Universally counting curves in Calabi–Yau threefolds

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Abstract

We show that curve enumeration invariants of complex threefolds with nef anti-canonical bundle are determined by their values on local curves. This statement and its proof are inspired by the proof of the Gopakumar–Vafa integrality conjecture by Ionel and Parker. The conjecture of Maulik, Nekrasov, Okounkov, and Pandharipande relating Gromov–Witten and Donaldson–Pandharipande–Thomas invariants is known for local curves by work of Bryan, Okounkov, and Pandharipande, hence holds for all complex threefolds with nef anti-canonical bundle (in particular, all Calabi–Yau threefolds).

1 Introduction

There are many ways of enumerating curves in complex threefolds [31]. These invariants turn out to satisfy some surprising relations which appear to have no straightforward explanation. In fact, according to Pandharipande–Thomas [30], the multitude of existing computations suggest that all reasonable curve enumeration theories for complex threefolds are equivalent, despite arising from quite varied geometric origins.

A folk conjecture offers an explanation of this phenomenon: a complex threefold should be 'enumeratively equivalent' to a linear combination of *local curves* (rank two vector bundles over smooth proper curves). We provide a precise formulation and proof of this conjecture for complex threefolds with nef anti-canonical bundle. That is, we define a certain *Grothendieck group of 1-cycles* in complex threefolds (with nef anti-canonical bundle), and we show that this group is freely generated by local curves. The proof is based on generic transversality, which explains the nef anti-canonical bundle hypothesis (it would be of exceptional interest to remove this hypothesis).

The main result and its proof are inspired by the proof of the Gopakumar–Vafa integrality conjecture by Ionel–Parker [13]. They showed that Gromov–Witten invariants of almost complex threefolds are integer linear combinations of Gromov–Witten invariants of local curves, which were known to satisfy Gopakumar–Vafa integrality by work of Bryan–Pandharipande [6]. Their argument may be interpreted as a proof that a certain Grothendieck group of

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1-cycles in *almost* complex threefolds with nef anti-canonical bundle is freely generated by local curves (after completing by genus, later removed by Doan–Ionel–Walpuski [9]). The setting of complex threefolds is more rigid, requiring a different Grothendieck group and a more delicate argument.

The main result opens a path to a number of conjectures relating different enumerative invariants of complex threefolds (under the assumption of nef anti-canonical bundle). We explain here how to deduce from it the conjecture of Maulik–Nekrasov–Okounkov–Pandharipande [22, 23] relating Gromov–Witten and Donaldson–Thomas/Pandharipande–Thomas invariants, in the case of primary insertions, for complex threefolds with nef anti-canonical bundle (given the calculations for local curves due to Bryan–Pandharipande [6] and Okounkov–Pandharipande [28]). The MNOP conjecture is interesting because there is no known or even proposed geometric relation between the moduli spaces giving rise to Gromov–Witten invariants and Donaldson–Thomas/Pandharipande–Thomas invariants. Of course, this is unlikely to be the only application. For example, the main result would apply to the invariants of Maulik–Toda [26] if they are shown to be (higher) deformation invariant. For another, see the work of Jockers–Mayr [14] and Chou–Lee [7] on quantum K-theory invariants.

1.1 Universal enumerative invariant

There is a (very tautological) *universal* curve enumeration invariant of complex threefolds. This invariant takes values in the group $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ (defined in §3.1), which is the homology of the double complex

$$C_*(\mathrm{Cpx}_3, C^*_c(\mathcal{Z})) = \bigoplus_{X\to\Delta^n} C^*_c(\mathcal{Z}(X/\Delta^n)) \tag{1.1}$$

in which the direct sum is over all (not necessarily proper) families $X \to \Delta^n$ of complex threefolds over a simplex, and $\mathcal{Z}(X/\Delta^n)$ denotes the space of compact (complex) 1-cycles in the fibers of $X \to \Delta^n$ (a 1-cycle $z \in \mathcal{Z}(X)$ is a formal non-negative integer linear combination $\sum_i m_i C_i$ of compact irreducible 1-dimensional subvarieties). If X is a projective threefold and $\beta \in H_2(X)$ is a homology class, then the 'universal count' of curves in X in homology class β in $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ is the class of the characteristic function $(\mathbf{1}_\beta : \mathcal{Z}(X) \to \mathbb{Z}) \in H^0_c(\mathcal{Z}(X))$ of the locus of 1-cycles with total homology class β (which is compact since X is projective). We call this group $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ the *Grothendieck group of 1-cycles in complex threefolds.* The chain-level dual $C^*(\mathrm{Cpx}_3, C^{\mathrm{rel}\infty}_*(\mathcal{Z}))$ of the Grothendieck group of 1-cycles classifies coherent 'virtual fundamental' cycles on each relative cycle space $\mathcal{Z}(X/\Delta^n)$. A class in its homology $H^{\mathrm{rel}\infty}_*(\mathcal{Z}/\mathrm{Cpx}_3)$ is thus a 'curve enumeration theory of complex threefolds which is deformation invariant up to coherent homotopy'. Such a class determines a homomorphism out of the Grothendieck group. The group $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ has a rich algebraic structure (see §3.2): it is a bi-algebra (product corresponds to disjoint union of cycles, while coproduct corresponds to sum of cycles). It also has bi-algebra endomorphisms corresponding to the 'multiply by d' operation on cycles.

This sort of 'universal' discussion is only useful to the extent that one can make nontrivial computations. Our main result is to compute (in virtual dimension ≤ 0) the Grothendieck group $H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ whose definition is identical to $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ except that it considers just the open 'semi-Fano' locus $\mathcal{Z}_{\mathrm{sF}} \subseteq \mathcal{Z}$ of 1-cycles $z = \sum_i m_i C_i$ all of whose components $C_i \subseteq X$ pair non-negatively with $c_1(TX)$.

Theorem 1.1. *In non-positive virtual dimension, the Grothendieck group* $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ *is freely generated as a ring by the equivariant local curve elements* $x_{g,1,k}$ *with* $k \geq 0$ *and* $x_{g,m,0}$*.*

We explain the statement. There is a natural bi-grading $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) = \bigoplus_{i,k} H_c^i(\mathcal{Z}(-,k)/\mathrm{Cpx}_3)$ by cohomological degree i and chern number k (pairing with $c_1(TX)$), and the 'total homological degree' (or 'virtual dimension') is $2k - i$. The equivariant local curve elements $x_{g,m,k}$ (defined in §4.3) have virtual dimension zero and correspond to the $\mathbb{C}^\times$-equivariant enumerative theory of degree m cycles on the total space of a rank two vector bundle $E \to C$ with $c_1(E) = k$ over a curve C of genus g. For example, Theorem 1.1 says that for any projective Calabi–Yau threefold X and any homology class $\beta \in H_2(X)$, the element $(X, \beta) \in H_c^0(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is equal to a unique polynomial in the variables $x_{g,m,0}$.

Remark 1.2. The Grothendieck group of 1-cycles in complex threefolds $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ is the homology of a naturally defined spectrum (see Remark 3.2). This spectrum thus has E-homology groups $E_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ for any spectrum E. Recall that a spectrum E is called *connective* when $E(\mathrm{pt})$ (homology or cohomology, they are the same) is supported in non-negative homological degree (that is $\pi_i E = 0$ for $i < 0$). It follows from Theorem 1.1 and the Atiyah–Hirzebruch spectral sequence that for any connective spectrum E, the group $E_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ in non-positive virtual dimension is freely generated as an $E_0(\mathrm{pt})$-algebra by the same equivariant local curve elements.

Theorem 1.1 is fundamentally a transversality statement, so the semi-Fano hypothesis appears necessary. We see no reason to expect that the group $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ is understandable. An analogue of Theorem 1.1 in almost complex geometry was proven by Ionel–Parker [13]. They showed, in particular, that $H_0(\mathrm{ACpx}_3, H_c^0(\mathcal{Z}_{\mathrm{CY}}))$ (one part of the E_2 term of the spectral sequence associated to the double complex whose total homology is $H_c^*(\mathcal{Z}/\mathrm{ACpx}_3)$) is generated by local curve elements $x_{g,m,0}$ (after completing by genus, later removed by Doan–Ionel–Walpuski [9]). Due to the rigidy of complex structures, we must work with the entire complex (1.1). The reason for this is that, while generic almost complex structures achieve transversality for all simple maps from curves, generic complex structures only achieve transversality for simple maps 'locally' on the space of cycles. Generic transversality for almost complex structures goes back to Gromov [12], while we are not aware of previous use of generic transversality in the complex setting. While one could probably prove Theorem 1.1 using a direct geometric argument, we actually only prove surjectivity geometrically and we deduce injectivity using the bi-algebra structure.

Theorem 1.1 is not the final word on the structure of enumerative invariants of complex threefolds with nef anti-canonical bundle. Specifically, one could ask for the product expansion of Ionel–Parker [13] in the complex setting (perhaps deducible from their result by comparing $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ and $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{ACpx}_3)$ via Theorem 1.1 and an almost complex analogue thereof), namely the following.

Conjecture 1.3. *For any complex projective Calabi–Yau threefold* X*, the element* $(X, t^{[\cdot]}) \in H_c^0(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)[[t^{H_2(X)}]]$ *is an infinite product* $\prod_\beta \prod_{g \geq 0} f_g(t^\beta)^{e_{\beta,g}(X)}$ *for unique integer invariants* $e_{\beta,g}(X) \in \mathbb{Z}$*, where* $f_g(t) = \sum_{m \geq 0} x_{g,m,0} t^m$*.*

In the absence of the semi-Fano hypothesis, calculating the Grothendieck group $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ appears intractable. Nevertheless, we may venture the following conjecture, which might at least allow almost complex methods to be used to study invariants of complex threefolds (whose anti-canonical bundle need not be nef).

Conjecture 1.4. *The map $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}/\mathrm{ACpx}_3)$ is an isomorphism.*

We expect the same to be true for $\mathcal{Z}_{\mathrm{sF}}$ in place of $\mathcal{Z}$, but it is less interesting given Theorem 1.1, which presumably remains valid for $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{ACpx}_3)$ with a similar (probably easier) proof.

The most interesting question is probably whether there exists a modification of the group $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ for which an analogue of Theorem 1.1 holds and which can be used to study enumerative invariants of general (not necessarily having nef anti-canonical bundle) complex threefolds. It would also be interesting to compute the class in $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ of specific complex threefolds (perhaps by lifting known computations of Gromov–Witten invariants).

1.2 MNOP correspondence

Theorem 1.1 implies that a curve enumeration invariant of complex threefolds with nef anti-canonical bundle is determined uniquely by its values on local curves. We now explain how this may be used to verify a conjecture of Maulik–Okounkov–Nekrasov–Pandharipande [22, 23] for such threefolds.

Maulik–Nekrasov–Okounkov–Pandharipande [22, 23] originally conjectured an equivalence between Gromov–Witten and Donaldson–Thomas invariants of projective threefolds. A similar conjecture relating Gromov–Witten and Pandharipande–Thomas invariants was proposed by Pandharipande–Thomas [30]. Work of Bridgeland [5] relates Donaldson–Thomas and Pandharipande–Thomas invariants, implying the two conjectures are equivalent. We will address the latter conjecture here (Pandharipande–Thomas invariants are easier to work with than Donaldson–Thomas invariants in many respects, and our work here is no exception).

We briefly recall the definition of Gromov–Witten and Pandharipande–Thomas invariants, leaving a more detailed discussion to §3.4. Given a complex projective threefold X, a homology class $\beta \in H_2(X)$, and cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X)$, these invariants have the form

$$\mathrm{GW}(X,\beta;\gamma_1,\ldots,\gamma_r) = \int_{[\overline{\mathcal{M}}'(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \pi_! \operatorname{ev}^* \gamma_i \cdot u^{-\chi} \in \mathbb{Q}((u)), \tag{1.2}$$

$$\mathrm{PT}(X,\beta;\gamma_1,\ldots,\gamma_r) = \int_{[P(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r \pi_!(\mathrm{ch}_2(\mathbb{F}) \cup \pi_X^*\gamma_i) \cdot q^n \in \mathbb{Z}((q)). \tag{1.3}$$

For Gromov–Witten invariants, $\overline{\mathcal{M}}'(X,\beta)$ is the moduli space of stable maps from (not necessarily connected) nodal curves to X, in homology class β, all of whose connected components are non-constant, and χ denotes the arithmetic Euler characteristic of the domain (locally constant, proper sublevel sets). For Pandharipande–Thomas invariants, $P(X,\beta)$ denotes the moduli space of stable pairs in homology class β, and n denotes the holomorphic Euler characteristic (locally constant, proper sublevel sets). The integrands are given by push/pull via the universal families over these moduli spaces.

Let us say that a pair of formal Laurent series $\mathrm{GW} \in \mathbb{Q}((u))$ and $\mathrm{PT} \in \mathbb{Z}((q))$ satisfies the *MNOP correspondence* when PT is a rational function of q whose evaluation at $q = -e^{iu}$ equals GW.

Conjecture 1.5 ([22, 23, 30])**.** *For any projective threefold X, any homology class $\beta \in H_2(X)$, and any tuple of cohomology classes $\gamma_1, \ldots, \gamma_r \in H^*(X)$, the invariants*

$$(-iu)^{\langle c_1(TX),\beta\rangle}\mathrm{GW}(X,\beta;\gamma_1,\ldots,\gamma_r) \text{ and } (-q)^{-\langle c_1(TX),\beta\rangle/2}\mathrm{PT}(X,\beta;\gamma_1,\ldots,\gamma_r) \tag{1.4}$$

satisfy the MNOP correspondence.

Conjecture 1.5 is known in many cases, essentially by computing both sides of the equality. The case of (equivariant invariants of) local curves holds by deep calculations of Bryan–Pandharipande [6] (of Gromov–Witten invariants) and Okounkov–Pandharipande [28] (of Donaldson–Thomas invariants and, by [25, Section 5], Pandharipande–Thomas invariants). Work of Maulik–Oblomkov–Okounkov–Pandharipande [24] established the conjecture for toric varieties by direct computation of both sides. Work of Pandharipande–Pixton [29] showed the result for many threefolds (e.g. complete intersections in products of projective spaces) by degeneration to the toric case.

Combining Theorem 1.1 with the known case of equivariant local curves [6, 28], we may conclude the following.

Theorem 1.6. *Conjecture 1.5 holds when the anti-canonical bundle of X is nef (that is, when $c_1(TX)$ pairs non-negatively with every curve $C \subseteq X$).*

Indeed, Gromov–Witten invariants and Pandharipande–Thomas invariants define ring homomorphisms

$$\mathrm{GW} : H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \to \mathbb{Q}((u)), \tag{1.5}$$

$$\mathrm{PT} : H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \to \mathbb{Z}((q)), \tag{1.6}$$

and Conjecture 1.5 amounts to the assertion that the homomorphisms $(-iu)^k\mathrm{GW}$ and $(-q)^{-k/2}\mathrm{PT}$ satisfy the MNOP correspondence when evaluated on the element

$$(X,\beta;\gamma_1,\ldots,\gamma_r) \in H_c^{(|\gamma_1|-2)+\cdots+(|\gamma_r|-2)}(\mathcal{Z}(-,\langle c_1(TX),\beta\rangle)/\mathrm{Cpx}_3). \tag{1.7}$$

represented by the product of $\mathbf{1}_\beta \in H_c^0(\mathcal{Z}(X))$ and the classes $\pi_! i^*\gamma_i \in H^{|\gamma_i|-2}(\mathcal{Z}(X))$. The results of [6, 28] imply that $(-iu)^k\mathrm{GW}$ and $(-q)^{-k/2}\mathrm{PT}$ satisfy the MNOP correspondence when evaluated on equivariant local curve elements. By Theorem 1.1, this implies they satisfy the MNOP correspondence on all of $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$. This approach to Conjecture 1.5 is similar in spirit to [29] in that in essence we are deforming to a simpler situation where the result is already known (and the strength of Theorem 1.1 allows us to obtain a stronger result from a weaker input).

The Grothendieck group formalism naturally encodes enumerative invariants of *families* of threefolds. The MNOP conjecture hence holds for Gromov–Witten and Pandharipande–Thomas invariants of families of threefolds with nef anti-canonical bundle. Note that equivariant invariants are a special case of family invariants (namely of the Borel construction). By taking arbitrary classes in $H^*(\mathcal{Z}(X))$ (instead of just primary cohomology insertions from X), we can also conclude that the invariants $\mathrm{GW} \in H_*(\mathcal{Z}(X,\beta);\mathbb{Q})((u))$

and PT $\in H_*(\mathcal{Z}(X,\beta);\mathbb{Q})((q))$ satisfy the MNOP correspondence for projective threefolds X with nef anti-canonical bundle (where rationality of PT means that it lies in $H_*(\mathcal{Z}(X,\beta);\mathbb{Q})[q]$ after multiplication by some element of $\mathbb{Q}[q]$).

The case of descendent invariants is conspicuously missing from this discussion. It would suffice to write down natural classes on $\mathcal{Z}(X)$ whose pullback to $\overline{\mathcal{M}}'$ and P are the respective descendent classes, but it is not clear this is possible.

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2 Spaces of 1-cycles

2.1 Background

A (compact holomorphic) 1-cycle z on a complex analytic manifold X is a formal non-negative integer linear combination of irreducible compact 1-dimensional subvarieties $C \subseteq X$. The set of such 1-cycles is denoted $\mathcal{Z}(X)$ (more systematically, this would be denoted $\mathcal{Z}_1(X)$, but we will not consider r-cycles $\mathcal{Z}_r(X)$ for any r other than 1 in this text, so we drop the subscript from the notation). Such a cycle will usually be written as a finite sum $z = \sum_i m_i C_i$ where it is implicitly assumed that the $C_i \subseteq X$ are distinct irreducible curves and all $m_i > 0$.

The (total) chern number of a cycle $z = \sum_i m_i C_i$ is the pairing $\langle c_1(TX), z\rangle = \sum_i m_i \langle c_1(TX), C_i\rangle$. A cycle $z = \sum_i m_i C_i$ is called semi-Fano when the pairing of *every* C_i with $c_1(TX)$ is non-negative (it bears emphasis that this is stronger than having non-negative chern number). We denote by $\mathcal{Z}(X,k) \subseteq \mathcal{Z}(X)$ the set of cycles with chern number k, and we denote by $\mathcal{Z}(X)_{\mathrm{sF}} \subseteq \mathcal{Z}(X)$ the set of semi-Fano cycles. We also denote by $\mathcal{Z}(X,\beta) \subseteq \mathcal{Z}(X)$ the set of cycles in homology class $\beta \in H_2(X)$ (which, we should warn, somewhat conflicts with the notation $\mathcal{Z}(X,k)$ for cycles of chern number k).

The set $\mathcal{Z}(X)$ has the structure of a separated reduced complex analytic space due to work of Barlet [2]. By definition, an analytic map $A \to \mathcal{Z}(X)$ from a reduced complex analytic space A is a family of 1-cycles $\{z_a \in \mathcal{Z}(X)\}_{a\in A}$ which satisfies a certain analyticity condition [2, Chapitre 1, §1, Définition fondamentale]. If the family $\{z_a \in \mathcal{Z}(X)\}_{a\in A}$ is analytic, then the union $\bigcup_{a\in A} z_a \subseteq X \times A$ is a closed analytic subset, proper over A, with fibers of pure dimension 1 and multiplicities which are constant on its irreducible components [2, Chapitre 1, §2, Théorème 1] (and the converse holds if A is normal). In particular, there is a 'universal family' $\mathcal{U}(X) \subseteq X \times \mathcal{Z}(X)$.

The homology class function $\mathcal{Z}(X) \to H_2(X)$ is locally constant; that is, the locus $\mathcal{Z}(X,\beta) \subseteq \mathcal{Z}(X)$ of cycles in homology class $\beta \in H_2(X)$ is open. In partcular, the subset $\mathcal{Z}(X,k) \subseteq \mathcal{Z}(X)$ of cycles with chern number k is open. The subset $\mathcal{Z}(X)_{\mathrm{sF}} \subseteq \mathcal{Z}(X)$ is also open.

This discussion generalizes readily to the relative setting. Given a holomorphic submersion $X \to B$, we define $\mathcal{Z}(X/B) = \bigcup_b \mathcal{Z}(X_b)$ to be the set of cycles in fibers of $X \to B$. It is an open subset of $\mathcal{Z}(X)$, so the basic properties of $\mathcal{Z}(X)$ pass easily to $\mathcal{Z}(X/B)$.

2.2 Semi-charts

Around each point $z = \sum_i m_i C_i \in \mathcal{Z}(X)$ is a *semi-chart* defined as follows. Let $\tilde{C}_i \to C_i$ denote the normalization of C_i, so $\tilde{C}_i$ is a compact smooth curve. We consider all local deformations of $\tilde{C} = \bigsqcup_i \tilde{C}_i \to X$ (including deformations of the complex structure on the domain), and we associate to such a nearby map $\tilde{C}' = \bigsqcup_i \tilde{C}_i' \to X$ the cycle $\sum_i m_i C_i'$. We denote by $(S_z, z) \to (\mathcal{Z}(X), z)$ (a germ) the semi-chart around z. The semi-chart $S_z \to \mathcal{Z}(X)$ need not be a (germ near z of) open embedding, since it does not take into account the possibility of the topology changing (as in $y^2 = x(x-t)(x+t)$ near $t = 0$) or of curves with multiplicities breaking apart (as in $y^2 = tx$ near $t = 0$). The locus of points z for which the semi-chart around z is an open embedding is evidently open.

Lemma 2.1. *The set of points $z \in \mathcal{Z}(X)$ whose semi-chart is an open embedding is dense.*

Proof. Begin with an arbitrary cycle $z = \sum_i m_i C_i \in \mathcal{Z}(X)$, and let us produce cycles arbitrarily close to z whose associated semi-charts are open embeddings.

A nearby cycle z' determines a partition μ_i of each m_i. Partially order the set $\Pi(m)$ of partitions of m by refinement: declare $\mu \geq \mu'$ when μ' is obtained from μ by replacing each of its constituents by a partition thereof. The map from a neighborhood of $z \in \mathcal{Z}(X)$ to $\prod_i \Pi(m_i)$ has local minima arbitrarily close to z since $\prod_i \Pi(m_i)$ satisfies the descending chain condition (since it is finite). We may thus assume wlog that z is itself a local minimum of this map. This means that if we write $z = \sum_{m \geq 1} m C_m$ (C_m not necessarily irreducible) then every nearby cycle z' has the form $\sum_{m \geq 1} m C_m'$ for C_m' nearby C_m. In other words, there is a factorization $(\mathcal{Z}(X), z) = \prod_{m \geq 1} (\mathcal{Z}(X), C_m)$ of germs. This factorization reduces us to the case $z = C$ for some not necessarily irreducible curve C.

The Euler characteristic function $\chi : \mathcal{Z}(X) \to \mathbb{Z}$ near $z = C$ is bounded below since the universal family $\mathcal{U}(X) \subseteq X \times \mathcal{Z}(X)$ is finite type. We may thus assume wlog that z is a local minimum of χ. Let us argue that this implies that the semi-chart at z surjects onto a neighborhood of z (hence is an open embedding). A nearby cycle z' is simply a curve C' nearby C. Near smooth points of C, the curve C' is a nearby smooth curve, hence may be (non-canonically) identified with C (as smooth manifolds) with nearby complex structure. Near a singular point of C (necessarily isolated), choose a ball B around it so that $\tilde{C} \cap B$ is a disjoint union of disks. Now a disjoint union of disks is the unique filling of a disjoint union of circles of maximal Euler characteristic, so since $\chi(\tilde{C}') = \chi(\tilde{C})$, we conclude that $\tilde{C}' \cap B$ is also a disjoint union of disks. This shows that $\tilde{C}' \to X$ is a small perturbation of $\tilde{C} \to X$, as desired. □

3 Grothendieck groups of 1-cycles

3.1 Definition

We now define the Grothendieck groups of 1-cycles which we will study.

We will consider families of complex threefolds over (real) simplices Δ^n. Such a family is, by definition, a submersion of complex manifolds with three-dimensional fibers over a(n unspecified) open neighborhood of $\Delta^n \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$, and an isomorphism of families is a germ of isomorphism defined in a neighborhood of Δ^n inside $\mathbb{C}^n$. In particular, if $X \to B$ is a family of complex threefolds over a smooth analytic base B and $\Delta^n \to B$ is any real analytic map, then the pullback $X \times_B \Delta^n \to \Delta^n$ is a family in the above sense, since $\Delta^n \to B$, being real analytic, extends uniquely to a germ near $\Delta^n \subseteq \mathbb{C}^n$.

Definition 3.1. The group $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ is the homology of (the total complex associated to) the double complex $C_*(\mathrm{Cpx}_3, C^*_c(\mathcal{Z}))$, illustrated below.

$$\begin{array}{ccccc} & \vdots & & \vdots & & \vdots \\ & \uparrow & & \uparrow & & \uparrow \\ \cdots \longrightarrow & \bigoplus_{X\to\Delta^2} C^2_c(\mathcal{Z}(X/\Delta^2)) & \longrightarrow & \bigoplus_{X\to\Delta^1} C^2_c(\mathcal{Z}(X/\Delta^1)) & \longrightarrow & \bigoplus_X C^2_c(\mathcal{Z}(X)) \\ & \uparrow & & \uparrow & & \uparrow \\ \cdots \longrightarrow & \bigoplus_{X\to\Delta^2} C^1_c(\mathcal{Z}(X/\Delta^2)) & \longrightarrow & \bigoplus_{X\to\Delta^1} C^1_c(\mathcal{Z}(X/\Delta^1)) & \longrightarrow & \bigoplus_X C^1_c(\mathcal{Z}(X)) \\ & \uparrow & & \uparrow & & \uparrow \\ \cdots \longrightarrow & \bigoplus_{X\to\Delta^2} C^0_c(\mathcal{Z}(X/\Delta^2)) & \longrightarrow & \bigoplus_{X\to\Delta^1} C^0_c(\mathcal{Z}(X/\Delta^1)) & \longrightarrow & \bigoplus_X C^0_c(\mathcal{Z}(X)) \end{array} \tag{3.1}$$

In this case, 'total complex' means we take the direct sum over the anti-diagonals. More precisely, we are considering here the complex $C_*(\mathrm{Cpx}_{3,\bullet}, C^*_c(\mathcal{Z}))$, where $\mathrm{Cpx}_{3,\bullet}$ denotes the simplicial groupoid (i.e. the simplicial object in the 2-category of groupoids) which assigns to each $[n] \in \mathbf{\Delta}$ the groupoid of all families of complex threefolds $X \to \Delta^n$ (and to a morphism of simplices the corresponding pullback functor). For more details on this notion, see §B.

The chain-level dual of the Grothendieck group of 1-cycles is $C^*(\mathrm{Cpx}_3, C^{\mathrm{rel}\infty}_*(\mathcal{Z}))$, namely the total complex (in this case the product along the anti-diagonals) of the following double complex.

$$\begin{array}{ccccc} & \vdots & & \vdots & & \vdots \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longleftarrow & \prod_{X\to\Delta^2} C^{\mathrm{rel}\infty}_2(\mathcal{Z}(X/\Delta^2)) & \longleftarrow & \prod_{X\to\Delta^1} C^{\mathrm{rel}\infty}_2(\mathcal{Z}(X/\Delta^1)) & \longleftarrow & \prod_X C^{\mathrm{rel}\infty}_2(\mathcal{Z}(X)) \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longleftarrow & \prod_{X\to\Delta^2} C^{\mathrm{rel}\infty}_1(\mathcal{Z}(X/\Delta^2)) & \longleftarrow & \prod_{X\to\Delta^1} C^{\mathrm{rel}\infty}_1(\mathcal{Z}(X/\Delta^1)) & \longleftarrow & \prod_X C^{\mathrm{rel}\infty}_1(\mathcal{Z}(X)) \\ & \downarrow & & \downarrow & & \downarrow \\ \cdots \longleftarrow & \prod_{X\to\Delta^2} C^{\mathrm{rel}\infty}_0(\mathcal{Z}(X/\Delta^2)) & \longleftarrow & \prod_{X\to\Delta^1} C^{\mathrm{rel}\infty}_0(\mathcal{Z}(X/\Delta^1)) & \longleftarrow & \prod_X C^{\mathrm{rel}\infty}_0(\mathcal{Z}(X)) \end{array} \tag{3.2}$$

A cycle in this complex may reasonably be called a 'coherent collection of cycles on all 1-cycle spaces of all complex threefolds'. A class in its homology $H^{\mathrm{rel}\infty}_*(\mathcal{Z}/\mathrm{Cpx}_3)$ will be called a *curve enumeration theory (for complex threefolds)*. Such a curve enumeration theory determines, via the tautological pairing, a homomorphism out of the Grothendieck group $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$.

There is a bigrading

$$H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) = \bigoplus_{i,k} H_c^i(\mathcal{Z}(-,k)/\mathrm{Cpx}_3) \tag{3.3}$$

by cohomological degree i (indexing the anti-diagonals of the double complex) and chern number k of the cycles. The 'total homological degree' is $2k - i$ (that is, k is 'half a homological grading').

Remark 3.2. The group $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ is the homology of the spectrum

$$\operatorname{colim}\Big(\cdots \substack{\to\\\to\\\to\\\to} \coprod_{X\to\Delta^2} D(\mathcal{Z}(X/\Delta^2)/\infty) \substack{\to\\\to\\\to} \coprod_{X\to\Delta^1} D(\mathcal{Z}(X/\Delta^1)/\infty) \rightrightarrows \coprod_{X} D(\mathcal{Z}(X)/\infty)\Big) \tag{3.4}$$

where by D we mean Spanier–Whitehead dual (more precisely, A/∞ denotes the inverse system $\{A/(A\setminus K)\}_{K\subseteq A \text{ compact}}$, and by $D(A/\infty)$ we mean the colimit of the directed system obtained by applying the contravariant Spanier–Whitehead duality functor D).

The definition of the Grothendieck group $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ applies without change to the open set $\mathcal{Z}_{\mathrm{sF}} \subseteq \mathcal{Z}$ in place of $\mathcal{Z}$, producing a group $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$. There is a tautological pushforward map $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$, using the fact that C_c^* is functorial under open embeddings. Let us briefly recall one specific model of C_c^* with the functoriality which we will require.

Definition 3.3 (Functoriality of compactly supported cochains C_c^*)**.** We fix here a model of compactly supported cochains which is functorial under open embeddings and proper maps. The singular cochains functor C^*_{sing} is a functor on topological spaces, and we consider its sheafification $C^{\#,*}_{\mathrm{sing}}$. Concretely, an element of $C^{\#,k}_{\mathrm{sing}}(X)$ is a k-cochain on X modulo equivalence: two k-cochains on X are equivalent when there exists an open covering $X = \bigcup_i U_i$ such that their restrictions to every U_i coincide. We take $C_c^*(X) \subseteq C^{\#,*}_{\mathrm{sing}}(X)$ to be the subcomplex of compactly supported sections (in the sheaf theoretic sense, namely the support of $\gamma \in C^{\#,*}_{\mathrm{sing}}(X)$ is the complement of the largest open set $U \subseteq X$ for which γ maps to zero in $C^{\#,*}_{\mathrm{sing}}(U)$, and a largest such U exists since $C^{\#,*}$ is a sheaf). Now C_c^* is a contravariant functor on the category whose objects are Hausdorff topological spaces and whose morphisms $X \dashrightarrow Y$ are correspondences $X \hookleftarrow U \to Y$ where $U \hookrightarrow X$ is an open embedding and $U \to Y$ is proper (composition of correspondences is via fiber product).

Given a simplicial set B and a family of threefolds $X \to B$ (equivalently, a map $B \to \mathrm{Cpx}_{3,\bullet}$), there is a tautological map (at least on homology)

$$C_*(B, C_c^*(\mathcal{Z}(X/-))) \to C_*(\mathrm{Cpx}_3, C_c^*(\mathcal{Z})), \tag{3.5}$$

and every element of $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ is in the image of this map for some family $X \to B$ over a finite simplicial set B.

The complex $C_*(B, C_c^*(\mathcal{Z}(X/-)))$ may be described more geometrically as follows. Form $\mathcal{Z}(X/B)$ as the evident gluing of $\mathcal{Z}(X_\sigma/\sigma)$ over $\sigma \subseteq B$. The dualizing complex ω_B is given by $\omega_B = \bigoplus_{\sigma\subseteq B} \mathbb{Z}_\sigma[\dim\sigma]$, which is the chain group $C_*(B;\mathbb{Z}_-)$ of B with respect to the coefficient system $\sigma \mapsto \mathbb{Z}_\sigma$ valued in sheaves on B. Pulling back to $\mathcal{Z}(X/B)$, we have

$$\pi^*\omega_B = C_*(B;\mathbb{Z}_{\mathcal{Z}(X/-)}) = \bigoplus_{\sigma\subseteq B} \mathbb{Z}_{\mathcal{Z}(X_\sigma/\sigma)}[\dim\sigma], \tag{3.6}$$

and taking compactly supported cohomology of this sheaf on B, we arrive at an identification

$$C_c^*(\mathcal{Z}(X/B), \pi^*\omega_B) = C_*(B, C_c^*(\mathcal{Z}(X/-))). \tag{3.7}$$

In other words, the complex $C_*(B, C_c^*(\mathcal{Z}(X/-)))$ calculates the group $H_c^*(\mathcal{Z}(X/B)/B)$ from Remark 3.4 (at least for finite simplicial sets B where there are no potential unboundedness issues for ω_B). This geometric description makes it clear that the cell decomposition of B is irrelevant; that is, there is a canonical map

$$H_c^*(\mathcal{Z}(X/B)/B) = H_c^*(\mathcal{Z}(X/B), \pi^*\omega_B) \to H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \tag{3.8}$$

independent of the choice of triangulation of B. Thus we have $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) = H_c^*(\mathcal{Z}(\mathrm{Cpx}_3), \pi^*\omega_{\mathrm{Cpx}_3})$.

In particular, we could consider a single real analytic manifold B as the base. A family of complex threefolds $X \to B$ then determines a map

$$H_c^{*+\dim B}(\mathcal{Z}(X/B); \mathfrak{o}_B) \to H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \tag{3.9}$$

obtained by triangulating B and summing over all top-dimensional simplices (here $\mathfrak{o}_B$ denotes the orientation local system of B, placed in degree zero). If we consider the restriction $X' \to B'$ of this family to a submanifold $i : B' \subseteq B$, then we have a commuting diagram

$$\begin{array}{ccc} H_c^{*+\dim B'}(\mathcal{Z}(X'/B'); \mathfrak{o}_{B'}) & \xrightarrow{\quad i_! \quad} & H_c^{*+\dim B}(\mathcal{Z}(X/B); \mathfrak{o}_B) \\ & \searrow \quad \swarrow & \\ & H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) & \end{array} \tag{3.10}$$

where horizontal map $i_!$ is the 'wrong way' map (defined, for example, by triangulating B so that B' is a subcomplex).

Remark 3.4. Fix a map of spaces $\pi : W \to B$, and let $a : B \to *$. We may consider the groups $H_c^*(W/B) = a_!\pi_!\pi^*a^!\mathbb{Z}$ ('homology of B with coefficients in fiberwise compactly supported cochains of $W \to B$') and $H_*^{\mathrm{rel}\infty}(W/B) = a_*\pi_*\pi^!a^*\mathbb{Z}$ ('cohomology of B with coefficients in fiberwise chains rel infinity of $W \to B$'). These groups are (chain-level) dual since duality exchanges $*$ and $!$. A class in $H_*^{\mathrm{rel}\infty}(W/B)$ is roughly analogous to what is often called a 'bivariant class' for the morphism $W \to B$.

Although there is no topological space Cpx_3 parameterizing all complex threefolds, this explains the notation $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ and $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$. The complex (3.2) may be familiar to those who have tried to write down explicitly the exceptional pullback functor $\pi^!$.

3.2 Algebraic structure

We now define a product and coproduct on the Grothendieck group $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$, forming the structure of a commutative and co-commutative bi-algebra. The product corresponds to 'disjoint union of cycles', while the coproduct corresponds to 'addition of cycles'. The Grothendieck group also has 'division by d' operations for integers $d \geq 1$.

To begin, let us understand the tensor product $C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))^{\otimes 2}$. The product $\mathrm{Cpx}_{3,\bullet} \times \mathrm{Cpx}_{3,\bullet}$ carries two families of threefolds, obtained by pulling back from the two factors, and we consider their disjoint union. The relative cycle space of the disjoint union $(X\times\Delta^m)\sqcup(\Delta^n\times Y)\to\Delta^n\times\Delta^m$ is the product of relative cycle spaces $\mathcal{Z}(X/\Delta^n)\times\mathcal{Z}(Y/\Delta^m)\to\Delta^n\times\Delta^m$. Taking compactly supported cochains on the relative cycle space of the disjoint union family defines a coefficient system $C^*_c(\mathcal{Z})$ over $\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$.

Lemma 3.5. *The tensor product* $C_*(\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))^{\otimes 2}$ *is canonically quasi-isomorphic to* $C_*(\mathrm{Cpx}_3\times\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))$.

Proof. Recalling the discussion of products of simplicial sets and coefficient systems in §B, the tensor product $C_*(\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))^{\otimes 2}$ is canonically quasi-isomorphic to $C_*(\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet},C^*_c(\mathcal{Z})_{(1)}\otimes C^*_c(\mathcal{Z})_{(2)})$, where $C^*_c(\mathcal{Z})_{(i)}$ denotes the coefficient system on $\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ given by pulling back the coefficient system $C^*_c(\mathcal{Z})$ on $\mathrm{Cpx}_{3,\bullet}$ under the projection to the ith factor. There is a natural 'cup product and restriction' map of coefficient systems $C^*_c(\mathcal{Z})_{(1)}\otimes C^*_c(\mathcal{Z})_{(2)}\to C^*_c(\mathcal{Z})$ over $\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ which is *not* a quasi-isomorphism. However, this map does induce an isomorphism on the cohomology of any product of simplices $\Delta^n\times\Delta^m\subseteq\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ rel boundary, and this implies (filtration and long exact sequence argument) that the induced map on cohomology of $\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ is also an isomorphism.

Alternatively (and equivalently), viewing $C_*(\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))$ as $C^*_c(\mathcal{Z}(\mathrm{Cpx}_3),\omega_{\mathrm{Cpx}_3})$, there is a Künneth quasi-isomorphism between $C^*_c(\mathcal{Z}(\mathrm{Cpx}_3),\omega_{\mathrm{Cpx}_3})^{\otimes 2}$ and

$$C^*_c(\mathcal{Z}(\mathrm{Cpx}_3)\times\mathcal{Z}(\mathrm{Cpx}_3),\omega_{\mathrm{Cpx}_3}\otimes\omega_{\mathrm{Cpx}_3})=C^*_c(\mathcal{Z}(\mathrm{Cpx}_3\times\mathrm{Cpx}_3),\omega_{\mathrm{Cpx}_3\times\mathrm{Cpx}_3}),\tag{3.11}$$

which is in turn quasi-isomorphic to $C_*(\mathrm{Cpx}_3\times\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))$. □

Definition 3.6 (Product on $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$)**.** The disjoint union family over $\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ is classifed by a map to $\mathrm{Cpx}_{3,\bullet}$, and the coefficient system $C^*_c(\mathcal{Z})$ over $\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ is the pullback of the coefficient system $C^*_c(\mathcal{Z})$ over $\mathrm{Cpx}_{3,\bullet}$, so this gives a map

$$C_*(\mathrm{Cpx}_3\times\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))\to C_*(\mathrm{Cpx}_3,C^*_c(\mathcal{Z})).\tag{3.12}$$

Appealing to Lemma 3.5, this defines a map

$$C_*(\mathrm{Cpx}_3,C^*_c(\mathcal{Z}))^{\otimes 2}\to C_*(\mathrm{Cpx}_3,C^*_c(\mathcal{Z})),\tag{3.13}$$

which defines the product on $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ upon taking cohomology.

A diagram chase shows that the product is associative and unital (the unit $\eta:\mathbb{Z}\to H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ is the constant function 1 on $\mathcal{Z}(\varnothing)$, which is indeed a cycle in $C_0(\mathrm{Cpx}_3,C^0_c(\mathcal{Z}))$). The product is also (graded) commutative: while cup product $\cup:C^*(A)^{\otimes 2}\to C^*(A)$ is not commutative on the cochain level, it is commutative up to Steenrod's $\cup_1$ operation which is a chain null-homotopy of $\alpha\otimes\beta\mapsto\alpha\cup\beta-(-1)^{|\alpha||\beta|}\beta\cup\alpha$ [40, Theorem 5.1].

Definition 3.7 (Coproduct on $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)/\mathrm{tors}$)**.** The addition map $\Sigma:\mathcal{Z}(X/\Delta^n)\times_{\Delta^n}\mathcal{Z}(X/\Delta^n)\to\mathcal{Z}(X/\Delta^n)$ is a map of (space valued) coefficient systems $\Delta^*(\mathcal{Z})\to\mathcal{Z}$ where $\Delta:\mathrm{Cpx}_{3,\bullet}\to\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet}$ denotes the diagonal embedding. Applying C^*_c thus determines a map of coefficient systems $C^*_c(\mathcal{Z})\to\Delta^*C^*_c(\mathcal{Z})$ on $\mathrm{Cpx}_{3,\bullet}$ since Σ is proper, hence a map of complexes

$$C_*(\mathrm{Cpx}_{3,\bullet},C^*_c(\mathcal{Z}))\to C_*(\mathrm{Cpx}_{3,\bullet}\times\mathrm{Cpx}_{3,\bullet},C^*_c(\mathcal{Z})).\tag{3.14}$$

Appealing to Lemma 3.5, this defines a map

$$C_*(\mathrm{Cpx}_3, C_c^*(\mathcal{Z})) \to C_*(\mathrm{Cpx}_3, C_c^*(\mathcal{Z}))^{\otimes 2}, \tag{3.15}$$

which defines the coproduct on $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)/\mathrm{tors}$ upon taking cohomology (note that homology does not commute with tensor product, but does modulo torsion).

A diagram chase shows that the coproduct is coassociative and counital (the counit $\varepsilon : H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \to \mathbb{Z}$ acts on $C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))$ by summing the 'evaluate at the empty cycle' map over all vertices). The coproduct is trivially cocommutative. A diagram chase shows that $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)/\mathrm{tors}$ is a bi-algebra, recalling that a bi-algebra $(R, \eta, \mu, \varepsilon, \Delta)$ means that:

- (R, η, μ) is an algebra (satisfies unitality and associativity).
- (R, ε, Δ) is a co-algebra (satisfies co-unitality and co-associativity).
- The maps η and Δ are algebra maps (equivalently, the maps ε and μ are co-algebra maps).

Definition 3.8 (Division)**.** For any $d \geq 1$, the 'multiply by d map' $\mathcal{Z}(X/B) \to \mathcal{Z}(X/B)$ determines, via pullback, a map of coefficient systems $C_c^*(\mathcal{Z}) \to C_c^*(\mathcal{Z})$ over $\mathrm{Cpx}_{3,\bullet}$. We denote by ρ_d the induced map on the chain group $C_*(\mathrm{Cpx}_{3,\bullet}, C_c^*(\mathcal{Z}))$ and on its homology $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$.

The maps ρ_d are bi-algebra morphisms by inspection.

By dualizing the above constructions, we obtain dual operations on $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$. The homomorphism out of $H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ associated to a curve enumeration theory $F \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$ is a ring homomorphism if F solves the equation $\Delta(F) = F \otimes F$ (modulo torsion). We will call such a curve enumeration theory *multiplicative*.

All these structures exist, with the same definition, on $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ (and dually on $H_*^{\mathrm{rel}\infty}(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$) as well. The tautological map $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ is a map of bi-algebras and commutes with ρ_d. To check compatibility with the coproduct, we should note a sum of cycles $z + z'$ is semi-Fano iff both z and z' are semi-Fano.

3.3 Virtual fundamental cycles

Let us now recall the practical origin of classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$ aka curve enumeration theories.

All known curve enumeration theories arise via proper pushforward $H_*^{\mathrm{rel}\infty}(\mathcal{E}/\mathrm{Cpx}_3) \to H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$ for some $\mathcal{E}$ associating to each family of threefolds $X \to B$ over a complex analytic base B an analytic space (or Deligne–Mumford stack) $\mathcal{E}(X/B) \to B$, compatible with pullback, with a natural transformation $\mathcal{E} \to \mathcal{Z}$ which is proper (hence has pushforward on homology rel infinity). Fix for now any such $\mathcal{E}$.

If $\mathcal{E}(X/B) \to B$ is smooth (i.e. submersive) for every $X \to B$, then its relative (i.e. vertical) fundamental class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}/\mathrm{Cpx}_3)$ gives a curve enumeration theory. It is almost never the case that $\mathcal{E}(X/B) \to B$ is smooth, rather it carries a weaker structure called a *perfect obstruction theory* in the sense of Behrend–Fantechi [3] (reviewed in §A). A perfect obstruction theory on $\mathcal{E}(X/B) \to B$ induces a relative 'virtual' fundamental class in

$H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B)$ (see Definition A.15). To go from a (functorial) perfect obstruction theory on $\mathcal{E}(X/B) \to B$ for complex analytic bases B to a relative virtual fundamental class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}/\mathrm{Cpx}_3)$ requires some discussion since the base $\mathrm{Cpx}_{3,\bullet}$ a simplicial set not a complex analytic space. In essence, the solution is simply to complexify the base.

Given a finite semi-simplicial set B, we may glue together copies of the complexifications $\Delta^n \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n = \Delta^n_{\mathbb{C}}$, to obtain the 'complexified geometric realization' $B_{\mathbb{C}}$, which is a complex analytic space. The hypothesis that B is a *finite semi*-simplicial set guarantees that this gluing exists as a complex analytic space. A family of complex threefolds $X \to B$ is, essentially by definition, the data of a submersion $X_{\mathbb{C}} \to B_{\mathbb{C}}$ over an open neighborhood of the (ordinary) geometric realization of B inside $B_{\mathbb{C}}$. Now a perfect obstruction theory on $\mathcal{E}(X_{\mathbb{C}}/B_{\mathbb{C}}) \to B_{\mathbb{C}}$ determines a class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X_{\mathbb{C}}/B_{\mathbb{C}})/B_{\mathbb{C}})$ hence, via pullback, a class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B)$. Since the given relative perfect obstruction theory on $\mathcal{E}$ is assumed compatible with pullback, it follows that for any map of finite semi-simplicial sets $B' \to B$ and $X' = X \times_B B'$, the pullback map $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B) \to H_*^{\mathrm{rel}\infty}(\mathcal{E}(X'/B')/B')$ sends the virtual fundamental class to the virtual fundamental class (since virtual fundamental classes are compatible with pullback, see Lemma A.16). This defines for any (possibly infinite) semi-simplicial set B, a virtual fundamental class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B)_{\mathrm{naive}}$ for any family of threefolds $X \to B$ (where the subscript $_{\mathrm{naive}}$ indicates the inverse limit over finite semi-simplicial subsets of the base B, which differs from the true $H_*^{\mathrm{rel}\infty}$ by a $\varprojlim^1$-term (B.3)). To define the virtual fundamental class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B)_{\mathrm{naive}}$ when the base B is a simplicial set (not a semi-simplicial set), we appeal to the isomorphism between this group and that for the pullback of B along $\mathbf{\Delta}^{\mathsf{inj}} \to \mathbf{\Delta}$ (which holds basically because the 'fat' realization and the 'reduced' realization have the same (co)homology, see §B). Note that the virtual fundamental class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B)_{\mathrm{naive}}$ is independent of the choice of triangulation of the base (consider a concordance of triangulations).

We have thus defined, for any family of threefolds $X \to B$ over a simplicial set B a 'virtual fundamental' class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}(X/B)/B)_{\mathrm{naive}}$, compatible with pullback, given a functorial relative perfect obstruction theory on $\mathcal{E}$. In particular, this defines a virtual fundamental class in $H_*^{\mathrm{rel}\infty}(\mathcal{E}/\mathrm{Cpx}_3)_{\mathrm{naive}}$ and thus a pushforward $[\mathcal{E}]^{\mathrm{vir}} \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)_{\mathrm{naive}}$. While this is not, strictly speaking, a curve enumeration theory in the sense of having a class in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$, this is of little importance to our present work since pairing with $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ factors through $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3) \to H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)_{\mathrm{naive}}$.

Now suppose $\mathcal{E}$ is *multiplicative* in the sense that for a pair of families of threefolds $X \to B \leftarrow X'$ over a complex analytic base B, there is a functorial isomorphism $\mathcal{E}(X \sqcup X'/B) = \mathcal{E}(X/B) \times_B \mathcal{E}(X'/B)$, compatible with perfect obstruction theories. It then follows from compatibility of virtual fundamental classes with (fiber) product (Lemmas A.16 and A.17) that $\Delta([\mathcal{E}]^{\mathrm{vir}}) = [\mathcal{E}]^{\mathrm{vir}} \otimes [\mathcal{E}]^{\mathrm{vir}}$ for the coproduct $\Delta : H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)_{\mathrm{naive}} \to ((H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)/\mathrm{tors})^{\otimes 2})_{\mathrm{naive}}$.

3.4 Enumerative invariants

We now define the Gromov–Witten and Pandharipande–Thomas virtual fundamental classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3;\mathbb{Q})((u))$ and $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)((q))$. We denote by

$$\mathrm{GW}: H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \to \mathbb{Q}((u)) \tag{3.16}$$
$$\mathrm{PT}: H_c^*(\mathcal{Z}/\mathrm{Cpx}_3) \to \mathbb{Z}((q)) \tag{3.17}$$

the resulting homomorphisms, which are in fact ring homomorphisms since these virtual fundamental classes solve $\Delta(\xi) = \xi \otimes \xi$.

Gromov–Witten and Pandharipande–Thomas invariants are defined using moduli spaces $\overline{\mathcal{M}}'(X/B)$ and $P(X/B)$ (respectively) over B associated to any family of threefolds $X \to B$ over a complex analytic space B. The moduli space $\overline{\mathcal{M}}'(X/B)$ is a Deligne–Mumford analytic stack representing stable maps from compact (not necessarily connected) nodal curves to fibers of $X \to B$, all of whose connected components are non-constant. The analytic space $P(X/B)$ parameterizes stable pairs on fibers of $X \to B$ (a stable pair is a coherent sheaf F of proper support of pure relative dimension one along with a section s whose cokernel has relative dimension zero [30]). There are locally constant maps

$$\chi : \overline{\mathcal{M}}'(X/B) \to \mathbb{Z} \tag{3.18}$$
$$n : \quad P(X/B) \to \mathbb{Z} \tag{3.19}$$

given by domain arithmetic Euler characteristic and holomorphic Euler characteristic, respectively.

Both $\overline{\mathcal{M}}'(X/B) \to B$ and $P(X/B) \to B$ carry a natural (relative) perfect obstruction theory, compatible with pullback. As reviewed in §3.3, there are hence induced virtual fundamental classes

$$[\overline{\mathcal{M}}'/\mathrm{Cpx}_3]^{\mathrm{vir}} = \prod_{X\to\Delta^k} [\overline{\mathcal{M}}'(X/\Delta^k)]^{\mathrm{vir}} \in H_*^{\mathrm{rel}\infty}(\overline{\mathcal{M}}'/\mathrm{Cpx}_3;\mathbb{Q}), \tag{3.20}$$
$$[P/\mathrm{Cpx}_3]^{\mathrm{vir}} = \prod_{X\to\Delta^k} [P(X/\Delta^k)]^{\mathrm{vir}} \quad \in H_*^{\mathrm{rel}\infty}(P/\mathrm{Cpx}_3). \tag{3.21}$$

Now the maps $\overline{\mathcal{M}}' \to \mathcal{Z}$ and $P \to \mathcal{Z}$ are proper when restricted to the sets on which χ and n are bounded above by a given $N < \infty$. Pushing forward $u^{-\chi} \cdot [\overline{\mathcal{M}}'/\mathrm{Cpx}_3]^{\mathrm{vir}}$ and $q^n \cdot [P/\mathrm{Cpx}_3]^{\mathrm{vir}}$ thus defines classes

$$\mathrm{GW} \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3;\mathbb{Q})((u)), \tag{3.22}$$
$$\mathrm{PT} \in H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)((q)), \tag{3.23}$$

which have virtual dimension zero since the virtual fundamental classes of $\overline{\mathcal{M}}'$ and P lie in relative virtual dimension $\langle c_1(T_{X/B}), \beta\rangle$. This defines the group homomorphisms (3.16)–(3.17).

The moduli spaces $\overline{\mathcal{M}}'$ and P are 'multiplicative' in the sense that $\overline{\mathcal{M}}'((X \sqcup Y)/B) = \overline{\mathcal{M}}'(X/B) \times_B \overline{\mathcal{M}}'(Y/B)$ compatibly with perfect obstruction theories (and the same for P). As reviewed in §3.3, it follows that the induced virtual fundamental classes $[\overline{\mathcal{M}}']^{\mathrm{vir}}$ and $[P]^{\mathrm{vir}}$

are also multiplicative in the sense of satisfying $\Delta(\xi)=\xi\otimes\xi$, implying (3.16)–(3.17) are ring homomorphisms.

Classical Gromov–Witten and Pandharipande–Thomas theory is interested in evaluating GW and PT on elements of $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ coming from projective threefolds. When X is projective, the space of cycles $\mathcal{Z}(X,\beta)$ in homology class β is compact, hence its characteristic function defines a class $(X,\beta)\in H^0_c(\mathcal{Z}(-,\langle c_1(TX),\beta\rangle)/\mathrm{Cpx}_3)$, which has virtual dimension $2\langle c_1(TX),\beta\rangle$. Thus when $\langle c_1(TX),\beta\rangle=0$, we may evaluate the homomorphisms GW and PT on this element to obtain invariants

$$\mathrm{GW}(X,\beta)=\int_{[\overline{\mathcal{M}}'(X,\beta)]^{\mathrm{vir}}} u^{-\chi}\in\mathbb{Q}((u)), \tag{3.24}$$

$$\mathrm{PT}(X,\beta)=\int_{[P(X,\beta)]^{\mathrm{vir}}} q^{n}\in\mathbb{Z}((q)). \tag{3.25}$$

More generally, given cohomology classes $\gamma_1,\ldots,\gamma_r\in H^*(X)$ (called 'insertions'), we may consider the class

$$(X,\beta;\gamma_1,\ldots,\gamma_r)\in H_c^{(|\gamma_1|-2)+\cdots+(|\gamma_r|-2)}(\mathcal{Z}(-,\langle c_1(TX),\beta\rangle)/\mathrm{Cpx}_3) \tag{3.26}$$

given by the cohomology class $\mathbf{1}_\beta\prod_{i=1}^r\pi_!i^*\gamma_i$ on $\mathcal{Z}(X)$, namely the result of push/pull via the universal family.

$$\begin{array}{ccc}\mathcal{U}(X) & \xrightarrow{\ i\ } & X\\ \big\downarrow{\scriptstyle\pi} & & \\ \mathcal{Z}(X) & & \end{array} \tag{3.27}$$

Evaluating GW and PT on this class produces Gromov–Witten invariants and Pandharipande–Thomas invariants of X in homology class β with insertions $\gamma_1,\ldots,\gamma_r$

$$\mathrm{GW}(X,\beta;\gamma_1,\ldots,\gamma_r)=\int_{[\overline{\mathcal{M}}'(X,\beta)]^{\mathrm{vir}}}\prod_{i=1}^r\pi_!\,\mathrm{ev}^*\gamma_i\cdot u^{-\chi}\in\mathbb{Q}((u)) \tag{3.28}$$

$$\mathrm{PT}(X,\beta;\gamma_1,\ldots,\gamma_r)=\int_{[P(X,\beta)]^{\mathrm{vir}}}\prod_{i=1}^r\pi_!(\mathrm{ch}_2(\mathbb{F})\cup\pi_X^*\gamma_i)\cdot q^n\in\mathbb{Z}((q)) \tag{3.29}$$

where the integrand involves push/pull for the universal families

$$\begin{array}{ccc}\overline{\mathcal{U}}'(X) & \xrightarrow{\ \mathrm{ev}\ } & X\\ {\scriptstyle\pi}\big\downarrow & & \\ \overline{\mathcal{M}}'(X) & & \end{array}\qquad\qquad \begin{array}{ccc}P(X)\times X & \xrightarrow{\ \pi_X\ } & X\\ {\scriptstyle\pi}\big\downarrow & & \\ P(X) & & \end{array} \tag{3.30}$$

and $\mathbb{F}$ denotes the universal stable pair on $P(X)\times X$ (note that the second chern character $\mathrm{ch}_2(\mathbb{F})$ is simply the fundamental cycle of the support of $\mathbb{F}$, a codimension four cohomology class on $P(X,\beta)\times X$). These invariants vanish for dimension reasons except when the virtual dimension $2\langle c_1(TX),\beta\rangle-\sum_i(|\gamma_i|-2)$ is zero.

4 Local curves

In the study of enumerative invariants of complex threefolds, the term *local curve* refers to (the total space of) a rank two vector bundle E over a smooth proper (usually connected) curve C. Given a local curve $E \to C$, one is then interested in enumerating curves *supported on the zero section* $C \subseteq E$; unfortunately, this has no meaning *a priori* since $\mathbb{Z}_{\geq 0} \cdot [C] \subseteq \mathcal{Z}(E)$ is usually not open. The goal of this section is to recall how to make sense of the enumerative theory of local curves by working equivariantly, and to show how this enumerative theory may be realized within the framework of the Grothendieck group $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$.

Remark 4.1. It is not hard to show that local curves are classified up to deformation by the pair of integers $g = g(C) \geq 0$ and $c = c_1(E) \in \mathbb{Z}$. The chern number of the zero section is given by $k = c_1(TE) = c_1(E) + c_1(TC) = 2 - 2g + c$ and is a more convenient index than c. We write $E_{g,k}$ for the (unique up to deformation) local curve of genus $g \geq 0$ and chern number k.

4.1 Equivariant homology

The flavor of equivariant homology relevant for our present discussion is called *co-Borel equivariant homology*, which measures 'homotopically S^1-invariant cycles' on an S^1-space. We will employ the following concrete definition of this homology theory.

Definition 4.2 (co-Borel equivariant homology)**.** Let X be an S^1-space with reasonable topology (say Hausdorff, paracompact, and locally homeomorphic to a finite CW-complex of uniformly bounded dimension). The co-Borel S^1-equivariant homology of X is the inverse limit

$$H^{cS^1}_*(X) = \varprojlim_n H_{*+2N}\left(\frac{X \times S^{2N+1}}{S^1}\right) \tag{4.1}$$

where $S^{2N+1} \subseteq \mathbb{C}^{N+1}$ is the unit sphere acted on by the unit circle $S^1 \subseteq \mathbb{C}$ by multiplication. The quotient $(X \times S^{2N+1})/S^1$ is a locally trivial fibration over $S^{2N+1}/S^1 = \mathbb{C}P^N$ with fiber X. The diagram

$$\begin{array}{ccc} \dfrac{X \times S^{2N+1}}{S^1} & \longrightarrow & \dfrac{X \times S^{2N+3}}{S^1} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}P^n & \longrightarrow & \mathbb{C}P^{N+1} \end{array} \tag{4.2}$$

thus determines maps $H_{*+2N+2}((X \times S^{2N+3})/S^1) \to H_{*+2N}((X \times S^{2N+1})/S^1)$ ('intersect with a hyperplane'), which are the structure maps of the inverse system in (4.1). These structure maps fit into a long exact sequence with third term $H_{*+2N+1}(X)$, so the inverse system is constant in degree d once $2N + 1 > \dim X - d$. This eventual constancy implies the inverse limit of homology (4.1) is well behaved (for example, the long exact sequence of the pair exists and is exact for H^{cS^1}).

Dually, the co-Borel S^1-equivariant cohomology of X is the direct limit

$$H^*_{cS^1}(X) = \varinjlim_N H^{*+2N}\left(\frac{X \times S^{2N+1}}{S^1}\right) \tag{4.3}$$

of the ‘wrong way’ maps $H^{*+2N}((X \times S^{2N+1})/S^1) \to H^{*+2N+2}((X \times S^{2N+3})/S^1)$.

It is evident that $H_*^{cS^1}(X)$ and $H^*_{cS^1}(X)$ are supported in degrees $\leq \dim X$, and they are typically nontrivial in arbitrarily negative degrees. For example, $H_*^{cS^1}(\mathrm{pt}) = \varprojlim_N H_{*+2N}(\mathbb{C}P^n) = \varprojlim_N H^{-*}(\mathbb{C}P^N) = \mathbb{Z}[t]$ (free polynomial algebra) where t is the class of a hyperplane and lies in homological degree -2 (cohomological degree 2). Intersection of cycles gives $H_*^{cS^1}(\mathrm{pt})$ the structure of a ring and gives each $H_*^{cS^1}(X)$ the structure of a module over it.

Definition 4.3 (Tate equivariant homology)**.** The Tate S^1-equivariant homology is the localization of co-Borel equivariant homology at $t \in H_{-2}^{cS^1}(\mathrm{pt})$, namely it is the direct limit

$$H_*^{tS^1}(X) = \varinjlim_i H_{*-2i}^{cS^1}(X) \tag{4.4}$$

where the transition maps are multiplication by t (compare Greenlees–May [11, Corollary 16.3]).

The key property of Tate equivariant homology is that it vanishes for (almost) free S^1-spaces (with rational coefficients), hence by the long exact sequence and excision, depends rationally only on the fixed set. This is known as the *equivariant localization theorem*, which originates in the work of Smith [37, 38, 39], was reformulated cohomologically by Borel [4], and was then formalized in its present form by Atiyah–Segal [1, 35] and Quillen [34].

Proposition 4.4. *The map $H_*^{tS^1}(X^{S^1}) \to H_*^{tS^1}(X)$ is an isomorphism over $\mathbb{Q}$.*

Proof. We assume that our spaces have a reasonable S^1-equivariant cell decomposition (which holds in the cases we care about by real analyticity). Precisely speaking, this means that X is glued out of cells of the form $(S^1/\Gamma) \times (D^k, \partial D^k)$ for subgroups $\Gamma \subseteq S^1$, where S^1 acts by multiplication on the first factor (and trivially on the second factor). Given such a cell decomposition of X, to show that $H_*^{tS^1}(X, X^{S^1}) = 0$, it suffices (by the long exact sequence and excision) to show that $H_*^{tS^1}((S^1/\Gamma) \times (D^k, \partial D^k)) = 0$ for $\Gamma \subsetneqq S^1$ a *proper* subgroup. We have $H_*^{tS^1}((S^1/\Gamma) \times (D^k, \partial D^k)) = H_{*-k}^{tS^1}(S^1/\Gamma)$, so we are reduced to showing that $H_*^{tS^1}(S^1/\Gamma) = 0$ for $\Gamma \subsetneqq S^1$. Since Γ is finite, there is a ‘transfer’ map $H_*^{tS^1}(S^1/\Gamma) \to H_*^{tS^1}(S^1)$ whose composition with the pushforward map $H_*^{tS^1}(S^1) \to H_*^{tS^1}(S^1/\Gamma)$ is multiplication by $\#\Gamma$ on $H_*^{tS^1}(S^1/\Gamma)$. It thus suffices to show that $H_*^{tS^1}(S^1) = 0$, which follows from calulating $H_*^{cS^1}(S^1) = \mathbb{Z}$. ∎

The significance of equivariant localization is the following. Given a class in $H_*^{cS^1,\mathrm{rel}\infty}(X)$, we may push forward to $H_*^{cS^1}(\mathrm{pt})$ provided X is compact. However, if we are satisfied with pushing forward to the Tate group $H_*^{tS^1}(\mathrm{pt})$ (over the rationals), then equivariant localization provides such a pushforward map when just the fixed set X^{S^1} is compact.

$$\begin{array}{ccc} H_*^{cS^1,\mathrm{rel}\infty}(X^{S^1}) & \longrightarrow & H_*^{tS^1,\mathrm{rel}\infty}(X^{S^1};\mathbb{Q}) \\ \downarrow & & \downarrow\sim \\ H_*^{cS^1,\mathrm{rel}\infty}(X) & \longrightarrow & H_*^{tS^1,\mathrm{rel}\infty}(X;\mathbb{Q}) \\ X\text{ compact}\downarrow & & \downarrow X\text{ compact} \\ H_*^{cS^1}(\mathrm{pt}) & \longrightarrow & H_*^{tS^1}(\mathrm{pt};\mathbb{Q}) \end{array} \qquad (X^{S^1}\text{ compact: } H_*^{cS^1,\mathrm{rel}\infty}(X^{S^1}) \to H_*^{cS^1}(\mathrm{pt}),\; H_*^{tS^1,\mathrm{rel}\infty}(X^{S^1};\mathbb{Q}) \to H_*^{tS^1}(\mathrm{pt};\mathbb{Q})) \tag{4.5}$$

4.2 Equivariant enumerative invariants

Curve enumeration theories, namely classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$, specialize to virtual fundamental classes in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}(X))$ for complex threefolds X. It turns out that a curve enumeration theory also determines S^1-equivariant virtual fundamental classes, namely classes in $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X))$ for X with a $\mathbb{C}^\times$-action. The fact that non-equivariant invariants determine their equivariant lifts may appear surprising at first, however the mechanism is very simple: equivariant invariants are non-equivariant family invariants of the Borel construction. Let us explain this in our particular case of interest $\mathbb{C}^\times \curvearrowright X$. We have

$$H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X)) = \varprojlim_N H_{*+2N}^{\mathrm{rel}\infty}\left(\frac{\mathcal{Z}(X)\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0}\right), \tag{4.6}$$

and $(\mathcal{Z}(X)\times(\mathbb{C}^{N+1}-0))/\mathbb{C}^\times$ is the relative cycle space of the family $(X\times(\mathbb{C}^{N+1}-0))/\mathbb{C}^\times \to \mathbb{C}P^N$, so we have

$$H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X)) = \varprojlim_N H_{*+2N}^{\mathrm{rel}\infty}\left(\mathcal{Z}\left(\frac{X\times(\mathbb{C}^{N+1}-0)}{\mathbb{C}-0}\;\Big/\;\mathbb{C}P^N\right)\right). \tag{4.7}$$

A curve enumeration theory gives rise to a coherent system of classes in this inverse system, hence to an 'equivariant virtual fundamental' class in $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X))$. In particular, this defines equivariant Gromov–Witten invariants $\mathrm{GW} \in H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X);\mathbb{Q})((u))$ and $\mathrm{PT} \in H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X))((q))$. Let us note that by equivariant localization (Proposition 4.4 and (4.5)), a class in $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(X))$ determines a class in $H^{tS^1,\mathrm{rel}\infty}(\mathcal{Z}(X)^{S^1};\mathbb{Q})$.

Now let us specialize to the case of a local curve $E = E_{g,k}$ equipped with the fiberwise scaling action of $\mathbb{C}^\times$. An equivariant virtual fundamental class thus lies in $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(E))$. Restricting to cycles $\mathcal{Z}(E,m)\subseteq\mathcal{Z}(E)$ of degree m (homology class $m[C]$ for $C\subseteq E$ the zero section), this class lies in degree $2km$. The fixed locus $\mathcal{Z}(E)^{S^1}$ is just $\mathbb{Z}_{\geq 0}\times[C]$ (multiples of the zero section). In particular, $\mathcal{Z}(E,m)^{S^1}$ is compact, so equivariant localization (Proposition 4.4 and (4.5)) provides a pushforward map $H_*^{cS^1,\mathrm{rel}\infty}(\mathcal{Z}(E,m)) \to H_*^{tS^1}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}[t,t^{-1}]$. The pushforward of the virtual fundamental class is an equivariant enumerative invariant in $H_{2km}^{tS^1}(\mathrm{pt};\mathbb{Q}) = \mathbb{Q}\cdot t^{-mk}$ which roughly speaking 'S^1-equivariantly count curves of degree m in $E_{g,k}$'. Specializing to the Gromov–Witten and Pandharipande–Thomas virtual fundamental classes, we obtain equivariant invariants

$$\mathrm{GW}_{S^1}(E_{g,k},m) \in \mathbb{Q}((u)) \cdot t^{-mk} \tag{4.8}$$

$$\mathrm{PT}_{S^1}(E_{g,k},m) \in \mathbb{Q}((q)) \cdot t^{-mk} \tag{4.9}$$

for any local curve $E_{g,k} \to C$ and integer multiplicity $m \geq 0$.

Theorem 4.5 ([6, 28])**.** *The power series $(-iu)^k \mathrm{GW}_{S^1}(E_{g,k},m)$ and $(-q)^{-k/2}\mathrm{PT}_{S^1}(E_{g,k},m)$ satisfy the MNOP correspondence.*

We explain the citation: Bryan–Pandharipande [6] compute the S^1-equivariant Gromov–Witten invariants of $E_{g,k}$, while Okounkov–Pandharipande [28] compute the S^1-equivariant Donaldson–Thomas invariants of $E_{g,k}$. It is explained in [25, Section 5] how to walk through the arguments of [28] to see that they apply equally well to Pandharipande–Thomas invariants.

4.3 Elements of the Grothendieck group

Let us now express, explicitly, the equivariant enumerative invariants of local curves $E_{g,m,k} \to C_g$ (defined just above) as the (non-equivariant) enumerative invariants of certain elements $x_{g,m,k} \in H_c^{2km}(\mathcal{Z}(-,km)/\mathrm{Cpx}_3)$ of virtual dimension zero which we call *equivariant local curve elements.*

Consider the map

$$\mathcal{Z}(E,m) \xrightarrow{\cap E_p} \mathrm{Sym}^m E_p \xrightarrow{\mathrm{Sym}^m \lambda} \mathrm{Sym}^m \mathbb{C} \xrightarrow{\beta_r} \mathbb{C} \tag{4.10}$$

associated to a point $p \in C$, a linear map $\lambda : E_p \to \mathbb{C}$, and a homogeneous symmetric polynomial $\beta_r : \mathrm{Sym}^m \mathbb{C} \to \mathbb{C}$ of degree $r \geq 1$. This map is $\mathbb{C}^\times$-equivariant for the weight r action on the target $\mathbb{C}$. It thus determines a section f of $\mathcal{L}^{\otimes r}$ over $(\mathcal{Z}(E,m) \times (\mathbb{C}^{N+1} - 0))/\mathbb{C}^\times$, where $\mathcal{L}$ denotes (the pullback of) the tautological line bundle on $\mathbb{C}P^N$. Let $\tau_{\mathcal{L}^{\otimes r}} \in H^2(\mathcal{L}^{\otimes r}, \mathcal{L}^{\otimes r} \setminus 0)$ denote the Thom class.

Given a tuple $f_1, \ldots, f_n$ of such sections, the product $f_1^* \tau_{\mathcal{L}^{\otimes r_1}} \cup \cdots \cup f_n^* \tau_{\mathcal{L}^{\otimes r_n}}$ is supported inside the common zero locus $f_1^{-1}(0) \cap \cdots \cap f_n^{-1}(0)$. We may choose such a tuple whose joint zero set is the single point $m[C] \in \mathcal{Z}(E,m)$ (in particular, is compact), hence giving us an element

$$\pi_{N,n} = \prod_{i=1}^n r_i^{-1} f_i^* \tau_{\mathcal{L}^{\otimes r_i}} \in H_c^{2n}\Big(\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N, m\Big)\Big). \tag{4.11}$$

Now each cocycle $r_i^{-1} f_i^* \tau_{\mathcal{L}^{\otimes r_i}}$ is cohomologous to the hyperplane class (pulled back from $\mathbb{C}P^N$), so if the joint zero set of $f_1, \ldots, f_{n-1}$ is compact, then the 'wrong way map'

$$i_! : H_c^{*+2N-2}\Big(\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^N - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^{N-1}\Big)\Big) \to H_c^{*+2N}\Big(\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N\Big)\Big) \tag{4.12}$$

sends $\pi_{N-1,n-1}$ to $H \cup \pi_{N,n-1} = \pi_{N,n}$. It follows that $\pi_{N,n}$ is independent of the choice of $f_1, \ldots, f_n$, and the collection of all $\pi_{N,N+a}$ (fixed integer a) gives rise to a well defined degree $2a$ element of the direct limit

$$H^*_{cS^1,c}(\mathcal{Z}(E,m)) = \varinjlim_N H_c^{*+2N}\Big(\frac{\mathcal{Z}(E,m) \times S^{2N+1}}{S^1}\Big) \tag{4.13}$$

$$= \varinjlim_N H_c^{*+2N}\Big(\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N, m\Big)\Big). \tag{4.14}$$

Now each term in this directed system maps to $H_c^*(\mathcal{Z}(-,km)/\mathrm{Cpx}_3)$, compatibly with the maps in the directed system (commutativity of (3.10)), thus determining a well defined map $H^*_{cS^1,c}(\mathcal{Z}(E,m)) \to H_c^*(\mathcal{Z}(-,km)/\mathrm{Cpx}_3)$. The image of the degree $2a$ element of $H^*_{cS^1,c}(\mathcal{Z}(E,m))$ defined by the set of $\pi_{N,N+a}$ is denoted

$$_\ell x_{g,m,k} \in H_c^{2km+2\ell}(\mathcal{Z}(-,km)/\mathrm{Cpx}_3) \tag{4.15}$$

in terms of the re-indexing $a = km + \ell$ (which is explained by the fact that $_\ell x_{g,m,k}$ has virtual dimension -2ℓ); we set $x_{g,m,k} = {}_0x_{g,m,k}$. For $k \geq 0$, the elements $_\ell x_{g,m,k} \in H_c^{2km+2\ell}(\mathcal{Z}(-,km)/\mathrm{Cpx}_3)$ evidently lift canonically to $H_c^{2km+2\ell}(\mathcal{Z}(-,km)_{\mathrm{sF}}/\mathrm{Cpx}_3)$ and are well defined there as well. We have $x_{g,0,k} = 1$ (take $n = N = 0$) and $_\ell x_{g,0,k} = 0$ for $\ell > 0$ (take $N = 0$ and $n = \ell$).

Proposition 4.6. *Given any curve enumeration theory (class in $H_*^{\mathrm{rel}\infty}(\mathcal{Z}/\mathrm{Cpx}_3)$), the resulting equivariant count of a local curve E in degree m is given by the pairing with $x_{g,m,k} \in H_c^{2km}(\mathcal{Z}(-,km)/\mathrm{Cpx}_3)$ times t^{-km}.*

Proof. Every class in $H^{cS^1}_{-2d}(\mathrm{pt})$ has the form $a \cdot t^d$ for some integer a. The coefficient a may be recovered by realizing the class inside some $H_{2N-2d}(S^{2N+1}/S^1)$ and pairing with H^{N-d} (power of the hyperplane class), provided $N \geq d$ so that this makes sense. We can apply the same recipe to find the image in $H_*^{cS^1}(\mathrm{pt})$ of a class in $H_*^{cS^1}(X)$ for any S^1-space X. Namely the image of a class in $H^{cS^1}_{-2d}(X) = \varprojlim_N H_{2N-2d}((X \times S^{2N+1})/S^1)$ in $H_*(\mathrm{pt}) = \mathbb{Z}[t]$ is given by t^d times its pairing with H^{N-d} for any $N \geq d$.

Now the equivariant enumerative invariants of $(E_{g,k}, m)$ are defined by pushing forward (after localizing at t) the virtual fundamental class in (the inverse limit wrt N of) $H^{\mathrm{rel}\infty}_{2km+2N}((\mathcal{Z}(E_{g,k},m) \times S^{2N+1})/S^1)$. This pushforward is only defined in Tate homology, that is we must multiply by t^n and lift to $H_{2km+2N-2n}((\mathcal{Z}(E_{g,k},m)^{S^1} \times S^{2N+1})/S^1)$ (which is guaranteed to be possible for n sufficiently large by Proposition 4.4) before pushing forward. This multiplication by t^n and lift is precisely realized by $(r_1 \cdot \cdots \cdot r_n)^{-1} f_1^* \tau_{\mathcal{L}^{\otimes r_1}} \cup \cdots \cup f_n^* \tau_{\mathcal{L}^{\otimes r_n}}$ for $f_1, \ldots, f_n$ with compact common zero locus. After multiplying by t^n, the pushforward to a point lies in degree $2km - 2n$, so following the above procedure we should cap with $H^{N-(n-km)}$ for $N \geq n - km$ to determine its image in $H^{cS^1}_{2km-2n}(\mathrm{pt})$. We can simply take $N = n - km$, so there is no cap with a power of H, and we conclude that the coefficient in front of t^{n-km} is the evaluation of our curve enumeration theory on $x_{g,m,k}$ as desired. $\square$

Corollary 4.7. *The power series $(-iu)^k \mathrm{GW}(x_{g,m,k})$ and $(-q)^{-k/2}\mathrm{PT}(x_{g,m,k})$ satisfy the MNOP correspondence.*

Proof. Combine Theorem 4.5 with Proposition 4.6. $\square$

Lemma 4.8. *We have* $\rho_d(x_{g,m,0}) = x_{g,m/d,0}$ *if* $d|m$.

Proof. Inspection: the pullback of a map f_i under the multiplication by d map is another such map of the same degree. $\square$

We now calculate the value of the coproduct Δ (Definition 3.7) applied to ${}_\ell x_{g,m,k}$. First let us note that ${}_\ell x_{g,m,k}$ vanishes for ℓ sufficiently large. Indeed, suppose that $x_{g,m,k}$ is represented by an expression (4.11) for $f_1,\ldots,f_n$ with compact joint zero locus. Now ${}_\ell x_{g,m,k}$ is obtained from this expression by adding ℓ more sections f, or equivalently by multiplying by the hyperplane class ℓ times, which results in zero once $\ell > N$.

Lemma 4.9. *We have* $\Delta({}_\ell x_{g,m,k}) = \sum_{\substack{a+b=m\\ a,b\geq 0}} \sum_{\ell_1+\ell_2=\ell} {}_{\ell_1}x_{g,a,k} \otimes {}_{\ell_2}x_{g,b,k}$.

Proof. Realize ${}_\ell x_{g,m,k}$ by the expression $\pi_{N,n}$ (4.11) for a local curve $E = E_{g,k}$ and some sections f_i of $\mathcal{L}^{\otimes r_i}$ as in (4.10) whose joint zero set $f_1^{-1}(0) \cap \cdots \cap f_n^{-1}(0)$ is compact, and $n = N + km + \ell$. The coproduct $\Delta({}_\ell x_{g,m,k})$ (Definition 3.7) is defined via the disjoint union family

$$\left(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \times \frac{\mathbb{C}^{N+1} - 0}{\mathbb{C} - 0}\right) \sqcup \left(\frac{(\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \times \frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\right) \to \mathbb{C}P^N \times \mathbb{C}P^N, \quad (4.16)$$

whose relative cycle space is the product of relative cycle spaces

$$\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N\Big) \times \mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N\Big) \to \mathbb{C}P^N \times \mathbb{C}P^N. \quad (4.17)$$

Over the diagonal $\Delta(\mathbb{C}P^N) \subseteq \mathbb{C}P^N \times \mathbb{C}P^N$, there is an addition map Σ from this relative cycle space to the relative cycle space of $(E \times (\mathbb{C}^{N+1} - 0))/\mathbb{C}^\times \to \mathbb{C}P^N$. The coproduct $\Delta({}_\ell x_{g,m,k})$ is represented by the disjoint union family (4.16) equipped with the cocycle $\Delta_!\Sigma^*\pi_{N,n}$.

Now let us consider the product $\Delta_!\Sigma^*\pi_{N,n} \cup p_1^*\pi_{N,p} \cup p_2^*\pi_{N,q}$ (where π_i denotes the projection to the ith factor), which we note is compactly supported if *either* the f's comprising $\pi_{N,n}$ have compact joint zero set *or* the f's comprising $\pi_{N,p}$ and the f's comprising $\pi_{N,q}$ both have compact joint zero set. Note that if we multiply this expression by p_i^*H, this can be described as incrementing p, but it can also be described as incrementing n since $(\Delta_!\alpha) \cup \beta = \Delta_!(\alpha \cup \Delta^*\beta)$ and $\Delta^*p_i^*H = H$. It follows that $\Delta_!\Sigma^*\pi_{N,n} = (\Delta_!\Sigma^*\mathbf{1}_m) \cup p_1^*\pi_{N,p} \cup p_2^*\pi_{N,q}$ for $n = p + q$ where $\mathbf{1}_m$ is the characteristic function of degree m cycles. Expanding $\Delta_!\Sigma^*\mathbf{1}_m = \sum_{\substack{a+b=m\\ a,b\geq 0}} \sum_{c+d=N} \mathbf{1}_a H^c \otimes \mathbf{1}_b H^d$, we can write the right hand side as

$$\sum_{\substack{a+b=m\\ a,b\geq 0}} \sum_{c+d=N} {}_{p+c-ka-N}x_{g,a,k} \otimes {}_{q+d-kb-N}x_{g,b,k}. \quad (4.18)$$

This is the desired result for $\Delta({}_\ell x_{g,m,k})$ since $[p+c-ka-N] + [q+d-kb-N] = (p+q) + (c+d) - k(a+b) - 2N = n + N - km - 2N = \ell$. $\square$

5 Transversality

We prove here a 'generic transversality' result, which says that *simple* (not multiply covered) maps from smooth curves to complex manifolds with generic (in a certain precise sense) complex structures are unobstructed (transverse). We derive from this that $H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is generated by certain equivariant local curve elements $x_{g,m,k}$.

5.1 Regularity

The deformation theory of a map $u : C \to X$ from a smooth proper curve C to a smooth complex analytic manifold X is controlled by $H^*(C, u^*TX)$. The deformation theory of C itself is controlled by $H^*(C, TC[1])$. The deformation theory of the pair (C, u) is controlled by $H^*(C, [TC[1] \to u^*TX])$. A deformation problem (in any of the above flavors) is said to be *unobstructed* when $H^{\geq 1} = 0$.

Given a complex analytic submersion $X \to B$, we may also consider the deformation theory of pairs $(b, u : C \to X_b)$, which is an extension of T_bB and the deformation theory of u. Note that this differs from the deformation theory of maps from C to the total space X (which we will not ever consider). If $X \to B$ is a pullback of a submersion $X' \to B'$, then a pair $(b, u : C \to X_b)$ in $X \to B$ which is unobstructed remains unobstructed when pushed forward to $X' \to B'$.

Definition 5.1 (Regular)**.** Let $u : C \to X$ be a holomorphic map from a compact smooth curve C. A point $x \in C$ will be called *special* (for u) when $du(x) = 0$ or $\#u^{-1}(u(x)) > 1$. The set $S \subseteq C$ of special points is finite provided $\dim_{\mathbb{C}} X \geq 2$, which we now assume. We now consider the deformation theory of the triple (C, S, u) subject to the constraint that the points S remain special with the same discrete data, meaning that all conditions $u(x) = u(x')$ and $(D^r u)(x) = 0$ which hold for u are preserved. We say that the map u is *regular* when this deformation problem is unobstructed.

To clarify the meaning of the point constraints ('remaining special with the same discrete data'), we note that the addition of the points S and their constraints adds to the (complex) index the quantity

$$|S| - \dim_{\mathbb{C}} X \cdot \Big(|S| - |u(S)| + \sum_{p\in S} \mathrm{ord}_p(du)\Big). \tag{5.1}$$

When $\dim_{\mathbb{C}} X \geq 3$, this quantity is < 0 unless $S = \varnothing$.

Regularity is also defined for curves in fibers of a family $X \to B$, meaning the deformation problem includes variation in the base parameter. If $X \to B$ is a pullback of $X' \to B'$, then regularity in $X \to B$ implies regularity of the pushforward to $X' \to B'$.

In contrast to curves and maps from curves, it is not so clear whether 1-cycles have a reasonable deformation theory. We will call a (possibly relative) 1-cycle $z = \sum_i m_i C_i$ *semi-regular* when the map $\bigsqcup_i \tilde{C}_i \to X$ is regular in the sense of Definition 5.1. Semi-regularity evidently measures properties of the semi-chart from §2.2. In particular, if $z \in \mathcal{Z}(X/B)$ is semi-regular, then the semi-chart through z is a smooth subvariety of $\mathcal{Z}(X/B)$ of dimension $\dim B + \sum_i \langle c_1(T_{X/B}), C_i\rangle$.

We denote by $\mathcal{Z}_{\mathrm{sr}} \subseteq \mathcal{Z}$ the locus of semi-regular cycles, and we call points in its interior $\mathcal{Z}^\circ_{\mathrm{sr}} \subseteq \mathcal{Z}_{\mathrm{sr}}$ *interior semi-regular*.

If $z \in \mathcal{Z}(X/B)$ is semi-regular, then it evidently remains semi-regular upon pushing forward to a family $X' \to B'$ of which $X \to B$ is a pullback. In contrast, interior semi-regularity need not be so preserved, which is a significant technical trip hazard!

Lemma 5.2. *$\mathcal{Z}(X/B)^\circ_{\mathrm{sF,sr}}$ has dimension $\leq \dim B + 2\langle c_1(TX), z\rangle$.*

Proof. The set of points of $\mathcal{Z}(X/B)^\circ_{\mathrm{sr}}$ whose associated semi-chart is an open embedding is dense by Lemma 2.1. At such a point, the dimension of $\mathcal{Z}(X/B)^\circ_{\mathrm{sr}}$ equals the dimension of the semi-chart. At semi-regular points $z = \sum_i m_i C_i$, this dimension is $\dim B + 2\sum_i \langle c_1(TX), C_i\rangle$. When z is semi-Fano (namely $\langle c_1(TX), C_i\rangle \geq 0$ for all i), this is bounded above by $\dim B + 2\sum_i m_i \langle c_1(TX), C_i\rangle = \dim B + 2\langle c_1(TX), z\rangle$. □

5.2 Interior semi-regular Grothendieck group

In §5.1, we introduced the notion of semi-regularity for points $z \in \mathcal{Z}(X/B)$ when B is smooth. Now for $B = \Delta^n$, we define a 1-cycle $z \in \mathcal{Z}(X/\Delta^n)$ to be semi-regular when it is semi-regular inside the minimal stratum of Δ^n containing it (i.e. consider variations in the base which are tangent to the stratum containing z). With this definition in hand, we will now define the *Grothendieck group of interior semi-regular 1-cycles*, denoted $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$.

For any injection $[i] \hookrightarrow [n]$ and any family $X \to \Delta^n$, we have $\mathcal{Z}(X \times_{\Delta^n} \Delta^i/\Delta^i)_{\mathrm{sr}} = \mathcal{Z}(X/\Delta^n)_{\mathrm{sr}} \times_{\Delta^n} \Delta^i$. There is thus a correspondence

$$\mathcal{Z}(X \times_{\Delta^n} \Delta^i/\Delta^i)^\circ_{\mathrm{sr}} \hookleftarrow \mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr}} \times_{\Delta^n} \Delta^i \to \mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr}} \tag{5.2}$$

in which the left arrow is an open embedding and the right arrow is a pullback of $\Delta^i \to \Delta^n$ (hence a closed embedding, but in particular proper). Beware that interior semi-regularity is *delicate*: the left arrow above need not be an isomorphism; this leads to some additional subtleties when working with $\mathcal{Z}^\circ_{\mathrm{sr}}$ instead of $\mathcal{Z}$.

Compactly supported cochains $C^*_c(\mathcal{Z}^\circ_{\mathrm{sr}})$ form a coefficient system on $\mathrm{Cpx}_{3,\bullet}$ since C^*_c is functorial under both proper maps and open embeddings (see Definition 3.3) as appear in (5.2) (by inspection, the composition of the correspondences (5.2) associated to $[\ell] \hookrightarrow [k] \hookrightarrow [n]$ is the correspondence associated the composition $[\ell] \hookrightarrow [n]$). The homology of the resulting chain complex $C_*(\mathrm{Cpx}_{3,\bullet}, C^*_c(\mathcal{Z})^\circ_{\mathrm{sr}})$ is the Grothendieck group of interior semi-regular 1-cycles $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$. There is an evident map of coefficient systems $C^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}) \to C^*_c(\mathcal{Z})$ on $\mathrm{Cpx}_{3,\bullet}$ (functoriality under open embeddings), inducing a map on Grothendieck groups $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3) \to H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$. In contrast to the case of $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ discussed in §3.1, note that due to the delicate nature of interior semi-regularity, we make no claim that $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$ is the relative compactly supported cohomology group of a 'total space' $\mathcal{Z}^\circ_{\mathrm{sr}} \to \mathrm{Cpx}_{3,\bullet}$.

5.3 Generation by local curves

Define a *geometric local curve element* in $H^*_c(\mathcal{Z}(\mathrm{Cpx}_3)^\circ_{\mathrm{sF,sr}})$ to be the Poincaré dual of a point of $\mathcal{Z}(X/B,k)^\circ_{\mathrm{sF,sr}}$ whose semi-chart is an open embedding of dimension $2k + \dim B$. Since the set of points whose semi-chart is an open embedding is dense (Lemma 2.1), the Poincaré dual of *any* dimension $2k + \dim B$ smooth point of $\mathcal{Z}(X/B,k)^\circ_{\mathrm{sF,sr}}$ is a geometric local curve element. The *topological type* of a point $z = \sum_i m_i C_i \in \mathcal{Z}(X/B, k)^\circ_{\mathrm{sF,sr}}$ is the collection of tuples (g_i, m_i, k_i) consisting of the genus g_i of C_i, the multiplicity m_i, and the chern number $k_i = \langle c_1(T_{X/B}), C_i \rangle \geq 0$ (since we are working with $\mathcal{Z}_{\mathrm{sF}}$); this is constant over any semi-chart. The dimension of the semi-chart at $z = \sum_i m_i C_i$ is $\dim B + 2\sum_i k_i \leq \dim B + 2\sum_i m_i k_i = \dim B + 2k$, so equality only occurs when $(m_i - 1)k_i = 0$ for all i (thus geometric local curve elements only come in such topological types). Every geometric local curve element has virtual dimension zero (it has chern number $\sum_i m_i k_i$ and cohomological degree $(\dim B + 2k) - \dim B = 2\sum_i m_i k_i$).

Proposition 5.3. *The group* $\bigoplus_{2k-i\leq 0} H^i_c(\mathcal{Z}(-, k)^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3)$ *is generated by geometric local curve elements.*

Proof. Represent a class in $H^i_c(\mathcal{Z}(-, k)^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3)$ by a finite semi-simplicial set $B_\bullet$, a map $B_\bullet \to \mathrm{Cpx}_{3,\bullet}$ (i.e. a family of complex threefolds $X \to B$), and a cycle $\lambda \in C_*(B_\bullet, C^*_c(\mathcal{Z}(X/-, k)^\circ_{\mathrm{sF,sr}}))$. The components $\lambda_\sigma \in C^{i+\dim\sigma}_c(\mathcal{Z}(X_\sigma/\sigma, k)^\circ_{\mathrm{sF,sr}})$ associated to top-dimensional simplices $\sigma \in B_\bullet$ are cocycles. Since $i + \dim\sigma \geq 2k + \dim\sigma = \dim \mathcal{Z}(X_\sigma/\sigma, k)^\circ_{\mathrm{sF,sr}}$ (Lemma 5.2), the cohomology class $[\lambda_\sigma] \in H^{i+\dim\sigma}_c(\mathcal{Z}(X_\sigma/\sigma, k)^\circ_{\mathrm{sF,sr}})$ is a linear combination of Poincaré duals of smooth points of dimension $2k + \dim\sigma$, namely geometric local curve elements (such smooth points lie over the interior of σ, hence their Poincaré duals are cycles in $C_*(B_\bullet, C^*_c(\mathcal{Z}(X/-, k)^\circ_{\mathrm{sF,sr}})))$. By subtracting these, we may reduce to the case that $[\lambda_\sigma] = 0$ in cohomology for top-dimensional simplices σ. Thus by adding a boundary to our cycle, we may reduce the dimension of $B_\bullet$. Iterating, we have reduced our class to zero by adding geometric local curve elements. □

Conjecture 5.4. *Every geometric local curve element coincides with the equivariant local curve element of the same topological type.*

Note that the almost complex version of Conjecture 5.4 is false by the analysis of Ionel–Parker [13, §7]: two geometric local curve elements of the same topological type need not coincide, rather they are related by a wall crossing formula. In the present complex analytic setting, we might expect any 'walls' to be of complex codimension one, hence real codimension two, meaning there is no wall crossing. This is nontrivial to make precise over real analytic bases B due to the delicate nature of interior semi-regularity.

5.4 Generic transversality

It is a standard result that for generic *almost* complex structures (on the target), all simple pseudo-holomorphic maps from closed Riemann surfaces are unobstructed (see [12, 27, 41] for precise statements). We now derive analogous results for complex structures. Since complex structures are much more rigid (for example, they have no nontrivial perturbations supported inside a small ball), these results are weaker than those in the almost complex setting: they only apply to a small neighborhood of a given compact 1-cycle.

We will describe complex structures and families thereof by gluing. To this end, let us introduce some notation. For complex manifolds U and V, denote by $\mathrm{An}(U, V)$ the space of analytic maps $U \to V$ with relatively compact image. If V admits an open embedding into some $\mathbb{C}^n$ (which will always be the case for us), then $\mathrm{An}(U, V)$ is a complex analytic Banach manifold, locally modelled on the space of n-tuples of bounded holomorphic functions on U. Given a complex manifold U, let $\mathcal{R}(U) = \mathrm{An}(U^-, U)$ (the space of 'regluings'), where $U^- \subseteq U$ denotes a(n unspecified) large relatively compact open subset. More formally, $\mathcal{R}(U)$ is a pro-object, namely the inverse system of all neighborhoods of the identity $\mathbf{1}_U \in \mathrm{An}(U^-, U)$ over all relatively compact open sets $U^- \subseteq U$. In all cases of interest to us, U will admit an open embedding into some $\mathbb{C}^n$, implying that $\mathcal{R}(U)$ is a (pro) complex analytic Banach manifold.

Definition 5.5. Given a complex manifold X with an open cover $X = A \cup B$, we may deform X by modifying the identification between open sets $A \supseteq A \cap B \subseteq B$. More formally, we consider the family $\tilde{X} \to \mathcal{R}(A \cap B)$ defined by taking the trivial families A and B over $\mathcal{R}(A \cap B)$ and gluing via the base parameter $A \times \mathcal{R}(A \cap B) \ni (a, \gamma) \sim (\gamma(a), \gamma) \in B \times \mathcal{R}(A \cap B)$. To make this construction precise, and to ensure the result is Hausdorff, we may fix compact sets $A^- \subseteq A$ and $B^- \subseteq B$ and glue $(A^- \sqcup B^-) \times \mathcal{R}(A \cap B)$ to obtain a proper map $\tilde{X}^- \to \mathcal{R}(A \cap B)$.

We will in fact only need a special case of the above construction, namely when the regluing takes place in a small neighborhood of a *divisor* (a closed complex submanifold of codimension one).

Definition 5.6 (Deforming complex structure near a divisor)**.** Let X be a complex manifold, and let $D \subseteq X$ be a smooth divisor. Regarding X as the gluing of $X \setminus D$ and $\mathrm{Nbd}\, D$ over their common intersection, Definition 5.5 provides a family $\tilde{X} \to \mathcal{R}(\mathrm{Nbd}\, D \setminus D)$. This family is smoothly trivial (analytic perturbations of the identity map on $\mathrm{Nbd}\, D \setminus D$ extend smoothly to $\mathrm{Nbd}\, D$), so a choice of smooth trivialization determines a family of complex structures on X parameterized by $\mathcal{R}(\mathrm{Nbd}\, D \setminus D)$. We will also denote this base space by $\mathcal{J}_D(X)$ (complex structures on X obtained by regluing near D). Of course, this isn't really a space but rather a family of spaces depending on choices of neighborhoods, etc. Sometimes we will need to fix a specific one, but we will do this at the relevant time.

The same construction applies to families $X \to B$ of complex manifolds. Given a *relative divisor* $D \subseteq X \to B$, meaning a divisor inside the total space which is submersive over B, we may consider the set $\mathcal{J}_D(X/B) = \mathcal{R}_B(\mathrm{Nbd}\, D \setminus D) = \bigcup_{b \in B} \mathcal{R}(\mathrm{Nbd}\, D_b \setminus D_b) \to B$, a holomorphic section α of which determines a 'vertical' (i.e. over B) regluing $X_\alpha \to B$ of X.

The tangent space to $\mathcal{R}(\mathrm{Nbd}\, D \setminus D)$ at the identity is the space of germs of holomorphic vector fields on $\mathrm{Nbd}\, D$ possibly singular along D. We denote this space by $H^0(D, TX(\infty D))$ (implicitly restricting the sheaf of holomorphic sections of TX over X to the divisor D). Such a vector field thus gives a first order deformation of the complex structure on X modulo gauge, that is an element of $H^1(X, TX)$. Concretely, this map $H^0(D, TX(\infty D)) \to H^1(X, TX)$ sends a holomorphic vector field v to (the Dolbeaut cohomology class represented by) $\bar{\partial}((1-\varphi) \cdot v)$ for a smooth function $\varphi : X \to [0, 1]$ supported inside an open set $U \subseteq X$ containing D such that v is defined on $U \setminus D$ and $\varphi \equiv 1$ in a neighborhood of D. Note that the choice of φ evidently does not matter since $\bar{\partial}((1-\varphi) \cdot v) - \bar{\partial}((1-\varphi') \cdot v) = \bar{\partial}((\varphi' - \varphi) \cdot v)$ is exact in the Dolbeat complex since $(\varphi' - \varphi) \cdot v$ is a smooth vector field on X (in contrast to $\varphi \cdot v$, which has singularities along D, or $(1-\varphi) \cdot v$, which is defined only on U). In terms of distributions, the map $H^0(D, TX(\infty D)) \to H^1(X, TX)$ is simply $v \mapsto \bar{\partial} v$, where $\bar{\partial} v$ is meant in the distributional sense and is supported on D since v is otherwise holomorphic (indeed, $\bar{\partial} v - \bar{\partial}((1-\varphi) \cdot v) = \bar{\partial}(\varphi \cdot v)$ is exact in the distributional Dolbeat complex).

We now identify the sort of maps which can be made transverse by deforming the complex structure near a divisor. Given a divisor $D \subseteq X$, a map $u : C \to X$ from a smooth proper curve C will be called *D-controlled* when $u^{-1}(D) \subseteq C$ is discrete and intersects every component of C. It is elementary to observe that being D-controlled is an open and condition on u. A cycle $z = \sum_i m_i C_i$ in X will be called D-controlled when $\bigsqcup_i \tilde{C}_i \to X$ is D-controlled.

Lemma 5.7. *The set of D-controlled cycles in $\mathcal{Z}(X/B)$ is open for any relative divisor $D \subseteq X \to B$.*

Proof. Suppose $z = \sum_i m_i C_i \in \mathcal{Z}(X/B)$ is D-controlled. Since C_i intersects D geometrically, the algebraic intersection number $C_i \cdot D$ is positive by positivity of intersection. If $z' = \sum_i m_i' C_i'$ is close to z, then every C_i' is homologous to a positive linear combination of some C_i's, hence also has positive algebraic intersection with D, thus *a fortiori* intersects it geometrically. □

For any family $X \to B$, the deformation complex of a map $u : C \to X_b$ maps to the deformation complex of the pair $(b, u : C \to X_b)$, with cokernel T_bB. This induces a map from T_bB to the obstruction space of the map u, whose cokernel is the obstruction space of the pair (b, u). Explicitly, this map is the Kodaira–Spencer map $T_bB \to H^1(X_b, TX_b)$ followed by restriction (e.g. of Dolbeaut representatives) from $H^1(X_b, TX_b)$ to $H^1(C, TX_b)$.

We now come to the key technical result underlying generic transversality, which says that the space of first order deformations (of a complex structure) associated to a divisor D by Definition 5.6 surjects onto the obstruction space of any D-controlled simple map u (via the map defined in the paragraph just above).

Lemma 5.8 (Enough first order deformations)**.** *Let $u : C \to X$ be a simple map from a smooth proper curve C to a complex manifold X, and let $D \subseteq X$ be a divisor. If u is D-controlled, then the map*

$$H^0(\mathrm{Nbd}(D \cap u(C)), TX(\infty D)) \to H^1(C, TX) \tag{5.3}$$

is surjective for every sufficienly small neighborhood of $D \cap u(C) \subseteq D$ inside X. In fact, it is surjective onto the obstruction space for the problem of deforming the map $u : C \to X$ subject to any finite number of point constraints (such as those appearing in the notion of 'regularity' in Definition 5.1).

Proof. Recall from above that the map in question sends a vector field v to $\bar\partial((1-\varphi)\cdot v)$ (note that the 'primitive' $(1-\varphi)\cdot v$ is not defined globally on X, so does not trivialize this element in cohomology), for any choice of cutoff function φ (supported near D and identically equal to 1 in a neighborhood of it). We will take φ to be (a smoothing of) the characteristic function of a small tubular $\mathrm{Nbd}(D)$ of D. Fix a local projection $\pi : X \to \mathbb{C}$ (defined near $u(C) \cap D$) with $D = \pi^{-1}(0)$, and let $\mathrm{Nbd}(D) = \pi^{-1}(D^2_\varepsilon)$. Now $\pi \circ u : C \to \mathbb{C}$ is a ramified cover near the origin, so the inverse image $u^{-1}(\partial \,\mathrm{Nbd}(D))$ is a union of circles, one going around each point of $u^{-1}(D)$. We will show that by varying v, we can make $u^*(\bar\partial((1-\varphi)\cdot v))$ approximate the delta function at any point of this union of circles $u^{-1}(\partial \,\mathrm{Nbd}(D))$. This implies the desired surjectivity of (5.3) since every nonzero element of $H^0(C, K_C \otimes T^*X) = H^1(C, TX)^*$

has nonzero restriction to $u^{-1}(\partial \operatorname{Nbd}(D))$ by holomorphicity and unique continuation (recall that $u^{-1}(\partial \operatorname{Nbd}(D))$ meets all components of C). Note that adding finitely many point constraints to the deformation problem of $u : C \to X$ (which leads to considering sections of $K_C \otimes T^*X$ which may have poles at said points) does not affect this argument, since we can always choose $u^{-1}(\partial \operatorname{Nbd}(D))$ to be disjoint from these finitely many special points (even if they happen to coincide with $u^{-1}(D)$).

It remains to prove that we can make $\bar{\partial}((1-\varphi)\cdot v)$ approximate a delta function at any point of $u^{-1}(\partial \operatorname{Nbd}(D))$. Fix local coordinates $X = \mathbb{C}_z \times \mathbb{C}^2_{x,y}$ near an (isolated, by hypothesis) intersection point $u(C) \cap D$ in which π is the z-coordinate, so $D = \{z = 0\} = 0 \times \mathbb{C}^2_{x,y}$ and $\operatorname{Nbd}(D) = D^2_\varepsilon \times \mathbb{C}^2_{x,y}$. Choose φ to be a smoothing of the characteristic function $H(\varepsilon - z\bar{z})$ of the ε-disk in $\mathbb{C}_z$, so that $\bar{\partial}(1-\varphi)$ is a smoothing of $\delta(\varepsilon - z\bar{z})z d\bar{z}$. Writing v in Laurent series expansion $v = \sum_k f_k(x,y) z^k \partial_z$, we calculate that $\bar{\partial}((1-\varphi)\cdot v) = \bar{\partial}(1-\varphi)\cdot v$ is a smoothing of $\delta(\varepsilon - z\bar{z}) \sum_k f_k(x,y) z^{k+1} d\bar{z}\partial_z$. Now the factor $\sum_k f_k(x,y) z^{k+1} d\bar{z}\partial_z$ can approximate any continuous function on $\partial D^2_\varepsilon \times \mathbb{C}^2_{x,y} = \partial \operatorname{Nbd}(D)$ which is holomorphic on fibers $e^{i\theta} \times \mathbb{C}^2_{x,y}$ (use approximation by Fourier polynomials in the ∂D^2_ε direction). The pullbacks of such functions to $u^{-1}(\partial \operatorname{Nbd}(D))$ are dense in continuous functions since u is simple and π is a ramified covering. □

Let us now explain how the existence of enough infinitesimal deformations (Lemma 5.8) implies various flavors of generic transversality. We say that (the complex structure on) X is *D-regular* when every D-controlled simple map is regular (Definition 5.1) and hence every D-controlled cycle is semi-regular.

Lemma 5.9 (Generic transversality)**.** *Fix a complex manifold X, a divisor $D \subseteq X$, and a finite set $A \subseteq D$. After possibly removing a closed subset of X contained in $D \setminus A$, generic elements of $\mathcal{J}_D(X)$ are D-regular.*

Proof. This is a typical argument based on Smale's infinite-dimensional Sard theorem [36].

We begin by fixing a precise space $\mathcal{J}_D(X)$ to consider. Let $D^2 \subseteq \mathbb{C}$ denote the unit disk. Fix coordinates $D^2 \times (D^2)^{n-1}$ on X near each point $a \in A$ with $a = (0,0)$ and $D = 0 \times (D^2)^{n-1}$, and *remove from X the part of D outside the interiors of the charts* $0 \times (D^2)^{n-1}$. We let $\mathcal{J}_D(X)$ consist of holomorphic sections f of the tangent bundle of $(D^2 \setminus 0) \times (D^2)^{n-1}$ with $\|f\|_2 < \varepsilon$ for some $\varepsilon > 0$, where the L^2-norm is weighted near $0 \times (D^2)^{n-1}$ so that meromorphic sections have finite norm (this space is most naturally identified with the Lie algebra of $\mathcal{R}((D^2 \setminus 0) \times (D^2)^{n-1})$, and is subsequently mapped to it via the exponential map). By smearing the Cauchy Integral Formula and appealing to Cauchy–Schwarz, we see that $\|f\|_\infty$ over any compact subset of the interior of $(D^2 \setminus 0) \times (D^2)^{n-1}$ is bounded linearly in ε. Thus for sufficiently small $\varepsilon > 0$, the reglued family (Definition 5.6) is defined over $\mathcal{J}_D(X)$. Using the L^2-norm here guarantees that the space $\mathcal{J}_D(X)$ is separable.

Now consider a compact smooth (not necessarily connected!) surface C and a smooth family of almost complex structures on C parameterized by a finite-dimensional smooth manifold $\mathcal{J}(C)$. Now $W^{k,2}(C, X)$ is a smooth Banach manifold for any integer $k \geq 2$ (note that $W^{k,2} \subseteq C^0$ for such k), whose product with $\mathcal{J}(C) \times \mathcal{J}_D(X)$ carries the smooth Banach bundle

$$\begin{array}{c} \mathcal{J}(C) \times W^{k,2}(C,X) \times_{W^{k-1,2}(C,X)} W^{k-1,2}(C,\overline{TC} \otimes_{\mathbb{C}} TX) \times \mathcal{J}_D(X) \\ {\scriptstyle\bar\partial}\big\uparrow\big\downarrow \\ \mathcal{J}(C) \times W^{k,2}(C,X) \times \mathcal{J}_D(X) \end{array} \tag{5.4}$$

with a section $\bar\partial$ measuring the failure of the map $C \to X$ to be holomorphic. Now the linearization (derivative) of $\bar\partial$ at a triple $(u : C \to X, j, J)$ is a map

$$W^{k,2}(C, u^*TX) \oplus T_j\mathcal{J}(C) \oplus T_J\mathcal{J}_D(X) \to W^{k-1,2}(C, \overline{TC} \otimes_{\mathbb{C}} u^*TX) \tag{5.5}$$

whose restriction to the first direct summand is the deformation complex of the map u. Lemma 5.8 guarantees that the restriction to $T_J\mathcal{J}_D(X)$ surjects onto the obstruction space $H^1(C,TX)$ if u is D-controlled. Thus $\bar\partial$ is transverse to zero at every D-controlled holomorphic triple (u, j, J) with u simple.

Now restrict to the clopen subset of $W^{k,2}(C,X)$ consisting of those maps whose restriction to every component of C has positive algebraic intersection with D (thus a holomorphic map is D-controlled iff it lies in this set). Over this clopen set, the section $\bar\partial$ is transverse to zero at simple, hence its zero set $\bar\partial^{-1}(0)_{\mathrm{simple}}$ (the open simple locus) is a smooth Banach manifold, and the projection map

$$\bar\partial^{-1}(0)_{\mathrm{simple}} \to \mathcal{J}_D(X) \tag{5.6}$$

is Fredholm by ellipticity of the deformation complex of the map u. Now Sard–Smale [36] implies that the fibers of this map over generic elements of $\mathcal{J}_D(X)$ are regular. We can cover all curves using countably many pairs $(C, \mathcal{J}(C))$, so we conclude that for generic elements of $\mathcal{J}_D(X)$, all D-controlled simple maps are unobstructed.

Now regularity is stronger than unobstructedness, since it involves a deformation problem with point constraints. To prove regularity of D-controlled simple maps with respect to generic elements of $\mathcal{J}_D(X)$, it suffices to apply the above argument to triples $(C, \mathcal{J}(C), \gamma)$ where γ is a finite set of point constraints (again, countably many such triples suffice to cover all possible situations). □

Lemma 5.10 (Generic transversality in a family)**.** *Fix a family of complex threefolds $X \to B$ over a finite semi-simplicial set B, a relative divisor $D \subseteq X \to B$, and a set $A \subseteq D$ whose map to B is proper with finite fibers. After subdividing B and removing a closed subset of X contained in $D \setminus A$, generic (simplex-wise) analytic sections of $\mathcal{J}_D(X/B) \to B$ are D-regular. More generally, for any semi-simplicial subset $B' \subseteq B$, generic analytic sections of $\mathcal{J}_D(X/B) \to B$ vanishing on B' are D-regular over $B \setminus B'$.*

Proof. The proof is parallel to that of Lemma 5.9. The main new task is to specify precisely the Banach space of sections of $\mathcal{J}_D(X/B) \to B$ we would like to consider.

Given a point $b \in B$, we may fix coordinates $D^2 \times (D^2)^{n-1}$ on X_b near each of the finitely many points $a \in A_b$, just as in the proof of Lemma 5.9. Furthermore, we may extend these to simplex-wise analytic coordinates $D^2 \times (D^2)^{n-1} \times M$ on X over a neighborhood M of $b \in B$ so that $D = 0 \times (D^2)^{n-1} \times M$ and $A \times_B M$ is contained in $0 \times (D^2_{1/2})^{n-1} \times M$ (this uses properness of $A \to B$). Fix a finite collection of points $b \in B$ and associated charts whose loci $0 \times (D^2_{1/2})^{n-1} \times M^\circ \subseteq X$ cover all of A. Subdivide B and shrink each chart so that $M \subseteq B$ is a finite collection of simplices (maintaining the property that the loci $0 \times (D^2_{1/2})^{n-1} \times M^\circ$ together cover A).

Now on each chart $D^2 \times (D^2)^{n-1} \times M$, we consider the space of complex/real analytic sections of $T_{X/B}$ over $(D^2 \setminus 0) \times (D^2)^{n-1} \times M$ which vanish over $\partial M = M \setminus M^\circ$ (and over B', if present). We consider the L^2-norm on this space which integrates over a small neighborhood inside the complexification $(D^2 \setminus 0) \times (D^2)^{n-1} \times M_{\mathbb{C}}$, with exponential weight near the puncture $0 \times (D^2)^{n-1} \times M_{\mathbb{C}}$ so that poles of arbitrary orders there are allowed.

Now we take our Banach space of (simplex-wise) analytic sections of $\mathcal{J}_D(X/B) \to B$ to be the direct sum of all these spaces. Any such sum of sections of $T_{X/B}$ can be exponentiated to reglue $X \setminus (D - \mathrm{Nbd}_D(A))$.

We can now follow the argument in the proof of Lemma 5.9 to show that generic elements of this Banach space of simplex-wise analytic sections of $\mathcal{J}_D(X/B) \to B$ are D-regular. $\square$

To get any mileage out of the above generic transversality results, we need a sufficiently rich collection of divisors. Given a 1-cycle z in a threefold X, it is trivial to observe that, after replacing X with a small neighborhood of z, there exists a divisor $D \subseteq X$ controlling z (namely, D is a union of transverse disks at a finite collection of smooth points on z). We now generalize this assertion to families, where a more subtle argument is required to keep the divisors disjoint.

Proposition 5.11 (Enough divisors)**.** *Let $X \to B$ be a family of complex threefolds over a finite semi-simplicial set, and let $K \subseteq \mathcal{Z}(X/B)$ be a compact analytic set whose projection map $K \to |B|$ is injective. After possibly removing a closed subset from X disjoint from K, there exists a finite collection of disjoint relative divisors $D_i \subseteq X \times_B U_i \to U_i$ ($U_i \subseteq |B|$ open) such that every $z \in K$ is D_i-controlled for some i.*

Proof. First, let us discuss how to construct (germs of) relative divisors $D \subseteq X \to B$ locally near a given point $x \in X$. Suppose x lies over the interior of a simplex $\sigma \subseteq B$. Define $D_\sigma \subseteq X_\sigma$ as the transverse zero set $D_\sigma = \pi_\sigma^{-1}(0)$ of a (germ of) holomorphic map $\pi_\sigma : (X_\sigma, x) \to (\mathbb{C}, 0)$ defined near x. To explain the term 'holomorphic' for π_σ, recall that $X_\sigma \to \sigma$ is the restriction of a given family $X_\sigma^{\mathbb{C}} \to \sigma^{\mathbb{C}} \cong \mathbb{C}^{\dim \sigma}$ over the complexification, so it makes sense to require that π_σ be the restriction to X_σ of a (necessarily unique) holomorphic function on $X_\sigma^{\mathbb{C}}$. For D_σ to be a *relative* divisor, we need it to be submersive over σ, which in terms of π_σ is the condition that $d\pi_\sigma | T_{X/B}$ is surjective. To extend D_σ to a neighborhood of x in the total space X, it suffices to extend π_σ to a (simplexwise) holomorphic map π (note that surjectivity of $d\pi | T_{X/B}$ is an open condition). Proceeding by induction on simplices, it suffices to address the question of extending (near the origin) an analytic function from $\partial \mathbb{R}^n_{\geq 0} \times \mathbb{C}^n$ to $\mathbb{R}^n_{\geq 0} \times \mathbb{C}^n$. This extension problem is solved by the standard formula $f(y_1, \ldots, y_n, z) = \sum_{\varnothing \neq S \subseteq \{1,\ldots,n\}} (-1)^{|S|-1} f(\{y_i\}_{i \notin S}, \{0\}_{i \in S}, z)$. Note that there is extra freedom to add any analytic function times $y_1 \cdots y_n$, which will be important below when we want to 'choose divisors generically'. We note that the resulting germ of relative divisor $D \subseteq X \to B$ can be promoted to a true (not germ) relative divisor over an open neighborhood of the image of x in B, by removing a suitable closed subset of X.

Given the local existence of relative divisors, compactness of K immediately produces a finite collection of relative divisors $D_i \subseteq X \times_B U_i \to U_i$ ($U_i \subseteq |B|$ open) such that every $z \in K$ is D_i-controlled for some i. These divisors, however, need not be disjoint. Note that it suffices to ensure that $D_i \cap D_j \cap z = \varnothing$ for all $z \in K$, as then $\bigcup_{i<j} D_i \cap D_j \subseteq X$ is closed and disjoint from K so we can simply remove it. To produce divisors with this property, we use an inductive argument, the key being that intersections $D \cap D' \cap z$ generically happen over a codimension two ($\dim(\mathcal{U}/\mathcal{Z}) - \operatorname{codim} D - \operatorname{codim} D' = -2$) subset of K.

Consider the following more general problem. In addition to the data of $X \to B$ and $K \subseteq \mathcal{Z}(X/B)$, fix a relative singular divisor $D^{\mathrm{prev}} \subseteq X \to B$ (by 'singular divisor', we simply mean a not necessarily disjoint union of not necessarily mutually transverse divisors) whose intersection with every cycle in K is discrete. We then ask for a finite collection of divisors D_i which together control all $z \in K$ and which are disjoint from each other and from D^{prev}. Our original problem is the special case $D^{\mathrm{prev}} = \varnothing$.

We now show how to reduce the problem for a given (D^{prev}, K) to that of another pair $(D^{\mathrm{prev}\prime}, K')$. Consider any choice of relative divisors D_i (not necessarily disjoint) controlling all $z \in K$. We claim that if the problem associated to

$$
\begin{multline}
(D^{\mathrm{prev}\prime}, K') = \Bigg(D^{\mathrm{prev}} \cup \bigcup_i D_i, \\
\pi_K \bigg(\Big(\mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K \Big) \cap \bigcup_i \Big(D_i \cap \big(D^{\mathrm{prev}} \cup \bigcup_{j \neq i} D_j \big) \Big) \bigg) \Bigg) \tag{5.7}
\end{multline}
$$

has a solution, then our original problem (D^{prev}, K) has a solution. Consider divisors D_i' solving the modified problem; they are disjoint from D^{prev} and from every D_i, and they control all $z \in K'$, hence all z in a neighborhood of this set (Lemma 5.7). For $z \in K \setminus \operatorname{Nbd} K'$, we use some D_i to control z. The intersections of these divisors with each other and with D^{prev} will be disjoint from $\mathcal{U}(X/B) \times_{\mathcal{Z}(X/B)} K$ by definition of K' hence can simply be removed from X.

We now claim that the more general problem has a solution. We argue by induction on $\dim K$, the case $K = \varnothing$ being trivial. For the inductive step, we simply note that in the construction above, the set K' will have at most complex codimension one inside K provided the D_i are chosen generically (this uses the fact that D^{prev} has only discrete intersection with cycles in K), which also ensures that $D^{\mathrm{prev}\prime}$ has only discrete intersection with cycles in K. □

We now use generic transversality to argue that the Grothendieck group of interior semi-regular 1-cycles coincides with that of all 1-cycles.

Proposition 5.12. *The map $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3) \to H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ is an isomorphism (and the same with $\mathcal{Z}_{\mathrm{sF}} \subseteq \mathcal{Z}$ in place of $\mathcal{Z}$).*

Proof. Fix a class in $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$, and let us show it is in the image of $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$. Represent our class by a finite semi-simplicial set $B_\bullet$, a family of threefolds $X \to B$ (i.e. a map $B_\bullet \to \mathrm{Cpx}_{3,\bullet}$ in the sense of §3.1), and a cycle $\lambda \in C_*(B_\bullet, C^*_c(\mathcal{Z}(X/-))$ consisting of cochains $\lambda_\sigma \in C^*_c(\mathcal{Z}(X_\sigma/\sigma))$ indexed by the simplices $\sigma \in B_\bullet$.

The pair (B, λ) is equivalent in $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ to its stabilization $(B \times \mathbb{R}^N, \pi_B^*\lambda \cup \pi_{\mathbb{R}^N}^*[0])$ (we will leave the choice of triangulation of $B \times \mathbb{R}^N$ implicit). It is also equivalent to the modified stabilization $(B \times \mathbb{R}^N, \lambda \cup (\pi_{\mathbb{R}^N} - i)^*[0])$ for any map $i : \mathcal{Z}(X/B) \to \mathbb{R}^N$. Now the product $\lambda \cup (\pi_{\mathbb{R}^N} - i)^*[0]$ is supported along the graph of i denoted $\Gamma_i \subseteq \mathcal{Z}(X/B) \times \mathbb{R}^N = \mathcal{Z}((X\times\mathbb{R}^N)/(B\times\mathbb{R}^N))$. Taking i to be a (simplexwise) analytic embedding, we may ensure that Γ_i is analytic and the projection $\Gamma_i \to \mathbb{R}^N$ (hence *a fortiori* the projection $\Gamma_i \to B\times\mathbb{R}^N$) is injective.

We have thus shown that (B, λ) is equivalent in $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$ to another pair, which we now rename as (B, λ), which comes with a compact analytic set $K \subseteq \mathcal{Z}(X/B)$ for which $K \to B$ is injective and with a lift of λ to a cycle

$$\lambda \in C_*(B_\bullet, C^*_K(\mathcal{Z}(X/-)) \tag{5.8}$$

where $C^*_Z(A) = C^*(A, A \setminus Z)$.

Now the fact that $K \to B$ is injective allows us to appeal to Proposition 5.11 to fix disjoint relative divisors $D_i \subseteq X \times_B U_i \to U_i$ for constructible closed sets $U_i \subseteq B$ (i.e. unions of closed simplices) whose restrictions to the respective $U_i^\circ = U_i \setminus \partial U_i$ together control K (this requires deleting a closed subset of X disjoint from K and subdividing B, neither of which change the class of (B, λ) in $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$).

Now finally we are in a situation which can be deformed to a semi-regular situation using generic transversality. Consider the family $\tilde{X} = X \times \mathbb{R} \to B \times \mathbb{R} = \tilde{B}$, and apply Lemma 5.10 (or rather its generalization to multiple divisors) to $\tilde{X} \to \tilde{B}$ to produce a collection Φ of analytic sections $\varphi_i : \tilde{B} \to \mathcal{J}_{\tilde{D}_i}(\tilde{X}/\tilde{B}) = \mathcal{J}_{D_i}(X/B)\times\mathbb{R}$ supported inside $U_i\times\mathbb{R}$ and vanishing on $B \times 0$ for which the resulting reglued family $\tilde{X}_\Phi \to \tilde{B}$ (Definition 5.6) is $(\bigsqcup_i D_i)$-regular over $B \times (\mathbb{R} \setminus 0)$.

Choose a local retraction $\rho : \mathcal{Z}(\tilde{X}/\tilde{B}) \to \mathcal{Z}(X/B)$ near K. Let $\tilde{B}_t$ denote the fiber of $\tilde{B}$ over $t \in \mathbb{R}$. For sufficiently small $t > 0$, the restriction $\rho_t : \mathcal{Z}(\tilde{X}_t, \tilde{B}_t) \to \mathcal{Z}(X/B)$ may be used to define a pullback cycle

$$\rho_t^*\lambda \in C_*(\tilde{B}_t, C^*_{\rho_t^{-1}(K)}(\mathcal{Z}(\tilde{X}_{\Phi,t}/-))) \tag{5.9}$$

which, by regularity of Φ, determines a class in $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$ for generic t. Now $\rho_t^*\lambda$ is homologous to λ by consideration of the pullback $\rho^*\lambda$ paired with the chain $[0, t]$, which is a chain in $C_*(\tilde{B}, C^*_{\rho^{-1}(K)}(\mathcal{Z}(\tilde{X}_\Phi/-)))$ with boundary $\rho_t^*\lambda - \lambda$. This shows surjectivity of the map $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3) \to H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$.

To prove injectivity of the map $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3) \to H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$, we just need a relative version of the same argument. Fix a class in $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$, and suppose it is sent to zero in $H^*_c(\mathcal{Z}/\mathrm{Cpx}_3)$. This means we have a finite semi-simplicial set B, a family of threefolds $X \to B$, a cycle $\lambda \in C_*(B, C^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}(X/-)))$, and a chain $\sigma \in C_*(B, C^*_c(\mathcal{Z}(X/-))$ such that $d\sigma = \lambda$. We would like to show that λ represents zero in $H^*_c(\mathcal{Z}^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$. By a mapping cylinder construction, we may assume that λ is supported on a subcomplex $B_0 \subseteq B$ over which σ vanishes.

The stabilization argument from above shows that we may fix wlog a compact analytic set $K \subseteq \mathcal{Z}(X/B)$ for which $K \to B$ is injective along with lifts of λ and σ to cochains supported inside K. Disjoint relative divisors $D_i \subseteq X \times_B U_i \to U_i$ as above exist again by Proposition 5.11.

Now choose Φ trivial over $B \times 0$ and $B_0 \times \mathbb{R}$ and $(\bigsqcup_i D_i)$-regular over $(B \setminus B_0) \times (\mathbb{R} \setminus 0)$. Choose retraction ρ which over $B_0 \times \mathbb{R}$ is simply projection to B_0 and over $(B \setminus B_0) \times \mathbb{R}$ maps inside $B \setminus B_0$. Now the pullback $\rho_t^*\lambda$ equals λ, and the pullback $\rho_t^*\sigma$ is interior semi-regular and satisfies $d(\rho_t^*\sigma) = \rho_t^*\lambda$. We have thus shown that λ represents zero in $H_c^*(\mathcal{Z}_{\mathrm{sr}}^\circ/\mathrm{Cpx}_3)$, and hence that $H_c^*(\mathcal{Z}_{\mathrm{sr}}^\circ/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}/\mathrm{Cpx}_3)$ is injective.

The same argument applies to $\mathcal{Z}_{\mathrm{sF}} \subseteq \mathcal{Z}$. ☐

Lemma 5.13. *$\rho_d(x_{g,m,k}) = 0$ in $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ for $k > 0$, $m > 0$, and $d > 1$.*

Proof. Combining Proposition 5.12 with Proposition 5.3, we see that $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ vanishes in negative virtual dimension. The virtual dimension of

$$\rho_d(x_{g,m,k}) \in H_c^{2km}(\mathcal{Z}(-, mk/d)_{\mathrm{sF}}/\mathrm{Cpx}_3) \tag{5.10}$$

is $2km/d - 2km$, which is negative for $k > 0$, $m > 0$, and $d > 1$. ☐

5.5 Filtration

We now compute generators for the virtual dimension ≤ 0 part of $H_c^*(\mathcal{Z}_{\mathrm{sF,sr}}^\circ/\mathrm{Cpx}_3)$ using a filtration argument.

The first step is to argue that to understand the virtual dimension ≤ 0 part of the group $H_c^*(\mathcal{Z}_{\mathrm{sF,sr}}^\circ/\mathrm{Cpx}_3)$, we can replace $\mathcal{Z}_{\mathrm{sF,sr}}^\circ$ with the interior of the subset consisting of cycles with *smooth support* (say $z = \sum_i m_i C_i$ has smooth support when $\bigcup_i C_i \subseteq X$ is smooth, equivalently when $\bigsqcup_i \tilde{C}_i \to X$ is a smooth embedding). The point will be that having non-smooth support is a codimension two phenomenon.

We indicate cycles with smooth support using the subscript $\mathcal{Z}_{\mathrm{sm}}$.

Lemma 5.14. *$\mathcal{Z}_{\mathrm{sm}} \subseteq \mathcal{Z}$ is an analytic constructible subset.*

Proof. The set of points $p \in \mathcal{U}$ in the universal family at which the fiber of $\mathcal{U} \to \mathcal{Z}$ is smooth is an analytic constructible subset. Since $\mathcal{U} \to \mathcal{Z}$ is proper, the image of a constructible subset is constructible. ☐

The Grothendieck group $H_c^*(\mathcal{Z}_{\mathrm{sr,sm}}^\circ/\mathrm{Cpx}_3)$ is defined just like $H_c^*(\mathcal{Z}_{\mathrm{sr}}^\circ/\mathrm{Cpx}_3)$ in §5.2. Functoriality of C_c^* under open embeddings gives a homomorphism $H_c^*(\mathcal{Z}_{\mathrm{sr,sm}}^\circ/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}_{\mathrm{sr}}^\circ/\mathrm{Cpx}_3)$.

$$\begin{array}{ccccc}
\mathcal{Z}(X \times_{\Delta^n} \Delta^k/\Delta^k)_{\mathrm{sr,sm}}^\circ & \hookleftarrow & \mathcal{Z}(X/\Delta^n)_{\mathrm{sr,sm}}^\circ \times_{\Delta^n} \Delta^k & \twoheadrightarrow & \mathcal{Z}(X/\Delta^n)_{\mathrm{sr,sm}}^\circ \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z}(X \times_{\Delta^n} \Delta^k/\Delta^k)_{\mathrm{sr}}^\circ & \hookleftarrow & \mathcal{Z}(X/\Delta^n)_{\mathrm{sr}}^\circ \times_{\Delta^n} \Delta^k & \twoheadrightarrow & \mathcal{Z}(X/\Delta^n)_{\mathrm{sr}}^\circ
\end{array} \tag{5.11}$$

Here we use $\hookrightarrow$ to indicate and open embedding and $\twoheadrightarrow$ to indicate a proper map. Note that the right square is a fiber product, so it is commutative in the category of correspondences on which C_c^* is a functor (see Definition 3.3) hence remains commutative upon applying C_c^*.

The homomorphism $H_c^*(\mathcal{Z}_{\mathrm{sr,sm}}^\circ/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}_{\mathrm{sr}}^\circ/\mathrm{Cpx}_3)$ fits into a long exact sequence with third term $H_c^*((\mathcal{Z}_{\mathrm{sr}}^\circ \setminus \mathcal{Z}_{\mathrm{sr,sm}}^\circ)/\mathrm{Cpx}_3)$, defined by the following diagram (rotated by $\pi/2$

for typesetting purposes).

$$
\begin{array}{ccc}
\mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr}} & \twoheadleftarrow & \mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr}} \setminus \mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr,sm}} \\
\uparrow & & \uparrow \\
\mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr}} \times_{\Delta^n} \Delta^k & \twoheadleftarrow & \big(\mathcal{Z}(X/\Delta^n)^\circ_{\mathrm{sr}} \times_{\Delta^n} \Delta^k\big) \setminus \mathcal{Z}(X \times_{\Delta^n} \Delta^k/\Delta^k)^\circ_{\mathrm{sr,sm}} \\
\downarrow & & \downarrow \\
\mathcal{Z}(X \times_{\Delta^n} \Delta^k/\Delta^k)^\circ_{\mathrm{sr}} & \twoheadleftarrow & \mathcal{Z}(X \times_{\Delta^n} \Delta^k/\Delta^k)^\circ_{\mathrm{sr}} \setminus \mathcal{Z}(X \times_{\Delta^n} \Delta^k/\Delta^k)^\circ_{\mathrm{sr,sm}}
\end{array}
\tag{5.12}
$$

Note that the bottom square is a fiber square, hence remains commutative upon applying C^*_c. Also note that the right column correspondence respects compositions $[\ell] \hookrightarrow [k] \hookrightarrow [n]$, ensuring that $C^*_c(\mathcal{Z}^\circ_{\mathrm{sr}} \setminus \mathcal{Z}^\circ_{\mathrm{sr,sm}})$ is indeed a coefficient system on $\mathrm{Cpx}_{3,\bullet}$.

The same considerations apply to $\mathcal{Z}_{\mathrm{sF}} \subseteq \mathcal{Z}$ in place of $\mathcal{Z}$.

Lemma 5.15. *The map*

$$H^*_c(\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}}/\mathrm{Cpx}_3) \to H^*_c(\mathcal{Z}^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3) \tag{5.13}$$

is an isomorphism in virtual dimension ≤ 0 *and surjective in virtual dimension* 1.

Proof. In view of the aforementioned long exact sequence, it suffices (and is in fact equivalent) to show that $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr}} \setminus \mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})/\mathrm{Cpx}_3)$ is supported in virtual dimension ≥ 2.

While $\mathcal{Z}(X/B,k)^\circ_{\mathrm{sF,sr}}$ has dimension $\leq 2k + \dim B$ by Lemma 5.2, the same argument shows that the complement of its subset $\mathcal{Z}(X/B,k)^\circ_{\mathrm{sF,sr,sm}}$ has dimension at most this quantity minus two, since singularities impose codimension two constraints, by the definition of regularity (Definition 5.1). It follows that $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr}} \setminus \mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})/\mathrm{Cpx}_3)$ is supported in virtual dimension ≥ 2. □

Having restricted to smooth cycles, we may now consider the following filtration.

Definition 5.16 (Multiplicity filtration)**.** Let $\mathbf{M} = \bigsqcup_{n\geq 0} \mathbb{Z}^n_{\geq 1}/S_n$ be the set of finite multi-sets of positive integers. There is a map $\mathcal{Z} \to \mathbf{M}$ associating to each cycle $z = \sum_i m_i C_i$ the multi-set $\mathbf{m}$ of multiplicities m_i. Partially order $\mathbf{M}$ by declaring that $\mathbf{m}' \leq \mathbf{m}$ whenever $\mathbf{m}$ may be obtained from $\mathbf{m}'$ by grouping together the multiplicities and replacing each group with some positive integer linear combination thereof. The map $\mathcal{Z} \to \mathbf{M}$ is not in general upper semi-continuous, however it is so at every point $\sum_i m_i C_i$ with all C_i disjoint. In particular, it is upper semi-continuous on $\mathcal{Z}_{\mathrm{sm}}$. Thus the loci

$$(\mathcal{Z}(-)_{\mathrm{sm}})_{\leq \mathbf{m}} \subseteq \mathcal{Z}(-)_{\mathrm{sm}} \tag{5.14}$$

are open.

We thus have groups $H^*_c((\mathcal{Z}^\circ_{\mathrm{sr,sm}})_{\leq \mathbf{m}}/\mathrm{Cpx}_3)$ (and similarly with $< \mathbf{m}$ in place of $\leq \mathbf{m}$). The tautological map

$$H^*_c((\mathcal{Z}^\circ_{\mathrm{sr,sm}})_{< \mathbf{m}}/\mathrm{Cpx}_3) \to H^*_c((\mathcal{Z}^\circ_{\mathrm{sr,sm}})_{\leq \mathbf{m}}/\mathrm{Cpx}_3) \tag{5.15}$$

fits into a long exact sequence with third term $H^*_c((\mathcal{Z}^\circ_{\mathrm{sr,sm}})_{= \mathbf{m}}/\mathrm{Cpx}_3)$.

Proposition 5.17. *The virtual dimension ≤ 0 part of $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ is generated by Poincaré duals of points in $(\mathcal{Z}(X/B)^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}$ (smooth B) whose combinatorial type (a multi-set of triples (g,m,k) of genus, multiplicity, and non-negative chern number, with total multiplicity $\mathbf{m}$) is* non-deficient *(meaning each triple (g,m,k) satisfies $(m-1)k=0$). The class in $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ of such a Poincaré dual depends only on its combinatorial type.*

Proof. The space $(\mathcal{Z}(X/B)^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}$ consists of cycles $\sum_i m_iC_i$ for $(m_1,\ldots,m_n)=\mathbf{m}$ and $\bigsqcup_i C_i\to X$ a smooth embedded curve unobstructed relative B. It is thus a manifold of dimension $\dim B+2\sum_i k_i$, hence has cohomology up to this degree. The map to the Grothendieck group reduces cohomological degree by $\dim B$, and the chern number is $\sum_i k_im_i$. Hence it contributes cohomology in virtual dimension $\geq 2\sum_i(m_i-1)k_i\geq 0$. Thus classes of virtual dimension ≤ 0 only exist when $(m_i-1)k_i=0$ for all i (i.e. non-deficient), and in this case they are generated by Poincaré duals of points.

To show that the class in $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ represented by the Poincaré dual of a non-deficient point of $(\mathcal{Z}(X/B)^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}$ depends only on the combinatorial type of this point, we would like to use a deformation argument. Smooth cycles of multiplicity $\mathbf{m}$ in threefolds are certainly classified up to deformation by their combinatorial type (deform to the normal cone and appeal to Remark 4.1). However, this deformation need not be everywhere semi-regular (let alone everywhere interior semi-regular smooth). To overcome this difficulty, what we will do is associate a class in $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ to *every* non-deficient multiplicity $\mathbf{m}$ smooth semi-Fano cycle z (not necessarily semi-regular) in a threefold X_0 (regarded as a germ around z), and show that this class is invariant under deformation of (X_0,z). The construction appears somewhat redundant and circuitious, but it is the simplest we have been able to come up with.

As a first step, let us associate a class in $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ to every semi-regular smooth point $z\in\mathcal{Z}(X/B)_{\mathrm{sF}}$ (real analytic base B) of non-deficient multiplicity $\mathbf{m}$ (invariant under deformation of $X\to B$ and z). Fix a relative divisor $D\subseteq X\to B$ controlling z (possibly replacing X with a neighborhood of z), and fix an analytic section $\Phi:B\times\mathbb{R}\to\mathcal{J}_D(X/B)$ vanishing on $B\times 0$ which is D-regular over $B\times(\mathbb{R}\setminus 0)$ (which exists by generic transversality, namely the argument of Lemma 5.10). Since $z\in\mathcal{Z}(X/B)$ is semi-regular, its semi-chart in $\mathcal{Z}((X\times\mathbb{R})_\Phi/B\times\mathbb{R})$ is smooth and submersive over $\mathbb{R}$ and has dimension $\dim(B\times\mathbb{R})+2\sum_i k_i=\dim(B\times\mathbb{R})+2\sum_i m_ik_i$. Over $\mathbb{R}\setminus 0$, this semi-chart is (near z) contained within the interior semi-regular locus since here Φ is D-regular. The closure of the non-smooth locus thus intersects it in (real) codimension at least two (see the dimension count in Lemma 5.15). Thus 'the Poincaré dual of a point of the semi-chart not in the closure of the non-smooth locus' is a well defined class in

$$H_c^{\dim(B\times\mathbb{R})+2\sum_i m_ik_i}((\mathcal{Z}((X\times\mathbb{R})_\Phi/B\times\mathbb{R})^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}),\tag{5.16}$$

hence its image in $H_c^{2\sum_i m_ik_i}((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ is also well defined. We claim that this class is independent of the choice of Φ. Given Φ_1 and Φ_2, we may choose analytic $\overline{\Phi}$ on $B\times\mathbb{R}\times\mathbb{R}$ whose restrictions to $B\times\mathbb{R}\times 0$ and $B\times 0\times\mathbb{R}$ are Φ_1 and Φ_2 and which is D-regular over $B\times(\mathbb{R}\setminus 0)\times(\mathbb{R}\setminus 0)$ (Lemma 5.10). Now an interior smooth point of the relative cycle space over $B\times(\mathbb{R}\setminus 0)\times 0$ need not remain interior smooth inside the relative cycle space over $B\times\mathbb{R}\times\mathbb{R}$, since it could be the limit of a sequence of non-smooth points over $B\times(\mathbb{R}\setminus 0)\times(\mathbb{R}\setminus 0)$. However, we have added only one more dimension, so the closure of the non-smooth locus in $B\times(\mathbb{R}\setminus 0)\times(\mathbb{R}\setminus 0)$ still has strictly smaller dimension than the semi-chart in $B\times(\mathbb{R}\setminus 0)\times 0$, so a generic point of the semi-chart of z inside $B\times(\mathbb{R}\setminus 0)\times 0$ will be interior smooth in $B\times(\mathbb{R}\setminus 0)\times(\mathbb{R}\setminus 0)$, hence we have the desired identity between Poincaré duals. Independence of D is simple: if $D'\supseteq D$ is a larger divisor (add disjoint components) then we can extend Φ by zero on $D'\setminus D$ to conclude that the invariants associated to D and D' are the same. Now given D_1 and D_2 (not necessarily disjoint), there exists D_3 controlling z which is disjoint from both D_1 and D_2, which allows us to relate the invariants associated to D_1, $D_1\sqcup D_3$, D_3, $D_3\sqcup D_2$, and D_2.

We have thus associated a class in $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ to every semi-regular smooth point $z\in\mathcal{Z}(X/B)_{\mathrm{sF}}$ (real analytic base B) of non-deficient multiplicity $\mathbf{m}$. We now claim that if $B\subseteq B'$ is a submanifold and X is the restriction of $X'\to B'$, then the class associated to such $z\in\mathcal{Z}(X/B)_{\mathrm{sF}}$ coincides with the class associated to its image in $\mathcal{Z}(X'/B')_{\mathrm{sF}}$ (note that this implies deformation invariance of the class associated to $X\to B$ and z). It suffices to consider the case that the inclusion $B\subseteq B'$ is codimension one. Now we choose D' for $X'\to B'$ (let $D\subseteq X$ be its restriction) and we choose Φ on $B\times\mathbb{R}$ and extend to Φ' on $B'\times\mathbb{R}$. Since the codimension of B inside B' equals one, the argument from the previous paragraph (the closure of the non-smooth locus over $B'\times(\mathbb{R}\setminus 0)$ having codimension one inside the semi-chart of z inside $B\times\mathbb{R}$) gives the desired result.

Now finally let us associate a class in $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ to a non-deficient multiplicity $\mathbf{m}$ smooth semi-Fano cycle z (not necessarily semi-regular) in a threefold X_0 (regarded as a germ around z). In view of the above constructions, it suffices to show that we can exhibit X_0 as a fiber of a family $X\to B$ for which the image of z in $\mathcal{Z}(X/B)$ is semi-regular, and to show that any two such families can be related by a zig-zag of inclusions. Given a family $X\to B$ in which $z\in\mathcal{Z}(X_0)\subseteq\mathcal{Z}(X/B)$ is semi-regular, we can choose a relative divisor $D\subseteq X\to B$ controlling z and a finite-dimensional vector space $V\to H^0(D,T_{X/B}(\infty D))$ which surjective onto the cokernel of the relevant linearized operator (Lemma 5.8) to ensure that the image of z in the relative cycle space of $(X_0)_V\to V$ (the reguling via the exponential of V as in Definition 5.6) is semi-regular. Now the families $X\to B$, $X_V\to B\times V$, and $(X_0)_V\to V$ are related by inclusions, hence induce the same element of $H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$. Any two such finite-dimensional vector spaces mapping to $H^0(D_0,TX_0(\infty D_0))$ can be related to their direct sum, so the resulting invariant depends at most on the divisor $D_0\subseteq X_0$ controlling z. Independence of the divisor may be argued as earlier. □

Lemma 5.15 and Proposition 5.12 show that the maps

$$H^*_c(\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}}/\mathrm{Cpx}_3)\to H^*_c(\mathcal{Z}^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3)\to H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3) \tag{5.17}$$

are bijective in virtual dimension ≤ 0 and bijective, respectively. The equivariant local curve elements $x_{g,m,k}\in H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ for $k\geq 0$ have degree zero, hence lift uniquely to $H^*_c(\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}}/\mathrm{Cpx}_3)$. We now follow the proofs of Proposition 5.12 and Lemma 5.15 to obtain an explicit description of these lifts and, moreover, define canonical lifts $\tilde{x}_{g,m,k}\in H^*_c((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\leq(m)}/\mathrm{Cpx}_3)$.

Represent $x_{g,m,k}$ via its definition (4.11), and use (the trace of) a fiber of $E\to C$ as relative divisor D (which controls all cycles in E). Use generic transversality Lemma 5.10 to produce a piecewise real analytic section $\Phi : \mathbb{C}P^N \times \mathbb{R} \to \mathcal{J}_D(((E \times (\mathbb{C}^{N+1} - 0))/\mathbb{C}^\times)/\mathbb{C}P^N)$ which is trivial over $\mathbb{C}P^N \times 0$ and D-regular over $\mathbb{C}P^N \times (\mathbb{R} \setminus 0)$. Pulling back the cocycle (4.11) under a local retraction from the relative cycle space over $\mathbb{C}P^N \times \mathbb{R}$ to that over $\mathbb{C}P^N \times 0$ and restricting to that over $\mathbb{C}P^n \times t$ for some generic small $t > 0$ defines a class in

$$H_c^{2n}\Big(\mathcal{Z}\Big(\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\Big)_{\Phi_t} \Big/ \mathbb{C}P^N, m\Big)^\circ_{\mathrm{sF,sr}}\Big) \tag{5.18}$$

whose image in $H_c^{2km}(\mathcal{Z}(-, km)^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3)$ lifts $x_{g,m,k}$. This element of $H_c^{2km}(\mathcal{Z}(-, km)^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3)$ is unique by Proposition 5.12; concretely, it can be seen to be independent of the choice of fiber D and section Φ by considering sections Φ'' over $\mathbb{C}P^N \times \mathbb{R}^2$. Now by the long exact sequence and dimension count in Lemma 5.15, the class in (5.18) lifts uniquely to

$$H_c^{2n}\Big(\mathcal{Z}\Big(\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\Big)_{\Phi_t} \Big/ \mathbb{C}P^N, m\Big)^\circ_{\mathrm{sF,sr,sm}}\Big), \tag{5.19}$$

which maps to $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le(m)}/\mathrm{Cpx}_3)$ since all cycles on E of total degree m have multiplicity tuple $\le (m)$. This defines the lift $\tilde{x}_{g,m,k} \in H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le(m)}/\mathrm{Cpx}_3)$, which is well defined for the same reason as above (consider sections over $\mathbb{C}P^N \times \mathbb{R}^2$).

Lemma 5.18. *For $k \ge 0$ and $(m-1)k = 0$, the lift $\tilde{x}_{g,m,k} \in H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le(m)}/\mathrm{Cpx}_3)$ maps to the generator of $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=m}/\mathrm{Cpx}_3)$ from Proposition 5.17 associated to the topological type (g, m, k).*

Proof. Recall that $x_{g,m,k} \in H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ (4.11) is given by

$$\prod_{i=1}^n r_i^{-1} f_i^* \tau_{\mathcal{L}^{\otimes r_i}} \in H_c^{2n}\Big(\mathcal{Z}\Big(\frac{E \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0} \Big/ \mathbb{C}P^N, m\Big)\Big) \to H_c^{2km}(\mathcal{Z}/\mathrm{Cpx}_3) \tag{5.20}$$

where $n = N + km$. The lift $\tilde{x}_{g,m,k} \in H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le(m)}/\mathrm{Cpx}_3)$ is defined by perturbation as detailed just above.

Now the cycles on E of degree m all have multiplicity $\le (m)$, and the multiplicity $= (m)$ locus inside $\mathcal{Z}(E, m)$ is canonically identified with $\mathcal{Z}(E, 1) = H^0(C, E)$. Let us take $E = L \oplus L'$ for generic line bundles L and L' of degrees $g - 1$ and $g - 1 + k$, respectively, which ensures that $h^0(C, L) = 0$, $h^0(C, L') = k$, and $h^1(C, L) = h^1(C, L') = 0$. In particular, the multiplicity m locus inside $\mathcal{Z}(E, m)$ is semi-regular and smooth.

Now let us consider the restriction of the cocycle (5.20) to the multiplicity $= (m)$ locus, which is thus a class in

$$H_c^{2n}\Big(\frac{H^0(C, E) \times (\mathbb{C}^{N+1} - 0)}{\mathbb{C} - 0}\Big). \tag{5.21}$$

Note that $(m-1)k = 0$ implies $mk = k$, so we have $n = N + k$. The restriction of a degree r_i function $f_i : \mathcal{Z}(E, m) \to \mathbb{C}$ of the form (4.10) to the multiplicity m locus identified as $\mathcal{Z}(E, 1)$ has the same form (4.10) of the same degree r_i. The restriction of (5.20) to the multiplicity $= (m)$ locus is thus a cocycle of precisely the same form, just with m replaced with 1. The argument used to show well definedness of ${}_\ell x_{g,m,k}$ shows that this class in $H^{2k}_{cS^1,c}(H^0(C, E))$ is independent of the choice of functions f_i. Taking k linear functions f_i which together define an isomorphism $\mathcal{Z}(E,1) = H^0(C,E) \xrightarrow{\sim} \mathbb{C}^k$, we see that this restricted class in (5.21) is the Poincaré dual of a point.

Now we are interested in the image of the lift $\tilde{x}_{g,m,k}$ in $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=m}/\mathrm{Cpx}_3)$. The lift $\tilde{x}_{g,m,k}$ is defined by perturbing the cocycle representing $x_{g,m,k}$. Since the multiplicity m locus is semi-regular and smooth, it remains so after the perturbation, and the point class remains the point class. It may not be *interior* smooth or *interior* semi-regular, but the loci where these fail are codimension two, and the point class lifts uniquely to the point class. □

A product of lifts $\tilde{x}_{g_i,m_i,k_i} \in H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le(m_i)}/\mathrm{Cpx}_3)$ is an element of $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le(m_i)_i}/\mathrm{Cpx}_3)$. Lemma 5.18 implies that its image in $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=(m_i)_i}/\mathrm{Cpx}_3)$ is the generator associated to the topological type of the set of triples $(g_i,m_i,k_i)_i$ (it is evident from the definition that products of generators from Proposition 5.17 are again such generators, with topological type the disjoint union of the topological types of the factors).

Corollary 5.19. *The group $H_c^*(\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}}/\mathrm{Cpx}_3)$ is generated in virtual dimension ≤ 0 by monomials in the equivariant local curve elements $x_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$.*

Proof. By a direct limit argument, it suffices to show that $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le\mathbf{m}}/\mathrm{Cpx}_3)$ is generated in virtual dimension ≤ 0 by monomials in the $\tilde{x}_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$ with multiplicity tuple $\le \mathbf{m}$. Now we prove this statement by induction.

Every element of $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{<\mathbf{m}}/\mathrm{Cpx}_3)$ is in the image of $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le\mathbf{m}'}/\mathrm{Cpx}_3)$ for some $\mathbf{m}' < \mathbf{m}$, so the induction hypothesis implies that $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{<\mathbf{m}}/\mathrm{Cpx}_3)$ is generated in virtual dimension ≤ 0 by monomials in the $\tilde{x}_{g,m,k}$ for $k \ge 0$ and $(m-1)k = 0$ with multiplicity tuple $< \mathbf{m}$. The monomials with multiplicity tuple $= \mathbf{m}$ generate $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3)$ by Lemma 5.18. Now appealing to the long exact sequence (5.15), we conclude the desired generation statement for $H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{\le\mathbf{m}}/\mathrm{Cpx}_3)$. □

Theorem 5.20. *The group $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is generated as a ring in virtual dimension ≤ 0 by the equivariant local curve elements $x_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$.*

Proof. The map $H_c^*(\mathcal{Z}^\circ_{\mathrm{sF,sr}}/\mathrm{Cpx}_3) \to H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is an isomorphism by Proposition 5.12. The map $H_c^*(\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}}/\mathrm{Cpx}_3) \to H_c^*((\mathcal{Z})^\circ_{\mathrm{sr}}/\mathrm{Cpx}_3)$ is an isomorphism in virtual dimension ≤ 0 by Lemma 5.15. By Corollary 5.19, the group $H_c^*(\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}}/\mathrm{Cpx}_3)$ is generated by monomials in equivariant local curve elements $x_{g,m,k}$ with $k \ge 0$ and $(m-1)k = 0$. □

The above argument is very close to giving a full proof of Theorem 1.1 (free generation by local curve elements) rather than just generation (Theorem 5.20). The missing ingredient is a proof that the connecting map

$$H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}/\mathrm{Cpx}_3) \xrightarrow{+1} H_c^*((\mathcal{Z}^\circ_{\mathrm{sF,sr,sm}})_{<\mathbf{m}}/\mathrm{Cpx}_3) \tag{5.22}$$

vanishes in virtual dimension 1 mapping to virtual dimension 0. This assertion appears reasonable over complex analytic bases (a generic path in $(\mathcal{Z}(X/B)^\circ_{\mathrm{sF,sr,sm}})_{=\mathbf{m}}$ would avoid the closure of $(\mathcal{Z}(X/B)^\circ_{\mathrm{sF,sr,sm}})_{<\mathbf{m}}$ since this subset has codimension two), but becomes less clear once we consider real simplices. Instead, we will prove the injectivity part of Theorem 1.1 using an algebraic argument in the next section.

Proof of Theorem 1.6. By Corollary 4.7, the ring homomorphisms $(-iu)^k\mathrm{GW}$ and $(-q)^{-k/2}\mathrm{PT}$ satisfy the MNOP correspondence when evaluated on all local curve elements $x_{g,m,k}$. These local curve elements generate $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ in virtual dimension zero by Theorem 5.20, so they satisfy the MNOP correspondence on all of $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$. We may thus evaluate on the element $(X, \beta; \gamma_1, \ldots, \gamma_r)$ (see §3.4) to obtained the desired result. □

6 Algebraic constraints

We now use the bi-algebra structure on $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)/\mathrm{tors}$ and the nontriviality of certain Gromov–Witten invariants to show that the sub-algebra generated by equivariant local curve elements $x_{g,m,k}$ for $k \geq 0$ and $(m-1)k = 0$ is free.

Consider the free polynomial ring $R = \mathbb{Z}[x_{g,m,k}]$ on formal variables $x_{g,m,k}$ indexed by integers $g \geq 0$, $m \geq 0$, and $k \geq 0$, satisfying $(m-1)k = 0$, modulo the relation that $x_{g,0,k} = 1$. Equip R with the co-unit and co-multiplication maps given by

$$\eta : R \to \mathbb{Z} \qquad\qquad \Delta : R \to R \otimes R \tag{6.1}$$

$$x_{g,m,k} \mapsto 0 \quad \text{for } m > 0 \qquad\qquad x_{g,m,k} \mapsto \sum_{\substack{a+b=m \\ a,b \geq 0}} x_{g,a,k} \otimes x_{g,b,k} \tag{6.2}$$

on generators and extended to be algebra maps. This makes R into a commutative and co-commutative bi-algebra. Sending $x_{g,m,k} \in R$ to the equivariant local curve element $x_{g,m,k} \in H_c^{2km}(\mathcal{Z}(-, mk)_{\mathrm{sF}}/\mathrm{Cpx}_3)$ defines a ring homomorphism $R \to H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ and a bi-algebra homomorphism $R \to H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)/\mathrm{tors}$ by Lemma 4.9 and the fact that $H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ vanishes in negative virtual dimension (note the virtual dimension of ${}_\ell x_{g,m,k}$ is -2ℓ) by Propositions 5.12 and 5.3.

Let $\rho_d : R \to R$ ($d \geq 1$) be given on generators by

$$\rho_d(x_{g,m,k}) = \begin{cases} x_{g,m/d,k} & m \text{ divisible by } d \text{ and } k = 0 \text{ or } d = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

and extended multiplicatively. This ρ_d is a map of bi-algebras (commutes with Δ and η) by inspection. The map $R \to H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)$ is compatible with the operations ρ_d by Lemma 4.8 and Lemma 5.13.

We now wish to analyze the kernel $A \subseteq R$ of the map $R \to H_c^*(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)/\mathrm{tors}$. Compatibility of this map with ρ_d implies that $\rho_d(A) \subseteq A$. Compatibility with Δ implies that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$ (at least rationally). Our goal is to prove that these constraints, along with a simple Gromov–Witten invariant calculation, forces $A = 0$.

The *weight* of a monomial in the variables $x_{g,m,k}$ is a function $w : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ defined by $w(ab) = w(a) + w(b)$ and $w(x_{g,m,k}) = m \cdot \mathbf{1}_{g,k}$. Given an arbitrary element $a \in R$, we denote by a_w its weight w part. A tensor product of monomials $a \otimes b$ has a bi-weight $(w(a), w(b))$ and a total weight $w(a) + w(b)$. The coproduct Δ preserves (total) weight.

Lemma 6.1 (Weight splitting). *Let $A \subseteq R$ be a subgroup with the property that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$. Let w be any nonzero weight. If A has an element with nonzero weight w part, then A has an element with nonzero weight $m\mathbf{1}_{g,k}$ part for some $(g,k) \in \operatorname{supp} w$.*

Proof. The idea is to use Δ to 'split' the weight w until its support becomes a singleton. We have $\Delta(a)_w = \Delta(a_w)$. If the support of w is not a singleton, then write $w = w_1 + w_2$ for nonzero w_1 and w_2 of disjoint support. Since the supports of w_1 and w_2 are disjoint, the result of applying Δ to a monomial of weight w will have a *unique* monomial of bi-weight (w_1, w_2). In particular, $a_w \neq 0$ implies that $\Delta(a)_{w_1,w_2} \neq 0$. Since $\Delta(a) \in (A \otimes R) + (R \otimes A)$, we conclude that A must have an element with nonzero weight w_1 part or weight w_2 part. Now we replace w with whichever of w_1 or w_2 it is and repeat until the support of w is a singleton. □

Lemma 6.2 (Weight purifying)**.** *Let $A \subseteq R$ be a subgroup with the property that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$. If A has an element with nonzero weight $m\mathbf{1}_{g,k}$ part, then A has an element containing the single variable monomial $x_{g,m',k}$ for some $m' \leq m$.*

Proof. The argument is similar to 'weight splitting' Lemma 6.1. Let $w = m\mathbf{1}_{g,k}$. Let $a \in A$ have nonzero weight w part. Among the weight w monomials appearing in a, consider the factor $x_{g,m',k}$ with m' the largest possible. Now consider the monomials in $\Delta(a)_w$ of the form $x_{g,m',k} \otimes -$. How can a given monomial in a_w contribute such a monomial to $\Delta(a)_w$? The factors $x_{g,m'',k}$ with $m'' < m'$ must go completely on the right. Of the factors of $x_{g,m',k}$, exactly one must go completely on the left, and the rest must go completely on the right. Thus the monomials in $\Delta(a)_w$ of the form $x_{g,m',k} \otimes -$ are in bijection with the monomials in a_w with at least one $x_{g,m',k}$ factor, and the effect of Δ is to multiply their coefficient by the number of such factors. In particular, $\Delta(a)_w$ contains monomials of the form $x_{g,m',k} \otimes -$. Appealing to $\Delta(a) \in (A \otimes R) + (R \otimes A)$, we have 'split' w unless $m' = m$, in which case we have proven the desired result. □

Lemma 6.3 (Weight dividing)**.** *Let $A \subseteq R$ be a subgroup with the property that $\rho_d(A) \subseteq A$. If A has an element containing the single variable monomial $x_{g,m,0}$, then A has an element containing the single variable monomial $x_{g,1,0}$ and no single variable monomials $x_{g,m',0}$ for $m' > 1$.*

Proof. Take an element of A containing single variable monomials $x_{g,m,0}$ for various m, and apply ρ_d to it where d is the maximum m among them. □

Order the pairs $(g, k) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ lexicographically, namely $(g, k) < (g', k')$ when either $g < g'$ or $g = g'$ and $k < k'$. This is evidently a well-ordering.

Proposition 6.4. *Let $A \subseteq R$ be a subgroup with the property that $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$ and $\rho_d(A) \subseteq A$. If $A \neq 0$, then there exists an element of A containing the single variable monomial $x_{g,1,k}$ and no single variable monomials $x_{g,m,k}$ with $m > 1$ or $x_{g',m',k'}$ with $(g', k') < (g, k)$.*

Proof. Consider the set of all weights of all monomials appearing in elements of A. Each such weight has a maximum pair (g, k) in its support. Fix (g, k) to be the minimum such pair. Let $a \in A$ have a monomial whose weight has (g, k) as the maximum element in its support. By Lemma 6.1, there exists $a \in A$ with a monomial of weight $m\mathbf{1}_{g,k}$. By Lemma 6.2, there exists $a \in A$ with a single variable monomial $x_{g,m,k}$ (possibly different m). By Lemma 6.3, there exists $a \in A$ with a single variable monomial $x_{g,1,k}$ and no single variable monomials $x_{g,m,k}$ with $m > 1$. Finally, a has no single variable monomials $x_{g',m',k'}$ with $(g', k') < (g, k)$ by choice of (g, k). □

Lemma 6.5. *Let $g, k \geq 0$. There exists a group homomorphism $\mathrm{GW}_{g,k} : H^*_c(\mathcal{Z}/\mathrm{Cpx}_3) \to \mathbb{Q}$ such that $\mathrm{GW}_{g,k}(x_{g,1,k}) = 1$, $\mathrm{GW}_{g,k}(x_{g',m',k'}) = 0$ for $(g', k') > (g, k)$, and $\mathrm{GW}_{g,k}$ evaluates to zero on any monomial of degree > 1.*

Proof. Let $\mathrm{GW}_{g,k}$ integrate over the virtual fundamental cycle of the moduli space $(\mathcal{M}'_g)_{c_1=k}$ of non-constant stable maps from connected nodal curves of arithmetic genus g representing a homology class with chern number k. Since the image of a connected space is connected, $\mathrm{GW}_{g,k}$ annihilates monomials of degree > 1.

We have $\mathrm{GW}_{g,k}(x_{g',m',k'}) = 0$ if $g' > g$, since there are no non-constant maps from a nodal curve of arithmetic genus g to a curve of genus $g' > g$. In the case $g = g'$, the map would have to have degree $d = k/k' < 1$, hence cannot exist.

Finally, let us calculate $\mathrm{GW}_{g,k}(x_{g,1,k}) = 1$. We can represent $x_{g,1,k}$ by a curve C of genus g and $E = L \oplus L'$ for $c_1(L) = g - 1$ and $c_1(L') = g - 1 + k$. Generically we have $h^1(L) = h^1(L') = 0$ and $h^0(L) = 0$ and $h^0(L') = k$. That is, C is part of a transversely cut out k-dimensional moduli space of sections (which locally coincides with $\mathcal{M}_g$), and the equivariant local curve element is by definition the Poincaré dual of a point in this space. □

Lemma 6.6. *The kernel A of any morphism $F : X \to Y$ of co-algebras over a field satisfies $\Delta(A) \subseteq (A \otimes X) + (X \otimes A)$.*

Proof. Compatibility of F with Δ implies that $\Delta(A) \subseteq \ker(F \otimes F)$. Now $\ker(F \otimes F) = (\ker F) \otimes X + X \otimes (\ker F)$ since indeed for any pair of morphisms of vector spaces $f : V \to W$ and $f' : V' \to W'$ we have $\ker(f \otimes f') = (\ker f) \otimes V' + V \otimes (\ker f')$. □

Proposition 6.7. *The map $R \to H^*_c(\mathcal{Z}_{\mathrm{sF}}/\mathrm{Cpx}_3)/\mathrm{tors}$ is injective.*

Proof. Since R is torsion free, it suffices to show that the map is injective after rationalizing. The kernel A of this map on rationalizations satisfies $\Delta(A) \subseteq (A \otimes R) + (R \otimes A)$ by Lemma 6.6 and $\rho_d(A) \subseteq A$. If this kernel is nonzero, then Proposition 6.4 produces an element of it on which $\mathrm{GW}_{g,k}$ is nonzero by Lemma 6.5, a contradiction (the preceding lemmas work just the same over $\mathbb{Q}$ as over $\mathbb{Z}$). □

Proof of Theorem 1.1. Combine Theorem 5.20 and Proposition 6.7. □

A Virtual fundamental classes

We give a brief exposition of the theory of the intrinsic normal cone, perfect obstruction theories, and virtual fundamental classes as pioneered by Behrend–Fantechi [3]. Naturally, we give particular emphasis to the properties of this theory which are used in the body of the paper. References include Behrend–Fantechi [3], Manolache [21], Qu [33], Khan [15, 16], Déglise–Jin–Khan [8], and Porta–Yu [32].

A morphism of complex analytic spaces is called a *closed embedding* when it is the inclusion of a closed analytic subspace, i.e. what is usually called a 'closed immersion' (a term which we will avoid since it conflicts with the meaning of the term 'immersion' in differential topoogy). If $X \to Y$ is a closed embedding, then $\mathrm{Bl}_X Y = \mathrm{Proj}_Y \bigoplus_{r\geq 0} I_X^r$ denotes the blow-up of Y along X. For any cartesian diagram of complex analytic spaces

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Y' & \longrightarrow & Y \end{array} \tag{A.1}$$

in which the vertical arrows are closed embeddings, there is a (functorial) closed embedding of blow-ups $\mathrm{Bl}_{X'}Y' \to \mathrm{Bl}_X Y \times_Y Y'$, corresponding to the surjection of graded rings $\bigoplus_{r\geq 0} I_X^r \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \twoheadrightarrow \bigoplus_{r\geq 0} I_{X'}^r$.

A *complex analytic stack* shall mean a stack (i.e. sheaf of groupoids) on the site of complex analytic spaces which admits a submersive atlas (hence has representable diagonal). A morphism of complex analytic stacks is called *Deligne–Mumford* when its diagonal is *unramified* (a morphism of complex analytic spaces is called unramified when it is, locally on the source, a closed embedding). The diagonal of any morphism of complex analytic spaces is a locally closed embedding, hence every morphism of complex analytic spaces is Deligne–Mumford.

Definition A.1 (Deformation to the normal cone)**.** Given $X \hookrightarrow Y$ a closed embedding of complex analytic spaces, one associates the following spaces [10, Chapter 5].

$$M_{X/Y} = \mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1) \tag{A.2}$$

$$M^\circ_{X/Y} = M_{X/Y} \setminus \mathrm{Bl}_{X\times 0}(Y \times 0) \tag{A.3}$$

$$C_{X/Y} = M^\circ_{X/Y} \times_{\mathbb{A}^1} 0 \tag{A.4}$$

Note that $\mathrm{Bl}_{X\times 0}(Y \times 0) \to \mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1)$ is a closed embedding; explicitly $M^\circ_{X/Y} = \mathrm{Spec}_Y\, \mathcal{O}_Y[t, I_X t^{-1}]$. The object $C_{X/Y}$ is called the *normal cone* of $X \hookrightarrow Y$, and the space $M^\circ_{X/Y}$ is called the *deformation to the normal cone* (it maps to $\mathbb{A}^1$ with fiber Y over $\mathbb{A}^1 - 0$ and fiber $C_{X/Y}$ over 0). Given $X \to Y$ a Deligne–Mumford morphism of complex analytic stacks, the above objects are defined by descent [3, 19, 18, 17][21, §2.2][33, §1.1].

Remark A.2. The normal cone $C_{X/Y}$ is an algebro-geometric analogue of the relative Spanier–Whitehead dual of X over Y.

Lemma A.3 ([21, Theorem 2.31][33, Proposition 1.2])**.** *For any cartesian square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ Y' & \longrightarrow & Y \end{array} \tag{A.5}$$

whose vertical arrows are Deligne–Mumford, there is an induced closed embedding $M^\circ_{X'/Y'} \to M^\circ_{X/Y} \times_Y Y'$ over $\mathbb{A}^1$, hence also a closed embedding $C_{X'/Y'} \to C_{X/Y} \times_Y Y'$.

Proof. For a closed embeddings $X \hookrightarrow Y$, this corresponds to the surjection of rings $\mathcal{O}_Y[t, I_X t^{-1}] \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \twoheadrightarrow \mathcal{O}_{Y'}[t, I_{X'} t^{-1}]$. Now apply descent. □

Lemma A.4. *For maps $X \to Y$ and $X' \to Y'$, there is a canonical isomorphism*

$$M^\circ_{X\times X'/Y\times Y'} \xrightarrow{\sim} M^\circ_{X/Y} \times_{\mathbb{A}^1} M^\circ_{X'/Y'} \tag{A.6}$$

which, in particular, specializes over 0 *to an isomorphism* $C_{X\times X'/Y\times Y'} \xrightarrow{\sim} C_{X/Y} \times C_{X'/Y'}$.

Proof. For closed embeddings $X \hookrightarrow Y$ and $X' \hookrightarrow Y'$, this map corresponds to the map of graded rings

$$\mathcal{O}_{Y\times Y'}[t, I_X\mathcal{O}_{Y'}t^{-1}, t', \mathcal{O}_Y I_{X'}(t')^{-1}]/(t-t') \to \mathcal{O}_{Y\times Y'}[t, (I_X\mathcal{O}_{Y'} + \mathcal{O}_Y I_{X'})t^{-1}], \tag{A.7}$$

which is an isomorphism by inspection (to see this, it is helpful to choose vector space splitings of the inclusions $I^r \supseteq I^{r+1}$). Now apply descent. □

Recall the derived category $D(X)$ of (unbounded) complexes of sheaves of $\mathbb{Z}$- or $\mathbb{Q}$-modules on a complex analytic stack X (references include [20]). At the level of ∞- or dg-categories, the category $D(X)$ associated to a stack X is the limit of $D(U)$ over all complex analytic spaces U with a map to X (equivalently, all those with a submersive map to X). For a map of complex analytic stacks $f : X \to Y$, we have pairs of adjoint functors (f^*, f_*) and $(f_!, f^!)$ (always derived) between $D(X)$ and $D(Y)$. Note that we need the exceptional pushforward/pullback operations for non-separated morphisms. These satisfy proper base change, in the sense that for a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha} & Y \end{array} \tag{A.8}$$

there is a canonical isomorphism $f'_!\beta^* = \alpha^* f_!$ (equivalently $\beta_* f'^! = f^!\alpha_*$). There is also a canonical natural transformation $f_! \to f_*$, which is an isomorphism for proper representable morphisms f. More generally, it is an isomorphism for proper Deligne–Mumford morphisms f provided we are using $\mathbb{Q}$-coefficients (to see the need for $\mathbb{Q}$-coefficients, note that for $f : */G \to *$ with G finite, the natural transformation $f_! \to f_*$ is the natural map $V_G \to V^G$ from co-invariants to invariants given by 'sum over G-orbits' for G-representations V).

Definition A.5. For a morphism of complex analytic stacks $f : X \to Y$, we define $H^{\mathrm{rel}\infty}_*(X/Y) = H^*(X, f^!\mathbb{Z}_Y)$.

Remark A.6. When $f : X \to Y$ is separated, the group $H^{\mathrm{rel}\infty}_*(X/Y)$ may be regarded as the 'cohomology of Y with coefficients in fiberwise chains rel infinity of $X \to Y$' (compare Remark 3.4 and the preceding discussion). A class in $H^{\mathrm{rel}\infty}_*(X/Y)$ is roughly analogous to what is often called a 'bivariant class' for the morphism $X \to Y$.

The groups $H^{\mathrm{rel}\infty}_*(X/Y)$ are functorial in various ways (references include [15, 32]). For a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\beta} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\alpha} & Y \end{array} \tag{A.9}$$

there is a pullback map $H_*^{\mathrm{rel}\infty}(X/Y) \to H_*^{\mathrm{rel}\infty}(X'/Y')$. Indeed, this amounts to wanting a map $f^!\mathbb{Z}_Y \to \beta_* f'^! \alpha^* \mathbb{Z}_Y$, and in fact there is a natural transformation $\beta^* f^! \to f'^! \alpha^*$ since this is (by adjunction) the same as a natural transformation $f'_! \beta^* \to \alpha^* f_!$ (and there is in fact here a natural isomorphism by proper base change). There is also a proper pushforward map $H_*^{\mathrm{rel}\infty}(X_1/Y) \to H_*^{\mathrm{rel}\infty}(X_2/Y)$ for $g : X_1 \to X_2$ proper. Indeed, this amounts to wanting a map $g_* f_1^! \mathbb{Z}_Y \to f_2^! \mathbb{Z}_Y$, and for this it is enough to have a natural transformation $g_* g^! \to 1$, which is the co-unit $g_! g^! \to 1$ combined with the identification $g_! \xrightarrow{\sim} g_*$ since g is proper.

Definition A.7. Fix a map $f : X \to Y$ and a map $\pi : D \to Y \times \mathbb{A}^1$ which is an isomorphism over $Y \times (\mathbb{A}^1 - 0)$ and whose fiber over $Y \times 0$ is identified (over Y) with X. Associated to this data is a canonical *specialization element* in $H_0^{\mathrm{rel}\infty}(X/Y)$ defined as follows (note that this definition is somewhat different from its analogue in the context of algebraic cycles [21, 33, 15]). In fact, there is an associated natural transformation of functors $1 \to f_* f^!$. Let $(\mathbb{A}^1 - 0)^\sim$ denote the universal cover, and consider the following diagram.

$$\begin{array}{ccccc} X & \xrightarrow{i} & D & \xleftarrow{j} & Y \times (\mathbb{A}^1 - 0)^\sim \\ \downarrow f & & \downarrow \pi & & \| \\ Y & \xrightarrow{i} & Y \times \mathbb{A}^1 & \xleftarrow{j} & Y \times (\mathbb{A}^1 - 0)^\sim \end{array} \tag{A.10}$$

There is an induced diagram of derived categories, which commutes by proper base change.

$$\begin{array}{ccccc} D(X) & \xrightarrow{i_*} & D(D) & \xleftarrow{j_*} & D(Y \times (\mathbb{A}^1 - 0)^\sim) \\ \uparrow f^! & & \uparrow \pi^! & & \| \\ D(Y) & \xrightarrow{i_*} & D(Y \times \mathbb{A}^1) & \xleftarrow{j_*} & D(Y \times (\mathbb{A}^1 - 0)^\sim) \end{array} \tag{A.11}$$

Now we combine this on the bottom with the natural transformation $j_* p^* \to i_*$ where $p : Y \times (\mathbb{A}^1 - 0)^\sim \to Y$, and we post-compose with the pushforward $D(D) \to D(Y)$. This produces the desired a natural transformation $1 \to f_* f^!$ by contractibility of $(\mathbb{A}^1 - 0)^\sim$.

Definition A.8. The *specialization element* in $H_0^{\mathrm{rel}\infty}(C_{X/Y}/Y)$ refers to that associated to the family $M_{X/Y}^\circ \to Y \times \mathbb{A}^1$ by Definition A.7.

Remark A.9. Continuing Remark A.2, the specialization element in $H_*^{\mathrm{rel}\infty}(C_{X/Y}/Y)$ is analogous to the Spanier–Whitehead dual (relative Y) of the map $X \to Y$.

Lemma A.10. *For a cartesian square*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array} \tag{A.12}$$

whose vertical arrows are Deligne–Mumford, the specialization elements in $H_*^{\mathrm{rel}\infty}(C_{X/Y}/Y)$ *and* $H_*^{\mathrm{rel}\infty}(C_{X'/Y'}/Y')$ *have the same image in* $H_*^{\mathrm{rel}\infty}(C_{X/Y} \times_Y Y'/Y')$.

Proof. Diagram chase and compatibility of the proper base change isomorphism with composition of pullback squares. □

Lemma A.11. *For maps $X \to Y$ and $X' \to Y'$, the specialization element of $H_*^{\mathrm{rel}\infty}(C_{X\times X'/Y\times Y'}/Y\times Y') = H_*^{\mathrm{rel}\infty}(C_{X/Y}\times C_{X'/Y'}/Y\times Y')$ is the image of the tensor product of the specialization elements of $H_*^{\mathrm{rel}\infty}(C_{X/Y}/X)$ and $H_*^{\mathrm{rel}\infty}(C_{X'/Y'}/Y')$ under the Künneth map.* □

Now we recall the notion of a *perfect obstruction theory* from Behrend–Fantechi [3], which allows to transfer the specialization element in $H_0^{\mathrm{rel}\infty}(C_{X/Y}/Y)$ to a class in $H_d^{\mathrm{rel}\infty}(X/Y)$ when the 'virtual relative tangent bundle' $T_{X/Y}^{\mathrm{vir}}$ has rank d.

Definition A.12. Given a two-term complex of vector bundles (aka perfect complex of amplitude $[-1\ 0]$) $E^\bullet = [E^{-1} \to E^0]$ over a complex analytic stack X (by which we mean an object in the sheafified 2-category of such), we may form the 'total space' $E^\bullet$ given locally in the case X is a space by the groupoid $E^{-1}\times_X E^0 \rightrightarrows E^0$ [3, §2].

Definition A.13. A *perfect obstruction theory* on a Deligne–Mumford morphism $X \to Y$ is an amplitude $[0\ 1]$ perfect complex $T_{X/Y}^{\mathrm{vir}}$ on X together with a closed embedding $C_{X/Y} \to T_{X/Y}^{\mathrm{vir}}[1]$ (i.e. into the 'total space' of $T_{X/Y}^{\mathrm{vir}}[1]$).

Remark A.14. Definition A.13 is non-standard and is made only for the sake of simplicity in the present discussion. A perfect obstruction theory (in the standard meaning as introduced by Behrend–Fantechi [3]) on a Deligne–Mumford morphism $X \to Y$ is an amplitude $[0\ 1]$ perfect complex $T_{X/Y}^{\mathrm{vir}}$ on X together with a linear map $T_{X/Y}[1] \to T_{X/Y}^{\mathrm{vir}}[1]$ which is a closed embedding on total spaces, where $T_{X/Y}[1]$ denotes the 'total space of the dual of $(\tau^{\geq -1}\mathbb{L}_{X/Y})[-1]$'. There is a canonical closed embedding $C_{X/Y} \to T_{X/Y}[1]$ (see [3, 21]), so a perfect obstruction theory in the sense of [3] induces one in the sense of Definition A.13.

Given (the total space of) a two-term complex $\pi : E^\bullet \to X$, the functor $\pi_*\pi^!$ is shift by $\dim E^\bullet = \dim E^0 - \dim E^{-1}$. In particular, for a morphism $X \to Y$ this identifies $H_{*+\dim E^\bullet}^{\mathrm{rel}\infty}(E^\bullet/Y)$ with $H_*^{\mathrm{rel}\infty}(X/Y)$.

Definition A.15. The *relative virtual fundamental class* $[X/Y]^{\mathrm{vir}} \in H_{\dim T_{X/Y}^{\mathrm{vir}}}^{\mathrm{rel}\infty}(X/Y)$ associated to a perfect obstruction theory $C_{X/Y} \to T_{X/Y}^{\mathrm{vir}}[1]$ is the image of the canonical degree zero element in $H_*^{\mathrm{rel}\infty}(C_{X/Y}/Y)$ under the proper pushforward map to $H_*^{\mathrm{rel}\infty}(T_{X/Y}^{\mathrm{vir}}[1]/Y)$, followed by the identification of this group with $H_{*+\dim T_{X/Y}^{\mathrm{vir}}}^{\mathrm{rel}\infty}(X/Y)$ from just above.

A perfect obstruction theory on $X \to Y$ determines one on $X' = X\times_Y Y' \to Y'$ for any morphism $Y' \to Y$, namely we take $T_{X'/Y'}^{\mathrm{vir}}$ to be the pullback of $T_{X/Y}^{\mathrm{vir}}$, and we consider the composition of the closed embedding $C_{X'/Y'} \to C_{X/Y}\times_Y Y'$ with the pullback to Y' of the map $C_{X/Y} \to T_{X/Y}^{\mathrm{vir}}[1]$.

Lemma A.16. *Fix a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array} \tag{A.13}$$

whose vertical arrows are Deligne–Mumford, and fix a perfect obstruction theory $C_{X/Y} \to T^{\mathrm{vir}}_{X/Y}[1]$ *with pullback* $C_{X'/Y'} \to C_{X\times Y} \times_Y Y' \to T^{\mathrm{vir}}_{X/Y}[1] \times_Y Y' = T^{\mathrm{vir}}_{X'/Y'}[1]$*. The pullback map* $H^{\mathrm{rel}\infty}_*(X/Y) \to H^{\mathrm{rel}\infty}_*(X'/Y')$ *sends* $[X/Y]^{\mathrm{vir}}$ *to* $[X'/Y']^{\mathrm{vir}}$*.*

Proof. Immediate from Lemma A.10. □

Given perfect obstruction theories on $X \to Y$ and $X' \to Y'$, their product is a perfect obstruction theory on $X \times X' \to Y \times Y'$ (recall the isomorphism $C_{X\times X'/Y\times Y'} \xrightarrow{\sim} C_{X/Y} \times C_{X'/Y'}$ of Lemma A.4).

Lemma A.17. *For Deligne–Mumford morphisms* $X \to Y$ *and* $X' \to Y'$ *with perfect obstruction theories, the virtual fundamental class* $[X \times X'/Y \times Y']^{\mathrm{vir}}$ *is the image of* $[X/Y]^{\mathrm{vir}} \otimes [X'/Y']^{\mathrm{vir}}$ *under the Künneth map* $H^{\mathrm{rel}\infty}_*(X/Y) \otimes H^{\mathrm{rel}\infty}_*(X'/Y') \to H^{\mathrm{rel}\infty}_*(X \times X'/Y \times Y')$*.*

Proof. Immediate from Lemma A.11. □

Lemmas A.16 and A.17 together imply the compatibility of virtual fundamental classes with *fiber* products.

B Chains on simplicial sets and groupoids

The simplex category $\mathbf{\Delta}$ consists of totally ordered sets $[n] = \{0 < \ldots < n\}$ and weakly order preserving ($x \le y$ implies $f(x) \le f(y)$) maps. There is a functor from $\mathbf{\Delta}$ to topological spaces sending $[n]$ to the standard simplex Δ^n (with vertices labelled $0, \ldots, n$) and a morphism $[n] \to [m]$ to the unique affine linear map with the corresponding action on vertices. A simplicial object $X_\bullet$ in a category C is a functor $X : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{C}$, and we write $X_n = X([n])$. A simplicial set $X : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Set}$ may be regarded as a combinatorial specification of how to glue together standard simplices Δ^n along simplicial maps preserving vertex order.

A coefficient system (valued in a category C) over a simplicial set $X_\bullet$ is a functor $(\mathbf{\Delta} \downarrow X_\bullet)^{\mathsf{op}} \to \mathsf{C}$ which sends surjections $[n] \twoheadrightarrow [m] \to X$ to isomorphisms. The complex of 'fat' chains on $X_\bullet$ with respect to a coefficient system A valued in the category Ab is given by

$$C_*(X_\bullet; A) = \bigoplus_{n\ge 0} \bigoplus_{\sigma \in X_n} A_\sigma \otimes \mathfrak{o}_n \tag{B.1}$$

where $\mathfrak{o}_n$ denotes the orientation group of Δ^n (which lies in homological degree n), and the boundary operator acts on $A_\sigma \otimes \mathfrak{o}_n$ via the usual sum over faces $d_{n,i} : [n-1] \hookrightarrow [n]$ for $0 \le i \le n$. Dually, we may define fat cochains $C^*(X_\bullet, A)$ with respect to any coefficient system A valued in $\mathsf{Ab}^{\mathsf{op}}$ (now involving direct product instead of direct sum). More generally, fat chains and cochains make sense for coefficient systems valued in the category of (unbounded) complexes of abelian groups (and its opposite); in this setting, the condition that surjections be sent to isomorphism can be relaxed to sending surjections to *quasi*-isomorphisms. A map of coefficient systems $A \to B$ over $X_\bullet$ induces a map $C_*(X_\bullet; A) \to C_*(X_\bullet; B)$, and a map $f : X_\bullet \to Y_\bullet$ of simplicial sets induces a map $C_*(X_\bullet, f^*A) \to C_*(Y_\bullet, A)$ for any coefficient system A over $Y_\bullet$.

Lemma B.1. *A quasi-isomorphism of coefficient systems* $A \to B$ *over* X *induces a quasi-isomorphism* $C_*(X; A) \to C_*(X; B)$*. The same holds for cohomology with coefficients.*

Proof. Let $X_{\leq k}$ denote the k-skeleton of X. The short exact sequence of complexes

$$0 \to C_*(X_{<k}; A) \to C_*(X_{\leq k}; A) \to \bigoplus_{\substack{\sigma \subseteq X \\ \dim \sigma = k}} A_\sigma \to 0 \tag{B.2}$$

induces a long exact sequence of cohomology groups. Applying the five lemma, we see that if $C_*(X_{<k}; A) \to C_*(X_{<k}; B)$ is a quasi-isomorphism then $C_*(X_{\leq k}; A) \to C_*(X_{\leq k}; B)$ is a quasi-isomorphism. Finally, note that $C_*(X; A)$ is the directed colimit of $C_*(X_{\leq k}; A)$ over k and that homology commutes with directed colimits.

To prove the analogous assertion for cohomology, we follow the same argument, except there is an apparent issue in that cohomology does not commute with inverse limits. However we actually just need to know that an inverse limit quasi-isomorphisms with surjective transition maps is a quasi-isomorphism. This follows from the fact that for an inverse system $\cdots \twoheadrightarrow A_2^\bullet \twoheadrightarrow A_1^\bullet \twoheadrightarrow A_0^\bullet$ of complexes with surjective transition maps, there is a functorial short exact sequence

$$0 \to \varprojlim_i{}^1 H^* A_i^{\bullet - 1} \to H^* \varprojlim_i A_i^\bullet \to \varprojlim_i H^* A_i^\bullet \to 0 \tag{B.3}$$

which yields the desired assertion upon applying the five lemma. □

The argument of Lemma B.1 is used all over the place. For example, the complex of 'reduced' chains is the quotient of the complex of 'fat' chains by its subcomplex of 'degenerate' chains (those for which the simplex σ in (B.1) is degenerate). The subcomplex of degenerate chains is acyclic: the argument of Lemma B.1 reduces us to checking the case of a single simplex σ rel boundary, where the degenerate chain group has the form

$$A_\sigma \xleftarrow{1} A_\sigma \xleftarrow{0} A_\sigma \xleftarrow{1} A_\sigma \xleftarrow{0} \cdots \tag{B.4}$$

which is acyclic by inspection. It is therefore irrelevant whether we take $C_*(X_\bullet; A)$ to mean fat chains or reduced chains. The same applies to cohomology with coefficients.

There are two reasonable notions of 'product' for simplicial sets X and Y. The 'external product' is the product functor $X \boxtimes Y : (\mathbf{\Delta} \times \mathbf{\Delta})^{\mathsf{op}} \to \mathsf{Set}$ (such a functor is called a bi-simplicial set). Just like simplicial sets, a bi-simplicial set may be regarded as a combinatorial specification of how to glue together products of standard simplices $\Delta^n \times \Delta^m$ along products of simplicial maps. A product of simplices $\Delta^n \times \Delta^m$ has a standard subdivision into $\binom{n+m}{n}$ copies of Δ^{n+m}, and this subdivision is moreover compatible with products of maps. This defines a functor from bi-simplicial sets to simplicial sets, which turns out to be pre-composition with the diagonal functor $\mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$. The subdivision of the external product $X \boxtimes Y$ thus coincides with the 'categorical product' $X \times Y : \mathbf{\Delta}^{\mathsf{op}} \to \mathsf{Set}$ given by $(X \times Y)_n = X_n \times Y_n$.

Given coefficient systems A and B on simplicial sets X and Y, the tensor product $C_*(X, A) \otimes C_*(Y, B)$ is naturally identified with the complex of chains $C_*(X \boxtimes Y, A \boxtimes B)$ on the external product $X \boxtimes Y$ equipped with the external tensor product of coefficient systems $A \boxtimes B$ (which assigns to a product of simplices the tensor product of what A and B assign to the two factors). There is a natural 'subdivision' map

$$C_*(X, A) \otimes C_*(Y, B) = C_*(X \boxtimes Y, A \boxtimes B) \to C_*(X \times Y, p_X^* A \otimes p_Y^* B), \tag{B.5}$$

which can be seen to be a quasi-isomorphism by using the argument of Lemma B.1 to reduce to the case of a product of simplices rel boundary, which can be verified explicitly.

Recall that a map of simplicial sets $A_\bullet \to B_\bullet$ is called a *trivial Kan fibration* when for every diagram of solid arrows

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & A_\bullet \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & B_\bullet \end{array} \tag{B.6}$$

there exists a dotted lift. If $A_\bullet \to B_\bullet$ is a trivial Kan fibration, then for any level-wise injection of simplicial sets $P_\bullet \to Q_\bullet$ and diagram of solid arrows

$$\begin{array}{ccc} P_\bullet & \longrightarrow & A_\bullet \\ \downarrow & \nearrow & \downarrow \\ Q_\bullet & \longrightarrow & B_\bullet \end{array} \tag{B.7}$$

there exists a dotted lift (construct the lift one simplex at a time).

Lemma B.2. *If $X \to Y$ is a trivial Kan fibration, then $C_*(X, f^*A) \to C_*(Y, A)$ is a quasi-isomorphism for any coefficient system A on Y.*

Proof. Solving the lifting problem

$$\begin{array}{ccc} \varnothing & \longrightarrow & X_\bullet \\ \downarrow & \overset{s}{\nearrow} & \downarrow f \\ Y_\bullet & \xrightarrow{\mathbf{1}_Y} & Y_\bullet \end{array} \tag{B.8}$$

produces a section $s : Y_\bullet \to X_\bullet$ of f. Solving the lifting problem

$$\begin{array}{ccc} X \times \partial\Delta^1 & \xrightarrow{sf \sqcup \mathbf{1}_X} & X_\bullet \\ \downarrow & \overset{H}{\nearrow} & \downarrow f \\ X \times \Delta^1 & \xrightarrow{f p_X} & Y_\bullet \end{array} \tag{B.9}$$

produces a homotopy H between $\mathbf{1}_X$ and $sf : X_\bullet \to X_\bullet$. It then follows from functoriality of C_* (and its behavior under products) that f_* is a homotopy equivalence. □

We may also define the (co)homology of simplicial *groupoids* with respect to coefficient systems. A *resolution* of a simplicial groupoid $X_\bullet$ is a trivial Kan fibration $\tilde{X}_\bullet \to X_\bullet$ from a simplicial set $\tilde{X}_\bullet$ (the notion of a trivial Kan fibration makes sense for simplicial groupoids). Every simplicial groupoid has a resolution: it may be constructed by induction on skeleta (or, more or less equivalently, by the small object argument). We define the (co)chain group of a simplicial groupoid $X_\bullet$ with coefficients in A to be the (co)chain group of any resolution $\tilde{X}_\bullet$ of $X_\bullet$ with coefficients in the pullback of A. This is well defined by Lemma B.2.

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