MAT 320 Foundations of Analysis — Jason Starr
Final Exam — Spring 2015
Wednesday 5/13 5:30pm – 8:00pm

MAT 320 Review Sheet for Final Exam with Solutions

Remark. The Final Exam will be **cumulative**, although there will be an emphasis on material covered since Midterm 2. Please review Midterm 1 and Midterm 2. If you are comfortable with the material from Midterms 1 and 2, as well as the following, then you will be well prepared for the final exam.

Exam Policies. You must show up on time for all exams. Please bring your student ID card: ID cards may be checked, and students may be asked to sign a picture sheet when turning in exams. Other policies for exams will be announced / repeated at the beginning of the exam.

If you have a university-approved reason for taking an exam at a time different than the scheduled exam (because of a religious observance, a student-athlete event, etc.), please contact your instructor as soon as possible. Similarly, if you have a documented medical emergency which prevents you from showing up for an exam, again contact your instructor as soon as possible.

All exams are closed notes and closed book. Once the exam has begun, having notes or books on the desk or in view will be considered cheating and will be referred to the Academic Judiciary.

It is not permitted to use cell phones, calculators, laptops, MP3 players, Blackberries or other such electronic devices at any time during exams. If you use a hearing aid or other such device, you should make your instructor aware of this before the exam begins. You must turn off your cell phone, etc., prior to the beginning of the exam. If you need to leave the exam room for any reason before the end of the exam, it is still not permitted to use such devices. Once the exam has begun, use of such devices or having such devices in view will be considered cheating and will be referred to the Academic Judiciary. Similarly, once the exam has begun any communication with a person other than the instructor or proctor will be considered cheating and will be referred to the Academic Judiciary.

Review Topics.
Definitions. Please know all of the following definitions. **Connected Metric Space. Path Connected Metric Space. Uniform Metric. Uniform Continuity. Uniformly Cauchy Sequence of Functions. Uniform Convergence / Uniform Limit of Sequence of Functions. Limit of a Function at a Accumulation / Limit Point of the Domain. Differentiability. Derivative. Partition and Tagged (or Marked) Partition. Refinement of**

Partitions. Upper / Lower Darboux Sum. Darboux Integral. Riemann Sum. Mesh. Riemann Integral. Power Series. Taylor Polynomials and Taylor Series.
Results. Please know all of the following lemmas, propositions, theorems and corollaries.

Connectedness of the Unit Interval. The unit interval with the Euclidean metric is a connected metric space.

Connectedness and Path Connectedness. Every path connected metric space is connected.

Uniform Continuity on Compact Domains. For every compact metric space (X, d_X), for every metric space (Y, d_Y), every continuous function f from (X, d_X) to (Y, d_Y) is uniformly continuous.

Completeness of $C(X)$ with Uniform Metric. For every metric space (X, d_X), for every complete metric space (Y, d_Y), the set $B((X, d_X), (Y, d_Y))$ of bounded functions from (X, d_X) to (Y, d_Y) endowed with the uniform metric is a complete metric space. The subset $C((X, d_X), (Y, d_Y))$ of bounded continuous functions is a closed subset. The subset $UC((X, d_X), (Y, d_Y))$ of uniformly continuous functions is also a closed subset.

Basic Properties of Differentiability. Differentiability is preserved by: scalar multiples, sums, pointwise products, pointwise quotients (where defined), composition. Moreover, $(cf)'(a) = c \cdot f'(a)$, $(f+g)'(a) = f'(a) + g'(a)$, $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$, $(f/g)'(a) = [g(a) \cdot f'(a) - f(a) \cdot g'(a)]/(g(a))^2$, $(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$.

Rolle's Theorem. A continuous (real-valued) function f on $[a, b]$ that is differentiable on (a, b) and has $f(a) = f(b)$ has a critical point in (a, b).

Mean Value Theorem. A continuous (real-valued) function f on $[a, b]$ that is differentiable on (a, b) has $f'(c) = [f(b) - f(a)]/(b - a)$ for some $c \in (a, b)$.

Increasing and Decreasing Functions. A continuous (real-valued) function f on $[a, b]$ that is differentiable on (a, b) is nondecreasing, resp. nonincreasing, if and only if $f'(c) \geq 0$, resp. $f'(c) \leq 0$, for every $c \in (a, b)$. If for every $c \in (a, b)$, $f'(c)$ is positive, resp. negative, then f is strictly increasing, resp. strictly decreasing.

Derivative of an Inverse Function. For a strictly increasing, surjective function $f : [a, b] \to [\widetilde{a}, \widetilde{b}]$, for $c \in (a, b)$ such that $f'(c)$ is defined and nonzero, then the inverse function f^{-1} is differentiable at $f(c)$ with $(f^{-1})'(f(c)) = 1/f'(c)$.

Darboux Sums of Refinements. For a bounded (real-valued) function f on $[a, b]$, for a partition P of $[a, b]$, for a refinement Q of P, the upper and lower Darboux sums satisfy

$$\inf(f) \cdot (b - a) \leq L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P) \leq \sup(f) \cdot (b - a).$$

Computation of Darboux Integral. For a bounded (real-valued) function f on $[a, b]$, f is (Darboux) integrable if and only if there exists a sequence of partitions $(P_n)_{n \in \mathbb{N}}$ such that $\lim_n U(f, P_n) - L(f, P_n) = 0$, in which case the (Darboux) integral equals $\lim_n L(f, P_n) = \lim_n U(f, P_n)$.

Riemann Integrals and Darboux Integrals. For a bounded (real-valued) function f on $[a, b]$, f is Darboux integrable if and only if it is Riemann integrable, in which case the Darboux integral equals the Riemann integral.

Properties of Integrals. For integrable functions f and g on $[a, b]$, every scalar multiple $c \cdot f$ is integrable with $\int_a^b c \cdot f(x)dx = c \cdot \int_a^b f(x)dx$, the sum $f + g$ is integrable with $\int_a^b (f(x) + g(x))dx = (\int_a^b f(x)dx) + (\int_a^b g(x)dx)$. If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$. The function $|f|$ is integrable with $\int_a^b f(x)dx \leq \int_a^b |f(x)|dx$. In particular, the Euclidean distance between $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ is bounded by $d_{\text{unif}}(f, g) \cdot (b - a)$.

Integrability of Monotone and Piecewise Continuous Functions. Every bounded monontone function on $[a, b]$ is integrable. Every bounded, piecewise continuous function on $[a, b]$ is integrable.

Fundamental Theorem of Calculus. For a continuous function F on $[a, b]$ that is differentiable on (a, b) and whose derivative f is integrable, $\int_a^b f(x)dx$ equals $F(b) - F(a)$. For every continuous function f on $[a, b]$, the function $F(x) = \int_a^x f(t)dt$ is differentiable on (a, b) with derivative f.

Integration by Parts. For continuous function f, g on $[a, b]$ that are differentiable on (a, b) and whose derivatives are integrable,

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(b)g(g) - f(a)g(a).$$

Change of Variables Formula. For a strictly increasing, surjective function $u : [a, b] \to [\widetilde{a}, \widetilde{b}]$ such that u is differentiable on (a, b), for every continuous, bounded function f on $[\widetilde{a}, \widetilde{b}]$,

$$\int_a^b f(u(x)) \cdot u'(x)dx = \int_{\widetilde{a}}^{\widetilde{b}} f(t)dt.$$

Re-expansion of Analytic Functions. For a power series $\sum_m a_m(x - x_0)^m$ with radius of convergence $R > 0$, for every $0 < R_1 < R$, the partial sums $\sum_{m=1}^n a_m(x - a_0)^m$ converge uniformly to a uniformly continuous function $F(x)$ on $[x_0 - R_1, x_0 + R_1]$. For every $x_1 \in (x_0 - R, x_0 + R)$, there is a power series $\sum_m b_m(x - x_1)^m$ with radius of convergence $R' \geq R - |x_0 - x_1| > 0$ that agrees with $f(x)$ on the common domain.

Infinite Differentiability of Analytic Functions. For a power series $F(x) = \sum_m a_m(x - x_0)^m$ with radius of convergence $R > 0$, the power series $f(x) = \sum_m (m + 1)a_{m+1}(x - x_0)^m$ has radius of convergence $R > 0$ and $\int_{x_0}^x f(t)dt$ equals $F(x) - F(0)$. Thus F is differentiable on $(x_0 - R, x_0 + R)$. It follows that F is infinitely differentiable on $(x_0 - R, x_0 + R)$. There exists $0 < R_1 < R$ such that $(|F^{(n)}(x)|/n!)_{n=0,1,2,...}$ is simultaneously uniformly bounded on $(x_0 - R_1, x_0 + R_1)$. Conversely, every infinitely differentiable function $F(x)$ such that $(|F^{(n)}(x)|/n!)_n$ is simulteneously, uniformly bounded on $(x_0 - R, x_0 + R)$ for some $R > 0$ is analytic on some interval $(x_0 - R_1, x_0 + R_1)$ with $0 < R_1 < R$.

Taylor's Theorem. For an n-times differentiable function f on (a,b), for $x_0 \in (a,b)$, for the Taylor polynomial $T_n(f,x_0,x) = \sum_{m=0}^{n-1} f^{(m)}(x_0)/m!(x-x_0)^m$ and with remainder $R_n(f,x_0,x) = f(x) - T_n(f,x_0,x)$, for every $x \in (a,b)$ there exists y between x_0 and x (inclusive) such that $R_n(f,x_0,x)$ equals $f^{(n)}(y)/n!(x-x_0)^n$. Also,

$$R_n(f,x_0,x) = \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

Thus there exists y between x_0 and x (inclusive) such that $R_n(f,x_0,x)$ equals $(x-y)^{n-1}/(n-1)!f^{(n)}(y)(x-x_0)$.

Please review all of the homework exercises. In addition the following theoretical problems are good practice for the new material.

Practice Problems.

(1) For a metric space (X,d_X), for a complete metric space (Y,d_Y), for a subset $A \subset X$, and for a uniformly continuous function $f : (A, d_X|_A) \to (Y,d_Y)$, prove that there exists a unique continuous function $\widetilde{f} : (\overline{A}, d_X|_{\overline{A}}) \to (Y,d_Y)$ whose restriction to $A \subset \overline{A}$ equals f. Moreover, prove that $\widetilde{f}$ is uniformly continuous.

Solution to (1) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A that is Cauchy. Since f is uniformly continuous, also the sequence $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y,d_Y): indeed, for every $\epsilon > 0$, since f is uniformly continuous, there exists $\delta > 0$ such that for every $a,a' \in A$ with $d_X(a,a') < \delta$, also $d_Y(f(a),f(a')) < \epsilon$. Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $N \in \mathbb{N}$ such that for every $m,n \in \mathbb{N}$ with $m \geq N$ and $n \geq N$, then $d_X(a_m,a_n) < \delta$. Thus, also $d_Y(f(a_m),f(a_n)) < \epsilon$.

Since (Y,d_Y) is complete and since $(f(a_n))_{n\in\mathbb{N}}$ is Cauchy, there exists a limit b of $(f(a_n))_{n\in\mathbb{N}}$ in Y, and this limit is unique.

Now let $\overline{a}$ be any element in $\overline{A}$. By the definition of $\overline{A}$, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $(a_n)_{n\in\mathbb{N}}$ converges to $\overline{a}$. Since $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, it is also a Cauchy sequence. Thus, by the previous paragraph, there exists $b \in Y$ such that $(f(a_n))_{n\in\mathbb{N}}$ converges to b. Define $\widetilde{f}(\overline{a})$ to be b. Note that if $\overline{a}$ is in A, then since f is continuous, $(f(a_n))_{n\in\mathbb{N}}$ converges to $f(a)$. Since the limit of a convergent sequence is unique, $\widetilde{f}(a)$ equals $f(a)$ for every $a \in A$.

Let $\epsilon > 0$ be a real number. Since f is uniformly continuous, there exists real $\delta > 0$ such that for every a and a' in A, if $d_X(a,a') < \delta$ then also $d_Y(f(a),f(a')) < \epsilon/3$. Let $\overline{a}$ and $\overline{a'}$ be elements of $\overline{A}$ such that $d_X(\overline{a},\overline{a'}) < \delta/3$. By the definition of $\overline{A}$, there exist sequences $(a_n)_{n\in\mathbb{N}}$ and $(a'_n)_{n\in\mathbb{N}}$ in A converging to $\overline{a}$ and $\overline{a'}$ respectively. By definition of $\widetilde{f}$, the sequence $(f(a_n))_{n\in\mathbb{N}}$ converges to $\widetilde{f}(\overline{a})$ and the sequence $(f(a'_n))_{n\in\mathbb{N}}$ converges to $\widetilde{f}(\overline{a}')$. Thus, there exist N and N' in $\mathbb{N}$ such that for every n in $\mathbb{N}$ with $n \geq N$, resp. for every n' in $\mathbb{N}$ with $n' \geq N'$, also $d_Y(f(a_n),\widetilde{f}(a)) < \epsilon/3$, resp. $d_Y(f(a'_{n'}),\widetilde{f}(a)) < \epsilon/3$. Since $(a_n)_{n\in\mathbb{N}}$ converges to $\overline{a}$, there exists M and M' in $\mathbb{N}$ such that for every n in $\mathbb{N}$ with $n \geq M$, resp. for every n' in $\mathbb{N}$ with $n' \geq M'$, also $d_X(a_n,\overline{a}) < \delta/3$, resp. $d_X(a'_{n'},\overline{a}') < \delta/3$. Let n be $\max(M,N)$, and let n' be $\max(M',N')$. Since $n \geq M$ and $n' \geq M'$, by the triangle inequality,

$$d_X(a_n, a'_{n'}) \leq d_X(a_n, \overline{a}) + d_X(\overline{a}, \overline{a}') + d_X(\overline{a}', a'_{n'}) < \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta.$$

Thus, $d_Y(f(a_n), f(a'_{n'})) < \epsilon/3$. Since $n \geq N$ and $n' \geq N'$, by the triangle inequality again,

$$d_Y(\widetilde{f}(\overline{a}), \widetilde{f}(\overline{a}')) \leq d_Y(\widetilde{f}(\overline{a}), f(a_n)) + d_Y(f(a_n), f(a'_{n'})) + d_Y(a'_{n'}, \widetilde{f}(\overline{a}')) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus, for every real $\epsilon > 0$, there exists real $\delta > 0$ such that for every $\overline{a}$ and $\overline{a}'$ in $\overline{A}$, if $d_X(\overline{a}, \overline{a'}) < \delta/3$ then $d_Y(\widetilde{f}(\overline{a}), \widetilde{f}(\overline{a}')) < \epsilon$. Therefore $\widetilde{f} : (\overline{A}, d_X|_{\overline{A}}) \to (Y, d_Y)$ is uniformly continuous and the restriction of $\widetilde{f}$ to A equals f.

Finally, let $\overline{f} : (\overline{A}, d_X|_{\overline{A}}) \to (Y, d_Y)$ be any continuous function whose restriction to A equals f. For every $\overline{a} \in \overline{A}$, let $(a_n)_{n\in\mathbb{N}}$ be the sequence in A converging to $\overline{a}$ used in the construction of $\widetilde{f}$. Since $\overline{f}$ is continuous, the sequence $(\overline{f}(a_n))_{n\in\mathbb{N}}$ converges to $\overline{f}(\overline{a})$. Since every a_n is in A, and since the restriction of $\overline{f}$ to A equals f, the sequence $(\overline{f}(a_n))_{n\in\mathbb{N}}$ equals the sequence $(f(a_n))_{n\in\mathbb{N}}$. By construction, this converges to $\widetilde{f}(\overline{a})$. Since the limit of a convergent sequence is unique, $\overline{f}(\overline{a})$ equals $\widetilde{f}(\overline{a})$. Thus the function $\overline{f}$ equals $\widetilde{f}$. Therefore $\widetilde{f}$ is the unique continuous function from $(\overline{A}, d_X|_{\overline{A}})$ to (Y, d_Y) whose restriction to A equals f.

(2) Inside $(C([a,b]), d_{\text{unif}})$, let A be the subset of continuous piecewise linear functions, i.e., the set of functions g such that there exists a partition P of $[a,b]$ such that the restriction of g to every P-interval is linear. Prove that A is dense, i.e., $\overline{A}$ equals the entire metric space. **Hint.** For a continuous f in $C([a,b])$, for every $\epsilon > 0$, since f is uniformly continuous, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Let P be a partition with mesh $< \delta$, and let f_P be the piecewise function that agrees with f on the endpoints of P-intervals. Prove that $d_{\text{unif}}(f, f_P) < 3\epsilon$.

Solution to (2) Let f be an element in $C([a,b])$. Let $\epsilon > 0$ be a real number. Since $[a,b]$ is bounded and closed, by the Heine-Borel Theorem, $[a,b]$ is compact. Since f is continuous an $[a,b]$ is compact, f is uniformly continuous. Thus, there exists real $\delta > 0$ such that for every $x, x' \in [a,b]$ with $|x' - x| < \delta$, also $|f(x') - f(x)| < \epsilon/2$. Let $P = \{x_0, x_1, \dots, x_n\}$ be any partition of $[a,b]$ with mesh $< \delta$. Let f_P be the unique piecewise linear function on $[a,b]$ such that for every $k = 0, 1, \dots, n$, $f_P(x_k)$ equals $f(x_k)$. The claim is that $d_{\text{unif}}(f, f_P) < \epsilon$, i.e., for every $x \in [a,b]$, $|f_P(x) - f(x)| < \epsilon$ (this uses that the continuous function $g(x) = |f_p(x) - f(x)|$ on the compact set $[a,b]$ attains its maximum).

For every x in $[a,b]$ there exists $k = 1, \dots, n$ such that x is in $[x_{k-1}, x_k]$. Since $|x_k - x_{k-1}| < \delta$, also $|f(x_k) - f(x_{k-1})| < \epsilon/2$. By the definition of f_P, also $|f_P(x_k) - f_P(x_{k-1})| = |f(x_k) - f(x_{k-1})| < \epsilon/2$. Since x is in $[x_{k-1}, x_k]$ and since f_P is linear on $[x_{k-1}, x_k]$, also $|f_P(x_k) - f_P(x)| < \epsilon/2$. Finally, since $|x_k - x| < \delta$, also $|f(x_k) - f(x)| < \epsilon/2$. Thus, by the triangle inequality,

$$|f(x) - f_P(x)| = |(f(x) - f(x_k)) + (f_P(x_k) - f_P(x))| \leq |f(x) - f(x_k)| + |f_P(x_k) - f_P(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since this holds for every $x\imath[a,b]$, $d_{\mathrm{unif}}(f, f_P) < \epsilon$. Since for every real $\epsilon > 0$ there exists f_P in A with $d_{\mathrm{unif}}(f, f_P) < \epsilon$, f is in the closure of A. Since every f in $C([a,b])$ is in the closure of A, the closure of A equals $C([a,b])$.

(3) With A as above, define $I : (A, d_{\mathrm{unif}}) \to (\mathbb{R}, d_{\mathrm{Eucl}})$ to be the integral of the piecewise linear function g by the "usual" formula, i.e., if g is linear on the P-intervals of a partition P, then

$$I(g) = \sum_{[x,y]\in P} \frac{(y-x)(f(y)+f(x))}{2}.$$

Prove that this is compatible with refinement of partitions. Use this to prove that for f and g in A, $|I(g) - I(f)| \leq d_{\mathrm{unif}}(f,g)(b-a)$. Conclude that I is uniformly continuous, and hence extends to a uniformly continuous function

$$I : (C([a,b]), d_{\mathrm{unif}}) \to (\mathbb{R}, d_{\mathrm{Eucl}}).$$

Conclude that this extension agrees with the usual Darboux and Riemann integrals.

Solution to (3) Since f is linear on $[x,y]$, $yf(x)$ equals $xf(y)$, i.e., $yf(x) - xf(y)$ equals 0. Let z be any element in (x,y), i.e., $z = tx + (1-t)y$ for some $t \in (0,1)$. Since f is linear on $[x,y]$, $f(tx + (1-t)y)$ equals $tf(x) + (1-t)f(y)$. Therefore,

$$(y-x)f(z) = t(y-x)f(x) + (1-t)(y-x)f(y) = (tx+(1-t)y)(f(y)-f(x)) + (yf(x)-xf(y)) = z(f(y)-f(x)).$$

Thus,

$$\frac{(y-z)(f(y)+f(z))}{2} + \frac{(z-x)(f(z)+f(x))}{2} =$$
$$\frac{(y-x)(f(y)+f(x))}{2} + \frac{(y-x)f(z)}{2} - \frac{z(f(y)-f(x))}{2} = \frac{(y-x)(f(y)+f(x))}{2}.$$

Therefore the contribution to $I(f)$ from the interval $[x,y]$ is unchanged by subdividing $[x,y]$. By induction on the number of subdivisions, $I(f)$ is compatible with arbitrary refinement of partitions.

Now let f and g be elements in A. There exists a common refinement P of the "linearizing partitions" of f and g. Thus, both f and g are linear on each P-interval $[x,y]$. By definition, the contribution to $I(g) - I(f)$ from $[x,y]$ is

$$\frac{(y-x)(g(y)+g(x))}{2} - \frac{(y-x)(f(y)+f(x))}{2} = \frac{y-x}{2}((g(y)-f(y)) + (g(x)-f(x))).$$

By the triangle inequality, the absolute value of this contribution is no greater than

$$\frac{y-x}{2}(|g(y)-f(y)| + |g(x)-f(x)|) \leq \frac{y-x}{2}(d_{\mathrm{unif}}(g,f) + d_{\mathrm{unif}}(g,f)) = (y-x)d_{\mathrm{unif}}(g,f).$$

Applying the triangle inequality once more,

$$|I(g) - I(f)| \leq \sum_{[x,y]\in P} (y-x)d_{\mathrm{unif}}(g,f) = (b-a)d_{\mathrm{unif}}(g,f).$$

Therefore I is Lipschitz with Lipschitz constant $b-a$. In particular, I is uniformly continuous. By (1), I extends uniquely to a continuous function $\widetilde{I}$ on the closure of A. By (2), the closure of A equals all of $C([a,b])$. Thus, there exists a unique continuous function

$$\widetilde{I} : (C([a,b]), d_{\text{unif}}) \to (\mathbb{R}, d_{\text{Eucl}})$$

whose restriction to A is the usual integral. Moreover, by construction $\widetilde{I}$ is Lipschitz with Lipschitz constant $b-a$. Since the usual Riemann integral also is Lipschitz with Lipschitz constant $b-a$, and since the Riemann integral equal I on A, it follows that the unique continuous extension $\widetilde{I}$ is the usual Riemann integral on $C([a,b])$.

(4) Define $f(x)$ on $\mathbb{R}$ by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that f is infinitely differentiable and all derivatives of f vanish at $x = 0$. Conclude that for every $R > 0$, the collection $(|f^{(n)}(x)|/n!)_{n=0,1,2,\ldots}$ is not uniformly bounded $[-R, R]$.

Solution to (4) Let $n \geq 0$ be any integer. Let $q(x)$ be any polynomial. Define $Q = Q_{n,q}$

$$Q(x) = \begin{cases} (q(x)/x^{3n})e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

Applying L'Hospital's Rule, the limit of $Q(x)$ at $x = 0$ exists 0 so that Q is continuous at $x = 0$. Moreover, the difference quotient computing the derivative of Q at $x = 0$ is of the form $Q_{n+1,x^2q(x)}$, so that also the limit of the difference quotient exists and equals 0. Thus Q is differentiable at $x = 0$ with derivative 0. The derivative of Q at all other points is straightforward to compute by the usual derivative rules,

$$Q'(x) = \frac{(1 - 3nx^2)q(x) + x^3q'(x)}{x^{3(n+1)}}e^{-1/x^2}.$$

Defining $r(x)$ to be the polynomial $(1 - 3nx^2)q(x) + x^3q'(x)$, the derivative is everywhere of the form $Q_{n+1,r(x)}$, which is continuous. Thus, by induction on n, Q is continuously differentiable to all orders and all derivatives of Q vanish at $x = 0$.

Thus, every Taylor polynomial of e^{-1/x^2} at $x = 0$ is the zero polynomial. For an infinitely differentiable function $f(x)$, if there exists $R > 0$ such that $(|f^{(n)}(x)|/n!)_{n\in\mathbb{N}}$ is uniformly bounded on $[-R, R]$, then the Taylor polynomials converge uniformly to $f(x)$ on $[-R, R]$. Since the Taylor polynomials are zero, and since the zero functions converge only to the zero function, this implies that e^{-1/x^2} is identically zero on $[-R, R]$. This is absurd since e^{-1/x^2} is positive for all $x \neq 0$. Therefore, by way of contradiction, the sequence $(|f^{(n)}(x)|/n!)_{n\in NN}$ is not uniformly bounded on $[-R, R]$.

(5) Let $b > 1$ be a real number. For every $n \in \mathbb{N}$, define $q_n = \sqrt[n]{b}$, and define P_n to be the partition of $[1, b]$ with $x_k = q_n^k$. For $r \geq 0$, for $f(x) = x^r$, compute $L(f, P_n)$ and $U(f, P_n)$. Prove that $\lim_n L(f, P_n) = \lim_n U(f, P_n) = (b^{r+1} - 1)/(r + 1)$.

Solution to (5) The interval $[x_{k-1}, x_k]$ is $[q_n^{k-1}, q_n^k]$. The length of this interval is $\Delta x_k = q_n^k - q_n^{k-1} = (q_n - 1)q_n^{k-1}$. Since $f(x)$ is nondecreasing, the supremum of f on this interval is $f(x_k) = q_n^{kr}$. The infimum is $f(x_{k-1}) = q_n^{(k-1)r}$. Thus the upper Darboux sum is

$$U(f, P_n) = \sum_{k=1}^{n} \sup_{[x_{k-1}, x_k]}(f) \cdot \Delta x_k = \sum_{k=1}^{n}(q_n - 1)q_n^r q_n^{(k-1)(r+1)} = (q_n - 1)q_n^r \sum_{k=1}^{n} q_n^{(k-1)(r+1)} =$$

$$(q_n - 1)q_n^r \frac{q_n^{n(r+1)} - 1}{q_n^{r+1} - 1} = \frac{q_n^r}{1 + q_n + \cdots + q_n^r}(b^{r+1} - 1).$$

Similarly, the lower Darboux sum is

$$L(f, P_n) = \sum_{k=1}^{n} \inf_{[x_{k-1}, x_k]}(f) \cdot \Delta x_k = \sum_{k=1}^{n}(q_n - 1)q_n^{(k-1)(r+1)} =$$

$$(q_n - 1)\sum_{k=1}^{n} q_n^{(k-1)(r+1)} = (q_n - 1)\frac{q_n^{n(r+1)} - 1}{q_n^{r+1} - 1} = \frac{1}{1 + q_n + \cdots + q_n^r}(b^{r+1} - 1).$$

Of course for every $d = 1, \dots, r$,

$$\lim_{n\to\infty} q_n^d = \lim_{n\to\infty} \sqrt[n]{b^d} = 1.$$

Thus, using the usual limit laws,

$$\lim_{n\to\infty} U(f, P_n) = \frac{1}{1 + 1 + \cdots + 1}(b^{r+1} - 1) = \frac{b^{r+1} - 1}{r + 1} = \lim_{n\to\infty} L(f, P_n).$$

Therefore f is integrable on $[1, b]$ and the integral equals $(b^{r+1} - 1)/(r + 1)$.

(6) Find an example of strictly increasing, differentiable function on $[-1, 1]$ such that the derivative is not always positive. Can you find such an example where the derivative is 0 for infinitely many $x \in (-1, 1)$? Can you find such an example where the derivative is 0 for uncountably many x?

Solution to (6) Probably the simplest example of a strictly increasing, differentiable function $f(x)$ with f' not strictly positive is $f(x) = x^3$. There are various ways to find an example whose derivative is 0 infinitely often, e.g.,

$$f(x) = \int_{t=-1}^{x} (1 - t^2)\sin(1/(1 - t^2))dt.$$

To construct an example such that f' is zero an uncountable set, let $K \subset [-1, 1]$ be a middle thirds Cantor set, let $g : [-1, 1] \to \mathbb{R}_{\geq 0}$ be the continuous function,

$$g(x) = \inf\{|x - y| : y \in K\}.$$

Since K is compact, g is a continuous function. By construction, g is zero on K. For every open interval in the complement of K, g is piecewise linear and positive on the interval. Thus the function,

$$f(x) = \int_{t=-1}^{x} g(t)dt,$$

is differentiable with derivative $g(x)$ by the Fundamental Theorem of Calculus. Since g is nonnegative, $f(x)$ is nondecreasing. Since g is positive on every open interval in the complement of K, f is strictly increasing on such an interval. However, for every pair of points $-1 \leq x < y \leq 1$, there exists x' and y' with $x < x' < y' < y$ such that (x', y') is in the complement of K. Thus, since f is nondecreasing and f is strictly increasing on (x', y'),

$$f(x) \leq f(x') < f(y') \leq f(y).$$

Therefore f is strictly increasing, f is everywhere differentiable with derivative g, and g is zero on the uncountable set K.

(7) Define $\ell(x)$ on $(0, \infty)$ by $\ell(x) = \int_1^x (1/t)dt$. For $1 < a$ and $1 < b$, use the change of variables formula to prove that $\int_a^{ab}(1/t)dt = \int_1^b (1/t)dt$. Conclude that $\ell(ab)$ equals $\ell(a) + \ell(b)$. Similarly, prove that $\ell(1) = 0$, $\ell(1/a) = -l(a)$, and $\ell(a^r) = r\ell(a)$. For every $a > 0$ with $a \neq 1$, conclude that $\log_a(b) = \ell(b)/\ell(a)$. In particular, for the unique real number $e > 0$ with $\ell(e) = 1$, prove that $\ell(b) = \log_e(b)$ (assuming we have already made sense of exponentiation and logarithms for arbitrary real numbers).

Solution to (7) Define $u(x)$ to be the strictly increasing linear function $u(x) = ax$ with derivative $u'(x) = a$. By the change of variables formula,

$$\int_{u(1)}^{u(b)} f(s)ds = \int_1^b f(u(x))u'(x)dx.$$

For $f(x) = 1/x$, $f(ax)a$ equals $a/(ax)$, i.e., $f(u(x))u'(x)$ equals $f(x)$. Therefore,

$$\int_a^{ab} (1/s)ds = \int_1^b (1/x)dx = \ell(b).$$

Thus, by properties of integrals,

$$\ell(ab) = \int_1^{ab}(1/s)ds = \int_1^a (1/s)ds + \int_a^{ab}(1/s)ds = \int_1^a (1/s)ds + \int_1^b (1/t)dt = \ell(a) + \ell(b).$$

The remaining identities are similar.

(8) Combine the previous problem with the derivative of the inverse function to conclude that there exists a strictly increasing function $E(x)$ such that $E'(x) = E(x)$, $E(0) = 1$, $E(x+y) = E(x) \cdot E(y)$, $E(-x) = 1/E(x)$, and $E(rx) = E(x)^r$. Finally, assuming we have already made sense of exponentiation for real numbers, prove that $E(x)$ equals e^x with e as above.

(9) Repeat the previous two exercises with the function

$$T^{-1}(x) := \int_0^x \frac{1}{1+t^2}dt$$

to make sense of the arctangent function T^{-1} and thus the tangent function $T(x)$. Use the identities $S(x) = T(x/2)/(1+(T(x/2))^2)$ and $C(x) = (1-(T(x/2))^2)/(1+(T(x/2))^2)$ to make sense of the sine and cosine functions. Which trigonometric identities can you prove directly from these definitions?

(10) Give an example of a bounded function $f(x)$ that is not integrable, yet such that $|f(x)|$ is integrable.

Solution to (10) Define $f(x)$ to be -1 if $x \in \mathbb{Q}$ and to be $+1$ if $x \notin \mathbb{Q}$. On every nonempty interval, by the Archimedean Property, $\inf(f)$ equals -1, and $\sup(f)$ equals $+1$. From this it is immediate that f is not integrable. On the other hand, $|f(x)|$ is the constant function 1, hence integrable.

(11) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of bounded, integrable functions on $[a,b]$, and let f be a bounded, integrable function on $[a,b]$. Assume that for every $x \in [a,b]$, $(f_n(x))_{n\in\mathbb{N}}$ is a nondecreasing sequence of real numbers that converges to $f(x)$. Prove that $(\int_a^b f_n(x)dx)_{n\in\mathbb{N}}$ is a nondecreasing sequence of real numbers that converges to $\int_a^b f(x)dx$.

(12) Find a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, integrable functions on $[a,b]$ that converges pointwise to a bounded, integrable function f on $[a,b]$ yet such that $(\int_a^b f_n(x)dx)_{n\in\mathbb{N}}$ does not converge to $\int_a^b f(x)dx$. (This is a challenging exercise, more challenging than would be asked on the final exam.)

Solution to (12) Let $[a,b]$ be $[-1,1]$. For every integer n, let $f_n(x)$ be the following continuous, piecewise linear function.

$$f_n(x) = \begin{cases} 0, & -1 \leq x \leq 0, \\ n^2x, & 0 < x \leq 1/n, \\ 2n - n^2x, & 1/n < x \leq 2/n, \\ 0, & 2/n < x \leq 1. \end{cases}$$

For every $x \leq 0$, for every $n \in \mathbb{N}$, $f_n(x)$ equals 0. Thus, the sequence $(f_n(x))_{n\in\mathbb{N}}$ is $(0)_{n\in\mathbb{N}}$, which converges to 0. For every $x > 0$, by the Archimedean property, there exists $N \in \mathbb{N}$ such that $2/N < x$. Thus, for every $n \in \mathbb{N}$ with $n \geq N$, $f_n(x)$ equals 0. Thus the tail of the sequence $(f_n(x))_{n\geq N}$ is $(0)_{n\in\mathbb{N}}$, which converges to 0. Thus, for every $x \in [-1,1]$, the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to 0. On the other hand, for every $n \in \mathbb{N}$, $\int_{-1}^1 f_n(x)dx$ equals 1, and $(1)_{n\in\mathbb{N}}$ does not converge to $\int_{-1}^1 0dx = 0$.

(13) Let F be a bounded, continuous function on $[a,b]$ such that for some partition P, the restriction of F to every P-interval is differentiable on the interior of the interval and the derivative f is uniformly continuous. Apply the usual integration by parts to each P-interval to prove that for every continuously differentiable function u on $[a,b]$,

$$\int_a^b f(x)u(x)dx + \int_a^b F(x)u'(x)dx = F(b)u(b) - F(a)u(a).$$

Thus, as far as integration by parts is concerned, it is as if F is everywhere differentiable with derivative f, i.e., f is a "weak derivative" of F.

Solution to (13) Let P be the partition $(x_0, x_1, \dots, x_k, \dots, x_{n-1}, x_n)$. Let $[x_{k-1}, x_k]$ be any P-interval. By integration by parts,

$$\int_{x_{k-1}}^{x_k} f(x)u(x)dx + \int_{x_{k-1}}^{x_k} F(x)u'(x)dx = F(x_k)u(x_k) - F(x_{k-1})u(x_{k-1}).$$

Thus, using properties of the integral,

$$\int_a^b f(x)u(x)dx + \int_a^b F(x)u'(x)dx =$$

$$\left(\sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)u(x)dx\right) + \left(\sum_{k=1}^n \int_{x_{k-1}}^{x_k} F(x)u'(x)dx\right) =$$

$$\sum_{k=1}^n \left(\int_{x_{k-1}}^{x_k} f(x)u(x)dx + \int_{x_{k-1}}^{x_k} F(x)u'(x)dx\right) =$$

$$\sum_{k=1}^n \left(F(x_k)u(x_k) - F(x_{k-1})u(x_{k-1})\right).$$

This telescoping sum equals

$$F(x_n)u(x_n) - F(x_0)u(x_0) = F(b)u(b) - F(a)u(a).$$

(14) Let (F, f) be a pair as in the previous exercise, and let (G, g) be another such pair. Prove that (cF, cf), $(F + G, f + g)$ and $(FG, fG + Fg)$ are also such pairs.

Solution to (14) As in the previous problem, due to the compatibility between the integral and the telescoping sum with respect to subdividing the interval, each of these claims can be checked on individual intervals $[x_{k-1}, x_k]$ on which F and G are continuously differentiable. In that case, by the usual rules for differentiation, $(cF)'(x)$ equals $cF'(x) = cf(x)$, $(F+G)'(x)$ equals $F'(x) + G'(x) = f(x) + g(x)$, and $(FG)'(x)$ equals $F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x)$.

(15) Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on $[a, b]$ that converges uniformly to F. Assume that every F_n is differentiable on (a, b) and the sequence $(F_n')_{n\in\mathbb{N}}$ is a sequence of continuous functions that converges uniformly to a continuous function f. Prove that F is differentiable and the derivative equals f.

Solution to (15) Fix some point x_0 in (a, b) and replace every function $F_n(x)$ by $G_n(x) = F_n(x) - F_n(x_0)$. Then the new sequence (G_n) satisfies the same properties as the original sequence, $G_n'(x)$ equals $F_n'(x)$, and $G_n(x_0)$ equals zero. Thus for the limit, $G(x)$, also $G(x_0)$ equals zero. By the Fundamental Theorem of Calculus,

$$G(x) = \lim_n G_n(x) = \lim_n \int_{x_0}^x F_n'(t)dt.$$

Because integration is compatible with **uniform** limits, also

$$\lim_n \int_{x_0}^x F_n'(t)dt = \int_{x_0}^x \lim_n F_n'(t)dt = \int_{x_0}^x f(t)dt.$$

Putting the pieces together,

$$G(x) = \int_{x_0}^x f(t)dt$$

for the continuous function $f(x)$. Thus, by the Fundamental Theorem of Calculus once more, $G(x)$ is differentiable with derivative equal to f. Therefore also $F(x) = G(x) + F(x_0)$ is differentiable with derivative equal to f.

(16) Find a uniformly convergent sequence $(F_n)_{n\in\mathbb{N}}$ of continuous functions that are differentiable and such that $(F_n')_{n\in\mathbb{N}}$ are continuous functions that converges pointwise to a function f, yet such that the limit F is not everywhere differentiable. Is your (F, f) a pair as in (13)? (As with (12), this is a challenging problem.)

Solution to (16) An example of such a sequence was given in lecture, namely the continuously differentiable, piecewise polynomial functions

$$F_n(x) = \begin{cases} -x, & x \leq -1/n, \\ (n^2x^2+1)/(2n), & -1/n < x < +1/n, \\ +x, & x \geq +1/n. \end{cases}$$

The pointwise limit of $(F_n'(x))_{n\in\mathbb{N}}$ is

$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ +1, & x > 0 \end{cases}$$

The uniform limit of $F_n(x)$ is the function $F(x) = |x|$. This is not differentiable at $x = 0$. This example does have a weak derivative.

(17) For every subset A of $[a, b]$, define

$$\mu(A) = \inf\{\int_a^b f(x)dx | f \text{ integrable }, f \geq 0, f|_A \geq 1\}.$$

Prove that for every finite union of intervals, $\mu(A)$ is the sum of the lengths of the intervals. Prove that $\mu([a, b] \cap \mathbb{Q})$ equals $\mu([a, b])$.

Solution to (17) For every subset $A \subset [a, b]$, define 1_A to be the function

$$1_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in [a, b] \setminus A, \end{cases}$$

Then, by definition,

$$\mu(A) = \inf\{\int_a^b f(x)dx | f \text{ integrable }, f \geq 1_A\}.$$

If A is an interval, or, more generally, a finite union of intervals, then 1_A is piecewise continuous, hence integrable. In particular, for every integrable f such that $f \geq 1_A$, also

$$\int_a^b f(x)dx \geq \int_a^b 1_A(x)dx.$$

Therefore, the minimum such integral is just $\int_a^b 1_A(x)dx$ itself, i.e.,

$$\mu(A) = \int_a^b 1_A(x)dx.$$

Moreover, if A is a union of finitely many intervals, then for every partition P such that every connected component of A is, up to taking the closure, a union of P-intervals, it is straightforward to compute that $L(1_A, P) = U(1_A, P)$ equals the sum of the lengths of the intervals of A. Using such partitions to compute the integral, $\mu(A)$ is simply the sum of the lengths of the intervals of A.

Now let A be $\mathbb{Q} \cap [a, b]$. Certainly $1_{[a,b]} \geq 1_A$, so that

$$\mu(A) \leq \int_a^b 1_{[a,b]}(x)dx = \mu([a, b]).$$

Let f be any integrable function such that $f \geq 1_A$. Let P be any partition of $[a, b]$. Since $f \geq 1_A$, for every P-interval I, the supremum of f on the interval I is at least as large as the supremum of 1_A on I. By the Archimedean property, the interval I contains an element of A. Thus the supremum of 1_A on I equals 1. Therefore,

$$U(f, P) = \sum_{I \in P} \sup_{x \in I} f(x)\mu(I) \geq \sum_{I \in P} 1\mu(I) = \mu([a, b]).$$

Therefore, the infimum of $U(f, P)$ over all P, namely the (Darboux) integral, is also $\geq \mu([a, b])$. Since this holds for every integrable f with $f \geq 1_A$, it follows that

$$\mu(A) = \mu([a, b]).$$