MAT534 Fall 2022 Midterm 2 Review Sheet

The topics tested on Midterm 2 will be among the following.

(i) Basic properties of fields, e.g., every finite subgroup of the multiplicative group of a field is cyclic.

(ii) Basic properties of vector spaces, e.g., every linearly independent set extends to a basis, and every generating set contains a basis as a subset.

(iii) Reduced row echelon form, i.e., classification of orbits of the (linear) left action of $\mathbf{GL}_n$ on the vector space of $n \times m$ matrices via reduced row echelon form.

(iv) Computation of bases for the kernel, the image, and the annihilator of the image of a linear transformation.

(v) Definition and basic properties of the determinant, e.g., polynomial involving signs of permutations.

(vi) Eigenspaces, generalized eigenspaces, and the Jordan Normal Form of a matrix whose characteristic polynomial factors into linear polynomials.

(vii) Basics of division for polynomials in one variable over a field: the Division Algorithm, the Principal Ideal Property, Unique Factorization.

(viii) The Rational Canonical Form for a matrix over a field.

(ix) Real inner product spaces and Hermitian inner product spaces. The Gram-Schmidt algorithm and the Spectral Theorem.

(x) Basic definitions of rings and modules, including Hom and tensor product.

Following are some practice problems. More practice problems are in the textbook as well as on the practice midterms.

Problem 1. For $q = p^e$, in the multiplicative group $\mathbb{F}_q^\times$ how many elements are there of order n for each divisor n of $q - 1 = p^e - 1$? How many elements are there whose order divides $p^d - 1$ for no strict divisor d?

Problem 2. For every commutative (unital, associative) $\mathbb{F}_p$-algebra A, prove that the set map $F_A : A \to A$ by $a \mapsto a^p$ is an $\mathbb{F}_p$-linear $\mathbb{F}_p$-algebra endomorphism. This is the (absolute) **Frobenius endomorphism**. Prove that this is natural for commutative $\mathbb{F}_p$-algebras. For a field extension E of $\mathbb{F}_p$, compare the number of roots of $x^p - x$ and $\#\mathbb{F}_p$ to conclude that the fixed field of F_E equals $\mathbb{F}_p$. Thus, also for every commutative $\mathbb{F}_p$-algebra A that is an integral domain, the fixed $\mathbb{F}_p$-subalgebra of F_A equals $\mathbb{F}_p$.

Problem 3. For the field $E = \mathbb{F}_q$, show that F_E is a field automorphism that fixes $\mathbb{F}_p$. Thus, it induces an action of the cyclic group of order e on E by field automorphisms, with fixed subfield equal to $\mathbb{F}_p$. How many orbits are there of size e?

Problem 4. For every field E, show that the torsion subgroup of $E^\times$ is isomorphic to a subgroup of the (infinite) torsion Abelian group $\mathbb{Q}/\mathbb{Z}$, and such a subgroup is uniquely determined by the data of whether or not there exists an element of order ℓ^m for each prime number ℓ and each integer $m \geq 0$. What is this subgroup for $\mathbb{R}$? What is this subgroup for $\mathbb{C}$?

Problem 5. For every field E, for every E-vector subspace U of a vector space V, prove that there exists an E-vector subspace W of V such that (U, W) is a direct sum decomposition of V.

Problem 6. For every E-linear transformation between finite dimensional F-vector spaces of the same dimension, prove that the dimension of the kernel equals the dimension of the cokernel. Prove that this can fail if the F-vector spaces have infinite dimension.

Problem 7. For every finite dimensional F-vector space V, for every F-vector space W, prove that the natural F-linear transformation from $V^* \otimes_F W$ to $\mathrm{Hom}_F(V, W)$ is an isomorphism of F-vector spaces. In particular, for an F-vector space V, prove that V has finite dimension if and only if the following F-linear transformation is surjective, $V^* \otimes_F V \to \mathrm{Hom}_F(V, V)$.

Problem 8. For a finite field $\mathbb{F}_q$, for integers $0 \leq m \leq n$, how many m-dimensional $\mathbb{F}_q$-vector subspaces are there in $\mathbb{F}_q^{\oplus n}$?

Problem 9. For every field endomorphism $\phi : F \to F$, for every F-vector space V, consider the F-vector space ${}^{\phi}V$ which equals V as an Abelian group, but with new scalar product $c \bullet \vec{v} := \phi(c) \cdot \vec{v}$. If ϕ is a field automorphism, prove that every basis for V is also a basis for ${}^{\phi}V$, hence the two F-vector spaces are isomorphic.

Problem 10. Continuing the previous problem, show that for two bases of V, the change of basis matrix for these as bases of ${}^{\phi}V$ are typically different from the change of basis matrix for V. What kind of map does this define from $\mathbf{GL}_n(F)$ to $\mathbf{GL}_n(F)$?

Problem 11. Work this out explicitly if F equals $\mathbb{C}$ and ϕ is complex conjugation, resp. $\mathbb{F}_q$ and the absolute Frobenius endomorphism.

Problem 12. Give an example of a field endomorphism ϕ of a field F such that the F-vector space F has different dimension from the F-vector space ${}^{\phi}F$.

Problem 13. For a set I and an indexed collection $(V_i)_{i \in I}$ of F-vector spaces, define a structure of F-vector space on the Cartesian product set $\prod_{i \in I} V_i$ by adding and scaling component-wise. This is the **product** F-vector space. If I is a finite set, prove that the dimension equals the sum of the dimensions of the factors V_i. However, if I is a countably infinite set and each V_i has dimension 2, prove that the dimension of the product is uncountable (rather than a countable sum of 2, which is again countably infinite).

Problem 15. Prove that the natural map $\prod_{i \in I} \mathrm{Hom}_F(U, V_i) \to \mathrm{Hom}_F(U, \prod_{i \in I} V_i)$ is an isomorphism of F-vector spaces (this is often taken as the definition of the product F-vector space, since it is a universal property).

Problem 15. Define $\bigoplus_{i \in I} V_i$ to be the subset of $\prod_{i \in I} V_i$ consisting of elements such that at most finitely many components are different from 0. Prove that this is an F-vector subspace whose dimension does equal the sum over all i of the dimension of V_i. This is the **direct sum** F-vector space. Also prove that the natural map $\mathrm{Hom}_F(\bigoplus_i V_i, U) \to \prod_i \mathrm{Hom}_F(V_i, U)$ is an isomorphism of F-vector spaces (this is often taken as the definition of the direct sum F-vector space, since it is a universal property).

Problem 16. Also show that the natural map $\bigoplus_i (V_i \otimes_F U) \to (\bigoplus_i V_i) \otimes_F U$ is an isomorphism of F-vector spaces.

Problem 17. For every integer $e \geq 1$, for every $\lambda \in F$, prove that there exists a polynomial $n_{\lambda,e}(x) \in F[x]$ (that will depend sensitively on whether or not λ is zero) such that for every F-linear operator L on a finite-dimensional F-vector space V whose minimal polynomial divides $(x-\lambda)^e$, then the F-linear operator $N := n_{\lambda,e}(L)\circ L$ is nilpotent, and the difference $S := L - N = (\mathrm{Id} - n_{\lambda,e}(L))\circ L$ is $\lambda\mathrm{Id}_V$. In particular, (S, N) is an ordered pair of commuting F-linear operators commutes, that sums to L and with S a diagonalizable operator and N a nilpotent operator.

Problem 18. For every F-linear operator L of a finite-dimensional F-vector space V whose minimal polynomial divides $(x-\lambda_1)^{e_1}\cdots(x-\lambda_r)^{e_r}$, prove that there exists a polynomial $n(x) \in F[x]$ such that $N := n(L)\circ L$ is nilpotent and $S := L - N = (\mathrm{Id}_V - n(L))\circ L$ is diagonalizable. In applications to Lie algebras, it is essential that the constant coefficients of $n(x)x$ and $(1-n(x))x$ both equal 0.

Problem 19. For a 2×2 matrix with variable entries, compute the square of this matrix. What are the conditions on the entries so that the square of the matrix is the zero matrix? What are the conditions on the entries so that the characteristic polynomial equals x^2.

Problem 20. Repeat the exercise above for a 3×3 matrix. Is it easier to explicitly compute the n^{th} power of an $n\times n$ matrix and set all entries equal to zero, or is it easier to demand that the non-leading coefficients of the characteristic polynomial all equals zero?

Problem 21. An F-linear operator L on an F-vector space V is **idempotent** if $L\circ L$ equals L. Prove that L is idempotent if and only if the kernel and fixed locus give a direct sum decomposition of V. What is the characteristic polynomial of an idempotent F-linear operator? If an idempotent is also nilpotent, what can you say about L? If an idempotent is also invertible, what can you say about L?

Problem 22. For a diagonalizable F-linear operator L on an F-vector space V, prove that an F-linear operator K commutes with L if and only if it *both* K preserves every generalized eigenspace $\mathrm{Ker}(L-\lambda\mathrm{Id}_V)^e$ *and* the restriction of K to each generalized eigenspace commutes with the nilpotent part N. In particular, conclude that K commutes with each power N^m, and thus K preserves the kernel and image of powers of N^m.

Problem 23. Let N be a nilpotent F-linear operator on an e-dimensional F-vector space V such that N^{e-1} is nonzero. Prove that for any vector $\vec{v}_1$ in $V\setminus \mathrm{Image}(N)$, there is a basis for V consisting of $(\vec{v}_1,\ldots,\vec{v}_e)$, where $\vec{v}_{i+1} := N(\vec{v}_i)$

for $i = 1, \ldots, e-1$. For each integer $1 \leq d \leq e$, show that $\text{Ker}(N^{e+1-d}) = \text{Image}(N^{d-1})$ equals $\text{span}(\vec{v}_d, \ldots, \vec{v}_e)$, and thus the dimension equals $e+1-d$. For every vector $\vec{w} \in V$, prove that there is a unique F-linear operator K on V that commutes with N and such that $K(\vec{v}_1) = \vec{w}$, and K maps every subspace $\text{Ker}(N^{e+1-d}) = \text{Image}(N^{d-1})$ back to itself. Conclude that the F-vector space of F-linear operators on V that commute with N has dimension e, and thus it has a basis consisting of $(\text{Id}_V, N, N^2, \ldots, N^{e-1})$. As an F-algebra under composition, this **centralizer** of N is F-algebra isomorphic to $F[x]/\langle x^e \rangle$ via the isomorphism that sends x to N. This is the F-algebra obtained as the quotient of $F[x]$ by the minimal polynomial of N.

Problem 24. Let N be a nilpotent F-linear operator on an n-dimensional F-vector space whose Jordan blocks have sizes $\underline{e} = (e_1, e_2, \ldots, e_s)$, where $e_1 \geq \cdots \geq e_s \geq 1$ and $e_1 + \cdots + e_s$ equals n, i.e., $\underline{e}$ is a **partition** of n into s parts. Prove that the minimal polynomial of N equals x^{e_1}. Since $\underline{e}$ determines the Jordan normal form of N, conclude that $\underline{e}$ uniquely determines the numerical function,

$$k : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}, \quad m \mapsto \text{dimKer}(N^m).$$

Problem 25. Continuing the previous problem, let W be the direct sum of all cyclic submodules in the Jordan decomposition (of dimensions $e_2, e_3, \ldots$) except for the first cyclic submodule of dimension e_1. Let N_W denote the restriction of N to W, and let $k_w : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be the numerical function $k_W(m) = \text{Ker}(N_W^m)$. Prove that the maximum value $k(m)$ of k is $k(m) = n$, and the smallest integer m for which $k(m)$ equals n is $m = e_1$. Prove that e_2 is the largest integer $\leq e_1$ such that $n - e_1 - 1 + e_2$ is not in the image of k. Finally, for every integer $d = 2, \ldots, e_2$, prove that $k_W(e_2 + 1 - d)$ equals $k(e_2 + 1 - d) - (e_2 + 1 - d)$; of course $k_W(e_2)$ is the maximum value $n - e_1$ of k_w. Thus the numerical function k_W is uniquely determined by k. By the induction hypothesis, the partition $(e_2, \ldots, e_s)$ of $n - e_1$ into $s - 1$ parts is uniquely determined by k_W, and thus also by k. Conclude by induction on s that $\underline{e}$ is uniquely determined by k.

Problem 26. Continuing the previous problem, for each of cyclic submodule of rank e_i, denote a cyclic generator of that submodule by $\vec{v}_{i,1}$. Prove that the vectors $\vec{v}_{i,m} = N^{m-1}\vec{v}_{i,1}$ for $i = 1, \ldots, s$ and $m = 1, \ldots, e_i$ form a basis for V with respect to which N is in Jordan canonical form. For each F-linear operator K that commutes with N, prove that each vector $\vec{w}_i = K(\vec{v}_{i,1})$ is an element of $\text{Ker}(N^{e_i})$, and K is uniquely determined by the sequence of vectors $(\vec{w}_i)_{i=1,\dots,s}$ with $\vec{w}_i \in \mathrm{Ker}(N^{e_i})$. Conversely, for every such sequence of vectors, prove that there exists a unique F-linear operator K of V that commutes with N and such that $K(\vec{v}_{i,1}) = \vec{w}_i$ for each $i = 1, \dots, s$. Thus, the dimension of the centralizer of N as an F-vector space equals $k(e_1)k(e_2)\cdots k(e_s)$.

Problem 27. For each of the following matrices, compute an ordered basis of the kernel, compute an ordered basis of the image of the matrix, compute an extension of that ordered basis of the image to an ordered basis for the entire target by adjoining elements of the standard basis, compute the change of basis matrix between this new ordered basis and the standard ordered basis, and compute an ordered basis for the annihilator of the image of the matrix.

$$L_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}, \quad L_4 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix},$$

$$L_5 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 3 & 6 & 1 \end{bmatrix}, \quad L_6 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 1 \end{bmatrix},$$

$$L_7 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{bmatrix}, \quad L_8 = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}.$$

Problem 28. For each change of basis matrix above, compute the trace and determinant of the change of basis matrix.

Problem 29. Let $a, b, c, d \in \mathbb{R}$ be real numbers such that $ad - bc$ is nonzero. Perform Gram-Schmidt on the following 2×2 matrix with real entries,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Problem 30. Perform Gram-Schmidt on the following 4×4 matrix with complex entries,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}.$$

Problem 31. Perform Gram-Schmidt on the following 4×4 matrix with complex entries,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2i & -2 & -i \\ 1 & -4 & 4 & -1 \\ 1 & -8i & -8 & i \end{bmatrix}.$$

Problem 32. Diagonalize the following 2×2 matrix with complex entries,

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

For what values of θ is the minimal polynomial over $\mathbb{R}$ irreducible? What is the corresponding field extension of $\mathbb{R}$?

Problem 33. Let F be an algebraically closed field. Let $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ be a bijection. Let L_σ be the corresponding permutation matrix, $[\delta_{\sigma(i),j}]_{1 \le i,j \le n}$. Assume that either F has characteristic 0 or the characteristic of F is prime to the order of each cyclic factor in the cycle decomposition of σ. Prove that L_σ is diagonalizable, and describe the diagonalization: what are the eigenvalues, and what are the dimensions of the corresponding eigenspaces?

Problem 34. Let $L : \text{Mat}_{n \times n}(F) \to \text{Mat}_{n \times n}(F)$ be the F-linear operator sending each matrix to its transpose. Diagonalize this F-linear operator.

Problem 35. Assume that the characteristic of F is prime to n. Let $L : \text{Mat}_{n \times n}(F) \to \text{Mat}_{n \times n}(F)$ be the F-linear operator that sends each matrix A to $(1/n)\text{Trace}(A) \cdot \text{Id}_{n \times n}$, i.e., the unique scalar multiple of the identity that has the same trace as A. Diagonalize this F-linear operator of the F-vector space $\text{Mat}_{n \times n}(F)$. Is this linear operator idempotent? Is the kernel preserved under taking commutators?

Problem 36. Repeat Problems 27 and 28 over a field of characteristic p.

Problem 37. Let L be an F-linear operator of a finite dimensional F-vector space that has a Jordan canonical form, i.e., the characteristic polynomial factors into a product of linear polynomials. In terms of the Jordan canonical form of L, what is the Jordan canonical form of $g(L)$ for each $g(x) \in F[x]$?

Problem 38. Find the diagonalization of the $\mathbb{R}$-linear operator on $\mathbb{C}$ given by complex conjugation.

Problem 39. For each diagonal $n \times n$ matrix a in $\mathbf{GL}_n(\mathbb{C})$, for the F-linear operator conj_a on $\text{Mat}_{n\times n}(\mathbb{C})$, compute the action on each elementary matrix $E_{k,l} = (\delta_{i,k}\delta_{j,l})_{1\le i,j\le n}$. Use this to diagonalize the operator conj_a. Compute the trace and determinant of this operator in terms of the trace and determinant of a.

Problem 40. Repeat the previous problem when a is the non-diagonalizable matrix that has 1 in every diagonal entry, and 1 in every entry directly about the main diagonal (and 0 in every other entry).

Problem 41. Let $(V, \langle \bullet, \bullet \rangle)$ be a finite dimensional Hermitian inner product space. For every $\mathbb{C}$-linear operator S of V, define

$$\langle \bullet, \bullet \rangle_S : V \times V \to \mathbb{C}, \quad (\vec{v}, \vec{w}) \mapsto \langle S\vec{v}, \vec{w} \rangle.$$

Show that $\langle \bullet, \bullet \rangle_S$ is a conjugate linear, sesquilinear pairing if and only if S is self-adjoint. In this case, prove that the pairing is positive definite if and only if the (real) eigenvalues of S are all positive.

Problem 42. Let $(V, \langle \bullet, \bullet \rangle)$ be a finite dimensional Hermitian inner product space. For every $\mathbb{C}$-linear operator A, define

$$\langle \bullet, \bullet \rangle^A : V \times V \to \mathbb{C}, \quad (\vec{v}, \vec{w}) \mapsto \langle A\vec{v}, A\vec{w} \rangle.$$

Show that this is a sequilinear, conjugate linear pairing that is positive semidefinite. Prove that it is positive definite (i.e., a Hermitian inner product) if and only if A is invertible.

Problem 43. Denote by A^* the adjoint of A with respect to $\langle \bullet, \bullet \rangle$. Define $S = A^* \circ A$. Prove that S is a self-adjoint $\mathbb{C}$-linear operator with real, nonnegative eigenvalues, and $\langle \bullet, \bullet \rangle^A = \langle \bullet, \bullet \rangle_S$. In particular, the trace of S is nonnegative. Prove that the kernel of S equals the kernel of A. Thus, S has positive real eigenvalues if and only if A is invertible, and the trace of S is zero only if A is zero.

Problem 44. For every pair (A, B) of $\mathbb{C}$-linear operators of V, define $\langle A, B\rangle_{\text{HS}}$ to be the trace of $B^* \circ A$. Prove that this $\mathbb{R}$-bilinear pairing on $\text{Hom}_{\mathbb{C}}(V, V)$ to $\mathbb{C}$ is a conjugate linear, sesquilinear pairing that is positive definite, i.e., it is a Hermitian inner product. This is the **Hilbert-Schmidt inner product**.

Problem 45. For a normal operator A of $(V, \langle \bullet, \bullet \rangle)$, prove that $\langle A, A\rangle_{\text{HS}}$ equals the sum of the squares of the complex norms of the eigenvalues of A. In particular, if A is a unitary operator, the Hilbert-Schmidt norm of A equals $\dim(V)$. (One could scale the Hilbert-Schmidt norm so that all unitary operators have norm equal to 1 rather than $\dim(V)$, but this is not the conventional normalization of this inner product.)

Problem 46. Show that the conjugate linear transformation from $\text{Hom}_{\mathbb{C}}(V, V)$ to itself sending A to A^* is an isometry for the Hilbert-Schmidt norm. Similarly, show that the $\mathbb{C}$-linear operator conj_a is an isometry for every $a \in \text{Isom}_{\mathbb{C}}(V, V)$.

Problem 47. For the standard Hermitian inner product on $V = \mathbb{C}^n$, prove that the elementary matrices $E_{k,\ell} = (\delta_{i,k}\delta_{j,\ell})_{1\leq i,j\leq n}$ form an orthonormal basis for the Hilbert-Schmidt inner product. Conclude that the adjointness isomorphism,

$$V \otimes_{\mathbb{C}} V^{\vee} \to \text{Hom}_{\mathbb{C}}(V, V),$$

is an isometry between Hermitian inner product spaces, where the Hermitian inner product on the tensor product $V \otimes_{\mathbb{C}} V^{\vee}$ is the one induced by the standard Hermitian inner product on V and on V^* via the isomorphism $V \to V^*$ induced by the standard Hermitian inner product. Deduce the analogous result for every finite dimensional Hermitian inner product space $(V, \langle \bullet, \bullet \rangle)$, since all of these have orthonormal bases.

Problem 48. For every invertible $\mathbb{C}$-linear operator A on a finite dimensional Hermitian inner product space $(V, \langle \bullet, \bullet \rangle)$, for the associated self-adjoint operator $S = A^* \circ A$ with real positive eigenvalues, prove that there exists a unique self-adjoint operator R with real positive eigenvalues such that S equals $R \circ R$ (same eigenspaces, eigenvalues equal to the positive square roots of the positive eigenvalues of S). In particular, R commutes with S.

Problem 49. Since $((R^*)^{-1} \circ A^*) \circ (A \circ R^{-1})$ equals $R^{-1} \circ S \circ R^{-1}$ equals the identity transformation, conclude that $U := A \circ R^{-1}$ is an invertible $\mathbb{C}$-linear operator whose adjoint $U^* = (R^*)^{-1} \circ A^*$ equals U^{-1}, i.e., U is a unitary operator. Thus, A equals $U \circ R$ where R is a self-adjoint operator with real positive eigenvalues that commutes with $A^* \circ A$, and where U is a unitary operator. This is the **polar decomposition** of A with respect to $\langle \bullet, \bullet \rangle$.

Problem 50. Continuing the previous problem, conclude that an invertible operator A is normal if and only if A commutes with $A^* \circ A$ if and only if A commutes with R if and only if A commutes with U. In this case, all of A, A^*, $A^* \circ A$, R and U pairwise commute. Moreover, the eigenvalues of R and U on each eigenspace gives the usual (complex) polar decomposition of the eigenvalues of A on that eigenspace.