MAT 322 Problem Set 1

Homework Policy. Please read through all the problems. Please write up solutions of the required problems. Please also read and attempt the extra problems, but please do not write up those solutions for grading. I will be happy to discuss the extra problems during office hours.

Each student is encouraged to work on problem sets with other students, but each submitted problem set must be in the student's own words and based on the student's own understanding. It is against university policy to copy answers from other students or from any other resource (such as a webpage).

Required Problems.

Problem 1. Let $b_{2,0}$, $b_{1,1}$ and $b_{0,2}$ be real numbers. Define a degree 2, homogeneous polynomial B in variables s, t with real coefficients by

$$B(s,t) = b_{2,0}s^2 + b_{1,1}st + b_{0,2}t^2.$$

The polynomial is *nonnegative*, respectively *positive*, if for every $(c,d) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $B(c,d)$ is nonnegative, respectively positive.

(a) Prove that B is nonnegative, resp. positive, if and only if all of the following real numbers are nonnegative, resp. positive: $b_{2,0} \geq 0$, $b_{0,2} \geq 0$, $4b_{2,0}b_{0,2} - b_{1,1}^2 \geq 0$.

(b) Define the function $\beta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as follows,

$$\beta\left(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 2b_{2,0} & b_{1,1} \\ b_{1,1} & 2b_{0,2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2b_{2,0}ux + b_{1,1}(uy + vx) + 2b_{0,2}vy.$$

Prove that β satisfies Axioms (1), (2) and (3) for an inner product, i.e., β is $\mathbb{R}$-bilinear and symmetric. **Prove** that β satisfies Axiom (4) if and only if all of the following real numbers are positive: $b_{2,0} > 0$, $b_{0,2} > 0$, $4b_{2,0}b_{0,2} - b_{1,1}^2 > 0$.

Problem 2. Let $(V, +, 0, \cdot)$ be an $\mathbb{R}$-vector space. Let $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$ be a function that satisfies Axioms (1), (2) and (3) for an inner product, i.e., it is $\mathbb{R}$-bilinear and symmetric. For every pair of vectors $(\mathbf{x}, \mathbf{y}) \in V \times V$, define the following degree 2, homogeneous polynomial

$$B_{(\mathbf{x},\mathbf{y})}(s,t) = \langle \mathbf{x}, \mathbf{x} \rangle s^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle st + \langle \mathbf{y}, \mathbf{y} \rangle t^2.$$

(a) Prove the following two *Polarization Identities*. For every $(\mathbf{x},\mathbf{y}) \in V \times V$,

$$2\langle \mathbf{x},\mathbf{y}\rangle = \langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle - \langle \mathbf{x},\mathbf{x}\rangle - \langle \mathbf{y},\mathbf{y}\rangle,$$

and also

$$\langle \mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle = 2\left(\langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle\right).$$

(b) Now assume that for every $\mathbf{z} \in V$, $\langle \mathbf{z},\mathbf{z}\rangle$ is nonnegative. Define $\| \mathbf{z} \| \in \mathbb{R}_{\geq 0}$ to be the unique nonnegative real number such that $\| \mathbf{z} \|^2 = \langle \mathbf{z},\mathbf{z}\rangle$. **Prove** that $B_{(\mathbf{x},\mathbf{y})}$ is nonnegative. Then **prove** the *Cauchy-Schwarz Inequality*,

$$\langle \mathbf{x},\mathbf{y}\rangle^2 \leq \langle \mathbf{x},\mathbf{x}\rangle\langle \mathbf{y},\mathbf{y}\rangle, \text{ i.e., } \langle \mathbf{x},\mathbf{y}\rangle \leq | \langle \mathbf{x},\mathbf{y}\rangle | \leq \| \mathbf{x} \| \| \mathbf{y} \|.$$

(c) Now use the Cauchy-Schwarz inequality and the polarization identities to **prove** the following *Triangle Inequality*,

$$\| \mathbf{x}+\mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|.$$

(d) Let $n \geq 2$ be an integer. Denote by $| \bullet | : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ the supremum norm,

$$\left| \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right| = \sup(| x_1 |, \dots, | x_n |).$$

For the vectors $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$, **prove** that the polarization identity above fails. Conclude that there does not exist any inner product $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $| \bullet |$ is the associated norm $\| \bullet \|$.

Problem 3 Following is a 4×6 matrix together with its reduced row echelon form.

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 & 2 & -1 \\ 2 & 1 & 1 & -1 & 2 & 0 \\ 2 & 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 & 2 & 0 \end{pmatrix}$$

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) Denote the column vectors of A by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$, $\vec{v}_5$, and $\vec{v}_6$. In other words,

$$A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3\ \vec{v}_4\ \vec{v}_5\ \vec{v}_6].$$

Find those integers $n = 1, \dots, 6$ such that $\vec{v}_n$ cannot be expressed as a linear combination of all vectors $\vec{v}_m$ with $m < n$; these are the *irredundant terms*, and the remaining terms are *redundant*. For every redundant term n, write $\vec{v}_n$ as a linear combination of the vectors $\vec{v}_m$ with m irredundant.

(b) Denote by $T_A : \mathbb{R}^6 \to \mathbb{R}^4$ the $\mathbb{R}$-linear transformation $T_A(\vec{x}) = A \cdot \vec{x}$. Write down a subset of $(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6)$ which is a basis for Image(T_A), i.e., a basis for the column space of A.

(c) Write down a basis for Ker(T_A), i.e, a basis for the null space of A.

(d) Find real numbers a_1, a_2, a_3, a_4 for which the equation

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + a_4 y_4 = 0$$

holds precisely for those vectors

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \text{ in Image}(T_A),$$

i.e., the kernel of the linear functional above precisely equals the column space of A.

Problem 4 For the following invertible 4×4 matrix A, find a sequence of elementary 4×4 matrices, $E_1, \dots, E_n$, such that for the invertible 4×4 matrix $E = E_n \cdots E_1$, the following identity holds for the augment 4×8 matrix $[A|I_4]$,

$$E \cdot [A|I_4] = [I_4|B]\,.$$

In particular, find B. What is $A \cdot B$, and what is $B \cdot A$? In terms of $E_1, \dots, E_n$, what is $\det(A)$?

$$A = \begin{bmatrix} 4 & 3 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & -4 \end{bmatrix}.$$

Problem 5 Let $(X, d_X : X \times X \to \mathbb{R}_{\geq 0})$ be a metric space. Let $Y \subset X$ be a subset with the restricted metric function $d_Y = d_X|(Y \times Y)$.

(a) If $Y \subset X$ is open with respect to d_X, and if $Z \subset Y$ is open with respect to d_Y, **prove** that $Z \subset X$ is open with respect to d_X. Also give an example of $Y \subset X$ that is not open with respect to d_X and a subset $Z \subset Y$ that is open with respect to d_Y, such that $Z \subset X$ is not open with respect to d_X.

(b) Let $Z \subset X$ be a subset. Assume that for every $z \in Z$, there exists an open neighborhood $Y \subset X$ of z (with respect to d_X) such that $Z \cap Y \subset Y$ is open with respect to d_Y. Prove that $Z \subset X$ is open with respect to d_X. In particular, if there exists an open covering $\{Y_\alpha\}$ of X (with respect to d_X) such that every $Z \cap Y_\alpha$ is open in Y_α (with respect to d_{Y_α}), then prove that $Z \subset X$ is open.

Problem 6 Let (X, d_X) and (T, d_T) be metric spaces. Let $f : X \to T$ be a function.

(a) Assume that for every $x \in X$ there exists an open neighborhood $Y \subset X$ of x such that $f|Y : Y \to T$ is continuous with respect to d_Y and d_T. Prove that f is continuous with respect to

d_X and d_T. In particular, if there exists an open covering $\{Y_\alpha\}$ of X (with respect to d_X) such that every $f|Y_\alpha : Y_\alpha \to T$ is continuous (with respect to d_{Y_α} and d_T), then prove that f is continuous.

(b) Let $C, D \subset X$ be closed subsets (with respect to d_X) such that X equals $C \cup D$. Assume both that $f|C : C \to T$ is continuous with respect to d_C and d_T, and $f|D : D \to T$ is continuous with respect to d_D and d_T. Prove that f is continuous with respect to d_X and d_T. On the other hand, give an example of X, T, f, and an *infinite* collection $\{C_\alpha\}$ of closed subsets $C_\alpha \subset X$ with $X = \cup_\alpha C_\alpha$ such that every $f|C_\alpha$ is continuous, yet f is not continuous.

Extra Problems. p. 10, Exercise 5; p. 24, Exercises 3, 5; pp. 30–31, Exercises 1, 5, 9.