MAT534 Fall 2022 Final Exam Review Sheet

The final exam will be cumulative, but there will be extra emphasis on material not already covered on Midterms 1 and 2, Chapters 7, 8, 9 (excluding Gröbner bases), 10, 11, and 12. Wedderburn's Theorem, Section 18.2, will **not** be covered on the final exam (it will be explained in lecture).

(i) Multiplicative subsets and fraction rings.

(ii) Maximal ideals and prime ideals.

(iii) The Chinese remainder theorem.

(iv) Finitely generated modules and Noetherian rings.

(v) Principal ideal domains.

(vi) Unique factorization domains.

(vii) The structure theorem for finitely generated modules over a principal ideal domain.

(viii) Gauss's Lemma.

(ix) Irreducibility criteria including Eisenstein's criterion.

(x) Hilbert's Basis Theorem.

Following are some practice problems. More practice problems are in the textbook as well as on the review sheet and practice midterm for Midterms 1 and 2.

Problem 1. Show that every nonzero ring with a unique proper ideal is a field.

Problem 2. Show that every nonzero integral domain with a unique prime ideal is a field.

Problem 3. Find an example of a nonzero ring that is not a field yet has a unique prime ideal.

Problem 4. For the integral domains $A = \mathbb{Z}$ and $A = F[x]$, show that the fraction field of A is **not** finitely generated as an A-algebra.

Problem 5. Find an example of a principal ideal domain A such that the fraction field of A is finitely generated as an A-algebra.

Problem 6. Find an example of a unique factorization domain that is not a principal ideal domain.

Problem 7. Find an example of an integral domain that is not a unique factorization domain yet is integrally closed in its fraction field. (This might be very challenging if you have never seen such rings before.)

Problem 8. Prove that for every prime $p > 1$ and every integer $e \geq 0$, every $\mathbb{F}_p$-subalgebra of $\mathbb{F}_{p^e}$ is a finite field.

Problem 9. For a field F, is the polynomial F-algebra in countably many variables a unique factorization domain?

Problem 10. Find an example of a finitely generated F-algebra that is a unique factorization domain yet that is not isomorphic to a polynomial F-algebra.

Problem 11. Let R be a commutative ring with 1. Let M be an R-module. Let M' be an R-submodule of M such that both M' and M/M' are torsion R-modules whose annihilator ideals are comaximal. Prove that there is an R-submodule N of M that projects isomorphically to M/M'.

Problem 12. Let R be a nonzero commutative ring with a unique prime ideal. Prove that the radical ideal of any proper principal ideal equals this unique prime ideal. Conclude that for every pair a, b of nonzero nonunits in R, there exists an integer $n \geq 1$ such that a divides b^n and b divides a^n.

Problem 13. In a Noetherian ring, prove that there are at most finitely many idempotent elements whose pairwise products equal zero.

Problem 14. Prove that every quotient ring of a Noetherian ring is Noetherian.

Problem 15. Find an example of a non-Noetherian subring of a Noetherian ring.

Problem 16. For a commutative integral domain R and a multiplicative subset $S \subset R \setminus \{0\}$, prove that if R is Noetherian, respectively a unique factorization domain, integrally closed in its fraction field, a principal ideal domain, then the same holds for $R[S^{-1}]$.

Problem 17. Let R be a principal ideal domain. Show that for every R-module M, not necessariy finitely generated, there exists an R-submodule N of M such that M is the direct sum of N and the torsion submodule of M.

Problem 18. Let F be a field, and let R be the F-subalgebra of $F[x, y]$ generated by x^2, xy, and y^2. Prove that R is not isomorphic to a polynomial F-algebra. Is R integrally closed in its fraction field? What is the fraction field, as a subfield of $F(x, y)$?

Problem 19. Let F be a field, and let R be the F-subalgebra of $F[x]$ generated by x^2 and x^3. Prove that R is not isomorphic to a polynomial F-algebra. Is R integrally closed in its fraction field?

Problem 20. Inside the polynomial F-algebra $R = F[x_1, \ldots, x_n]$, denote by R_1 the F-vector subspace generated by $x_1, \ldots, x_n$. Let G be a finite subgroup of $\mathrm{Aut}_F(V)$ with its natural action on R by F-algebra automorphisms, $(g, f) \mapsto g \cdot f$. Denote by R^G the graded F-subalgebra of all G-invariant elements of R. Assume that the order $\#G$ is invertible, so that the following set map is well-defined,

$$a_G : R \to R^G, \quad f \mapsto \frac{1}{\#G} \sum_{g \in G} g \cdot f.$$

Prove that this is a homomorphism of R^G-modules that determines a direct sum decomposition of R as an R^G-module, $R \cong R^G \oplus (R/R^G)$.

Problem 21. Continuing the previous problem, denote by $I \subset R$ the ideal generated by R^G_+, i.e., generated by all G-invariant homogeneous elements of positive degree. Use Hilbert's Basis Theorem to prove that there exist finitely many elements $f_1, \ldots, f_m \in R^G_+$ that generate I. Prove that $f_1, \ldots, f_m$ also generate R^G as an F-subalgebra of R. Thus, the problem of finding a finite list of generators of R^G as an F-algebra is reduced to the problem of finding a finite list of generators of I as an ideal. (There is a theorem of Emmy Noether that makes all of this algorithmic.)

Problem 22. Let R be an integral domain. Let $f(x) \in R[x]$ be a monic polynomial of degree n that is irreducible as an element in $\mathrm{Frac}(R)[x]$. Inside the field extension $L = \mathrm{Frac}(R)[x]/\langle f(x) \rangle$, show that the R-subalgebra S generated by x is a finite free R-module of rank n with free basis $(1, \overline{x}, \dots, \overline{x}^{n-1})$.

Problem 23. Give an example of a principal ideal domain R and $f(x) \in R[x]$ as in the previous exercises such that the integral closure of R in L is strictly larger than S. If R is a finite type F-algebra that is integrally closed in its fraction field ("normal"), then the integral closure of R in L is a finite R-module. In many cases, Emmy Noether gave an algorithmic construction of this "normalization" of R in L based on her own Noether Normalization Theorem.

Problem 24. For every integral domain R, prove that every finite subgroup of the multiplicative group $R^\times$ is cyclic.

Problem 25. Find an example of a division ring D and a non-Abelian subgroup of the multiplicative group $D^\times$.

Problem 26. For an associative, unital ring R, prove that every direct sum of flat left R-modules is again a flat left R-module.

Problem 27. For a commutative ring R that is a finite product of fields (not necessarily the same field each time), classify all of the ideals. In particular, prove that every prime ideal is maximal, and the corresponding quotient rings are just the fields in the product. (This is **not** correct for an infinite product of fields, and the amazing properties of ideals in infinite products of fields are a crucial component in the model-theoretic results of James Ax and many others.)

Problem 28. Show that in a Noetherian ring R, for every proper ideal I, there are only finitely many primes containing I that are minimal among primes containing I, i.e., there are only finitely many **minimal primes over** I. By definition, every prime ideal has this property (the unique minimal prime over that ideal is the prime itself). By the ascending chain condition for Noetherian rings, if there is any ideal with infinitely many minimal primes over it, then there is a maximal such ideal I. Since I is not prime, there exist elements $a, b \in R \setminus I$ such that $ab \in I$. Now prove that the primes containing

I are precisely those primes that contain at least one of the strictly bigger ideals $(I : a)$ and $(I : b)$. Since there are only finitely many minimal primes over each of $(I : a)$ and $(I : b)$, this gives a contradiction.

Problem 29. Show that in an infinite product of fields, the kernels of the quotient homomorphisms to the factor fields are minimal prime ideals over $\langle 0 \rangle$. Conclude that there are infinitely many minimal prime ideals over $\langle 0 \rangle$.

Problem 30. Show that a finite product of Noetherian rings is again a Noetherian ring, but this can fail for an infinite product of Noetherian rings.