MAT 320 Practice for Midterm 2

Remark. If you are comfortable with all of the following problems, you will be well prepared for Midterm 2.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Review Topics.
Definitions. Please know all of the following definitions. **Monotone Sequence. Nondecreasing / Nonincreasing Sequence. Bounded Sequence. Limit Supremum / Limit Infimum. Cauchy Sequence. Subsequence. Subsequential Limit. Metric Space and Distance Function. Ball of Radius ϵ Centered at x_0 in a Metric Space. Sequence, Subsequence, Boundedness, Convergence, Cauchy Sequence in a Metric Space. Complete Metric Space. Interior of a Subset of a Metric Space. Open Subset / Closed Subset of a Metric Space. Closure of a Subset of a Metric Space. Boundary of a Subset of a Metric Space. Decreasing Sequence of Closed Bounded Subsets of a Metric Space. Open Cover / Subcover of an Open Cover. Compactness. Series. Convergence of a**

Series / Divergence of a Series. Absolute Convergence of a Series. Geometric Series. Cauchy Criterion for Convergence of a Series. Continuity at x_0 for a Real-Valued Function. Continuous Function / Discontinuous Function.

Results. Please know all of the following lemmas, propositions, theorems and corollaries.

Monotone Convergence Theorem. Every bounded monotone sequence of real numbers converges. Every unbounded monotone sequence either diverges to $+\infty$ or diverges to $-\infty$.

Test for Convergence Via Lim Inf and Lim Sup. A bounded sequence of real numbers converges if and only if the lim inf equals the lim sup, in which case both equal the limit of the sequence.

Cauchy Convergence Theorem. A sequence of real numbers converges if and only if it is a Cauchy sequence.

Convergence and Subsequences. Every subsequence of a convergent sequence converges to the same limit as the original sequence.

Subsequential Limits and Lim Inf / Lim Sup. Every subsequential limit is bounded below by lim inf and bounded above by lim sup. If lim inf is finite, resp. if lim sup is finite, then it is the limit of a monotone subsequence.

Bolzano-Weierstrass Theorem. Every bounded sequence of real numbers has a convergent subsequence.

Convergence in a Product Metric Space. A sequence of elements of $\mathbb{R}^k$ converges with the Euclidean metric if and only if each of the k component sequences converges in $\mathbb{R}$.

Cauchy Convergence Theorem for $\mathbb{R}^k$. Every Cauchy sequence in $\mathbb{R}^k$ (with the Euclidean metric) converges. Said differently, $\mathbb{R}^k$ with the Euclidean metric is a complete metric space.

Bolzano-Weierstrass Theorem for $\mathbb{R}^k$. Every bounded sequence in $\mathbb{R}^k$ (with the Euclidean metric) has a convergent subsequence.

Axioms for a Topology. For a metric space (S, d), the empty set and S are both open subsets. The intersection of any finite collection of open subsets is an open subset. The union of any collection of open subsets is an open subset. Equivalently, both the empty set and S are closed subsets, the union of finitely many closed subsets is a closed subset, and the intersection of an arbitrary collection of closed subsets is a closed subset.

Interiors and Closures. For a subset T of a metric space (S, d), the interior of T equals the maximal open subset of S that is contained in T. The closure of T equals the minimal closed subset of S that contains T. In particular, S is open if and only if S equals its interior, resp. S is closed if and only if S equals its closure.

Countable Compactness and Sequential Compactness. For every metric space (S, d) that satisfies the Bolzano-Weierstrass theorem (i.e., bounded, closed subsets are "sequentially compact"), every decreasing sequence of nonempty, bounded, closed subsets has nonempty intersection (i.e., bounded, closed subsets are "countably compact").

MAT 320 Foundations of Analysis **Jason Starr**
Midterm 2 **Spring 2015**
Thursday 4/9 11:30am – 12:50pm

Heine-Borel Theorem. Every bounded metric space (S, d) that satisfies the Bolzano-Weierstrass theorem is compact: every open covering has a finite subcovering. In particular, every bounded, closed subset of $\mathbb{R}^k$ is compact.

Geometric Series Test. For $a \neq 0$, the geometric series $\sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$, in which case it converges absolutely.

p-Series Test. For $p > 0$, the series $\sum_{n=1}^{\infty}(1/n^p)$ converges if and only if $p > 1$.

Cauchy Criterion for Convergence. A series converges if and only if the sequence of partial sums is a Cauchy sequence.

Comparison Test. A series that is bounded above termwise in absolute value by a convergent series is also convergent. A nonnegative series that is bounded below by a divergent nonnegative series is also divergent.

Ratio Test. A series of nonzero real numbers is absolutely convergent if the lim sup of the absolute values of successive ratios is less than 1. The series diverges if the lim inf of the absolute values of the successive ratios is greater than 1.

Root Test. A series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\limsup \sqrt[n]{|a_n|}$ is less than 1. The series diverges if $\limsup \sqrt[n]{|a_n|}$ is greater than 1.

Integral Test. If $f(x)$, $x > 0$ is an integrable function such that $f(x) \geq a_n \geq 0$ for every $n \in \mathbb{N}$ and for every $x \in [n-1, n]$, and if $\int_{x=0}^{\infty} f(x)dx$ converges, then also $\sum_{n=0}^{\infty} a_n$ converges and is bounded above by the improper integral. Conversely, if $a_n \geq f(x) \geq 0$ for every $n \in \mathbb{N}$ and for every $x \in [n-1, n]$, and if $\int_{x=0}^{\infty} f(x)dx$ diverges, then also $\sum_{n=0}^{\infty} a_n$ diverges.

Alternating Series Test. For a nondecreasing sequence (a_n) of nonnegative real numbers, the alternating series $\sum(-1)^n a_n$ converges if and only if (a_n) converges to 0.

Sequential Convergence and Convergence. A function f defined at x_0 satisfies the $\epsilon - \delta$ definition of continuity if and only if it satisfies the sequential definition of continuity.

Properties of Continuous Functions. The class of continuous functions (defined on a specified set, continuity measured at a specified point) is preserved by absolute values, scaling, sum, difference, product, and nonzero division. The composition of continuous functions is continuous.

Extremal Value Theorem. Every bounded, closed subset of $\mathbb{R}$ has a maximum, and it has a minimum. Every continuous, real-valued function defined on a compact set has a maximum value and a minimum value on that compact set.

Intermediate Value Theorem. Every continuous function defined on a bounded, closed interval takes on every value between its maximum value and its minimum value.

Strictly Increasing Functions. A strictly increasing function defined on an interval is continuous if and only if the image is an interval. In this case, the inverse function is also continuous.

Please review all of the homework exercises on Sections 10 - 18. In addition the following theoretical problems are good practice.

Practice Problems.

(1) For a sequence of real numbers, if every monotone subsequence converges to a common limit, prove that the sequence converges to that limit.

(2) Prove that every subsequence of a Cauchy sequence (in a specified metric space) is a Cauchy sequence. Prove that every subsequence of a convergent sequence is a convergent sequence, and the limits are equal.

(3) For a metric space (S, d) and a subset C, if C with its induced metric is a complete metric space, prove that C is a closed subset of S.

(4) Give an example of a sequence of real numbers such that no subsequence is convergent. Prove your answer.

(5) For a metric space (S, d) and complete subsets C, C', prove that the union $C \cup C'$ is again complete. For every collection (C_i) of complete subsets, prove that the common intersection $\cap_i C_i$ is complete.

(6) For a metric space (S, d), prove that every compact subset is a closed subset. For compact subsets C, C', prove that the union $C \cup C'$ is again compact. For every collection (C_i) of compact subsets, prove that the common intersection $\cap_i C_i$ is a compact subset.

(7) For a metric space (S, d) and subsets A, B, prove that $(A \cup B)^-$ equals $A^- \cup B^-$. Give an example proving that $(A \cap B)^-$ may be strictly contained in $A^- \cap B^-$. Similarly, prove that the interior of $A \cap B$ equals the intersection $A^o \cap B^o$, yet the interior of $A \cup B$ may strictly contain $A^o \cup B^o$.

(8) If a series of nonnegative real numbers $\sum_{n=0}^{\infty} a_n$ converges, prove that also the series $\sum_{n=0}^{\infty} a_n^2$ converges. Does the series $\sum_{n=0}^{\infty} \sqrt{a_n}$ necessarily converge?

(9) For a series (a_n), if the series $\sum_{n=1}^{\infty} a_n$ converges, does it necessarily follow that for every subsequence a_{n_k}, the series $\sum_{k=1}^{\infty} a_{n_k}$ converges? If $\sum_{n=1}^{\infty} a_n$ converges absolutely, does $\sum_{k=1}^{\infty} a_{n_k}$ converge absolutely?

(10) For a continuous function f defined on $\mathbb{R}$, if f is constant on the terms of a convergent sequence, prove that f takes the same value at the limit of the sequence. Conclude that if f is constant on a set T, then f is also constant on the closure T^- of T.