# Midterm 1 Solutions

MAT 118

March 3, 2009

| **Name:** (please print) | **ID #:** |
|---|---|

| | 1 | 2 | 3 | 4 | **Total** |
|---|---|---|---|---|---|
| | 20 pts | 20 pts | 15 pts | 20 pts | 75 pts |
| *Grade* | | | | | |

**Directions:** There are 4 problems on 5 pages (including this one) in this exam. Please make sure you have all the pages.

Do all of your work in this exam booklet, and cross out any work that should be ignored. **Show your reasoning and computations — not just the answer.**

You may not use any books, notes, calculators, or discussions with friends during this exam.

You have 1 hr 20 minutes.

Good luck!

**DO NOT OPEN THE EXAM**
**UNTIL INSTRUCTED BY THE PROCTOR!**

**1.** An election is held with four candidates: A, B, C, D. The table below shows the preferences schedule.

| Percentage of voters | 40% | 25% | 20% | 15% |
|---|---|---|---|---|
| $1^{st}$ place | A | C | B | B |
| $2^{nd}$ place | D | D | C | C |
| $3^{rd}$ place | B | B | D | A |
| $4^{th}$ place | C | A | A | D |

(a) Which candidate would win a plurality election?
*Solution*: A has 40% votes, B has 35%, C has 25%, and D, 0%, so A wins.

(b) Which candidate would win using plurality with elimination method?
*Solution*: First, eliminate D; after that, C has fewest number of votes, so eliminate C; after this, A has 40% and B has 60%, so we eliminate A and B wins.

(c) Which candidate would win using the method of pairwise comparisons?
*Solution*:
A vs B: B wins
A vs C: C wins
A vs D: A wins
B vs C: B wins
B vs D: D wins
C vs D: C wins
Thus, A gets 1 pt, B has 2 pts, C has 2 pts, and D has 1 pt. So it is a tie between B and C.

(d) Which candidate, if any, is the Condorcet winner?
*Solution*: From pairwise comparisons in previous part, there is no Condorcet winner: no candidate beats all others in pariwise comparisons.

**2.**

Three members of the council have 16, 8, and 4 votes respectively; a decision requires at least 2/3 of the votes.

(a) Find the Banzhaf power index of each player. (You can leave the answers in the fraction form, e.g. $\frac{6}{11}$ — you are not required to convert it to percents.)
*Solution*: First, $16+8+4=28$, and $28\times\frac{2}{3}=18.66$, so $q=19$.
Here are the winning coalitions (with critical players underlined):

$$\{\underline{P_1},\underline{P_2}\},\quad \{\underline{P_1},\underline{P_3}\},\quad \{\underline{P_1},P_2,P_3\}$$

Thus, $B_1=3$, $B_2=B_3=1$, $T=5$, and $\beta_1=\frac{3}{5}=60\%$, $\beta_2=\beta_3=\frac{1}{5}=20\%$

(b) Are there any players who are dictators? have veto power? dummies? Justify your answer.
*Solution*: No player has more than 2/3 of total vote, so there are no dictators. Every winning coalition must contain $P_1$, so he has veto power. Every player is critical in some coalition, so there are no dummies.

(c) Find the Shapley-Shubik power index of each player. (You can leave the answers in the fraction form, e.g. $\frac{6}{11}$ — you are not required to convert it to percents.)
*Solution*: Here are sequential winning coalitions, with pivotal player underlined:

$$\begin{array}{ll} \{P_1,\underline{P_2},P_3\}, & \{P_1,\underline{P_3},P_2\} \\ \{P_2,\underline{P_1},P_3\}, & \{P_2,P_3,\underline{P_1}\} \\ \{P_3,\underline{P_1},P_2\}, & \{P_3,P_2,\underline{P_1}\} \end{array}$$

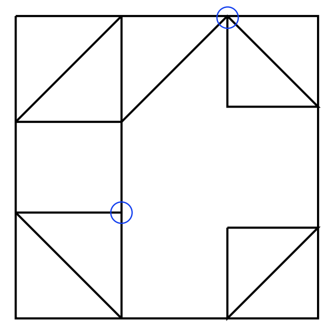
So $SS_1=4$, $SS_2=SS_3=1$, $T=6$, and $\sigma_1=\frac{4}{6}$, $\sigma_2=\sigma_3=\frac{1}{6}$.

(d) Suppose we add one more board member, with 1 vote. Find the new Banzhaf power index of each player. (You can leave the answers in the fraction form, e.g. $\frac{6}{11}$ — you are not required to convert it to percents.)
*Solution*: Afer we add one more player, total number of votes becomes 29, and $29\times\frac{2}{3}=19\frac{1}{3}$, so $q=20$, and we have weighted voting scheme $[20:16,8,4,1]$. (Note: in fact, the answer below does not depend on whether we take $q=19$ or $q=20$, so those who solved the problem assuming $q=19$ got full credit.)
Winning coalitions are

$$\begin{array}{lll} \{\underline{P_1},\underline{P_2}\}, & \{\underline{P_1},\underline{P_3}\}, & \{\underline{P_1},P_2,P_3\} \\ \{\underline{P_1},\underline{P_2},P_4\}, & \{\underline{P_1},\underline{P_3},P_4\}, & \{\underline{P_1},P_2,P_3,P_4\} \end{array}$$

(first line are same coalitions as in part (a); second line are same coalitions with addition of $P_4$).
Thus, $B_1=6$, $B_2=B_3=2$, $B_4=0$, $T=10$, and $\beta_1=\frac{6}{10}=60\%$, $\beta_2=\beta_3=\frac{2}{10}=20\%$, $\beta_4=0$.

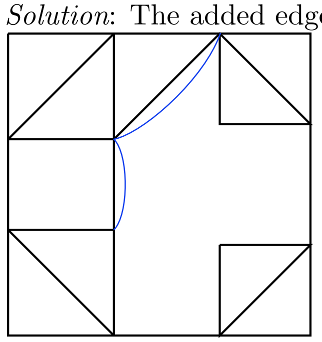
**3.** Consider the graph shown to the right.



(a) Does this graph have Euler circuits? Euler paths? Justify your answer. If there are any odd vertices, mark them in the figure above.
*Solution*: This graph has 2 odd vertices, circled above; thus, it has no Eulerian circuit.

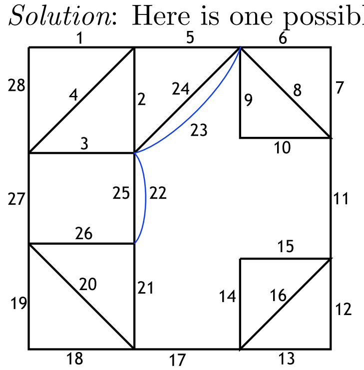
(b) Find the minimal Eulerization of this graph, by doubling some of the edges. Mark these edges on the copy of the figure below.

*Solution*: The added edges are shown in blue below.

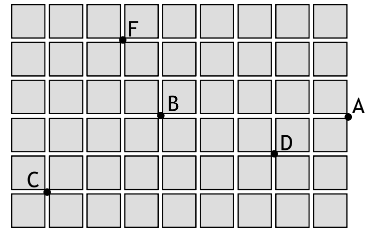


(c) Use Fleury's algorithm to find an Euler circuit in the graph you had constructed in part (b). The circuit should begin and end in the upper left corner. Mark the circuit, labeling edges by numbers 1, 2,. . . in order they should be traveled.
For your convenience, here is a copy of the graph once again; do not forget to copy to it the edges you added in part (b).

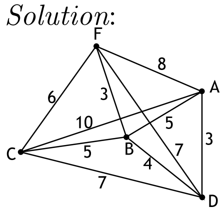
*Solution*: Here is one possible circuit:



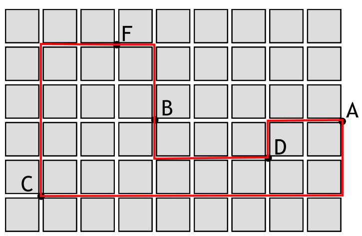
**4.** The figure to the right shows the map of a section of a city. A Federal Express driver has to start at the FedEx office (marked by letter F on the map), deliver packages to locations marked by letters A, B, C, D, and return back to the office.




(a) Draw the weighted graph corresponding to this situation (assuming that all city blocks are of the same size, and all streets are two-way).
*Solution*:

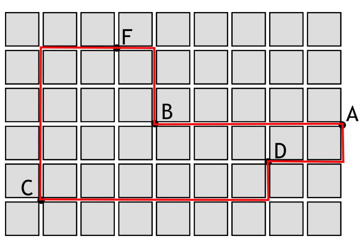



(b) Find a Hamiltonian circuit and its length using the nearest neighbor algorithm. Write your answers in the spaces below. You can use the figure below to mark your path, but you are not required to do so.



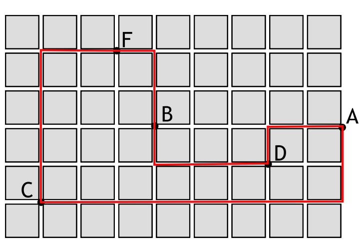

F, B, D, A, C, F.
Length (in blocks): 3+4+3+10+6=26

(c) Find a Hamiltonian circuit using nearest neighbor algorithm **starting at point D**. Rewrite the circuit you found as a route starting and ending at point F (as is done in repetitive nearest neighbor algorithm). Write your answers in the spaces below. You can use the figure below to mark your path, but you are not required to do so.




Starting from D: D, A, B, F, C, D
Rewriting starting from F: F, C, D, A, B, F.
Length (in blocks): 6+7+3+5+3=24

(d) Find a Hamiltonian circuit and its length using the cheapest link algorithm. Write your answers in the spaces below. You can use the figure below to mark your path, but you are not required to do so.




Segments in the order we select them: FB, DA, BD, FC, CA.
Complete circuit:
F, B, D, A, C, F. (same as in part (a))
Length (in blocks): 26