Central limit theorems for uniform model random polygons

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Abstract

We show how a central limit theorem for Poisson model random polygons implies a central limit theorem for uniform model random polygons. To prove this implication, it suffices to show that in the two models, the variables in question have asymptotically the same expectation and variance. We use integral geometric expressions for these expectations and variances to reduce the desired estimates to the convergence $(1+\frac{\alpha}{n})^n \to e^\alpha$ as $n \to \infty$.

1 Introduction

Given a convex set $K \subset \mathbb{R}^2$ of unit area, we may define two random polygon models. For the *Poisson model*, we consider a Poisson process of intensity λ inside K, and define $\Pi_{K,\lambda}$ to be the convex hull of the points of this process. For the *uniform model*, we take n independent random points distributed uniformly in K and let $P_{K,n}$ be their convex hull. For a polygon $\mathcal{P}$ inside K, we let $N(\mathcal{P})$ denote the number of vertices of $\mathcal{P}$, and we let $A(\mathcal{P})$ denote the area of $K \setminus \mathcal{P}$. When one wants to prove central limit theorems for N and A in either of the two models of random polygons, it is often the case that the Poisson model is easier than the uniform model. Indeed, in general, results are proved first for the Poisson model, and then more arguments are needed to deduce a corresponding result for the uniform model.

Recently, the author [4] studied the Poisson model of random polygons and proved the following central limit theorem for N and A:

Theorem 1.1 ([4]). *As $\lambda \to \infty$, the following estimates for $\Pi_{K,\lambda}$ hold uniformly over all K of unit area:*

$$\sup_x \left| P\left(\frac{N-\mathbb{E}[N]}{\sqrt{\operatorname{Var} N}} \le x\right) - \Phi(x)\right| \ll \frac{\log^2 \mathbb{E}[N]}{\sqrt{\mathbb{E}[N]}} \tag{1.1}$$

$$\sup_x \left| P\left(\frac{A-\mathbb{E}[A]}{\sqrt{\operatorname{Var} A}} \le x\right) - \Phi(x)\right| \ll \frac{\log^2 \mathbb{E}[N]}{\sqrt{\mathbb{E}[N]}} \tag{1.2}$$

Here $\Phi(x) = P(Z \le x)$ where Z is the standard normal distribution.

In this paper, our goal is to show how to derive the following corollary for the uniform model:

Corollary 1.2. *As $n \to \infty$, the following estimates for $P_{K,n}$ hold uniformly over all K of unit area:*

$$\sup_x \left| P\left(\frac{N - \mathbb{E}[N]}{\sqrt{\operatorname{Var} N}} \leq x \right) - \Phi(x) \right| \to 0 \tag{1.3}$$

$$\sup_x \left| P\left(\frac{A - \mathbb{E}[A]}{\sqrt{\operatorname{Var} A}} \leq x \right) - \Phi(x) \right| \to 0 \tag{1.4}$$

Here $\Phi(x) = P(Z \leq x)$ where Z is the standard normal distribution.

From the estimates derived in this paper, a secondary result from [4] (Theorem 1.3 below) also carries over immediately to the uniform model. We should say that Theorem 1.3 and the consequence derived here, Corollary 1.4, have both been proven independently by Imre Bárány and Matthias Reitzner.

Theorem 1.3 ([4])**.** *As $\lambda \to \infty$, the following estimates for $\Pi_{K,\lambda}$ hold uniformly over all K of unit area:*

$$\mathbb{E}[N] \asymp \operatorname{Var} N \asymp \lambda \mathbb{E}[A] \asymp \lambda^2 \operatorname{Var} A \tag{1.5}$$

Corollary 1.4. *As $n \to \infty$, the following estimates for $P_{K,n}$ hold uniformly over all K of unit area:*

$$\mathbb{E}[N] \asymp \operatorname{Var} N \asymp n \mathbb{E}[A] \asymp n^2 \operatorname{Var} A \tag{1.6}$$

Though both Theorem 1.1 and Corollary 1.2 have been known for quite some time in the case that either K is a polygon or ∂K is of class C^2, the proof given here of Theorem 1.1 $\implies$ Corollary 1.2 for *arbitrary* convex K appears to be new. Corollary 1.2 answers a question of Van Vu [1].

We will see below that in order to prove that Theorem 1.1 implies Corollary 1.2, it suffices to show that when $n = \lambda$, the random variables $N(P_{K,n})$ and $N(\Pi_{K,\lambda})$ (as well as $A(P_{K,n})$ and $A(\Pi_{K,\lambda})$) have the same expectation and variance up to a small enough error. This is essentially the same strategy used by Van Vu in [7] to derive a similar implication for a special case of random polytopes.

However, in contrast to [7], we will use relatively down to earth integral geometry to establish our estimates, instead of sophisticated arguments from probability theory. The approach we take is conceptually very simple. We write down integral geometric expressions for the expectation and variance for the Poisson and uniform models, and then estimate their difference in the limit $n = \lambda \to \infty$. In this formulation, the "reason" that the desired convergence holds is completely transparent: it is essentially reduced to the convergence $(1 + \frac{\alpha}{n})^n \to e^\alpha$ as $n \to \infty$. Also, the variables N and A are treated simultaneous with an identical proof for each (c.f. [7] where the case of f_i, the number of i-simplices, is harder than the case of the volume and requires a new idea). An admitted disadvantage of this approach is that one has to actually write down these integrals explicitly, however once this is done, no further manipulations are necessary. It is interesting to observe that using integral geometry is almost never the "right" way to prove statements along the lines of Theorem 1.1 or even Theorem 1.3, essentially because the expressions quickly become too complicated to deal with either conceptually or theoretically. However for our applications here, the desired estimates become simple when written in terms of the integral geometry, so we in fact believe that these integral geometric expressions do in some sense give the "right" proof of our main lemmas.

One expects that our results and the proofs given here will admit straightfoward generalization to higher dimensions. For random polytopes in dimension $d \geq 3$, Theorems 1.1–1.4 are all known in the case of fixed K whose boundary is C^2 and has nonvanishing Gauss curvature, due to Reitzner [5] and Vu [7]. In the case that K is a polytope, the analogue of Theorem 1.1 was proven very recently by Bárány and Reitzner [3] (one expects that an analogue of Theorem 1.3 also follows from their methods). We expect that if applied to higher dimensions, the methods in this paper would show that Corollaries 1.2 and 1.4 follow in any situation in which Theorems 1.1 and 1.3 respectively are known to hold (in particular for the case that K is a polytope). It is conjectured that Theorems 1.1–1.4 hold for $d \geq 3$ with no restriction on K.

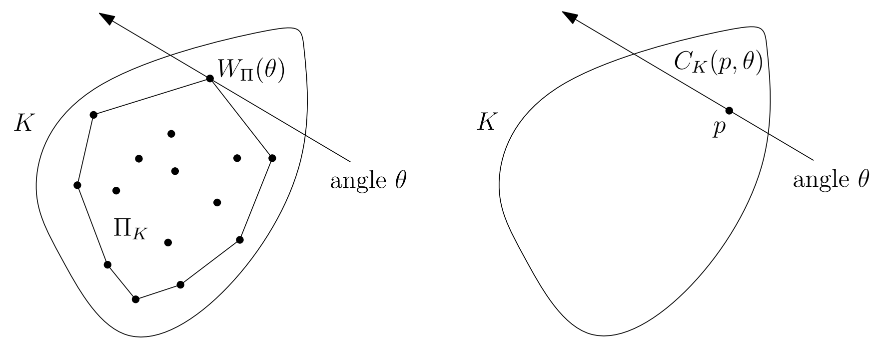
1.1 Acknowledgement

We thank the referee for useful comments and in particular for making us realize an error in the original proof of Lemma 3.1.

2 Notation and definitions

We now review some definitions and two basic lemmas from [4].

In this paper, K will always denote a (bounded) convex set in $\mathbb{R}^2$. Any constants implied by the symbols $\ll$, $\gg$, or $\asymp$ are absolute; in particular they are not allowed to depend on K.



(a) Illustration of $W(\theta)$ (b) Illustration of $C_K(p, \theta)$

Figure 2.1: Illustration of some definitions.

Many of the following definitions are illustrated in Figure 2.1. We may leave out the subscript K later when doing so is unambiguous.

Definition 2.1. We define the random variable $W_{\mathcal{P}}(\theta)$ to be the vertex of $\mathcal{P}$ which has an oriented tangent line at angle θ. This is illustrated in Figure 2.1(a).

Definition 2.2. A *cap at angle* θ is the intersection of K with a halfplane H_θ at angle θ. We may specify a cap at angle θ by giving either its area r or a point $p \in \partial H_\theta$. These are denoted $C_K(r,\theta)$ and $C_K(p,\theta)$ respectively; the latter is illustrated in Figure 2.1(b).

Definition 2.3. We define the real number $A_K(p,\theta)$ to be the area of the cap $C_K(p,\theta)$.

Lemma 2.4. *The random variable* $W_{\Pi,\lambda}(\theta)$ *has probability distribution given by* $\lambda \exp(-\lambda A_K(p,\theta))\, dp$ *where* dp *is the Lebesgue measure. This has total mass* $1 - e^{-\lambda \operatorname{Area}(K)}$*, as* $\Pi_{K,\lambda}$ *is empty with probability* $e^{-\lambda \operatorname{Area}(K)}$.

Proof. This follows directly from the definition of a Poisson point process. The probability that no point lands in $C_K(p,\theta)$ is $\exp(-\lambda A_K(p,\theta))$, and we multiply this by $\lambda\, dp$, which is the density of the Poisson point process.

Alternatively, we may differentiate $\exp(-\lambda A_K(p,\theta))$ with respect to the direction orthogonal to θ and divide by the length of $\partial H_\theta \cap K$. This also yields $\lambda \exp(-\lambda A_K(p,\theta))\, dp$. $\square$

Definition 2.5. We define the function $f_K(x,\theta) : [0,1] \times \mathbb{R}/2\pi \to \mathbb{R}$ as follows:

$$f_K(x,\theta) = \begin{cases} \text{length of } (\partial H_\theta) \cap K \text{ where } C_K(\log \frac{1}{x}, \theta) = H_\theta \cap K \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x > \exp(-\operatorname{Area}(K)) \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \leq \exp(-\operatorname{Area}(K)) \end{cases} \tag{2.1}$$

It will be important to have the following bound on the growth of f:

Lemma 2.6. *If* $y \leq x$*, then:*

$$\frac{f(y)}{\sqrt{-\log y}} \leq \frac{f(x)}{\sqrt{-\log x}} \tag{2.2}$$

The bound above is sharp, for instance $f(x) = \text{const} \cdot \sqrt{-\log x}$ for $K = \{x, y \geq 0\}$ (i.e. the first quadrant).

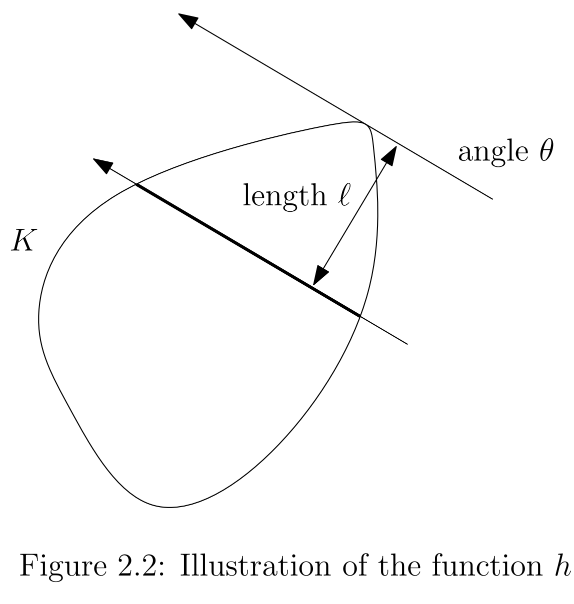


Figure 2.2: Illustration of the function h.

Proof. Project K along the lines at angle θ to get a height function $h : [0, \infty) \to \mathbb{R}_{\geq 0}$; in Figure 2.2, $h(\ell)$ is the length of the thick segment. Now if $A(\ell) = \int_0^\ell h(\ell')\, d\ell'$ then $f(\exp(-A(\ell))) = h(\ell)$. Thus we see that it suffices to show that the function:

$$\frac{h(\ell)}{\sqrt{A(\ell)}} \tag{2.3}$$

is decreasing. Differentiating with respect to ℓ, we see that it suffices to show that:

$$h(\ell)^2 - 2h'(\ell)A(\ell) \geq 0 \tag{2.4}$$

For $\ell = 0$, the left hand side is clearly nonnegative, and the derivative of the left hand side equals $-2h''(\ell)A(\ell)$, which is ≥ 0 by concavity of h. □

When proving central limit theorems, it is important to decompose N and A into local pieces. Thus we define $N(\alpha, \beta)$ to equal the number of edges with angle in the interval (α, β). Then it is easy to see that:

$$N = N(\alpha_1, \alpha_2) + N(\alpha_2, \alpha_2) + \cdots + N(\alpha_L, \alpha_1) \tag{2.5}$$

A similar decomposition is valid for A, where $A(\alpha, \beta)$ is best explained graphically in Figure 2.3.

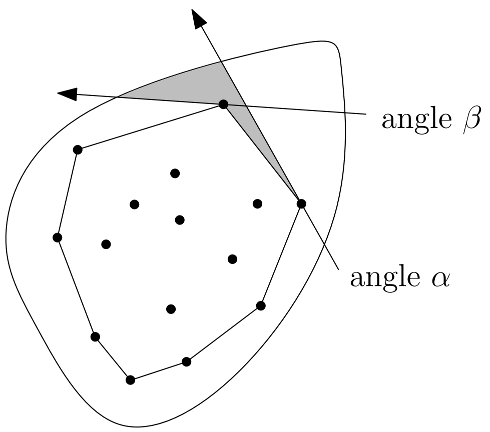


Figure 2.3: Illustration of $A(\alpha, \beta)$

Consider for the moment the Poisson model, and let X denote N or A. In [4], it is shown that if one chooses the partition so that each interval $[\alpha_i, \alpha_{i+1}]$ has constant *affine invariant measure* (a notion from [4] which will not concern us here), then $X(\alpha_i, \alpha_{i+1})$ has constant expectation and variance, and the correlation between $X(\alpha_i, \alpha_{i+1})$ and $X(\alpha_j, \alpha_{j+1})$ is exponentially decreasing in $|i-j|$ (specifically, an α-mixing estimate is proved). From these facts, along with a general lower bound on the variance of X due to Bárány and Reitzner [2] (their result also holds in higher dimensions), it follows on general principles that a central limit theorem holds for X in the Poisson model.

3 Proofs

Let us begin by stating the lemmas which we will prove.

Lemma 3.1. *As $n \to \infty$, we have:*

$$\sup_x |P(A(\Pi_{K,n}) \le x) - P(A(P_{K,n}) \le x)| \to 0 \tag{3.1}$$

$$\sup_x |P(N(\Pi_{K,n}) \le x) - P(N(P_{K,n}) \le x)| \to 0 \tag{3.2}$$

uniformly over all convex K of unit area.

Lemma 3.2. *As $n \to \infty$, we have:*

$$|\mathbb{E}[N(\Pi_{K,n})] - \mathbb{E}[N(P_{K,n})]| = o(\sqrt{\operatorname{Var} N(\Pi_{K,n})}) \tag{3.3}$$

$$|\mathbb{E}[A(\Pi_{K,n})] - \mathbb{E}[A(P_{K,n})]| = o(\sqrt{\operatorname{Var} A(\Pi_{K,n})}) \tag{3.4}$$

uniformly over all convex K of unit area.

Lemma 3.3. *As $n \to \infty$, we have:*

$$\operatorname{Var} N(\Pi_{K,n}) \sim \operatorname{Var} N(P_{K,n}) \tag{3.5}$$

$$\operatorname{Var} A(\Pi_{K,n}) \sim \operatorname{Var} A(P_{K,n}) \tag{3.6}$$

uniformly over all convex K of unit area.

Lemma 3.1 says essentially that as $n \to \infty$, the functionals of $P_{K,n}$ and $\Pi_{K,n}$ have asymptotically the same distributions. Lemmas 3.2 and 3.3 give us equivalence of the expectation and variance in the two models. It is elementary to observe that these lemmas combine to give Corollary 1.2 (and also Corollary 1.4). We note that Van Vu has observed (3.4) in [7, p224, Proposition 3.1 and Remark 3.5]; our proof is different.

3.1 Proof of Lemma 3.1

Proof. We follow and slightly correct an argument of Reitzner [5, p492–3]. We thank the referee for asking us to clarify our use of Reitzner's argument, since it was this that led us to realize the error. The argument below becomes fallacious if we write $P(X \le x|E)$ (as Reitzner does) everywhere we have $P(X \le x \,\&\, E)$. The problem is that the the first equality in equation (3.9) (which is Reitzner's equation (13)) is false in this case (note our E is Reitzner's A). Thus the proof in [5] is typographically very close to being correct; there is no problem once we replace every $P(X \le x|E)$ with $P(X \le x \,\&\, E)$.

First, fix $\epsilon > 0$ and let $S_\epsilon = \bigcup_{\theta \in \mathbb{R}/2\pi} C(\theta, \epsilon)$ be the union of all caps of area ϵ. Let E_P and E_Π be the events that ∂P and $\partial \Pi$ are completely contained in S_ϵ respectively. Now trivially, we have that $P(E_P), P(E_\Pi) \to 1$ as $n \to \infty$, uniformly over all K of unit area.

If B is any event, then $|P(B) - P(B \,\&\, E)| \le 1 - P(E)$. Letting B be $X \le x$, we have:

$$\sup_K \sup_x |P(X(\Pi_{K,n}) \le x) - P(X(\Pi_{K,n}) \le x \,\&\, E_\Pi)| \to 0 \tag{3.7}$$

$$\sup_K \sup_x |P(X(P_{K,n}) \le x) - P(X(P_{K,n}) \le x \,\&\, E_P)| \to 0 \tag{3.8}$$

as $n \to \infty$, where X denotes either N or A.

Now consider $P(X(\Pi_{K,n}) \le x \,\&\, E_\Pi)$ and $P(X(P_{K,n}) \le x \,\&\, E_P)$. Observe that if we condition both probabilities on the number of points of the process in S_ϵ, then they become equal. Let us call this probability $P(X \le x \,\&\, E|k)$. In other words, suppose we place k points uniformly at random in S_ϵ. Then $P(X \le x \,\&\, E|k)$ is defined to equal the probability that the boundary of their convex hull is contained in S_ϵ and $X \le x$. Thus setting $p = \text{Area}(S_\epsilon)$, we find:

$$\begin{aligned}
&|P(X(\Pi_{K,n}) \le x \,\&\, E_\Pi) - P(X(P_{K,n}) \le x \,\&\, E_P)| \\
&\quad = \left| \sum_{k=0}^{\infty} \frac{(np)^k}{k!} e^{-np} P(X \le x \,\&\, E|k) - \binom{n}{k} p^k (1-p)^{n-k} P(X \le x \,\&\, E|k) \right| \\
&\quad \le \sum_{k=0}^{\infty} \left| \frac{(np)^k}{k!} e^{-np} - \binom{n}{k} p^k (1-p)^{n-k} \right| \le 2p
\end{aligned} \tag{3.9}$$

where $\binom{n}{k} = 0$ if $k > n$. The last bound is due to Vervaat [6]. Combining (3.7) and (3.8) with (3.9), we find that:

$$\limsup_{n\to\infty} \sup_K \sup_x |P(X(\Pi_{K,n}) \le x) - P(X(P_{K,n}) \le x)| \le 2 \sup_K \text{Area}(S_\epsilon) \tag{3.10}$$

But we may choose $\epsilon > 0$ arbitrarily, so we are done. □

It may indeed be possible to take a similar strategy to prove Lemmas 3.2 and 3.3. However, in this case bounding the sum in equation (3.9) becomes harder, since we have expectations instead of probabilities, and the former are not bounded by 1. Also, proving analogues of equations (3.7) and (3.8) becomes nontrivial. Since we need good estimates for Lemmas 3.2 and 3.3, choosing ϵ correctly as a function of n and estimating $P(E_\Pi)$ and $P(E_P)$ becomes an issue.

3.2 Proofs of Lemmas 3.2 and 3.3

The proofs of Lemmas 3.2 and 3.3 will make use of some simple integral geometric expressions for the expectations and variances in question. The derivation of these expressions is completely elementary. The integrals appear complicated, though the point is not their exact form, but rather that they are almost identical for P_K and Π_K. With the appropriate integrals in hand, the desired convergence essentially reduces to the fact that $(1 + \frac{\alpha}{n})^n \to e^\alpha$ as $n \to \infty$. So, before, we begin the proofs, we make some elementary observations about this convergence. If $e^{-n} < x \le 1$, then $0 < 1 + \frac{\log x}{n} \le 1$, so:

$$n \log\left(1 + \frac{\log x}{n}\right) \le n \frac{\log x}{n} = \log x \implies \frac{1}{x}\left(1 + \frac{\log x}{n}\right)^n \le 1 \tag{3.11}$$

If additionally it holds that $(\log x)^2 \leq \frac{n}{2}$, then:

$$n \log\left(1 + \frac{\log x}{n}\right) = n\left(\frac{\log x}{n} + O\left(\frac{(\log x)^2}{n^2}\right)\right) = \log x + O\left(\frac{(\log x)^2}{n}\right)$$
$$\implies \frac{1}{x}\left(1 + \frac{\log x}{n}\right)^n = 1 + O\left(\frac{(\log x)^2}{n}\right) \quad (3.12)$$

For the proofs of Lemmas 3.2 and 3.3, it is most convenient to use the normalization $\mathrm{Area}(K) = n$ and $\lambda = 1$ (breaking from our previous convention). Thus n will be a positive integer, K will have area n, and we let $\Pi_K = \Pi_{K,1}$ and $P_K = P_{K,n}$.

Proof of Lemma 3.2. Let X denote either N or A. In the derivation of the integral geometric expressions, we treat Π_K and P_K simultaneously.

The following formula is tautological:

$$\mathbb{E}[X] = \int_{\mathbb{R}/2\pi} \int_K I_X(p, \theta)\, d\theta \quad (3.13)$$
$$I_X(p, \theta) = \frac{d}{dh} \mathbb{E}[X(\theta, \theta + h) | W(\theta) = p]\Big|_{h=0} dP(W(\theta) = p) \quad (3.14)$$

Now let us derive expressions for $I_X(p, \theta)$ for the uniform and Poisson models respectively. It will be convenient to let $y_{\theta,p}$ equal $f(p, \theta)^{-1}$ times the distance from p to ∂K in the positive θ direction. For every angle θ, the coordinates $x_{p,\theta} := \exp(-A(p, \theta))$ and $y_{p,\theta}$ give a bijection between K and $[e^{-n}, 1] \times [0, 1]$. It will prove very useful to express points in K in terms of these coordinates, mostly because $dP(W_\Pi(\theta) = p) = \exp(-A(p, \theta))\, dp = dx\, dy$.

First, let us observe that $\frac{d}{dh} \mathbb{E}[X(\theta, \theta + h)|W(\theta) = p]\big|_{h=0}$ is equal to:

$$\text{For } \Pi_K \text{ and } X = N: \quad \frac{1}{2} y_{p,\theta}^2 f(p, \theta)^2 \quad (3.15)$$
$$\text{For } P_K \text{ and } X = N: \quad \frac{1}{2} y_{p,\theta}^2 f(p, \theta)^2 \frac{n-1}{n - A(p, \theta)} \quad (3.16)$$
$$\text{For } \Pi_K \text{ and } X = A: \quad \frac{1}{2} y_{p,\theta}^2 f(p, \theta)^2 \quad (3.17)$$
$$\text{For } P_K \text{ and } X = A: \quad \frac{1}{2} y_{p,\theta}^2 f(p, \theta)^2 \quad (3.18)$$

And we also observe that $dP(W(\theta) = p)$ equals:

$$\text{For } \Pi_K: \quad \exp(-A(p, \theta))\, dp \quad (3.19)$$
$$\text{For } P_K: \quad \left(1 - \frac{A(p, \theta)}{n}\right)^n \frac{n}{n - A(p, \theta)}\, dp \quad (3.20)$$

Using coordinates x and y in the integral (3.13) and substituting our expressions for $I_X(p, \theta)$, we observe that we can integrate out $y_{p,\theta}$ in every case. The reader can check that the final expressions are:

$$\mathbb{E}[X] = \int_{\mathbb{R}/2\pi} \int_{e^{-n}}^1 I_X(x, \theta)\, dx\, d\theta \quad (3.21)$$

where $I_X(x,\theta)$ equals:

$$\text{For } \Pi_K \text{ and } X = N: \quad \frac{1}{6} f(p,\theta)^2 \tag{3.22}$$

$$\text{For } P_K \text{ and } X = N: \quad \frac{1}{6} f(p,\theta)^2 \frac{1}{x}\left(1+\frac{\log x}{n}\right)^{n-2}\left(1-\frac{1}{n}\right) \tag{3.23}$$

$$\text{For } \Pi_K \text{ and } X = A: \quad \frac{1}{6} f(p,\theta)^2 \tag{3.24}$$

$$\text{For } P_K \text{ and } X = A: \quad \frac{1}{6} f(p,\theta)^2 \frac{1}{x}\left(1+\frac{\log x}{n}\right)^{n-1} \tag{3.25}$$

We now proceed to use the representations (3.21) and (3.22)–(3.25) to show that the expectations of $X(\Pi_K)$ and $X(P_K)$ are the same up to a relative error of $O(n^{-1+\epsilon})$.

First, observe that our estimate on the growth of f (Lemma 2.6) shows that cutting off the integral (3.21) to $x \geq n^{-B}$ for some large fixed B incurs a relative error of no more than $n^{-B+\epsilon}$. Now for $x \in [n^{-B}, 1]$, we may use (3.12) to see that the relative error incurred by replacing $\frac{1}{x}\left(1+\frac{\log x}{n}\right)^n$ by 1 is no more than $\frac{(\log n)^2}{n}$. Observe also that for $x \in [n^{-B}, 1]$, we know that replacing $1 + \frac{\log x}{n}$ with 1 incurs a relative error of no more than $\frac{\log n}{n}$. These operations suffice to transform between the expressions for $\mathbb{E}[X(\Pi_K)]$ and $\mathbb{E}[X(P_K)]$, so we have shown that they are equal up to a relative error of $O(n^{-1+\epsilon})$.

Thus to finish the proof, we just need to show that:

$$\mathbb{E}[X(\Pi)]n^{-1+\epsilon} = o(\sqrt{\operatorname{Var} X(\Pi)}) \tag{3.26}$$

By a result of Bárány and Reitzner [2], $\operatorname{Var} X(\Pi) \gg \mathbb{E}[X(\Pi)]$, so it suffices to show that $\sqrt{\mathbb{E}[X(\Pi)]} = o(n^{1-\epsilon})$. It is trivial to see that $\mathbb{E}[X(\Pi)] \leq n$, so we are done. □

Proof of Lemma 3.3. This proof follows the same outline, so we will be a little less explicit; the interested reader can write down the long integrals if they so desire. Again, we let X denote either N or A.

Our plan is to show that $\mathbb{E}[X^2]$ is the same in the two cases up to a relative error of $O(n^{-1+\epsilon})$.

We think of X as being the integral of a random measure μ_X on $\mathbb{R}/2\pi$. This random measure is just given by the family of variables $X(\alpha,\beta)$ (explicitly, the measure of the interval $[\alpha,\beta]$ is $X(\alpha,\beta)$). Now X^2 is just the total mass of $\mu_X \otimes \mu_X$ on $(\mathbb{R}/2\pi)^2$. Using linearity of expectation, we just need to take the $d\theta\, d\psi$ integral of the expectation of $X(\theta,\theta + d\theta)X(\psi,\psi+d\psi)$. This expectation in turn, we condition on $W(\theta)$ and $W(\psi)$, writing it as an integral $dP((W(\theta),W(\psi)) = (p,q))$ over $K \times K$. We will often implicitly use the fact that the integrand is *positive*.

The first step is to show that we may remove the region where either $\exp(-A(p,\theta)) < n^{-B}$ or $\exp(-A(q,\psi)) < n^{-B}$ and incur a relative error of $O(n^{-1+\epsilon})$. By symmetry, let us deal with the region where $\exp(-A(p,\theta)) < n^{-B}$. Then the contribution to the total integral representing $\mathbb{E}[X^2]$ is just:

$$\int_{\mathbb{R}/2\pi} \int_{\{p \in K : A(p,\theta) \geq B \log n\}} \frac{d}{dh}\mathbb{E}[X(\theta,\theta+h)\cdot X | W(\theta) = p]\bigg|_{h=0} dP(W(\theta) = p)\, d\theta \tag{3.27}$$

Now the X in the expectation contributes at most a multiplicative factor of $n+2$. With this X removed, the integral becomes something we already estimated in the proof of Lemma 3.2 as being $O(n^{-B+\epsilon})$. Thus we are done.

The second step is to show that on the region where $A(p,\theta)$ and $A(q,\psi)$ are both $\leq B\log n$, the integrands (corresponding to $\mathbb{E}[X(\theta,\theta+d\theta)X(\psi,\psi+d\psi)]$ in the respective models) are equal up to a relative error of $O(n^{-1+\epsilon})$. As before, this splits up into two problems:

First, we need to show that the probability densities $dP((W(\theta),W(\psi))=(p,q))$ in the cases of Π_K and P_K are the same up to a relative error of $O(n^{-1+\epsilon})$. As before, we may express $dP((W(\theta),W(\psi))=(p,q))$ elementarily in terms of $A(p,q,\theta,\psi):=\mathrm{Area}(C(p,\theta)\cup C(q,\psi))$. Then the fact that this quantity is $O(\log n)$ means we may apply (3.12) to see that the densities are equal up to a relative error of $O(n^{-1+\epsilon})$ (note that this is true even for the singular part of the measure $dP((W(\theta),W(\psi))=(p,q))$ occurring on the diagonal $p=q$).

Second, we need to show that the incremental expectations $\mathbb{E}[X(\theta,\theta+d\theta)X(\psi,\psi+d\psi)]$ are the same up to a relative error of $O(n^{-1+\epsilon})$. Again, this just involves writing equations such as (3.15)–(3.18). Then the fact that $A(p,q,\theta,\psi)=O(\log n)$ shows easily that they coincide up to a relative error of $O(n^{-1+\epsilon})$. Though it presents no difficulty in the proof, one should note that when $X=N$, there is a singular component to the measure $\mathbb{E}[X(\theta,\theta+d\theta)X(\psi,\psi+d\psi)]$ on the diagonal $\theta=\psi$.

We have shown that:

$$|\mathbb{E}[X(\Pi_K)^2]-\mathbb{E}[X(P_K)^2]|\ll n^{-1+\epsilon}\mathbb{E}[X(\Pi_K)^2] \tag{3.28}$$

Now $\mathbb{E}[X(\Pi_K)^2]=\mathbb{E}[X(\Pi_K)]^2+\mathrm{Var}\,X(\Pi_K)$. Thus we have:

$$|\mathbb{E}[X(\Pi_K)^2]-\mathbb{E}[X(P_K)^2]|\ll n^{-1+\epsilon}\max(\mathbb{E}[X(\Pi_K)]^2,\mathrm{Var}\,X(\Pi_K)) \tag{3.29}$$

In the proof of Lemma 3.2, we showed that $\mathbb{E}[X(\Pi_K)]$ and $\mathbb{E}[X(P_K)]$ are the same up to a relative error of $O(n^{-1+\epsilon})$. This implies then that:

$$|\mathbb{E}[X(\Pi_K)]^2-\mathbb{E}[X(P_K)]^2|\ll n^{-1+\epsilon}\mathbb{E}[X(\Pi_K)]^2 \tag{3.30}$$

Thus it follows that:

$$|\mathrm{Var}\,X(\Pi_K)-\mathrm{Var}\,X(P_K)|\ll n^{-1+\epsilon}\max(\mathbb{E}[X(\Pi_K)]^2,\mathrm{Var}\,X(\Pi_K)) \tag{3.31}$$

Thus to finish the proof, we just need to show that $n^{-1+\epsilon}\mathbb{E}[X(\Pi)]^2=o(\mathrm{Var}[X(\Pi)])$. By a result of Bárány and Reitzner [2], $\mathrm{Var}\,X(\Pi)\gg\mathbb{E}[X(\Pi)]$, so it suffices to show that $\mathbb{E}[X(\Pi)]=o(n^{1-\epsilon})$. It is a well known estimate (see [4]) that $\mathbb{E}[X(\Pi)]\ll n^{1/3}$, so we are done. □

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