is a direct summand of a finitely generated, free left R -module if and only if, for every left R -module N , the following natural homomorphism of Abelian groups is an isomorphism,

$$\text{Hom}_{R\text{-Mod}}(N, R) \otimes_R M \rightarrow \text{Hom}_{R\text{-Mod}}(N, M).$$

(27) For every associative, unital ring R , for every left ideal \mathfrak{a} , for every right R -module M , prove that $\text{Tor}_1^R(M, R/\mathfrak{a})$ is naturally isomorphic to the kernel of the natural surjective homomorphism of Abelian groups from $M \otimes_R \mathfrak{a}$ to the Abelian subgroup $M \cdot \mathfrak{a}$ of M .

(28) For every associative, unital ring R , for every left ideal \mathfrak{a} , for every right ideal \mathfrak{b} , prove that $\text{Tor}_1^R(\mathfrak{b} \setminus R, R/\mathfrak{a})$ is naturally isomorphic to $(\mathfrak{b} \cap \mathfrak{a})/(\mathfrak{b} \cdot \mathfrak{a})$.

(29) For every associative, unital ring R , for every short exact sequence of left R -modules,

$$\Sigma: 0 \rightarrow M' \xrightarrow{q} M \xrightarrow{q} M'' \rightarrow 0,$$

define obs_Σ to be the image $\delta_\Sigma^0(\text{Id}_{M''})$ in $\text{Ext}_{R\text{-Mod}}^1(M'', M')$ of the element $\text{Id}_{M''}$ in $\text{Ext}_{R\text{-Mod}}^0(M'', M'') = \text{Hom}_{R\text{-Mod}}(M'', M'')$. For short exact sequences Σ and $\tilde{\Sigma}$ as above, with outer terms M' and M'' , prove that obs_Σ equals $\text{obs}_{\tilde{\Sigma}}$ if and only if Σ is isomorphic to $\tilde{\Sigma}$ as a short exact sequence with M' and M'' fixed.

(30) Continuing the previous exercise, conversely prove that for every element of $\text{Ext}_{R\text{-Mod}}^1(M'', M')$, there exists a short exact sequence Σ of left R -modules with M' and M'' fixed such that $\delta_\Sigma^0(\text{Id}_{M''})$ is that element. Deduce that $\text{Ext}_{R\text{-Mod}}^1(M'', M')$ is naturally bijective with the set of isomorphism classes of short exact sequences of left R -modules with M' and M'' fixed.

(31) For every associative, unital ring R , the objects of the additive **arrow category** $\text{Arr}(R\text{-Mod})_{ff}$ of finite, free left R -modules are morphisms $R^{\oplus n_1} \xrightarrow{d_1} R^{\oplus n_0}$ of finitely generated, free, left R -modules (i.e., an ordered triple (n_1, n_0, d_1) of nonnegative integers n_1 and n_0 and an $n_1 \times n_0$ -matrix d_1 whose entries are elements of R). The Abelian group of morphisms in the arrow category from an object $(R^{\oplus n_1} \xrightarrow{d_1} R^{\oplus n_0})$ to an object $(R^{\oplus \tilde{n}_1} \xrightarrow{\tilde{d}_1} R^{\oplus \tilde{n}_0})$ is the Abelian group of all ordered pairs (u_1, u_0) of morphisms of left R -modules u_1 from $R^{\oplus n_1}$ to $R^{\oplus \tilde{n}_1}$ and u_0 from $R^{\oplus n_0}$ to $R^{\oplus \tilde{n}_0}$ such that $u_0 \circ d_1$ equals $\tilde{d}_1 \circ u_1$. In particular, u_1 induces a unique morphism $Z^{-1}(u_1, u_0)$ of left R -modules from $\text{Ker}(d_1)$ to $\text{Ker}(\tilde{d}_1)$, and u_0 induces a unique morphism \bar{u}^0 from $\text{Coker}(d_1)$ to $\text{Coker}(\tilde{d}_1)$.

Prove that there is an essentially surjective, fully faithful functor \mathbf{F} from $\text{Arr}(R\text{-Mod})_{ff}$ to the full subcategory of $\text{Ch}^{[-2,0]}(R\text{-Mod})$ of all acyclic cochain complexes of left R -modules concentrated in degrees $\{-2, -1, 0\}$ whose terms in degrees 0 and -1 are finitely generated and free. The functor adjoins to each complex $(R^{\oplus n_1} \xrightarrow{d_1} R^{\oplus n_0})$ concentrated in degrees $\{-1, 0\}$ the term $\text{Ker}(d_1)$ in degree -2 with its inclusion morphism q_{d_1} as the differential. The functor sends each pair (u_1, u_0) to the morphism of cochain complexes that is u_0 in degree 0, that is u_1 in degree -1 , and that is $Z^{-1}(u_1, u_0)$ in degree -2 . These complexes are **acyclic** in the sense that H^n of the complex is zero except possibly for $n = 0$.

(32) Continuing the previous problem, prove that $\text{Coker} = H^0$ is an essentially surjective, full, additive functor from $\text{Arr}(R\text{-Mod})_{ff}$ to the additive category $R\text{-Mod}_{fp}$ of finitely presented left R -modules associating to each arrow $(R^{\oplus n_1} \xrightarrow{d_1} R^{\oplus n_0})$ the left R -module $\text{Coker}(d_1)$, and associating to every morphism (u_1, u_0) the induced morphism \bar{u}^0 of cokernels.

(33) The previous functor is not faithful. Prove that if \bar{u}_0 is a zero morphism of cokernels then the morphism $(Z^{-1}(u_1, u_0), u_1, u_0)$ is null homotopic. More precisely, prove that there exists a morphism s_0 of left R -modules from $R^{\oplus n_0}$ to $R^{\oplus \tilde{n}_1}$ such that $p_{\tilde{d}_1} \circ s_0$ equals u_0 . Moreover, for every such s_0 , prove that there exists a unique morphism s_1 of left R -modules from $R^{\oplus n_1}$ to $\text{Ker}(\tilde{d}_1)$ such that u_1 equals $q_{\tilde{d}_1} \circ s_1$ and $Z^{-1}(u_1, u_0)$ equals $s_1 \circ q_{d_1}$. Deduce that Coker factors through the image of $\text{Arr}_{ff}(R\text{-Mod})$ in the homotopy category $\mathbf{K}(R\text{-Mod})$, and this induced functor is a weak equivalence of categories to $R\text{-Mod}_{fp}$.

(34) For every morphism (u_1, u_0) in $\text{Arr}(R\text{-Mod})_{ff}$, prove that $\text{Coker}(u_1, u_0)$ is also a finitely presented, left R -module with one finite presentation given by $(R^{\oplus(n_0+\tilde{n}_1)} \xrightarrow{(u_0, \tilde{d}_1)} R^{\oplus\tilde{n}_0})$.

(35) For every finite presentation of a right R -module of N , use that presentation to describe an additive, right exact functor between arrow categories that induces the functor $N \otimes_R -$ on the category $R\text{-Mod}_{fp}$.

(36) Let p and q be distinct prime integers (strictly greater than 1). Let d and e be positive integers. Compute $M \otimes_{\mathbb{Z}} N$ when (M, N) is each of the following pairs $(\{0\}, N)$, $(\mathbb{Z}/p^e\mathbb{Z}, \mathbb{Z}/p^d\mathbb{Z})$, $(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$, $(\mathbb{Z}/p^e\mathbb{Z}, \mathbb{Z})$, (\mathbb{Z}, \mathbb{Z}) , $(\mathbb{Z}/p^e\mathbb{Z}, \mathbb{Q})$, (\mathbb{Z}, \mathbb{Q}) , $(\mathbb{Z}/p^e\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, $(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, and $(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$.

(37) Repeat the previous problem to compute $\text{Hom}_{\mathbb{Z}\text{-Mod}}(M, N)$ and $\text{Hom}_{\mathbb{Z}\text{-Mod}}(N, M)$.

(38) In each of the cases above, also compute $\text{Tor}_1^{\mathbb{Z}}(M, N)$, $\text{Ext}_{\mathbb{Z}}^1(M, N)$ and $\text{Ext}_{\mathbb{Z}}^1(N, M)$.

(39) Let R be a principal ideal domain that is properly contained in its fraction field F . Compute $(F/R) \otimes_R (F/R)$, and compute $\text{Tor}_1^R(F/R, F/R)$.

(40) Let R be a principal ideal domain that is properly contained in its fraction field F . Let f and g be noninvertible, nonzero elements of R that are irreducible (hence prime). Compute $(R/fR) \otimes_R (R/gR)$ and $\text{Tor}_1^R(R/fR, R/gR)$ both when f equals g and when the prime ideals fR and gR are distinct.

(41) Continuing the previous problem, also compute $\text{Hom}_R(R/fR, R/gR)$ and $\text{Ext}_R^1(R/fR, R/gR)$, both when f equals g and when the prime ideals fR and gR are distinct.

(42) Let p be a prime integer (strictly greater than 1). For the commutative ring $R = \mathbb{Z}/p^2\mathbb{Z}$, for the canonical surjection of R -modules from R to the quotient ring $R/pR = \mathbb{Z}/p\mathbb{Z}$, what is the kernel? What is one free resolution of $\mathbb{Z}/p\mathbb{Z}$ as an R -module? Use this to compute $\text{Tor}_n^R(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ for every nonnegative integer n .

(43) Let p be a prime integer (strictly greater than 1). For the commutative ring $R = \mathbb{Z}/p^2\mathbb{Z}$, also compute $\text{Ext}_R^n(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ for every nonnegative integer n .

(44) Let p be a prime integer (strictly greater than 1). For the commutative ring $S = (\mathbb{Z}/p\mathbb{Z})[x]/x^2(\mathbb{Z}/p\mathbb{Z})[x]$, for the canonical surjection of S -modules from S to the quotient ring $S/xS = \mathbb{Z}/p\mathbb{Z}$, what is the kernel? What is one free resolution of $\mathbb{Z}/p\mathbb{Z}$ as an S -module? Use this to compute $\text{Tor}_n^S(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ for every nonnegative integer n .

(45) Let p be a prime integer (strictly greater than 1). For the commutative ring $S = (\mathbb{Z}/p\mathbb{Z})[x]/x^2(\mathbb{Z}/p\mathbb{Z})[x]$, also compute $\text{Ext}_S^n(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ for every nonnegative integer n . Can you distinguish the ring $(\mathbb{Z}/p\mathbb{Z})[x]/x^2(\mathbb{Z}/p\mathbb{Z})[x]$ from the ring $\mathbb{Z}/p^2\mathbb{Z}$ only in terms of Ext and Tor applied to simple (i.e., irreducible) R -modules?

(46) Let R be a principal ideal domain that is properly contained in its fraction field F . Let M be a finitely generated R -module. If we know the R -module $M \otimes_R F$ and each of the R -modules $M \otimes_R (R/f^d R)$ for every noninvertible, nonzero element f of R that is irreducible and for every positive integer d , can we determine M up to isomorphism?

(47) Given your answer to the previous problem, for each finitely generated R -module M and each finitely generated R -module N , can you determine the R -modules $M \otimes_R N$, $\text{Tor}_1^R(M, N)$, $\text{Hom}_R(M, N)$ and $\text{Ext}_R^1(M, N)$ up to isomorphism in terms of the data for M and N from the previous problem?

(48) Let k be a field, let V be a k -vector space of finite dimension, and let $T \in \text{Hom}_{k\text{-Mod}}(V, V)$ be a k -linear endomorphism of V . Let R be the k -subalgebra of $\text{Hom}_{k\text{-Mod}}(V, V)$ generated by T (and the identity matrix). Thus V is a left R -module. Assume that V is a semisimple R -module, i.e., a direct sum of finitely many simple (i.e., irreducible) R -modules. Let S be the k -subalgebra $\text{Hom}_{R\text{-Mod}}(V, V)$ of $\text{Hom}_{k\text{-Mod}}(V, V)$, so that again V is a left S -module. What is the k -subalgebra $\text{Hom}_{S\text{-Mod}}(V, V)$ of $\text{Hom}_{k\text{-Mod}}(V, V)$? (This is part of the “Double Centralizer Theorem” which will be the key step in proving Schur-Weyl duality later in the semester.)

(49) Let k be a field, let V be a k -vector space of finite dimension, and let $T \in \text{Hom}_{k\text{-Mod}}(V, V)$ be a k -linear endomorphism of V . If T is diagonalizable, prove that also $T^n = T \circ \dots \circ T$ is diagonalizable for every positive integer n . However, find an example where $T \circ T$ is diagonalizable even though T is not diagonalizable. In your example, for R and S as above, what is $\text{Hom}_{S\text{-Mod}}(V, V)$?

(50) Continuing the previous problems, find an example where V is a simple R -module, yet for the algebraic closure \bar{k} of k with the induced \bar{k} -linear map $\text{Id}_{\bar{k}} \otimes T$ of the \bar{k} -vector space $\bar{k} \otimes_k V$, this is not a semisimple module over the subring $\bar{k} \otimes_k R$ of $\text{Hom}_{\bar{k}\text{-Mod}}(\bar{k} \otimes_k V, \bar{k} \otimes_k V)$.

(51) For the category **Set** of all sets, what are all of the natural transformations from the identity functor to itself?

(52) For the category **Set** of all sets, what are all of the natural transformations from the identity functor to the functor that sends every set S to the power set $\mathcal{P}(S)$ of S and that sends every function f from a set S to a set \tilde{S} to the function $\mathcal{P}(f)$ from $\mathcal{P}(S)$ to $\mathcal{P}(\tilde{S})$ sending each subset A of S to the f -image subset $f(A)$ of \tilde{S} ?

(53) For each field k , for the category $k\text{-Mod}_f$ of k -vector spaces that have finite dimension, for each k -vector space W that has finite dimension, what are all of the natural transformations from the (exact, additive) identity functor to the (exact, additive) functor $W \otimes_k -$ that sends each k -vector space V of finite dimension to the k -vector space $W \otimes_k V$ and that sends each k -linear transformation f from a k -vector space V to a k -vector space \tilde{V} (both having finite dimension) to the k -linear transformation $\text{Id}_W \otimes_k f$?

(54) For each commutative ring R , for the category $R\text{-Mod}_{fp}$ of finitely presented R -modules, for each finitely presented R -module N , what are all of the natural transformations from the (exact, additive) identity functor to the (right exact, additive) functor $N \otimes_R -$ that sends each finitely presented R -module M to the finitely presented R -module $N \otimes_R M$ and that sends each morphism f of R -modules from a finitely presented R -module M to a finitely presented R -module \widetilde{M} to the morphism $\text{Id}_N \otimes_R f$ of finitely presented R -morphisms?

(55) For each Noetherian commutative ring R , can you recover the ring R from the Abelian category $R\text{-Mod}_{fp}$? If you know about compact objects in a category, can you recover R from the bigger Abelian category $R\text{-Mod}$?

(56) For each Noetherian commutative ring R , the Abelian category $R\text{-Mod}_{fp}$ has enough projective objects. For each finitely presented R -module N , do we have universality of the δ -functor that is the restriction to $R\text{-Mod}_{fp}$ from $R\text{-Mod}$ of the universal δ -functor $\text{Tor}_{\bullet}^R(N, -)$ of $N \otimes_R -$, respectively $\text{Ext}_{R\text{-Mod}}^{\bullet}(-, N)$ of $\text{Hom}_{R\text{-Mod}}(-, N)$?

(57) For each Noetherian commutative ring R , the Abelian category $R\text{-Mod}_{fp}$ typically does not have enough injective objects. For each finitely presented R -module N , do we have universality of the δ -functor that is the restriction to $R\text{-Mod}_{fp}$ from $R\text{-Mod}$ of the universal δ -functor $\text{Ext}_{R\text{-Mod}}^{\bullet}(N, -)$ of $\text{Hom}_{R\text{-Mod}}(N, -)$? (The shortest proof that I know uses that the Ext functors are “balanced”, which is not something we have studied.)

(58) For each algebraically closed field k , for each k -vector space of finite dimension k , for each k -linear endomorphism T of V , what is the relationship between the Jordan normal form of T and the Jordan normal form of the induced k -linear endomorphism of precomposition by T on $V^{\vee} := \text{Hom}_{k\text{-Mod}}(V, k)$?

(59) For each algebraically closed field k , for each k -vector space of finite dimension, also each of the tensor powers $V^{\otimes_k n} = V \otimes_k \cdots \otimes_k V$ is a k -vector space of finite dimension. What is the relation between the Jordan normal form of a k -linear endomorphism T of V and the Jordan normal form of the k -linear endomorphisms $T^{\otimes_k n}$ of each $V^{\otimes_k n}$? In particular, define $\Lambda_k^{\bullet}(V)$ to be the quotient associative, unital k -algebra of the associative, unital tensor k -algebra $V^{\otimes_k \bullet} := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V^{\otimes_k n}$ by the two sided ideal generated by all elements $a \otimes v \otimes b \otimes v \otimes c$ for elements v of V and elements a, b and c of $V^{\otimes_k \bullet}$. What are the Jordan normal forms of the k -linear endomorphisms $\Lambda_k^n(T)$ of $\Lambda_k^n(V)$?

(60) Let n be a positive integer, and let R be the commutative integral domain $\mathbb{Z}[t_{i,j}]_{1 \leq i, j \leq n}$, i.e., a polynomial ring in n^2 variables over \mathbb{Z} . Denote the fraction field by F . Let T be the $n \times n$ matrix with entries in R whose (i, j) entry is $t_{i,j}$. Denote the characteristic polynomial $\det(x\text{Id}_{n \times n} - T)$ of T by

$$\chi_T(x) = x^n + \cdots + (-1)^r c_r x^{n-r} + \cdots + (-1)^n c_n,$$

where each c_r is an element of R . For the Cayley-Hamilton Theorem applied to the F -linear transformation T of $F^{\otimes n}$, deduce that $\chi_T(T)$ equals the zero $n \times n$ matrix. Now, for every commutative

ring S , for every $n \times n$ matrix M with entries in S , for the associated unique morphism f_M of commutative rings from R to S that sends each element $t_{i,j}$ to the (i,j) -entry of M , show that $f(T)$ equals M . Deduce that $f(c_r)$ equals the corresponding coefficient of the characteristic polynomial $\chi_M(x)$ of M , i.e., $f(\chi_T(x))$ equals $\chi_M(x)$. What is $\chi_M(M)$? Does it matter if S is an integral domain, or even if S is a reduced ring (i.e., do nonzero nilpotent elements of S affect this proof)? This is one instance of a general principle in algebra: if the “universal” instance of some algebraic object has good properties, then results deduced for that universal instance can be “specialized” to instances that do not necessarily have good properties.

(61) For each algebraically closed field k , for each positive integer n , prove that the subset of $\mathbf{Mat}_{n \times n, k} := \mathbf{Hom}_{k\text{-Mod}}(k^{\oplus n}, k^{\oplus n})$ of all nilpotent k -linear endomorphisms is the common zero locus of the n homogeneous polynomial functions from $\mathbf{Mat}_{n \times n, k}$ to k given by the n non-leading coefficients of the characteristic polynomial. What if the field is not algebraically closed?

(62) For every commutative ring R , prove that every right exact, additive functor \mathbf{F} from $R\text{-Mod}_{fp}$ to itself is naturally isomorphic to $N \otimes_R -$ for some finitely presented R -module N .

(63) For every commutative ring R , prove that every right exact, additive functor \mathbf{F} from $R\text{-Mod}$ to itself that commutes with arbitrary colimits (e.g., as holds for left adjoint functors of additive functors) is naturally isomorphic to $N \otimes_R -$ for some R -module N .

(64) For every commutative ring R , prove that every additive functor from $(R\text{-Mod})_{fp}^{\text{opp}}$ to $(R\text{-Mod})_{fp}$ that sends right exact sequences to left exact sequences is naturally isomorphic to $\mathbf{Hom}_{R\text{-Mod}}(-, N)$ for some finitely presented R -module N .

(65) For every commutative ring R , prove that every left exact additive functor from $(R\text{-Mod})^{\text{opp}}$ to $R\text{-Mod}$ that commutes with arbitrary (inverse) limits (e.g., as holds for right adjoint functors of additive functors) is naturally isomorphic to $\mathbf{Hom}_{R\text{-Mod}}(-, N)$ for some R -module N .

(66) Let R be the commutative ring that is a countable product of copies of \mathbb{C} , i.e., $\prod_{n \in \mathbb{Z}} \mathbb{C}$. For which nonnegative integers n do there exist R -modules M and N such that $\text{Ext}_R^n(M, N)$ is nonzero? The dependence of the answer on our foundations for set theory is one of (many) arguments in favor of *condensed mathematics* as developed by Clausen and Scholze.

(67) Let k be a field, and let Γ be a group. A k -**representation** of Γ is an ordered pair (V, ρ) of a k -vector space V and a group homomorphism ρ from Γ to the group $\mathbf{Aut}_{k\text{-Mod}}(V)$ of k -linear automorphisms of V . For k -representations (V, ρ) and $(\tilde{V}, \tilde{\rho})$, the subset $\mathbf{Hom}_{k\text{-Mod}}(V, \tilde{V})^\Gamma$ of **morphisms** of k -representations of Γ from (V, ρ) to $(\tilde{V}, \tilde{\rho})$ is a k -linear transformation f from V to \tilde{V} that commutes with ρ and $\tilde{\rho}$ is the set of all elements f of $\mathbf{Hom}_{k\text{-Mod}}(V, \tilde{V})$ such that, for every element γ of Γ , the composite $f \circ \rho_\gamma$ equals the composite $\tilde{\rho}_\gamma \circ f$. Prove that this subset is a k -vector subspace. Prove that composition of k -linear transformations preserves the subsets of morphisms of k -representations. Prove that Id_V is a morphism of k -representations of Γ from (V, ρ)

to itself, for every k -representation (V, ρ) . Deduce that the category of k -representations of Γ is an additive category with a faithful, additive functor to $k\text{-Mod}$ sending every k -representation (V, ρ) of Γ to V and acting as the inclusion on sets of morphisms. Show that the kernel k -vector space and cokernel k -vector space of every morphism f of k -representations of Γ have unique structures of k -representations of Γ such that both q_f and p_f are morphisms of k -representations of Γ . Deduce that the additive category of k -representations of Γ is an Abelian category, and the faithful, additive functor is exact. Also the full subcategory of k -representations of Γ that are finite dimensional is a Serre subcategory.

(68) Let $k[\Gamma]$ be the k -vector space of all set functions u from Γ to k such that the **support** $u^{\text{pre}}(k \setminus \{0\})$ of u is a finite subset of Γ . For such functions u and \tilde{u} , since the support of $u + \tilde{u}$ is contained in the union of the supports of u and \tilde{u} , deduce that $u + \tilde{u}$ is another such function. Similarly, for every element c of k , also the support of $c \cdot u$ is contained in the support of u , hence $c \cdot u$ is another such function. Altogether, $k[\Gamma]$ is a k -vector space. Prove that also it is an associative, unital k -algebra for the **convolution product**,

$$- * - : k[\Gamma] \times k[\Gamma] \rightarrow k[\Gamma], \quad (u * \tilde{u})(\tilde{\gamma}) = \sum_{\gamma \in \Gamma} u(\gamma) \tilde{u}(\gamma^{-1} \tilde{\gamma}).$$

Also prove that one k -vector space basis consists of the functions δ_γ such that $\delta_\gamma(\tilde{\gamma})$ equals 1 if $\tilde{\gamma}$ equals γ , and otherwise it equals 0.

(69) For every k -representation (V, ρ) of Γ , prove that the following function makes V into a left $k[\Gamma]$ -module,

$$\cdot_\rho : k[\Gamma] \times V \rightarrow V, \quad u \cdot \vec{v} = \sum_{\gamma \in \Gamma} u(\gamma) \rho_\gamma(\vec{v}).$$

Conversely, for every left $k[\Gamma]$ -module V considered as a k -vector space, prove that each endomorphism $\delta_\gamma \cdot -$ from V to V is a k -linear automorphism (with $\delta_{\gamma^{-1}} \cdot -$ as its inverse), and the induced function ρ from Γ to $\text{Aut}_{k\text{-Mod}}(V)$ makes V into a k -representation of Γ . Altogether, this gives an equivalence of categories from the category of k -representations of Γ to the category $k[\Gamma]\text{-Mod}$ of left $k[\Gamma]$ -modules.

(70) Prove that for every k -representation (V, ρ) of Γ and for every element \vec{v} of V , there exists a unique morphism of k -representations of Γ from $k[\Gamma]$ (with its left regular representation) to (V, ρ) sending δ_e to \vec{v} . Deduce that the k -algebra of natural transformations from the identity functor to itself for the category $k[\Gamma]\text{-Mod}$ of k -representations of Γ equals the center $Z(k[\Gamma])$ of $k[\Gamma]$. In particular, if Γ is an Abelian group, this recovers $k[\Gamma]$ from the Abelian category of k -representations of Γ .

(71) For every k -representation (V, ρ) of Γ , define $p_{V, \rho} : V \twoheadrightarrow V_\Gamma$ to be the largest quotient k -representation of (V, ρ) on which Γ acts trivially, i.e., the quotient by the k -subspace spanned by

all elements $\rho_\gamma(\vec{v}) - \vec{v}$ for all elements γ of Γ and all elements \vec{v} of V . For every morphism f of k -representations of Γ from (V, ρ) to $(\tilde{V}, \tilde{\rho})$, since $p_{\tilde{V}, \tilde{\rho}} \circ f$ is a morphism to a trivial k -representation of Γ , it factors uniquely through the quotient $p_{V, \rho}$, i.e., there exists a unique k -linear transformation f_Γ from V_Γ to \tilde{V}_Γ such that $f_\Gamma \circ p_{V, \rho}$ equals $p_{\tilde{V}, \tilde{\rho}} \circ f$. Prove that this defines a right exact, additive functor from $k[\Gamma] - \mathbf{Mod}$ to $k - \mathbf{Mod}$.

(72) Show that the previous functor is left adjoint to the functor Triv from $k - \mathbf{Mod}$ to $k[\Gamma] - \mathbf{Mod}$ sending every k -vector space V to the k -representation $(V, \text{const}_{\text{Id}_V})$ of Γ that sends every element γ of Γ to the identity k -linear transformation Id_V of V .

(73) Find a right $k[\Gamma]$ -module N such that the functor $(-)_\Gamma$ is naturally isomorphic to $N \otimes -$. Is N also a left $k[\Gamma]$ -module? Is N a quotient of $k[\Gamma]$ by a two sided ideal (so that N is a quotient k -algebra)? Is this quotient k -algebra commutative?

(74) Find an example of a field k and a group Γ such that $(-)_\Gamma$ is not an exact functor, i.e., such that the $k[\Gamma]$ -module N is not flat. Can you find an example where Γ is a finite group?

(75) For every k -representation (V, ρ) of Γ , define $q_{V, \rho} : V^\Gamma \hookrightarrow V$ to be the largest k -subrepresentation of (V, ρ) on which Γ acts trivially. For every morphism f of k -representations of Γ from (V, ρ) to $(\tilde{V}, \tilde{\rho})$, since the composition $f \circ q_{V, \rho}$ is a morphism from a trivial k -representation of Γ , it factors uniquely through $q_{\tilde{V}, \tilde{\rho}}$, i.e., there exists a unique k -linear transformation f^Γ from V^Γ to \tilde{V}^Γ such that $q_{\tilde{V}, \tilde{\rho}} \circ f^\Gamma$ equals $f \circ q_{V, \rho}$. Prove that this defines a left exact, additive functor from $k[\Gamma] - \mathbf{Mod}$ to $k\mathbf{Mod}$.

(76) Show that the previous functor is right adjoint to the functor Triv from $k - \mathbf{Mod}$ to $k[\Gamma] - \mathbf{Mod}$.

(77) Find a left $k[\Gamma]$ -module N such that the functor $(-)^{\Gamma}$ is naturally isomorphic to $\text{Hom}_{k[\Gamma] - \mathbf{Mod}}(N, -)$.

(78) Assume that Γ is a finite group whose order is an invertible element of k . Prove that the quotient module homomorphism from $k[\Gamma]$ to N splits. Deduce that for every k -representation (V, ρ) , the natural k -linear transformation from V^Γ to V_Γ is an isomorphism, so that (V, ρ) is a direct sum of V^Γ and the k -subrepresentation $\text{Ker}(p_{V, \rho})$. Note that $\text{Ker}(p_{V, \rho})^\Gamma$ is a zero subspace.

(79) For every k -representation (V, ρ) of Γ , for the k -vector space $\text{Hom}_{k - \mathbf{Mod}}(V, k)$, define the k -representation structure ρ^\vee by $\rho_\gamma^\vee(\chi) = \chi \circ \rho_{\gamma^{-1}}$ for every k -linear functional χ of V . Prove that this is a left exact, additive functor from $(k - \mathbf{Mod})^{\text{opp}}$ to $k - \mathbf{Mod}$.

(80) For the previous functor, find a k -algebra homomorphism S from $k[\Gamma]$ to $k[\Gamma]^{\text{opp}}$ such that the functor is naturally isomorphic to $\text{Hom}_{k[\Gamma] - \mathbf{Mod}}(-, k[\Gamma]^{\text{opp}})$ where we produce a left module structure for $k[\Gamma]$ from the natural right $k[\Gamma]^{\text{opp}}$ -module structure (arising from the fact that $k[\Gamma]^{\text{opp}}$ is a $k[\Gamma] - (k[\Gamma])^{\text{opp}}$ -bimodule via S and the right regular representation). The k -algebra homomorphism is the **antipode**.

(81) For every ordered pair $((V, \rho), (\tilde{V}, \tilde{\rho}))$ of k -representations of Γ , define a structure $\rho \otimes \tilde{\rho}$ of k -representation of Γ on the k -vector space $V \otimes_k \tilde{V}$ by $(\rho \otimes \tilde{\rho})_\gamma$ equals $\rho_\gamma \otimes \tilde{\rho}_\gamma$. Prove that this is bifunctorial.

(82) For the previous bifunctor, find a morphism Δ of associative, unital k -algebras from $k[\Gamma]$ to $k[\Gamma] \otimes_k k[\Gamma]$ such that for the binatural structure of left $k[\Gamma] \otimes_k k[\Gamma]$ -module on $V \otimes_k \tilde{V}$, the left $k[\Gamma]$ -module structure $\rho \otimes \tilde{\rho}$ is obtained by precomposing that binatural structure with Δ . The k -algebra homomorphism Δ is the **comultiplication**.

(83) Now assume that Γ is finite and the order is invertible in k . Read the definition of **Hopf k -algebra** (not the graded analogue), and check that the structures defined above on $k[\Gamma]$ form a Hopf k -algebra (which happens to be cocommutative). Notice that we recover this Hopf k -algebra from the Abelian category of finite dimensional k -representations of Γ with its natural enrichment over the category of k -vector spaces, from the **fiber functor** to $k\text{-Mod}$ sending each (V, ρ) to V , from the functor of Γ -invariants (which is equivalent to the functor of Γ -coinvariants by our hypotheses and which determines the counit of the Hopf algebra), from the dual k -representation (contravariant) endofunctor of this Abelian category (which determines the antipode), and from the tensor product operation on this category (which determines the comultiplication), altogether making this category into a rigid monoidal category with fiber functor. This is the template for other examples, e.g., the category of finite dimensional representations over a characteristic 0 field k of a finite dimensional Lie k -algebra (the Hopf algebra produced in the analogous way is then the universal enveloping algebra), or the Drinfeld-Jimbo deformations of these categories when the Lie k -algebra is semisimple (the corresponding deformation of the Hopf algebra is one approach to quantum groups).

(84) For every commutative ring R , for every ideal \mathfrak{a} , and for every R -module M , the R -submodule $q_{M, \mathfrak{a}} : H_{\mathfrak{a}}^0(M) \rightarrow M$ consists of all elements x of M such that for some nonnegative integer ℓ , the product set $\mathfrak{a}^\ell \cdot \{x\}$ equals $\{0\}$, i.e., x is annihilated by the ideal \mathfrak{a}^ℓ . For every R -module homomorphism u from M to an R -module \tilde{M} , for every element x of $H_{\mathfrak{a}}^0(M)$, since $\mathfrak{a}^\ell \cdot \{x\}$ equals $\{0\}$ for some integer ℓ , also $\mathfrak{a}^\ell \cdot \{u(x)\} = u(\mathfrak{a}^\ell \cdot \{x\})$ equals $\{0\}$. So there is a unique R -module homomorphism $H_{\mathfrak{a}}^0(u)$ from $H_{\mathfrak{a}}^0(M)$ to $H_{\mathfrak{a}}^0(\tilde{M})$ such that $q_{\tilde{M}, \mathfrak{a}} \circ H_{\mathfrak{a}}^0(u)$ equals $u \circ q_{M, \mathfrak{a}}$. Prove that this defines a left exact, additive functor $H_{\mathfrak{a}}^0$ from $R\text{-Mod}$ to itself.

(85) Prove that there exists a universal δ -functor $H_{\mathfrak{a}}^\bullet$ from $R\text{-Mod}$ to itself concentrated in degrees ≥ 0 that extends $H_{\mathfrak{a}}^0$. This is called **local cohomology** with respect to \mathfrak{a} .

(86) If you know about colimits (i.e., direct limits), prove that $H_{\mathfrak{a}}^0(M)$ equals the colimit over all nonnegative integers ℓ of $\text{Hom}_{R\text{-Mod}}(\mathfrak{a}^\ell, M)$. Since the colimit is an exact functor for filtering directed systems, it follows that $H_{\mathfrak{a}}^q(M)$ equals the colimit of $\text{Ext}_{R\text{-Mod}}^q(\mathfrak{a}^\ell, M)$.

(87) In defining the colimit above, show that we get the same answer if instead of the (decreasing) sequence of ideal $(\mathfrak{a}^\ell)_{\ell \in \mathbb{Z}_{\geq 0}}$ we use the sequence of all ideals $\tilde{\mathfrak{a}}$ (with the subset partial order) such

that there exists an integer ℓ with $\mathfrak{a} \subseteq \tilde{\mathfrak{a}} \subseteq \mathfrak{a}^\ell$. In particular, assume that \mathfrak{a} is generated by a regular sequence $u = (u_1, \dots, u_n)$. Then also $u^\ell = (u_1^\ell, \dots, u_n^\ell)$ is a regular sequence for every positive integer ℓ . Denoting by \mathfrak{a}_ℓ the ideal generated by this regular sequence, show that we can compute $H_{\mathfrak{a}}^q(M)$ as the colimit over all ℓ of $\text{Ext}_R^q(\mathfrak{a}_\ell, M)$. Of course these equal the degree $(q-1)$ -cohomology modules of the complex $\text{Hom}_R(K^\bullet(u_\ell), M)$ obtained by applying $\text{Hom}_R(-, M)$ to the Koszul complex.

(88) As a very special case, for every commutative ring k , for each polynomial ring $R = k[x_0, \dots, x_r]$, for the regular sequence $u = (x_0, \dots, x_r)$ that generates the maximal ideal \mathfrak{a} , for every finitely generated, \mathbb{Z} -graded module M , the local cohomology groups $H_{\mathfrak{a}}^q(M)$ are the same as the sheaf cohomology groups $\bigoplus_{\ell \geq 0} H^{q-1}(\mathbb{P}_k^n, \widetilde{M}(\ell))$, where \widetilde{M} is the coherent sheaf \widetilde{M} associated to M on the projective space \mathbb{P}_k^n with homogeneous coordinate ring R . In particular, the natural self-duality of the Koszul complex establishes Serre duality for the invertible sheaves $\mathcal{O}(\ell)$ on projective space (and then the “standard” proof of Serre duality for projective space bootstraps from there, interpreting the theorem as an equivalence of two δ -functors, and using Serre duality for invertible sheaves to establish coexactness).

(89) Let R be an integral domain that is properly contained in its fraction field Q . Prove that the elements of R that are nonzerodivisors are precisely the elements of $R \setminus \{0\}$. Denoting by R^\times the subgroup of multiplicatively invertible elements of R (which always contains the multiplicative identity 1), prove that the set of noninvertible, nonzerodivisors equals $R \setminus (\{0\} \sqcup R^\times)$.

(90) Let R be an integral domain that is properly contained in its fraction field Q . For all elements f_1, \dots, f_n of R that are noninvertible nonzerodivisors, if $f_i R + f_j R$ equals all of R for each $i \neq j$, prove that there exist elements noninvertible nonzerodivisors $u(f_i, f_j)$ and $u(f_j, f_i)$ in R such that $u(f_i, f_j)f_i + u(f_j, f_i)f_j$ equals 1. Moreover, $u(f_i, f_j)$ and $u(f_j, f_i)$ are well-defined modulo the intersection ideal $(f_i R) \cap (f_j R)$.

(91) Continuing the previous problem, for all ordered n -tuples of integers (a_1, \dots, a_n) and (b_1, \dots, b_n) such that $\min(a_i, b_i) = 0$ for every $i = 1, \dots, n$, for the products $f = f_1^{a_1} \dots f_n^{a_n}$ and $g = f_1^{b_1} \dots f_n^{b_n}$, prove that there exist noninvertible nonzerodivisors $u(f, g)$ and $u(g, f)$ in R such that $u(f, g)f + u(g, f)g$ equals 1.

(92) For all (noninvertible and nonzero) irreducible elements f_1, \dots, f_n of a principal ideal domain R such that $f_i R$ is distinct from $f_j R$ for all $i \neq j$, for all ordered n -tuples of integers (d_1, \dots, d_n) and (e_1, \dots, e_n) such that $\min(d_i, e_i) = 0$ for every $i = 1, \dots, n$, for the product $f = f_1^{e_1} \dots f_n^{e_n}$ and $g = f_1^{d_1} \dots f_n^{d_n}$, prove that there exist elements $u(f, g)$ and $u(g, f)$ of R such that $u(f, g)f + u(g, f)g$ equals 1.

(93) For all (noninvertible and nonzero) irreducible elements f_1, \dots, f_n of a principal ideal domain R such that $f_i R$ is distinct from $f_j R$ for all $i \neq j$, for every ordered n -tuple (e_1, \dots, e_n) of positive integers, for the product $f = f_1^{e_1} \dots f_n^{e_n}$, prove that there exist elements $\epsilon_1, \dots, \epsilon_n$ of R , unique modulo

fR , such that $\epsilon_1 + \dots + \epsilon_n - 1$, $\epsilon_i^2 - \epsilon_i$, $\epsilon_i \epsilon_j$, and $f_i^{\epsilon_i} \epsilon_i$ are all in the ideal fR for every i and for every $j \neq i$. Thus the R -module $\epsilon_i R/fR$ is isomorphic to $R/f_i^{\epsilon_i} R$ for every i . Altogether, deduce that the R -algebra R/fR is isomorphic to the product R -algebra $(R/f_1^{\epsilon_1} R) \times \dots \times (R/f_n^{\epsilon_n} R)$.

(94) Continuing the previous problem, for every R -module M , prove that M/fM is isomorphic to the product module $(M/f_1^{\epsilon_1} M) \times \dots \times (M/f_n^{\epsilon_n} M)$ functorially in M . In particular, if fM equals $\{0\}$, deduce that M is isomorphic to the product module. Also, every R -submodule M' of M is a direct product of the R -submodules $M'_i = M'/f_i M'$ of the R -module $M_i = M/f_i M$.

(95) For every commutative ring R , for every prime element f of R (that is a noninvertible nonzerodivisor), an R -module M is f -**primary** if (and only if) every element of M is annihilated by f^d for some nonnegative integer d (depending on the element). For a set of generators $\{m_j\}_j$ of M as an R -module, prove that M is f -primary if and only if, for every j , there exists a nonnegative integer d_j such that $f^{d_j} m_j$ equals 0.

(96) Continuing the previous problem, assuming that $\{m_j\}_j$ is a finite set of generators of an f -primary R -module M , for the least nonnegative integers d_j with $f^{d_j} m_j$ equal to 0, prove that $d = \max\{d_j\}_j$ is the least nonnegative integer such that $f^d M$ equals $\{0\}$.

(97) Continuing the previous problem, for every d_i that equals d , use induction on the number of generators to prove that there exists an R -submodule M' of M such that the R -module homomorphism from M' to M/Rm_i is an isomorphism, and thus M is isomorphic to $M' \oplus (R/m_i)$. Deduce that every finitely generated, f -primary R -module M is isomorphic to a direct sum $(R/f^{d_1} R) \times \dots \times (R/f^{d_\ell} R)$ of cyclic f -primary R -modules for a sequence of positive integers $d = d_1 \geq \dots \geq d_\ell \geq 1$.

(98) For every commutative ring R , for every irreducible element f (that is a noninvertible nonzerodivisor), for every positive integer $d - 1 > 0$, deduce that the R -modules $\text{Hom}_R(R/f^{d-1} R, R/fR)$, $\text{Ext}_R^1(R/f^{d-1} R, R/fR)$, $\text{Hom}_R(R/fR, R/f^{d-1} R)$ and $\text{Ext}_R^1(R/fR, R/f^{d-1} R)$ are all isomorphic to R/fR . Also $\text{Ext}_R^q(R/f^{d-1} R, R/fR)$ and $\text{Ext}_R^q(R/fR, R/f^{d-1} R)$ are zero modules for $q > 1$.

(99) For every principal ideal domain R that is properly contained in its fraction field Q , for every irreducible element f of R (that is noninvertible and nonzero), for every finitely generated, f -primary R -module M whose annihilator equals f^d for a positive integer d , prove that the annihilator R -submodules $A_b^f(M) = \text{Ker}(M \xrightarrow{f^{b,-}} M)$ for $b = 0, \dots, d$, form an increasing filtration of M by R -submodules whose associated graded R -modules $A_b^f(M)/A_{b-1}^f(M)$ are each finite dimensional vector spaces over the **residue field** R/fR . Also prove that the image R -submodule $I_f^b(M) = \text{Image}(M \xrightarrow{f^{b,-}} M)$ form a decreasing filtration of M by R -submodules whose associated graded R -modules I_f^b/M_f^{b+1} are each finite dimensional vector spaces over R/fR . Prove that each R -submodule $A_b^f(M)$ contains the R -submodule $I_f^{d-b}(M)$, and $A_b^f(M)$ equals $I_f^{d-b}(M)$ for all $b = 0, \dots, d$ if and only if M equals a direct sum of R -submodules each isomorphic to $R/f^d R$. More generally, each

$(A_b^r(M) \cap I_f^c(M)) / ((A_{b-1}^r(M) \cap I_f^c(M)) + (A_b^r(M) \cap I_f^{c+1}(M)))$ is a finite dimensional R/fR -vector space. The dimensions of these vector spaces for each b and c give numerical invariants of the finitely generated, f -primary R -module M .

(100) Continuing the previous problem, prove that the R -module $M = M/f^d M$ is isomorphic to $(R/f^{d_1} R) \times \cdots \times (R/f_i^{d_i})$ for nonnegative integers $d = d_1 \geq \cdots \geq d_\ell$. In terms of this sequence of integers, compute the dimensions in the previous problem.

(101) When R is the principal ideal domain \mathbb{Z} properly contained in its fraction field \mathbb{Q} , deduce that every finitely generated, torsion Abelian group M is isomorphic to $(M/f_1^{e_1} M) \times \cdots \times (M/f_n^{e_n} M)$ for pairwise distinct prime integers $f_1, \dots, f_n > 1$ and positive integers e_1, \dots, e_n such that $f_i^{e_1-1} M / f_i^{e_i} M$ is nonzero. Moreover, each $M/f_i^{e_i} M$ is isomorphic to $(\mathbb{Z}/f_1^{d_{i,1}} \mathbb{Z}) \times \cdots \times (\mathbb{Z}/f_i^{d_{i,\ell_i}} \mathbb{Z})$ for positive integers $e_i = d_{i,1} \geq \cdots \geq d_{i,\ell_i} \geq 1$.

(102) For every field F , for the principal ideal domain $F[t]$, deduce that a left R -module M has finite dimension as an F -vector space if and only if M is isomorphic to $(M/f_1^{e_1} M) \times \cdots \times (M/f_n^{e_n} M)$ for pairwise distinct, irreducible, monic polynomials $f_i \in F[t]$ of degrees $\deg(f_i) = r_i \geq 1$ and for positive integers e_i , called the **geometric multiplicity** of f_i , such that each R -module $M/f_i^{e_i} M$ is isomorphic to a product $(R/f_i^{d_{i,1}}) \times \cdots \times (R/f_i^{d_{i,\ell_i}})$ for a positive integer ℓ_i and for integers $e_i = d_{i,1} \geq \cdots \geq d_{i,\ell_i} \geq 1$. In particular, the minimal polynomial of the F -linear operator T of multiplication by t on M equals $f_1^{e_1} \cdots f_n^{e_n}$, and the characteristic polynomial equals $f_1^{d_1} \cdots f_n^{d_n}$ for the positive integers $d_i = d_{i,1} + \cdots + d_{i,\ell_i}$, called the **algebraic multiplicity** of f_i . Thus the minimal polynomial divides the characteristic polynomial, and they have the same irreducible factors.

(103) For every field F , for every irreducible, monic polynomial f in $F[t] \setminus F$ of degree r , for every positive integer d , for the ordered F -basis $(\bar{1}, \bar{t}, \dots, \bar{t}^{d-1})$ of $F[t]/f^d F[t]$, compute the matrix representative of the F -linear operator of multiplication by t . This matrix is the **companion matrix** of the primary, monic polynomial f^d .

(104) Continuing the previous problem, if $f = t - \lambda$ prove that the companion matrix is conjugate to a **Jordan block**, i.e., a $d \times d$ matrix with λ in each diagonal entry, with 1 in each entry directly above the main diagonal, and with 0 in every other entry. Show that one ordered basis giving this matrix representative is $(\overline{t - \lambda}^{d-1}, \dots, \overline{t - \lambda}, 1)$.

(105) For every field F , for $F[t]$ -modules M and N that have finite dimension as F -vector spaces, for every irreducible polynomial f that is either a factor of the minimal polynomial of M or of the minimal polynomial of N , prove that the f -primary component $(M \oplus N)/f^d(M \oplus N)$ of $M \oplus N$ (for d equal to the geometric multiplicity of f in $M \oplus N$) equals the direct sum of the f -primary component of M and the f -primary component of N . Thus, the decomposition into primary components for $M \oplus N$ equals the direct sum of the decompositions for M and N .

(106) Continuing the previous problem, for the direct sum decompositions of the f -primary components into cyclic, f -primary components, say $M/f^d M = (R/f^{d_1} R) \times \cdots \times (R/f^{d_\ell} R)$ and $N/f^d N = (R/f^{c_1} R) \times \cdots \times (R/f^{c_m} R)$ for positive integers ℓ and m and for integers $d \geq d_1 \geq \cdots \geq d_\ell \geq 1$ and $d \geq e_1 \geq \cdots \geq e_m \geq 1$, prove that one direct sum decomposition of $(M \oplus N)/f^d(M \oplus N)$ into cyclic f -primary components is obtained by concatenating, i.e., $(R/f^{d_1} R) \times \cdots \times (R/f^{d_\ell} R) \times (R/f^{c_1} R) \times \cdots \times (R/f^{c_m} R)$. (These are not in the usual order, but we can permute summands to put them in the correct order.)

(107) For every field F , for every $F[t]$ -module M that has finite dimension as an F -vector space, for every F -linear endomorphism U of M that commutes with the F -linear endomorphism T of multiplication by t , prove that U maps every f -primary submodule of M back to itself for every irreducible factor f of the minimal polynomial.

(108) For every field F , for every monic, irreducible element f of $F[t]$, for every f -primary $F[t]$ -module M that has finite dimension as an F -vector space, for every F -linear endomorphism U of M that commutes with T , prove that U maps each piece of each of the two filtrations back to itself (quite possibly into a deeper piece of the filtration).

(109) Continuing the previous problems, if every irreducible factor f of the minimal polynomial of T is linear and if every geometric multiplicity equals the algebraic multiplicity, prove that U commutes with T if and only if U maps every f -primary submodule of M back to itself. Find examples where this fails if some of the hypotheses do not hold.

(110) For every field F , for every left $F[t]$ -module M that has finite F -dimension, if every irreducible factor f_i of the minimal polynomial is linear, i.e., $f_i(t) = t - \lambda_i$, so that also the minimal polynomial f equals $(t - \lambda_1)^{e_1} \cdots (t - \lambda_n)^{e_n}$, prove that there exists an ordered F -basis for M with respect to which the matrix representative of T is in block form, with Jordan blocks of size $d_{i,j}$ with eigenvalue λ_i for every $i = 1, \dots, n$ and for every $j = 1, \dots, \ell_i$. Compute that the characteristic polynomial of T equals $(t - \lambda_1)^{d_1} \cdots (t - \lambda_n)^{d_n}$ for the integers $d_i := d_{i,1} + \cdots + d_{i,\ell_i}$. For the linear transformation S that is in block form for the same ordered basis whose block is $\lambda_i \text{Id}_{d_{i,j} \times d_{i,j}}$ for each Jordan $d_{i,j} \times d_{i,j}$ block with eigenvalue λ_i , deduce that S is diagonalizable, that $S \circ T$ equals $T \circ S$, and the difference $N = T - S$ is a nilpotent F -linear operator whose Jordan block is a $d_{i,j} \times d_{i,j}$ Jordan block with diagonal entries 0 for each Jordan $d_{i,j} \times d_{i,j}$ block of T with eigenvalue λ_i . Thus T equals $S + N$ with N nilpotent, with S diagonalizable, and with $S \circ N$ equals $N \circ S$. This is the **Jordan-Chevalley decomposition** of T , often called the **semisimple-nilpotent decomposition** of T .

(111) For every polynomial $f = (t - \lambda_1)^{e_1} \cdots (t - \lambda_n)^{e_n}$ in $F[t]$ with pairwise distinct elements λ_i of F , use the Chinese Remainder Theorem to prove that there exists a unique element $\lambda_i \epsilon_i$ in $F[t]/f \cdot F[t]$ whose restriction to $F[t]/(t - \lambda_i)^{e_i} F[t]$ is λ_i and whose restriction to every other factor is zero. Thus, $(t - \lambda_i) \epsilon_i$ restricts to the nilpotent part of T on $F[t]/(t - \lambda_i)^{e_i} F[t]$ and restricts

to zero on every other factor. Deduce that there exist unique elements $\sigma(t)$ in $F[t]/f \cdot F[t]$ and $\nu(t)$ in $F[t]/f \cdot F[t]$ such that $\sigma(T)$ equals S and $\nu(T)$ equals N .

(112) For every field F , for every $F[t]$ -module M that has finite F -dimension and whose minimal polynomial f has a factorization into linears $(t - \lambda_1)^{e_1} \cdots (t - \lambda_n)^{e_n}$, for every F -linear endomorphism U of M , deduce that U commutes with T if and only if both U commutes with $S = \sigma(T)$ and U commutes with $N = \nu(T)$.

(113) For every field F , for every $F[t]$ -module M that has finite F -dimension such that $T = S$ is diagonalizable, for every F -linear endomorphism U of M , prove that U commutes with $T = S$ if and only if U maps every eigenspace of $T = S$ back to itself.

(114) For every field F , for every $F[t]$ -module M that has finite F -dimension such that the minimal polynomial f has a factorization $(t - \lambda_1)^{e_1} \cdots (t - \lambda_n)^{e_n}$ for pairwise distinct elements λ_i of F and for positive integers e_i , for every F -linear endomorphism U of M , deduce that U commutes with T if and only if both U maps every generalized eigenspace back to itself and U commutes with N .

(115) For every field F , for every $F[t]$ -module M that has finite F -dimension such that $T = N$ is a nilpotent endomorphism with minimal polynomial t^e for some positive integer e , prove that M is isomorphic as an $F[t]$ -module to a direct sum $(F[t]/t^{d_1}F[t])\mathbf{e}_1 \times \cdots \times (F[t]/t^{d_\ell}F[t])\mathbf{e}_\ell$ for elements $\mathbf{e}_1, \dots, \mathbf{e}_\ell$ of M and for integers $e = d_1 \geq \cdots \geq d_\ell \geq 1$, for every ordered ℓ -tuple $(\mathbf{u}_1, \dots, \mathbf{u}_\ell)$ of elements of M such that $T^{d_i}(\mathbf{u}_i)$ is zero for every $i = 1, \dots, \ell$, prove that there exists a unique F -linear transformation U of M that commutes with N and such that $U(\mathbf{e}_i)$ equals \mathbf{u}_i for each $i = 1, \dots, \ell$. In this sense, altogether we have a characterization of the centralizer of every F -linear endomorphism T of a finite dimensional F -vector space M (in particular, we can describe the F -algebra of elements commuting with T in terms of the integers $d_{i,j}$ introduced earlier).

(116) A field F is **perfect** if (and only if) either F has characteristic 0 or both F has characteristic $p > 0$ and the p -power function from F to itself is a bijection. For every perfect field F , for every F -linear endomorphism T of a finite dimensional F -vector space N , there exist F -linear endomorphisms S and N of M such that $S \circ N$ equals $N \circ S$, such that T equals $S + N$, such that N is nilpotent, and such that the base change of S from F to the algebraic closure \overline{F} is diagonalizable. This is the **Jordan-Chevalley decomposition**, and it is unique. Conversely, for every element a of $F \setminus F^p$, prove that the $F[t]$ -module $M = F[t]/(t^p - a)F[t]$ has no Jordan-Chevalley decomposition.

(117) More generally, for every field F of characteristic $p > 0$ and every irreducible, monic element f of $F[t]$, prove that the $F[t]$ -module $M = F[t]/fF[t]$ has a Jordan-Chevalley decomposition if and only if f is separable, i.e., if and only if the formal derivative $f'(t)$ of $f(t)$ is nonzero. In this case, prove that N is zero and the base change of $T = S$ to the algebraic closure \overline{F} has one eigenvalue λ_i of algebraic multiplicity one (thus also geometric multiplicity one) for each root λ_i of $f(t)$ in \overline{F} (the number of which precisely equals the degree of f).

(118) Even more generally, for every field F of characteristic $p > 0$ and every $F[t]$ -module M having finite F -dimension, prove that there exists a Jordan-Chevalley decomposition of T if and only if every irreducible factor of f is separable.

(119) For every field F , for every finite dimensional F -vector space M , for all F -linear endomorphisms T and U of N such that $U \circ T$ equals $T \circ U$, if both T and U have Jordan-Chevalley decompositions, prove that the semisimple and nilpotent parts of T and U commute in pairs. Deduce that for all elements $f(t, u)$ and $g(t, u)$ of the polynomial F -algebra $F[t, u]$, also the F -linear endomorphisms $f(T, U)$ and $g(T, U)$ have Jordan-Chevalley decompositions whose parts commute in pairs.

(120) For every field F , for every finite dimensional F -vector space M , for every F -linear endomorphism T of M , for every positive integer d that is not a multiple of the characteristic of F (automatic if the characteristic is 0), if T^d equals the identity map on M , prove that T has a Jordan-Chevalley decomposition with zero nilpotent part, $N = 0$.

(121) For every field F of positive characteristic $p > 0$, find an F -vector space M of dimension $n > 1$ and an F -linear endomorphism T of M such that T^p equals the identity, yet T is not semisimple (i.e., the base change of T to the algebraic closure of F is not diagonalizable).

(122) For the polynomials

$$e_\ell(t_1, \dots, t_n) = \sum_{1 \leq i_1 < \dots < i_\ell \leq n} t_{i_1} \cdots t_{i_\ell}$$

in the invariant subring $\mathbb{Z}[t_1, \dots, t_n]^{\mathfrak{S}_n}$ of $\mathbb{Z}[t_1, \dots, t_n]$ (for the action of they symmetric group \mathfrak{S}_n by permuting variables), for every field F , for every diagonalizable $n \times n$ matrix with n pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_n$ in F , prove that the characteristic polynomial equals the following,

$$t^n - e_1(\lambda_1, \dots, \lambda_n) + \dots + (-1)^\ell e_\ell(\lambda_1, \dots, \lambda_n) + \dots + (-1)^n e_n(\lambda_1, \dots, \lambda_n).$$

In particular, the trace is $e_1(\lambda_1, \dots, \lambda_n)$ and the determinant is $e_n(\lambda_1, \dots, \lambda_n)$.

(123) Prove that $\mathbb{Z}[t_1, \dots, t_n]^{\mathfrak{S}_n}$ equals the \mathbb{Z} -polynomial algebra on the generators

$$e_1(t_1, \dots, t_n), \dots, e_n(t_1, \dots, t_n)$$

(which are algebraically independent over \mathbb{Q}). In particular, deduce that the square $\Delta = V^2$ of the following **Vandermonde polynomial** equals a polynomial $D_n(e_1, \dots, e_n)$ for a polynomial $D_n(s_1, \dots, s_n)$ (called the **discriminant**),

$$V(t_1, \dots, t_n) = \prod_{1 \leq i < j \leq n} (t_j - t_i).$$

(124) For every field F , for every $n \times n$ matrix with entries in F and with characteristic polynomial $t^n - c_1 t^{n-1} + \dots + (-1)^\ell c_\ell t^{n-\ell} + \dots + (-1)^n c_n$, prove that the base change of the matrix over the algebraic closure \overline{F} is diagonalizable with n pairwise distinct eigenvalues if and only if $D(c_1, \dots, c_n)$ is nonzero.

(125) For every field F , for every diagonalizable F -linear endomorphism T of an n -dimensional F -vector space M with n pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_n$ in F , for every integer ℓ , prove that the trace of $\wedge_F^\ell(T)$ acting on the exterior power F -vector space $\wedge_F^\ell(M)$ equals $e_\ell(\lambda_1, \dots, \lambda_n)$.

(126) Use the technique from lecture, working with the fraction field F of the polynomial ring $\mathbb{Z}[a_{i,j}]_{1 \leq i,j \leq n}$ and the universal $n \times n$ matrix with entries equal to n^2 algebraically independent variables $a_{i,j}$ to deduce that for every field F and every F -linear endomorphism of an n -dimensional F -vector space M , the coefficient $c_\ell(T)$ of the characteristic polynomial of T equals the trace of $\wedge_F^\ell(T)$ acting on $\wedge_F^\ell(M)$.

(127) For every field F , for every diagonalizable F -linear endomorphism T of an n -dimensional F -vector space M with n pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_n$ in F , for every nonnegative integer d , prove that the coefficient $c_\ell(T^d)$ equals $e_\ell(\lambda_1^d, \dots, \lambda_n^d)$. Since $e_\ell(t_1^d, \dots, t_n^d)$ is an element of $\mathbb{Z}[t_1, \dots, t_n]^{\otimes n}$, deduce that there exists a unique polynomial $p_{\ell,d}(s_1, \dots, s_n)$ in $\mathbb{Z}[s_1, \dots, s_n]$ such that for every field F and every diagonalizable F -linear endomorphism T of an n -dimensional F -vector space M with n pairwise distinct eigenvalues, the coefficient $c_\ell(T^d)$ of the characteristic polynomial of T^d equals $p_{\ell,d}(c_1(T), \dots, c_n(T))$. Compute $p_{\ell,d}$ in a few small cases.

(128) Use the same method as in lecture to deduce that for every field F and for every F -linear endomorphism T of an n -dimensional F -vector space M , the coefficient $c_\ell(T^d)$ of the characteristic polynomial of T^d equals $p_{\ell,d}(c_1(T), \dots, c_n(T))$.

(129) For each positive integer n , and for each integer $\ell = 1, \dots, n$, prove that there exists a polynomial $q_{n,\ell}(\tau_1, \dots, \tau_n)$ in $\mathbb{Q}[\tau_1, \dots, \tau_n]$ (whose denominator divides $n!$) such that $e_\ell(t_1, \dots, t_n)$ equals $q_{n,\ell}(t_1^1 + \dots + t_n^1, \dots, t_1^n + \dots + t_n^n)$.

(130) For every field F of characteristic 0 field (or of characteristic $p > n$), for every diagonalizable F -linear endomorphism T of an n -dimensional F -vector space M with n pairwise distinct eigenvalues, deduce that the coefficient $c_\ell(T)$ equals $q_{n,\ell}(c_1(T^1), \dots, c_1(T^n))$. Use the same method as in lecture to extend this to the case that T is an arbitrary F -linear endomorphism of an n -dimensional F -vector space M .

(131) For every field F , for every diagonalizable F -linear endomorphism S of an n -dimensional F -vector space L with ℓ pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_\ell$, for every diagonalizable F -linear endomorphism U of an m -dimensional F -vector space M with m pairwise distinct eigenvalues μ_1, \dots, μ_m , for the F -linear endomorphism $T = S \otimes_F U$ of the F -vector space $L \otimes_F M$ of dimension $n = \ell \cdot m$, prove that T is diagonalizable with eigenvalues $(\lambda_i \mu_j)_{1 \leq i \leq \ell, 1 \leq j \leq m}$. For every integer $k = 1, \dots, \ell \cdot m$, deduce that there exist polynomials $r_{\ell,m}^k$ in the polynomial ring $\mathbb{Z}[a_1, \dots, a_\ell, b_1, \dots, b_m]$

such that, for every field F , for every S and for every U as above, the coefficient $c_k(S \otimes_F U)$ equals $r_{\ell,m}^k(c_1(S), \dots, c_\ell(S), c_1(U), \dots, c_m(U))$.

(132) Use the same method as in lecture to prove that for every field F , for every F -linear endomorphism S of an ℓ -dimensional F -vector space L , for every F -linear endomorphism U of an m -dimensional F -vector space M , for every integer $k = 1, \dots, \ell \cdot m$, the coefficient $c_k(S \otimes_F U)$ equals $r_{\ell,m}^k(c_1(S), \dots, c_\ell(S), c_1(U), \dots, c_m(U))$.

(133) Use induction to prove, for every positive integer d , for every ordered d -tuple (n_1, \dots, n_d) of positive integers, for every integer $k = 1, \dots, n_1 \cdots n_d$, there exists a unique polynomial

$$r_{n_1, \dots, n_d}^k(a_{1,1}, \dots, a_{1,n_1}, \dots, a_{i,1}, \dots, a_{i,n_i}, \dots, a_{d,1}, \dots, a_{d,n_d}) \in \mathbb{Z}[a_{i,j}]_{1 \leq i \leq k, 1 \leq j \leq n_i},$$

such that for every field F , for every ordered r -tuple (T_1, \dots, T_r) of F -linear endomorphisms T_i of n_i -dimensional F -vector spaces M_i , the coefficient $c_k(T_1 \otimes_F \cdots \otimes_F T_d)$ equals

$$r_{n_1, \dots, n_d}^k(c_1(T_1), \dots, c_{n_1}(T_1), \dots, c_1(T_i), \dots, c_{n_i}(T_i), \dots, c_1(T_d), \dots, c_{n_d}(T_d)).$$

(134) As a special case of the above, for every positive integer d , for every positive integer n , for every integer $k = 1, \dots, n^d$, prove that there exists a unique polynomial $r_{n,d}^k(s_1, \dots, s_n)$ such that for every field F , for every F -linear endomorphism T of an n -dimensional F -vector space M , the coefficient $c_k(T^{\otimes d})$ of the F -linear endomorphism $T^{\otimes d}$ of the d -fold tensor product F -vector space $M^{\otimes d}$ equals $r_{n,d}^k(c_1(T), \dots, c_n(T))$.

(135) The F -vector space $M^{\otimes d}$ has a natural action of the symmetric group \mathfrak{S}_d permuting the tensor factors, and $T^{\otimes d}$ commutes with this action. Thus, for every F -linear \mathfrak{S}_d -subrepresentation of $M^{\otimes d}$, the restriction of $T^{\otimes d}$ is an F -linear endomorphism. If the characteristic of F does not divide $d!$ (i.e., it is 0 or $p > d$), then the irreducible F -linear \mathfrak{S}_d -representations V_γ are indexed by partitions γ of d , i.e., γ is a nonincreasing sequence $(\gamma_k)_{k \in \mathbb{Z}_{>0}}$ of nonnegative integers such that there exists ℓ in $\mathbb{Z}_{>0}$ with γ_ℓ positive, with $\gamma_{\ell+m}$ equal to 0 for all positive integers m , and with $|\gamma| := \gamma_1 + \cdots + \gamma_\ell$ equal to d . The corresponding irreducible F -linear \mathfrak{S}_d -representation (V_γ, ρ_γ) is a **Specht module**, and the corresponding subrepresentation of $M^{\otimes d}$ is the injective image $\mathbb{S}^\gamma M$ of the morphism of F -linear \mathfrak{S}_d -representations,

$$V_\gamma \otimes_F \text{Hom}_F(V_\gamma, M^{\otimes d})^{\mathfrak{S}_d} \hookrightarrow M^{\otimes d},$$

where the action of \mathfrak{S}_d on the Hom F -vector space is the conjugation action (so that the \mathfrak{S}_d -invariant subspace is the F -vector space of morphisms of F -linear \mathfrak{S}_d -representations). The subspace $\mathbb{S}^\gamma M$ is the **Schur functor** of M associated to γ . Prove that there exists a polynomial $s_\gamma(s_1, \dots, s_n)$ in $\mathbb{Z}[s_1, \dots, s_n]$ so that the trace of $T^{\otimes d}$ acting on $\mathbb{S}^\gamma M$ equals $s_\gamma(c_1(T), \dots, c_n(T))$. Combined with the decomposition of each exterior power of $\mathbb{S}^\gamma M$ into tensor products of Schur

functors of M , these traces are sufficient to recover the entire characteristic polynomial of each $\mathbb{S}^\gamma T$.

(136) For $d = 2$, compute the two Schur functors $\mathbb{S}^2 M$ and $\mathbb{S}^{1,1} M$. Also compute the polynomials s_2 and $s_{1,1}$ of $c_1(T)$ and $c_2(T)$.

(137) For T equal to the identity map, deduce that the polynomials compute the dimension of each F -vector space $\mathbb{S}^\gamma M$. For T equal to a diagonalizable F -linear endomorphism with eigenvalues $\lambda_1, \dots, \lambda_n$, the polynomial expressed in terms of $\lambda_1, \dots, \lambda_n$ is the **character** of the Schur functor $\mathbb{S}^\gamma M$. The Schur character formula, which at the combinatorial level is essentially Jacobi's bialternant formula, is a formula for this character polynomial as a fraction of certain determinant polynomials (the denominator is a Vandermonde polynomial). Read more about this to help compute the character of the first "new" Schur functor, $\mathbb{S}^{2,1}(M)$.