PROBLEM SET 9

(1) Let F be a field and let A be an associative F-algebra with 1. Let S be a subset of A which commutes, i.e., for every pair s, t in S, st equals ts. Let B be the smallest F-subalgebra of B which contains S and 1. Prove that B commutes. Deduce the claim from the exercise in the middle of p. 9 of the notes on the spectral theorem.

(2) For the following linearly indepdendent subset of $\mathbb{R}^3$, $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, find the orthonormal basis $(\hat{u}_1, \hat{u}_2, \hat{u}_3)$ satisfying the conditions of the Gram-Schmidt theorem.

$$\vec{v}_1 = \begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 5 \\ 3 \\ -8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 25 \end{bmatrix}.$$

(3) Let $n \geq 2$ be an integer, and let a, b, c be integers. Define an $\mathbb{R}$-linear operator,

$$T_{a,b,c} : \mathbb{R}^n \to \mathbb{R}^n$$

by

$$T_{a,b,c}(\mathbf{e}_i) = \begin{cases} b\mathbf{e}_1 + c\mathbf{e}_2, & i = 1, \\ a\mathbf{e}_{i-1} + b\mathbf{e}_i + c\mathbf{e}_{i+1}, & 2 \leq i \leq n-1, \\ a\mathbf{e}_{n-1} + b\mathbf{e}_n, & i = n \end{cases}$$

Prove that $T_{a,b,c}$ is normal if and only if $a^2 = c^2$. And when $a = c$ and $n = 2, 3$, diagonalize this matrix.

(4) Polar decomposition of normal operators. Let V be a finite dimensional, complex Hermitian space and let T be an invertible, normal operator on T. Prove that there exists a unique factorization

$$T = |T|U$$

of T into a product of commuting operators $|T|$ and U on V such that

(i) $|T|$ is a *positive operator*, i.e., $\langle |T|\vec{v}, \vec{v}\rangle$ is a positive real number for every nonzero $\vec{v}$ in V,
(ii) and U is unitary.

Hint. For such a factorization, relate T^*T and $|T|$. Use this to define $|T|$ and then prove the factor $(|T|)^{-1}T$ is unitary.