Name: ______________________ **Problem 1:** __________ /25

Problem 1. Mandatory Problem.(25 points) Let (S,d) and (S^*,d^*) be metric spaces, and let $f:(S,d)\to(S^*,d^*)$ satisfy the $\epsilon-\delta$ definition of continuity (at all points of S). Do all of the following.
(a) Define the notion of **open subset** of S with respect to d.
(b) Define **compactness** of (S,d).
(c) Without quoting from the book, prove that for every subset U of S^* that is open (with respect to d^*), also the preimage subset $f^{-1}(U)\subset S$ is open (with respect to d).

(a) An open subset of (S,d) is a subset $U\subseteq S$ (possibly empty) such that for every $s_0\in U$, there exists positive real ε such that U contains the open ball of radius ε about s_0, i.e., for every $s\in S$, if $d(s_0,s)<\varepsilon$, then $s\in U$.

(b) The metric space (S,d) is compact if for every open covering, i.e., collection $(U_i)_{i\in I}$ of open subsets $U_i\subset S$ such that S equals $\bigcup_{i\in I} U_i$, there is a finite subcovering, i.e., there exists a finite subset $J\subseteq I$ such that $\bigcup_{i\in J} U_i$ equals S.

(c). Let s_0 be an element of $f^{-1}(U)$. Thus $f(s_0)$ is an element of U. Since U is open, there exists positive real ε such that U contains the open ball $B^*_\varepsilon(f(s_0)) := \{s^*\in S^* : d^*(s^*,f(s_0))<\varepsilon\}$. Since f is ε-δ continuous, there exists positive real δ such that $f^{-1}(B^*_\varepsilon(f(s_0)))$ contains the open ball $B_\delta(s_0) := \{s\in S : d(s,s_0)<\delta\}$, i.e., if $d(s,s_0)<\delta$, then $d^*(f(s),f(s_0))<\varepsilon$. Since $B_\delta(s_0)$ is contained in $f^{-1}(B^*_\varepsilon(f(s_0)))$, and since $f^{-1}(B^*_\varepsilon(f(s_0)))$ is contained in $f^{-1}(U)$, $B_\delta(s_0)$ is contained in $f^{-1}(U)$. Thus, for every s_0 in $f^{-1}(U)$, there exists positive real δ such that $B_\delta(s_0)$ is contained in $f^{-1}(U)$, i.e., $f^{-1}(U)$ is an open subset of S.

Name: ______________________________ **Problem 2:** ______________ /25

Problem 2(25 points) For each of the following series, determine whether the series converges or diverges with justification.

(a) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n!}}$.
(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
(c) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$.
(d) $\sum_{n=1}^{\infty} \frac{1}{(1+(1/n))^n}$.

(a) Every term $a_n = \frac{1}{\sqrt{n!}}$ is positive, thus nonzero. By the Ratio Test,
$\frac{|a_{n+1}|}{|a_n|} = \frac{\sqrt{n!}}{\sqrt{(n+1)!}} = \frac{\sqrt{n!}}{\sqrt{(n+1)\cdot n!}} = \frac{\sqrt{n!}}{\sqrt{n+1}\cdot\sqrt{n!}} = \frac{1}{\sqrt{n+1}}$, which is less than $\frac{1}{\sqrt{3}} < 1$
for all $n \geq 2$. Thus, $\limsup \frac{|a_{n+1}|}{|a_n|} \leq \frac{1}{\sqrt{3}} < 1$. So the series converges.

(b) The terms are $(-1)^n a_n$ with $(a_n)_n = (\frac{1}{n})_n$ a positive sequence.
Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$, by the Alternating Series Test, the series converges.

(c) Every term $a_n = \frac{2^n}{n!}$ is positive, thus nonzero. Also,
$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{\sqrt{(n+1)!}} \cdot \frac{\sqrt{n!}}{2^n} = \frac{2\cdot 2^n}{\sqrt{n+1}\cdot\sqrt{n!}} \cdot \frac{\sqrt{n!}}{2^n} = \frac{2}{\sqrt{n+1}}$. For every $n \geq 4$,
$\frac{2}{\sqrt{n+1}} \leq \frac{2}{\sqrt{9}} < 1$. Thus, $\limsup \frac{|a_{n+1}|}{|a_n|} \leq \frac{2}{\sqrt{5}} < 1$. By the Ratio Test, the series converges.

(d) Recall from calculus (by L'Hospital's rule), $\lim_{n\to\infty} (1+\frac{1}{n})^n$ equals e. Thus, since $\lim_{n\to\infty} \frac{1}{(1+(\frac{1}{n}))^n} = \frac{1}{e}$, which is nonzero, the series diverges.

Name: ______________________________ Problem 3: ____________ /25

Problem 3(25 points) For each of the following subsets of $\mathbb{R}$, with respect to the usual metric, $d(x,y) = |y-x|$, find the interior and the closure. State whether the closure is compact. Justify your answers.
(a) $[0,\infty)$,
(b) $\mathbb{Q} \cap [0,\sqrt{2}]$,
(c) $\cup_{n=2}^{\infty}[1/n^2, 1/(n^2-1)]$

(a) The interval $\mathcal{U} = (0,\infty)$ is an open subset of $A \subset [0,\infty)$. For the unique element $0 \in A \setminus \mathcal{U}$, for every positive real ε, $-\varepsilon/2$ is an element of $B_\varepsilon(0)$ that is not in A. Thus 0 is not an interior point. Hence interior(A) equals $\mathcal{U} = (0,\infty)$. Already A is a closed interval, hence A is a closed set. Thus A^- equals $A = [0,\infty)$. Finally, since A^- is not bounded above, by Heine-Borel, A^- is not compact.

(b) For every nonempty open interval $B_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$ in $\mathbb{R}$, since $B_\varepsilon(x)$ is uncountable yet $\mathbb{Q} \cap B_\varepsilon(x)$ is countable, there exists $y \in B_\varepsilon(x)$ such that $y \notin \mathbb{Q}$. In particular, $B_\varepsilon(x)$ is not contained in $\mathbb{Q}$. Thus interior($\mathbb{Q}$) is the empty set. Since $\mathbb{Q} \cap [0,\sqrt{2}]$ is a subset of $\mathbb{Q}$, also interior ($\mathbb{Q} \cap [0,\sqrt{2}]$) is empty.
On the other hand, for every nonempty open interval (a,b), $a<b$, by the density of $\mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $q \in (a,b)$. Thus, for every nonempty open interval $B_\varepsilon(x)$, if $B_\varepsilon(x) \cap [0,\sqrt{2}]$ is nonempty, then $B_\varepsilon(x) \cap [0,\sqrt{2}]$ contains a nonempty interval (a,b), and hence contains $q \in \mathbb{Q} \cap [0,\sqrt{2}]$. Thus closure$(\mathbb{Q} \cap [0,\sqrt{2}]) = [0,\sqrt{2}]$. Since $[0,\sqrt{2}]$ is a closed, bounded subset of $\mathbb{R}$, by Heine-Borel, $[0,\sqrt{2}]$ is compact.

(c) For $E_n = \left[\frac{1}{n^2}, \frac{1}{n^2-1}\right]$, interior$(E_n)$ equals $\mathcal{U}_n = \left(\frac{1}{n^2}, \frac{1}{n^2-1}\right)$. Since E_n is a subset of $E = \bigcup_{n \geq 2} E_n$, also $\mathcal{U}_n \subseteq$ interior(E). Thus $\mathcal{U} := \bigcup_{n\geq 2} \left(\frac{1}{n^2}, \frac{1}{n^2-1}\right)$ is contained interior(E). The relative complement is $\left\{\frac{1}{n^2}, \frac{1}{n^2-1} \,\middle|\, n \in \mathbb{N}, n \geq 2\right\}$. For every $n \in \mathbb{N}$, $n \geq 2$, for every positive real $\varepsilon \leq \frac{1}{n^2} - \frac{1}{(n+1)^2-1} = \frac{2}{n^2(n+2)}$, the set $\left(\frac{1}{n^2} - \varepsilon, \frac{1}{n^2}\right) \subset B_\varepsilon\left(\frac{1}{n^2}\right)$ is nonempty and disjoint from E. Hence $\frac{1}{n^2} \notin$ interior(E). Similarly, $\frac{1}{n^2-1} \notin$ interior(E). Thus interior(E) equals $\bigcup_{n=2}^{\infty} \left(\frac{1}{n^2}, \frac{1}{n^2-1}\right)$.
For $x \in (-\infty, 0)$, since $(-\infty,0) \cap E = \phi$, x is not in E^-. For $x \in (0,+\infty)$, there exists $N \in \mathbb{N}, N \geq 2$ with $\frac{1}{N^2-1} < x$. Thus $\left(\frac{1}{N^2-1}, \infty\right) \cap E = \bigcup_{2 \leq n < N} \left[\frac{1}{n^2}, \frac{1}{n^2-1}\right]$ is closed, and contains x if & only if $x \in E$. Finally $0 = \lim_{n\to\infty} \frac{1}{n^2}$. Hence $\boxed{\text{closure}(E) \text{ equals } \{0\} \cup E = \{0\} \cup \bigcup_{n=2}^{\infty} \left[\frac{1}{n^2}, \frac{1}{n^2-1}\right].}$ Since this is closed and bounded, it is compact by Heine-Borel.

Name: ______________________________ Problem 4: ____________ /25

Problem 4(25 points) Let (S, d) be a compact metric space, let (S^*, d^*) be a metric space, and let $f : (S, d) \to (S^*, d^*)$ be a continuous function.
(a) If f is onto, prove that (S^*, d^*) is a compact metric space.
(b) Assume that (S^*, d^*) is $\mathbb{R}$ with the usual metric, and assume that S is nonempty. Prove that $f(S)$ has a maximum, and prove that $f(S)$ has a minimum. You may now assume (a).
(c) Now for (S^*, d^*) a general metric space, assume that f is not onto, and let y be an element in $S^* \setminus f(S)$. Prove that $\{d^*(f(x), y) | x \in S\}$ has a minimum that is positive. You may now assume both (a) and (b).

(a) Let $(V_i)_{i\in I}$ be an open covering of S^*. Since f is continuous, every $f^{-1}(V_i)$ is an open subset of S. For every $s \in S$, $\exists i$ with $f(s) \in V_i$. Hence s is in $f^{-1}(V_i)$. Thus $(f^{-1}(V_i))_{i\in I}$ is an open covering of S. Since (S, d) is compact, there exists a finite subset $J \subseteq I$ such that $(f^{-1}(V_j))_{j\in J}$ is an open covering of S. For every $s^* \in S^*$, since f is onto, there exists $s \in S$ such that $s^* = f(s)$. Since $(f^{-1}(V_j))_{j\in J}$ is an open cover of S, there exists $j \in J$ with $f(s) \in V_j$. Thus $s^* = f(s)$ is in V_j. Thus $(V_j)_{j\in J}$ is an open subcovering of $(V_i)_{i\in I}$ that is finite. Thus (S^*, d^*) is compact.

(b) By (a), $f(S) \subseteq \mathbb{R}$ with the usual, Euclidean metric forms a compact metric space. By Heine-Borel, $f(S) \subseteq \mathbb{R}$ is a closed, bounded set. By the Completeness Axiom, every nonempty, closed, bounded subset of $\mathbb{R}$ contains a maximum and contains a minimum.

(c) Consider $g : (S^*, d^*) \to (\mathbb{R}, d_{Eucl})$, $g(z) = d^*(z, y)$. This is continuous (by the triangle inequality for d^*). Thus the composition $g \circ f : (S, d) \to (\mathbb{R}, d_{Eucl})$, $g(f(x)) = d^*(f(x), y)$, is also continuous. By (b), $g \circ f(S)$ has a minimum $d^*(f(x), y)$. Since $f(x) \neq y$ (since $y \notin f(S)$), by the positive definiteness of d^*, $d^*(f(x), y)$ is positive.

Name: ______________________ Problem 5: ____________ /35

Problem 5(25 points) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Prove that if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then also for every subsequence $(a_{n_k})_{k\in\mathbb{N}}$, the series $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely. Also give an example of (a_n) and a subsequence (a_{n_k}) such that the series $\sum_{n=1}^{\infty} a_n$ converges, yet the series $\sum_{k=1}^{\infty} a_{n_k}$ does not converge.

Define $(b_n)_{n\in\mathbb{N}}$ by $b_n = \begin{cases} a_{n_k}, & n=n_k \text{ some } k\in\mathbb{N} \\ 0, & n\notin\{n_k \mid k\in\mathbb{N}\} \end{cases}$. Then for every $n\in\mathbb{N}$, $0\leq |b_n| \leq |a_n|$. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then by the Comparison Test, also $\sum_{n=1}^{\infty} b_n$ converges absolutely, i.e., the sequence $(S_m)_{m\in\mathbb{N}} = \left(\sum_{n=1}^{m} |b_n|\right)_{m\in\mathbb{N}}$ converges. Thus the subsequence $(S_{n_k})_{k\in\mathbb{N}}$ also converges. Of course $S_{n_k} = \sum_{n=1}^{n_k} |b_n|$ equals $\sum_{\ell=1}^{k} |a_{n_\ell}|$. Thus, also $\sum_{\ell=1}^{\infty} a_{n_\ell}$ converges absolutely.

On the other hand, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges (conditionally), yet for the subsequence $(a_{2k})_{k\in\mathbb{N}} = \left(\frac{(-1)^{2k}}{2k}\right)_{k\in\mathbb{N}} = \left(\frac{1}{2k}\right)_{k\in\mathbb{N}}$, the series $\sum_{k=1}^{\infty} a_{2k} = \sum_{k=1}^{\infty} \frac{1}{2k}$ diverges to $+\infty$ by the Integral Test comparing the sequence with the divergent integral $\lim_{N\to\infty} \int_{x=1}^{N} \frac{2}{x}\,dx = \lim_{N\to\infty} 2\ln(N) = +\infty$.