Central limit theorems for random polygons in an arbitrary convex set

John Pardon

19 March 2010; Revised 11 May 2010

Abstract

We study the probability distribution of the area and the number of vertices of random polygons in a convex set $K \subset \mathbb{R}^2$. The novel aspect of our approach is that it yields uniform estimates for all convex sets $K \subset \mathbb{R}^2$, without imposing any regularity conditions on the boundary ∂K. Our main result is a central limit theorem for both the area and the number of vertices, settling a well known conjecture in the field. We also obtain asymptotic results relating the growth of the expectation and variance of these two functionals.

1 Introduction

Consider a Poisson point process in a convex set $K \subset \mathbb{R}^2$ of intensity equal to the Lebesgue measure. We denote by Π_K the convex hull of the points of this process; Π_K is called a *random Poisson polygon*. We denote by $N = N(\Pi_K)$ the number of vertices of Π_K and by $A = A(\Pi_K)$ the area of $K \setminus \Pi_K$. In this paper, we develop techniques to study the distributions of these random variables. Our main result is a central limit theorem, which is *uniform* over the set of all convex $K \subset \mathbb{R}^2$:

Theorem 1.1. *As* $\operatorname{Area}(K) \to \infty$*, we have the following central limit theorems for* Π_K*:*

$$\sup_x \left| P\left(\frac{N - \mathbb{E}[N]}{\sqrt{\operatorname{Var} N}} \leq x\right) - \Phi(x) \right| \ll \frac{\log^2 \mathbb{E}[N]}{\sqrt{\mathbb{E}[N]}} \tag{1.1}$$

$$\sup_x \left| P\left(\frac{A - \mathbb{E}[A]}{\sqrt{\operatorname{Var} A}} \leq x\right) - \Phi(x) \right| \ll \frac{\log^2 \mathbb{E}[A]}{\sqrt{\mathbb{E}[A]}} \tag{1.2}$$

Here $\Phi(x) = P(Z \leq x)$ *where* Z *is the standard normal distribution.*

The novel aspect of our approach is that we require no regularity on ∂K; it is this that enables us to obtain bounds which are uniform over all convex sets. Previous results on random polygons analogous to Theorems 1.1 have been confined to two cases: (i) K a polygon [7, 5] and (ii) ∂K of class C^2 with nonvanishing curvature [8]. The key part of our argument is our use of a new compactness result for various types of local configuration spaces of convex boundaries.

As a consequence of our techniques, we also prove the following:

Theorem 1.2. *As* Area$(K) \to \infty$*, we have the following estimates for* Π_K*:*[1]

$$\mathbb{E}[N] \asymp \operatorname{Var} N \asymp \mathbb{E}[A] \asymp \operatorname{Var} A \tag{1.3}$$

In other words, there is (up to a constant factor) only one parameter, say $\mathbb{E}[A]$, which controls the asymptotics of the distributions of N and A. Thus, for example, the error terms in Theorem 1.1 could have instead been stated in terms of the variances.

For completeness, we should mention what is known about the growth of (say) $\mathbb{E}[A]$, which can be effectively estimated using elementary geometric and combinatorial techniques. In dimension two, one has:

$$\log[\text{Area}(K)] \ll \mathbb{E}[A] \ll [\text{Area}(K)]^{1/3} \tag{1.4}$$

(In particular, the error terms in Theorem 1.1 go to zero as Area$(K) \to \infty$). The estimate (1.4) is a consequence of the *economic cap covering lemma* of Bárány and Larman [2] in combination with other estimates in [2] and those of Groemer [6] (in fact, their results apply to higher dimensions as well). We remark that the lower asymptotic is achieved when K is a polygon and the upper asymptotic is achieved when ∂K is C^2 with nonvanishing curvature.

We conclude by remarking that in recent years there has been significant progress in the study of random *polytopes*, but again most results deal only with the cases when (i) K is a polytope [4] and (ii) ∂K is C^2 with nonvanishing Gauss curvature [10, 15]. We believe that an approach similar to ours should be possible in higher dimensions as well. This would shed new light on problems in that setting, and ultimately show that there is no qualitative difference between the cases (i) and (ii).

1.1 The uniform model random polygons

A model related to Π_K is $P_{K,n} := \text{conv. hull.}(X_1, \ldots, X_n)$ where X_i are i.i.d. uniformly in K; $P_{K,n}$ is called a *random polygon*. This is often referred to as the "uniform model" whereas Π_K is the "Poisson model". Morally they are the same process in the limit Area$(K) = n \to \infty$ (though making this precise is often difficult). It has been a well known open problem to prove central limit theorems for functionals of $P_{K,n}$. For instance, Van Vu [1] has asked the question of whether a central limit theorem holds for $A(P_{K,n})$, though the problem is a very natural one in the study of random polygons, a subject that began with work of Rényi and Sulanke [11, 12]. Theorems 1.1 and 1.2 both carry over to the setting of $P_{K,n}$, thus answering this question in the affirmative:

Corollary 1.3. *As* $n \to \infty$*, we have the following central limit theorems for* $P_{K,n}$*:*

$$\sup_x \left| P\left(\frac{N - \mathbb{E}[N]}{\sqrt{\operatorname{Var} N}} \le x \right) - \Phi(x) \right| \to 0 \tag{1.5}$$

$$\sup_x \left| P\left(\frac{A - \mathbb{E}[A]}{\sqrt{\operatorname{Var} A}} \le x \right) - \Phi(x) \right| \to 0 \tag{1.6}$$

uniformly over all convex K*. Here* $\Phi(x) = P(Z \le x)$ *where* Z *is the standard normal distribution.*

[1] After this paper was written, we learned that Imre Bárány and Matthias Reitzner have independently proved this result, as well as the closely related Corollary 1.4.

Corollary 1.4. *As $n \to \infty$, we have the following estimates for $P_{K,n}$:*

$$\mathbb{E}[N] \asymp \operatorname{Var} N \asymp \frac{n}{\operatorname{Area}(K)} \mathbb{E}[A] \asymp \left(\frac{n}{\operatorname{Area}(K)} \right)^2 \operatorname{Var} A \tag{1.7}$$

uniformly over all convex K.

As in the case of the Poisson model, these results are well known in the field in the two cases (i) K a polygon and (ii) ∂K of class C^2 with nonvanishing curvature. The innovation in this paper is that all K are treated uniformly.

A detailed derivation of Corollaries 1.3 and 1.4 from Theorems 1.1 and 1.2 will appear elsewhere [9]. Suffice it to say here that they are almost immediate consequences of the corresponding results on the Poisson model once one proves that when $n = \operatorname{Area}(K)$, the variables $N(P_{K,n})$ and $N(\Pi_K)$ (as well as $A(P_{K,n})$ and $A(\Pi_K)$) have the same expectation and variance up to a small enough error.

Acknowledgements

The author thanks Yakov Sinai for conversations about this work as well as opportunities to speak about it in seminars. Thanks also go to Imre Bárány and Matthias Reitzner for posting [3] on Bárány's website. The author is also grateful to the referee for many helpful comments.

2 The basic decomposition

In this section, we illustrate our basic approach. We will aim for Theorem 1.1, and Theorem 1.2 will be a corollary of our methods.

First, we observe that the functionals N and A both enjoy decompositions into local pieces. We define $N(\alpha, \beta)$ to equal the number of edges of Π whose angle lies in the interval $[\alpha, \beta] \subset \mathbb{R}/2\pi$. The definition of $A(\alpha, \beta)$ is best explained graphically (see Figure 2.1). Thus for any fixed sequence of angles $\alpha_1 < \alpha_2 < \cdots < \alpha_L$, we have the following decompositions:

$$N = N(\alpha_1, \alpha_2) + \cdots + N(\alpha_L, \alpha_1) \tag{2.1}$$

$$A = A(\alpha_1, \alpha_2) + \cdots + A(\alpha_L, \alpha_1) \tag{2.2}$$

During the proof, we often do not need to distinguish between whether we are dealing with N or A. Thus we will use $X(\Pi)$ to denote either N or A when a statement holds for both.

A central limit theorem will follow if we can find a choice of $\{\alpha_i\}$ such that the moments of $X(\alpha_i, \alpha_{i+1})$ are bounded uniformly, and such that the dependence between $X(\alpha_i, \alpha_{i+1})$ and $X(\alpha_j, \alpha_{j+1})$ becomes small as $|i-j| \to \infty$. Our construction is to choose $\{\alpha_i\}$ so that the intervals $[\alpha_i, \alpha_{i+1}]$ have constant *affine invariant measure* (a measure depending on K). In this paper, we give a more or less explicit description of the affine invariant measure, which in practice should allow its easy estimation for any given class of convex sets, and thus a complete description of the behaviour of random Poisson polygons and random polygons.

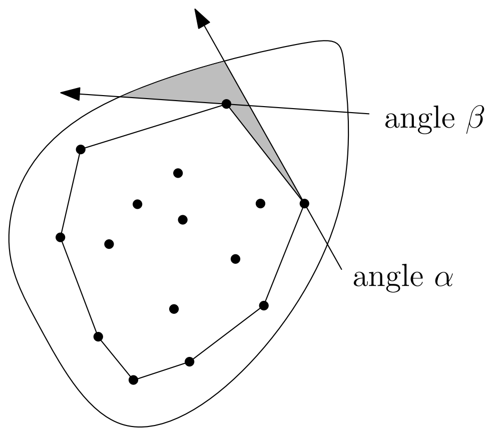


Figure 2.1: Illustration of $A(\alpha, \beta)$

As we remarked in the introduction, a key result is the compactness of various configuration spaces.

After fixing notation in Section 3, we define the affine invariant measure in Section 4. Section 5 is devoted to the crucial step of proving the compactness of the configuration spaces. Using the information coming from compactness:

- In Section 6, we estimate the moments of X (Proposition 6.1).
- In Section 7, we estimate the long range dependence of X (Proposition 7.5).
- In Section 8, we recall an estimate the variance of X due to Imre Bárány and Matthias Reitzner (Proposition 8.1).

The remainder of the paper contains the explicit deduction of Theorems 1.1 and 1.2.

3 Notation and definitions

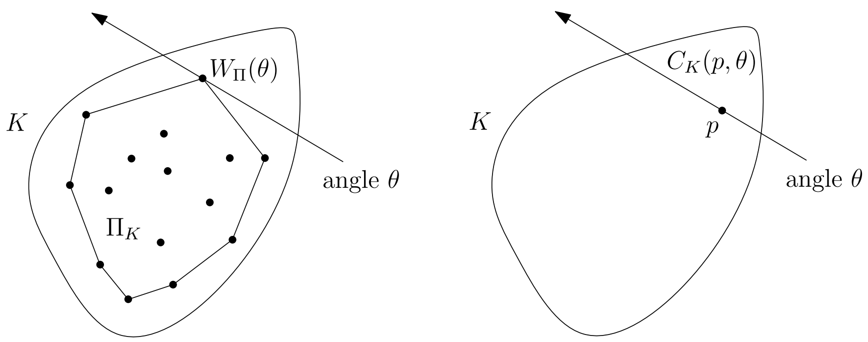
In this paper, K will always denote a (bounded) convex set in $\mathbb{R}^2$.

We warn the reader that in most of the literature, one fixes $\mathrm{Area}(K) = 1$ and then considers a Poisson process of intensity $\lambda \to \infty$. We have chosen instead to use the normalization $\lambda = 1$ and let $\mathrm{Area}(K) \to \infty$. This is convenient for us because it makes many of our formulas simpler to state.

Any constants implied by the symbols $\ll$, $\gg$, or $\asymp$ are absolute; in particular they are not allowed to depend on K. There will be times when we require $\mathrm{Area}(K) \gg 1$; this is no real restriction to us since in the end we will take $\mathrm{Area}(K) \to \infty$. The group $\mathrm{Aff}(2) = \mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ is the group of (oriented) area preserving affine transformations of $\mathbb{R}^2$; it acts naturally on the entire problem studied here.

Many of the following definitions are illustrated in Figure 3.1. We may leave out the subscript K later when doing so is unambiguous.

Definition 3.1. We define the random variable $W_K(\theta)$ to be the vertex of Π_K which has an oriented tangent line at angle θ. This is illustrated in Figure 3.1(a).



(a) Illustration of $W_K(\theta)$ (b) Illustration of $C_K(p,\theta)$

Figure 3.1: Illustration of some definitions.

Definition 3.2. A *cap at angle* θ is the intersection of K with a halfplane H_θ at angle θ. We may specify a cap at angle θ by giving either its area r or a point $p \in \partial H_\theta$. These are denoted $C_K(r,\theta)$ and $C_K(p,\theta)$ respectively; the latter is illustrated in Figure 3.1(b).

Definition 3.3. We define the real number $A_K(p,\theta)$ to be the area of the cap $C_K(p,\theta)$.

Lemma 3.4. *The random variable* $W_K(\theta)$ *has probability distribution given by* $\exp(-A_K(p,\theta))\,dp$ *where* dp *is the Lebesgue measure.*

Proof. This follows directly from the definition of a Poisson point process. □

Definition 3.5. We define the function $f_K(x,\theta) : [0,1] \times \mathbb{R}/2\pi \to \mathbb{R}$ as follows:

$$f_K(x,\theta) = \begin{cases} \text{length of } (\partial H_\theta) \cap K \text{ where } C_K(\log \frac{1}{x}, \theta) = H_\theta \cap K \\ \qquad\qquad\qquad\qquad \text{if } x > \exp(-\operatorname{Area}(K)) \\ 0 \qquad\qquad\qquad\quad\;\; \text{if } x \le \exp(-\operatorname{Area}(K)) \end{cases} \tag{3.1}$$

It will be important to have the following bound on the growth of f:

Lemma 3.6. *If* $y \le x$*, then:*

$$\frac{f(y)}{\sqrt{-\log y}} \le \frac{f(x)}{\sqrt{-\log x}} \tag{3.2}$$

The bound above is sharp; for instance $f(x) = \text{const} \cdot \sqrt{-\log x}$ for $K = \{x, y \ge 0\}$ (i.e. the first quadrant).

Proof. Project K along the lines at angle θ to get a height function $h : [0,\infty) \to \mathbb{R}_{\ge 0}$; in Figure 3.2, $h(\ell)$ is the length of the thick segment. Now if $A(\ell) = \int_0^\ell h(\ell')\,d\ell'$ then $f(\exp(-A(\ell))) = h(\ell)$. Thus we see that it suffices to show that the function:

$$\frac{h(\ell)}{\sqrt{A(\ell)}} \tag{3.3}$$

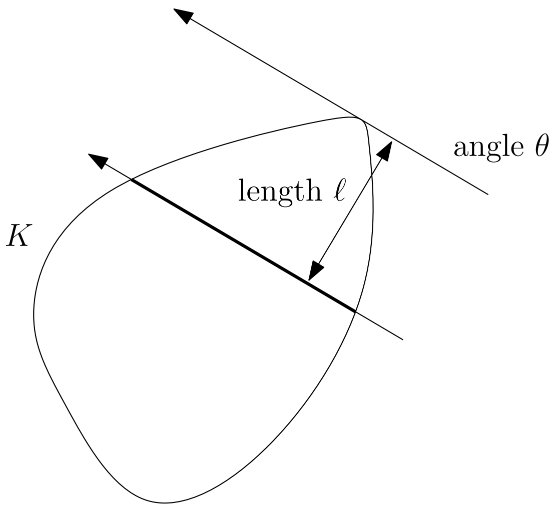


Figure 3.2: Illustration of the function h.

is decreasing. Differentiating with respect to ℓ, we see that it suffices to show that:

$$h(\ell)^2 - 2h'(\ell)A(\ell) \geq 0 \tag{3.4}$$

For $\ell = 0$, the left hand side is clearly nonnegative, and the derivative of the left hand side equals $-2h''(\ell)A(\ell)$, which is ≥ 0 by concavity of h. $\square$

Lemma 3.7. *If* $\mathrm{Area}(K) \geq 2\log\frac{1}{x}$*, then* $f(y) \leq 2f(x)$ *for* $y \geq x$*.*

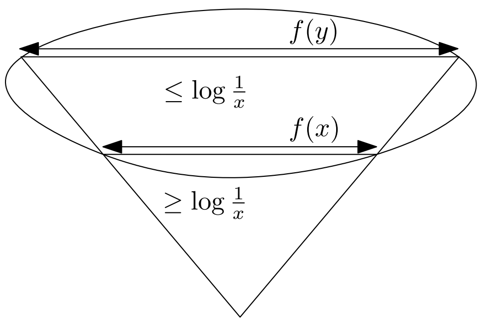


Figure 3.3: Illustration of an inequality.

Proof. Refer to Figure 3.3. The area of the upper trapezoid is $\leq \log\frac{1}{x}$ since it is contained in $C(\log\frac{1}{x}, \theta)$. The area of the lower triangle is $\geq \log\frac{1}{x}$ since it contains $K \setminus C(\log\frac{1}{x}, \theta)$ and $\mathrm{Area}(K) \geq 2\log\frac{1}{x}$. Similar triangles gives the following inequality:

$$\frac{f(y) - f(x)}{\log\frac{1}{x}} \leq \frac{f(x)}{\log\frac{1}{x}} \tag{3.5}$$

Simplifying yields $f(y) \leq 2f(x)$. $\square$

4 The affine invariant measure

Proposition 4.1. *For every* $g \in \mathrm{Aff}(2)$*, we have:*

$$r_g^* \left[f_{gK}(x, \theta)^2 \, d\theta \right] = f_K(x, \theta)^2 \, d\theta \tag{4.1}$$

where $r_g : \mathbb{R}/2\pi \to \mathbb{R}/2\pi$ *is the action of* g *on line slopes. We say "*μ_x *is affine invariant".*

Proof. Define $\mathbf{v}(\theta)$ to be the vector of length $f(x, \theta)$ parallel to the chord whose length gives $f(x, \theta)$. Then we have:

$$\int_{\theta_1}^{\theta_2} f(x, \theta)^2 \, d\theta = \int_{\theta_1}^{\theta_2} \mathbf{v}(\theta) \times d\mathbf{v}(\theta) \tag{4.2}$$

The right hand side is invariant under the action of Aff(2), so the result follows. □

Definition 4.2. We define the *affine invariant measure* to be $\mu_K := f_K(e^{-1}, \theta)^2 \, d\theta$.

The ϵ*-wet part* of K is defined as the union of all caps of area ϵ. In the literature, estimates for random polygons are frequently expressed in terms of the area of the ϵ-wet part of K. It is perhaps not surprising that our notion of the affine invariant measure is related to the area of the wet part in the following manner:

Lemma 4.3. *One has the following relation:*

$$\mathrm{Area} \left(\bigcup_{\gamma \in [\alpha, \beta]} C_K(1, \gamma) \right) = 1 + \frac{1}{8} \mu_K([\alpha, \beta]) \tag{4.3}$$

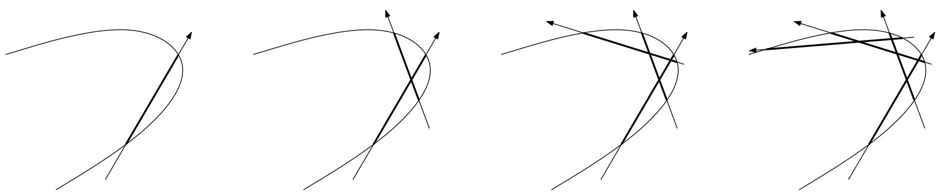
Proof. Consider the area swept out by the line segments bounding the caps of area 1 at angles $\gamma \in [\alpha, \beta]$ (area covered twice is counted twice). On the one hand, this area just equals:

$$2 \, \mathrm{Area} \left(\bigcup_{\gamma \in [\alpha, \beta]} C_K(1, \gamma) \right) - \mathrm{Area}(C_K(1, \alpha)) - \mathrm{Area}(C_K(1, \beta)) \tag{4.4}$$

On the other hand, we may express the area as an integral $d\theta$. Each line segment rotates about its midpoint (since the area of the caps is constant), so the area covered is just the $d\theta$ integral of $\int_{-f(e^{-1},\theta)/2}^{f(e^{-1},\theta)/2} |y| \, dy = \frac{1}{4} f(e^{-1}, \theta)^2$. Comparing this with (4.4) yields the result. □

5 Compactness of configuration spaces

Definition 5.1. Define a configuration space $\mathcal{C}(r)$ for $r > 0$ as follows. The objects of $\mathcal{C}(r)$ are convex subsets of $\mathbb{R}^2$ of area r with a distinguished line segment on their boundary. As a set, $\mathcal{C}(r)$ is equal to everything of the form $(H \cap K, (\partial H) \cap K)$, where K is any convex set of area $\geq 2r$ and H is a halfplane such that $H \cap K$ has area r. A typical member of $\mathcal{C}(r)$ is illustrated in Figure 5.1(a). We emphasize that the space $\mathcal{C}(r)$ does not depend on any choice of convex set K, rather it is the space of *all* caps of area r that come from some convex set of area $\geq 2r$.



(a) (b) (c) (d)

Figure 5.1: A series of caps.

We call $\mathcal{C}(r)$ the configuration space of caps of area r. If $c \in \mathcal{C}(r)$, then we call the distinguished part of its boundary its *flat boundary* and the undistinguished part of its boundary its *convex boundary*. We let the *halfplane of* c equal the unique halfplane which contains c and whose boundary contains the flat boundary of c (this is exactly the H appearing above).

We topologize $\mathcal{C}(r)$ by using the Hausdorff metric to compare both the set and its distinguished subset. Explicitly, $d((A, A_0), (B, B_0)) = d(A, B) + d(A_0, B_0)$. Let us observe that there is a natural action of $\operatorname{Aff}(2)$ on $\mathcal{C}(r)$; it is continuous. Certainly $\mathcal{C}(r)$ is not compact, since the group $\operatorname{Aff}(2)$ is noncompact. However, we will show directly that $\mathcal{C}(r)/\operatorname{Aff}(2)$ is compact. This simple fact will be an essential tool in virtually all of the estimates in the remainder of this paper.

Lemma 5.2. *The space* $\mathcal{C}(r)/\operatorname{Aff}(2)$ *is compact.*

Proof. Let $c_1, c_2, \ldots$ be a sequence of elements of $\mathcal{C}(r)/\operatorname{Aff}(2)$. Pick representatives $\tilde{c}_1, \tilde{c}_2, \ldots$ in $\mathcal{C}(r)$ so that the flat part of $\partial\tilde{c}_i$ is the unit line segment on the x-axis, $\tilde{c}_i$ is contained in the upper half plane, and the highest y-coordinate of any point in $\tilde{c}_i$ is attained at $(\frac{1}{2}, h_i)$. This is illustrated in Figure 5.2.

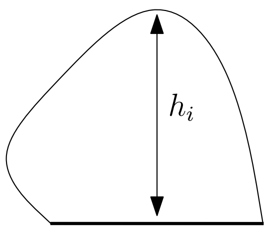


Figure 5.2: Compactness of $\mathcal{C}(r)/\operatorname{Aff}(2)$.

By Lemma 3.7, we conclude that $\operatorname{Area}(K) \geq 2r$ implies that every horizontal chord in $\tilde{c}_i$ has length ≤ 2. This implies that $-\frac{3}{2} \leq x \leq \frac{5}{2}$ for any x-coordinate of a point in $\tilde{c}_i$. On the other hand, $\tilde{c}_i$ contains a triangle of base 1 and height h_i, so by comparing areas we must have $\frac{1}{2}h_i \leq r$. Thus we conclude that $\tilde{c}_i \subseteq [-\frac{3}{2}, \frac{5}{2}] \times [0, 2r]$. It is well known that the space of convex sets of fixed volume in some bounded region of $\mathbb{R}^d$ given the Hausdorff topology is compact (this is the so-called Blaschke selection theorem). Thus we conclude that there exists a subsequence of $\tilde{c}_i$ that converges. □

Definition 5.3. We define the complex configuration space $\mathcal{C}(r_1, \epsilon, r_2)$ for $r_1, r_2 > 0$ and $0 < \epsilon < \min(r_1, r_2)$ as follows. We let $\mathcal{C}(r_1, \epsilon, r_2)$ denote a particular subset of $\mathcal{C}(r_1) \times \mathcal{C}(r_2)$. An ordered pair $(c_1, c_2) \in \mathcal{C}(r_1) \times \mathcal{C}(r_2)$ is in $\mathcal{C}(r_1, \epsilon, r_2)$ if and only if it satisfies the following:

- $\text{Area}(c_1 \cap c_2) = \epsilon$.
- If H_1 is the halfplane of c_1, then $H_1 \cap c_2 = c_1 \cap c_2$.
- If H_2 is the halfplane of c_2, then $c_1 \cap H_2 = c_1 \cap c_2$.
- $\text{angle}(H_1) < \text{angle}(H_2) < \text{angle}(H_1) + \pi$.

We then give $\mathcal{C}(r_1, \epsilon, r_2)$ the subspace topology.

One can see that the middle two conditions taken together just mean that c_1 and c_2 coincide on $H_1 \cap H_2$, and the last condition just says that c_1 precedes c_2 if we traverse their convex boundary counterclockwise. Examples appear in Figure 5.1(b) and in Figure 6.1.

Lemma 5.4. *The space* $\mathcal{C}(r_1, \epsilon, r_2)/\operatorname{Aff}(2)$ *is compact.*

Proof. Let $(c_1, d_1), (c_2, d_2), \ldots$ be a sequence of elements of the quotient $\mathcal{C}(r_1, \epsilon, r_2)/\operatorname{Aff}(2)$. Lift these to a sequence $(\tilde{c}_1, \tilde{d}_1), (\tilde{c}_2, \tilde{d}_2), \ldots$ in $\mathcal{C}(r_1, \epsilon, r_2)$ where we assume (after passing to a subsequence using Lemma 5.2) that $\tilde{c}_1, \tilde{c}_2, \ldots$ is convergent to $\tilde{c} \in \mathcal{C}(r_1)$.

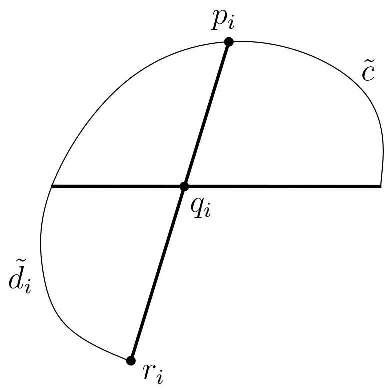


Figure 5.3: Compactness of $\mathcal{C}(r_1, \epsilon, r_2)/\operatorname{Aff}(2)$.

Now refer to Figure 5.3. Label the intersection of the flat boundary of $\tilde{d}_i$ with the convex boundary of $\tilde{c}_i$ as p_i. Label the intersection of the flat boundaries of $\tilde{d}_i$ and $\tilde{c}_i$ as q_i. Label the intersection of the flat boundary of $\tilde{d}_i$ with its convex boundary other than p_i as r_i. Clearly we can extract a subsequence for which p_i converges to a point p on the convex boundary of $\tilde{c}$, and then extract a further subsequence for which q_i converges to a point q on the flat boundary of $\tilde{c}$. The only subtlety in this proof is to observe that $0 < \epsilon < r_1$ shows that p and q are not on the corners of $\tilde{c}$.

Given p and q, the boundedness of the area of $\tilde{d}_i$ implies that r_i is bounded, so we extract another subsequence for which additionally r_i converges to a point r. Now it is easy to see that the fixing of $\tilde{c}, p, q, r$ provide only a bounded set for $\tilde{d}_i$ to range over, so compactness follows again using the Blaschke selection theorem. $\square$

Lemma 5.5. *There exists an absolute constant $M_0 < \infty$ such that if we are given K and angles $\alpha < \beta$ with $\mu_K([\alpha, \beta]) \geq M_0$, then we can find a sequence $\alpha \leq \gamma_0 < \gamma_1 < \cdots < \gamma_L \leq \beta$ so that $(C_K(\gamma_{i-1}, 1), C_K(\gamma_i, 1)) \in \mathcal{C}(1, \frac{1}{2}, 1)$ and $L \asymp \mu_K([\alpha, \beta])$.*

Proof. Let $\gamma_0 = \alpha$. Now define γ_i inductively for $i \geq 1$ as follows. The function:

$$\text{Area}(C(1, \gamma_{i-1}) \cap C(1, \gamma)) \quad \text{for } \gamma \in [\gamma_{i-1}, \gamma_{i-1} + \pi] \tag{5.1}$$

is strictly decreasing until it reaches zero, where it remains constant. Thus there exists a unique γ_i so that $\text{Area}(C(1, \gamma_{i-1}) \cap C(1, \gamma_i)) = \frac{1}{2}$. We now have an infinite chain of angles $\alpha = \gamma_0 < \gamma_1 < \gamma_2 < \cdots$ so that $C(1, \gamma_i) \cap C(1, \gamma_{i+1})$ has area $\frac{1}{2}$ for $i \geq 0$. This is illustrated in Figure 5.1.

Let L be the maximum index such that $\gamma_L \leq \beta$. Note that since $\mathcal{C}(1, \frac{1}{2}, 1)$ is compact, there exist absolute constants $0 < Y_1 < Y_2 < \infty$ (not depending on K) such that:

$$Y_1 < \mu_K([\gamma_i, \gamma_{i+1}]) < Y_2 \tag{5.2}$$

for all i. Thus we conclude that:

$$Y_1 L < \mu_K([\gamma_0, \gamma_L]) \leq \mu_K([\alpha, \beta]) < Y_2(L + 1) \tag{5.3}$$

which is sufficient. □

6 A moment estimate

An ingredient in the central limit theorems for the polygonal case is a moment estimate [7, p341 Lemma 2.5][5, p36 Lemma 2.1]. Here, we prove an analogous estimate in general.

Proposition 6.1. *Let X denote either N or A. There exist absolute constants $M_0 < \infty$ and $\epsilon > 0$ such that for any convex K and interval $[\alpha, \beta]$ with $\mu_K([\alpha, \beta]) \geq M_0$, we have the following estimate:*

$$\mathbb{E} \exp(\lambda X_K(\alpha, \beta)) \ll 1 \quad \textit{for all } |\lambda| < \epsilon / \mu_K([\alpha, \beta]) \tag{6.1}$$

Proof. We can split up $[\alpha, \beta]$ into subintervals of small affine invariant measure, and use Cauchy's inequality:

$$\mathbb{E} \exp(\lambda [A + B]) \leq \sqrt{[\mathbb{E} \exp(2\lambda A)][\mathbb{E} \exp(2\lambda B)]} \tag{6.2}$$

so it suffices to show that there exist $\delta > 0$ and $\epsilon > 0$ so that for all K and $[\alpha, \beta]$ satisfying $\mu_K([\alpha, \beta]) \leq \delta$, it holds that the moment generating function $\mathbb{E} \exp(\lambda X_K(\alpha, \beta))$ is $\ll 1$ for all $|\lambda| < \epsilon$.

Since $\mathcal{C}(1, \frac{1}{2}, 1)$ is compact, the affine invariant measure of the interval between the angles of c_1 and c_2 is bounded below. Thus we conclude that it suffices to show that for every $(c_1, c_2) \in \mathcal{C}(1, \frac{1}{2}, 1)$, the moment generating function of $X_K(\alpha, \beta)$ is defined in a neighborhood of zero where α is the angle of c_1 and β is the angle of c_2.

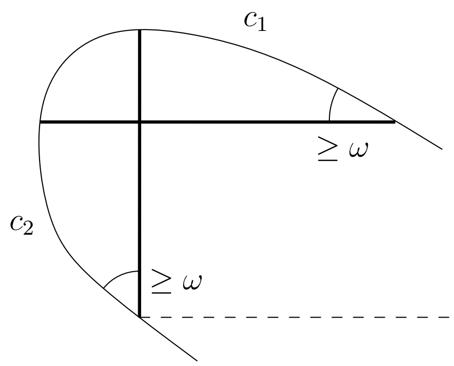


Figure 6.1: Two adjacent caps.

Now we may put such an element $(c_1, c_2) \in \mathcal{C}(1, \frac{1}{2}, 1)$ in a standard position in $\mathbb{R}^2$ by requiring that both boundary segments have equal length, and that the angles of c_1 and c_2 are 0 and $\frac{\pi}{2}$ respectively (see Figure 6.1).

Thus, given the configuration in Figure 6.1, we would like to show that for sufficiently small $\lambda > 0$, we have $\mathbb{E}\exp(\lambda X_K(0, \frac{\pi}{2})) \ll 1$. First, write:

$$\mathbb{E}\exp(\lambda X_K(0, \frac{\pi}{2})) = \int_K \mathbb{E}[\exp(\lambda X_K(0, \frac{\pi}{2}))|W(0) = p]\, dP(W(0) = p) \tag{6.3}$$

If $X = N$, then $X_K(0, \frac{\pi}{2})$ is bounded by the number of the Poisson process in the region $C(W(0), \frac{\pi}{2}) \setminus C(W(0), 0)$. An elementary calculation shows that $\mathbb{E}\exp(\lambda \Xi(k)) = \exp(k[e^\lambda - 1])$, where $\Xi(k)$ is a Poisson distribution of parameter k. We may assume $|\lambda| < 1$, so $e^\lambda - 1 < 2|\lambda|$. Thus in this case:

$$\mathbb{E}[\exp(\lambda X_K(0, \frac{\pi}{2}))|W(0) = p] \leq \exp(2|\lambda| \operatorname{Area}(C(p, \frac{\pi}{2}) \setminus C(p, 0))) \tag{6.4}$$

If $X = A$, then $X_K(0, \frac{\pi}{2})$ is bounded by $C(W(0), \frac{\pi}{2}) \setminus C(W(0), 0)$, so we have:

$$\mathbb{E}[\exp(\lambda X_K(0, \frac{\pi}{2}))|W(0) = p] \leq \exp(|\lambda| \operatorname{Area}(C(p, \frac{\pi}{2}) \setminus C(p, 0))) \tag{6.5}$$

Thus in both cases, we have the estimate:

$$\mathbb{E}\exp(\lambda X_K(0, \frac{\pi}{2})) \leq \int_K \exp(2|\lambda| \operatorname{Area}(C(p, \frac{\pi}{2}) \setminus C(p, 0))) \exp(-A(p, 0))\, dp \tag{6.6}$$

recalling Lemma 3.4.

By compactness of $\mathcal{C}(1, \frac{1}{2}, 1)/\operatorname{Aff}(2)$, the angle where the convex part of c_i meets the flat boundary of c_i is bounded below by an absolute constant (say by ϖ, see Figure 6.1). Similarly, the lengths of the flat parts of c_1 and c_2 are bounded above absolutely (say by $R \geq 1$). Thus the area above the dotted line in Figure 6.1 is bounded above absolutely, say by $B = 2 + R^2 + R^2 \cot \varpi$.

Now we claim that:

$$\operatorname{Area}(C(p, \frac{\pi}{2}) \setminus C(p, 0)) \leq B + f(p, 0)^2 \cot \omega \tag{6.7}$$

(recall that $f(p,0)$ is the length of $\ell \cap K$ where ℓ is the horizontal line passing through p). If $p \in c_1$, then the area of $C(p, \frac{\pi}{2}) \setminus C(p,0)$ is $\leq B$ by definition. If $p \notin c_1$, then argue as follow: the area of $C(p, \frac{\pi}{2}) \setminus C(p,0)$ above the dotted line is certainly less than B, and the area of $C(p, \frac{\pi}{2}) \setminus C(p,0)$ below the dotted line is bounded by $f(p,0)^2 \cot \omega$.

Thus we have:

$$\mathbb{E} \exp(\lambda X_K(0, \frac{\pi}{2})) \leq e^{2|\lambda|B} \int_K \exp(2|\lambda| f(p,0)^2 \cot \omega) \exp(-A(p,0))\, dp \tag{6.8}$$

If we substitute $x = \exp(-A(p,0))$, then the integral becomes:

$$\mathbb{E} \exp(\lambda X_K(0, \frac{\pi}{2})) \leq e^{2|\lambda|B} \int_0^1 \exp(2|\lambda| f(x,0)^2 \cot \omega)\, dx \tag{6.9}$$

Now $f(e^{-1}, 0) \leq R$, so $f(x,0) \leq R\sqrt{-\log x}$ for $x \leq e^{-1}$ by Lemma 3.6, and Area$(K) \gg 1$ implies $f(x,0) \leq 2R$ for $x \geq e^{-1}$ by Lemma 3.7. Thus we conclude that:

$$\mathbb{E} \exp(\lambda X_K(0, \frac{\pi}{2})) \leq e^{2|\lambda|B} \int_0^{e^{-1}} x^{-2|\lambda|R^2 \cot \omega}\, dx + e^{2|\lambda|B} \int_{e^{-1}}^1 e^{8|\lambda|R^2 \cot \omega}\, dx \tag{6.10}$$

which is bounded absolutely for small enough $|\lambda|$. □

7 A dependence estimate

Definition 7.1. If $S \subset \mathbb{R}/2\pi$ is an interval, then we let $\mathcal{F}_S^{(K)}$ be the σ-algebra which keeps track of $W_K(\theta)$ for $\theta \in S$.

For example, Π_K is $\mathcal{F}_S^{(K)}$-measurable if and only if $S = \mathbb{R}/2\pi$.

The type of dependence estimate we prove will be an α-mixing estimate, that is an estimate on $|P(A \cap B) - P(A)P(B)|$ where A and B are events that are supposed to be almost independent. This type of estimate has been used previously in studying random polygons; we were motivated to prove our estimate by a similar result in [7, p341, Theorem 2.3].

Lemma 7.2. *Let $[\theta_1, \theta_2]$ and $[\psi_1, \psi_2]$ be two disjoint intervals in $\mathbb{R}/2\pi$. Let $A \in \mathcal{F}_{[\theta_1,\theta_2]}$ and $B \in \mathcal{F}_{[\psi_1,\psi_2]}$. Then:*

$$|P(A \cap B) - P(A)P(B)| \ll \sum_{i,j \in \{1,2\}} \int_K \exp(-A(p,\theta_i)) \exp(-A(p,\psi_j))\, dp \tag{7.1}$$

The proof is an elementary calculation and is given in Appendix A. The object of this section is to reexpress the right hand side of (7.1) in terms of the affine invariant measure.

Lemma 7.3. *There exists an absolute constant $\delta > 0$ such that if $\theta \leq \psi \leq \theta + \pi$, then area of $C(1,\theta) \cap C(1,\psi)$ is $\ll \exp(-\delta \mu_K([\theta, \psi]))$.*

Proof. We use Lemma 5.5 to construct a sequence $\theta = \gamma_0 < \gamma_1 < \cdots < \gamma_L \leq \psi$ so that $\text{Area}(C(\gamma_i, 1) \cap C(\gamma_{i+1}, 1)) = \frac{1}{2}$ and $L \asymp \mu_K([\theta, \psi])$. From this decompostion, we see that it suffices to show that there exists $\delta > 0$ such that for all i:

$$\text{Area}(C(\gamma_i, 1) \cap C(\gamma_0, 1)) \leq (1 - \delta)\,\text{Area}(C(\gamma_{i-1}, 1) \cap C(\gamma_0, 1)) \tag{7.2}$$

Now we know that $C(\gamma_0, 1) = K \cap H$ for some halfplane H and that additionally $\text{Area}(C(\gamma_{i-1}, 1) \cap C(\gamma_0, 1)) = \text{Area}(C(\gamma_{i-1}, 1) \cap H) \leq \frac{1}{2}$. Hence it suffices to show that:

$$\text{Area}(C(\gamma_i, 1) \cap H) \leq (1 - \delta)\,\text{Area}(C(\gamma_{i-1}, 1) \cap H) \tag{7.3}$$

whenever $\text{Area}(C(\gamma_{i-1}, 1) \cap H) \leq \frac{1}{2}$ and $\text{angle}(H) \in (\gamma_i - \pi, \gamma_{i-1})$.

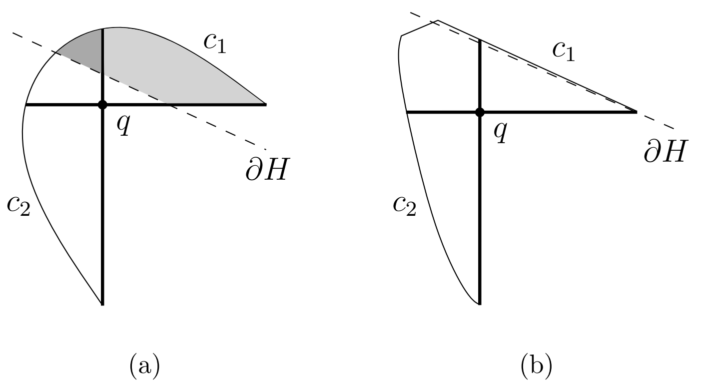


Figure 7.1: Intersecting caps.

Remember that $C(\gamma_i, 1)$ and $C(\gamma_{i-1}, 1)$ have intersection $\frac{1}{2}$. Thus it suffices to show that for every $(c_1, c_2) \in \mathcal{C}(1, \frac{1}{2}, 1)$, the following is true:

$$\frac{\text{Area}(c_2 \cap H)}{\text{Area}(c_1 \cap H)} < 1 - \delta \tag{7.4}$$

whenever $\text{Area}(c_1 \cap H) \leq \frac{1}{2}$ and $\text{angle}(H) \in (\text{angle}(c_2) - \pi, \text{angle}(c_1))$ (see Figure 7.1(a)). Here, if we put c_1 and c_2 in standard position (i.e. as in Figure 7.1, with both flat boundaries of equal length), then ∂H has negative slope. Denote by q the intersection of the flat boundaries of c_1 and c_2. Then since $\text{Area}(c_1 \cap H) \leq \frac{1}{2}$, we must have $q \notin H$. From this, we see that $c_2 \cap H \subseteq c_1 \cap H$, so we may rewrite (7.4) as:

$$\frac{\text{Area}((c_1 \setminus c_2) \cap H)}{\text{Area}(c_2 \cap H)} > \delta \tag{7.5}$$

The minimum of this expression is clearly a continuous function on $\mathcal{C}(1, \frac{1}{2}, 1)$, and is by definition invariant under the action of $\text{Aff}(2)$. We know that $\mathcal{C}(1, \frac{1}{2}, 1)/\text{Aff}(2)$ is compact, so it suffices to show that for any fixed configuration (c_1, c_2), the expression (7.5) is bounded below away from zero. Certainly, if this ratio were approacing zero, then $\text{Area}((c_1 \setminus c_2) \cap H) \to 0$. However in this case, the situation is illustrated in Figure 7.1(b), where it is clear that the ratio (7.5) in fact does *not* approach zero, but rather some appropriate ratio of lengths of the boundaries of the caps. Thus we are done. ☐

Lemma 7.4. *There exists an absolute constant $\delta > 0$ such that if $\theta \le \psi \le \theta + \pi$:*

$$\int_K \exp(-A(p,\theta))\exp(-A(p,\psi))\,dp \ll \exp(-\delta\mu_K([\theta,\psi])) \tag{7.6}$$

Proof. We pick the unique θ_1, ψ_1 so that $\theta < \theta_1 < \psi_1 < \psi$ and $\mu_K([\theta,\theta_1]) = \mu_K([\theta_1,\psi_1]) = \mu_K([\psi_1,\psi])$.

Define:

$$S_p = C(p,\theta) \cup C(p,\psi) = \bigcup_{\theta\le\alpha\le\psi} C(p,\alpha) \tag{7.7}$$

so $\text{Area}(S_p) \le A(p,\theta) + A(p,\psi)$.

Now if $A(p,\alpha) \ge 1$ for all $\alpha \in [\theta,\theta_1]$, then by Lemma 4.3, the area of S_p is $\gg \mu_K([\theta,\theta_1]) = \frac{1}{3}\mu_K([\theta,\psi])$. The same applies if $A(p,\alpha) \ge 1$ for $\alpha \in [\psi_1,\psi]$. Thus in both of these cases, we conclude that $A(p,\theta) \gg \mu_K([\theta,\psi])$ or $A(p,\psi) \gg \mu_K([\theta,\psi])$.

If $A(p,\theta_2) < 1$ for some $\theta_2 \in [\theta,\theta_1]$ and $A(p,\psi_2) < 1$ for some $\psi_2 \in [\psi_1,\psi]$, then necessarily $p \in C(1,\theta_1) \cap C(1,\psi_1)$. Thus we know that for all $p \in K$, at least one of the following is true:

- $p \in C(1,\theta_1) \cap C(1,\psi_1)$
- $A(p,\theta) \gg \mu_K([\theta,\psi])$
- $A(p,\psi) \gg \mu_K([\theta,\psi])$

By elementary integration, the integral over the second and third regions is $\ll \exp(-\delta\mu_K(\theta,\psi))$. The area of the first region is $\ll \exp(-\delta\mu_K(\theta,\psi))$ by Lemma 7.3, so we are done. ☐

Proposition 7.5. *There exists an absolute constant $\delta > 0$ so that if $[\theta_1,\theta_2]$ and $[\psi_1,\psi_2]$ are two disjoint intervals in $\mathbb{R}/2\pi$ and we have events $A \in \mathcal{F}_{[\theta_1,\theta_2]}$ and $B \in \mathcal{F}_{[\psi_1,\psi_2]}$, then:*

$$|P(A\cap B) - P(A)P(B)| \ll \sum_{i,j\in\{1,2\}} \exp(-\delta\mathfrak{d}_K(\theta_i,\psi_i)) \tag{7.8}$$

where $\mathfrak{d}_K(\alpha,\beta)$ denotes $\mu_K([\alpha,\beta])$ if $\alpha \le \beta \le \alpha+\pi$ and $\mu_K([\beta,\alpha])$ if instead $\beta \le \alpha \le \beta+\pi$.

The reader may wonder exactly what follows from an α-mixing estimate. We won't answer that here, though we will record here two lemmas that will be useful later whose hypotheses are α-mixing estimates.

Lemma 7.6 ([13, p115, Lemma 1(6)])**.** *Suppose X and Y are random variables taking values in $\mathbb{R}$ such that:*

$$|P(X\in A \& Y \in B) - P(X\in A)P(Y\in B)| < \alpha \tag{7.9}$$

for all $A, B \subseteq \mathbb{R}$. Then we have:

$$|\operatorname{Cov}(X,Y)| \le 6(\mathbb{E}|X|^3)^{1/3}(\mathbb{E}|Y|^3)^{1/3}\alpha^{1/3} \tag{7.10}$$

Lemma 7.7. *Suppose X and Y are random variables taking values in $\mathbb{R}$ such that:*

$$|P(X \in A \& Y \in B) - P(X \in A)P(Y \in B)| < \alpha \tag{7.11}$$

for all $A, B \subseteq \mathbb{R}$. Let $Z = X + Y$, and let $\tilde{Z}$ equal the sum of independent copies of X and Y. Then we have:

$$\sup_x |P(Z \leq x) - P(\tilde{Z} \leq x)| \ll \sqrt{\alpha} \tag{7.12}$$

Proof. Let $-\infty = x_0 < x_1 < \cdots < x_N = \infty$ be any finite increasing sequence of real numbers. Then we have:

$$\begin{aligned} P(Z \leq 0) &\geq \sum_{i=1}^{N} P(X \in (x_{i-1}, x_i] \& Y \leq -x_i) \\ &\geq -N\alpha + \sum_{i=1}^{N} P(X \in (x_{i-1}, x_i])P(Y \leq -x_i) \end{aligned} \tag{7.13}$$

Now using the definition of $\tilde{Z}$, we can bound this below by:

$$\geq -N\alpha + P(\tilde{Z} \leq 0) - \sum_{i=1}^{N} P(X \in (x_{i-1}, x_i))P(Y \in (-x_i, -x_{i-1})) \tag{7.14}$$

Thus we find that:

$$P(Z \leq 0) - P(\tilde{Z} \leq 0) \geq -N\alpha - \sum_{i=1}^{N} P(X \in (x_{i-1}, x_i))P(Y \in (-x_i, -x_{i-1})) \tag{7.15}$$

Now choose $K - 1$ real numbers $-\infty = u_0 < u_1 < \cdots < u_K = \infty$ so that the probability that X falls in the open interval (u_{i-1}, u_i) is $\leq K^{-1}$ for all i. Do the same for Y to get v_i's. Then let the x_i's be the union of the u_i's and $-v_i$'s (so $N \leq 2K$). With this choice, we see that each of the probabilities in the last sum of (7.15) is $\leq K^{-1}$, so their product is $\leq K^{-2}$. Hence the right hand side is $\geq -2K\alpha - 2KK^{-2}$. Now choosing K to equal the nearest integer to $\alpha^{-1/2}$, we conclude that $P(Z \leq 0) - P(\tilde{Z} \leq 0) \geq -\text{const} \cdot \alpha^{1/2}$. By a symmetric argument, we get the other inequality, so $|P(Z \leq 0) - P(\tilde{Z} \leq 0)| \ll \alpha^{1/2}$, which is sufficient. ☐

8 A variance estimate

The task of providing a lower bound on the variance of N and A has already been completed by Imre Bárány and Matthias Reitzner [3, p4 Theorem 2.1]. They prove the following theorem:

Proposition 8.1. *Provided $\mu_K([\alpha, \beta]) \gg 1$, we have the estimates:*

$$\operatorname{Var} N(\alpha, \beta) \gg \mu_K([\alpha, \beta]) \tag{8.1}$$

$$\operatorname{Var} A(\alpha, \beta) \gg \mu_K([\alpha, \beta]) \tag{8.2}$$

In fact, Bárány and Reitzner's result is valid for random polytopes as well. They only state this estimate in the case $[\alpha, \beta] = \mathbb{R}/2\pi$, though their proof is valid in general. We also note that they phrase their result in terms of the area of the ϵ-wet part of K; we have replaced this with the affine invariant measure using Lemma 4.3.

9 Proof of Theorem 1.2

Proof of Theorem 1.2. Let X denote either N or A.

From linearity of the expectation, one immediately observes that $\mathbb{E}[X] \asymp \mu_K(\mathbb{R}/2\pi)$. Proposition 8.1 implies that:

$$\mu_K(\mathbb{R}/2\pi) \ll \operatorname{Var} X \tag{9.1}$$

Thus it suffices to show the reverse inequality. For this, simply decompose $\mathbb{R}/2\pi$ into L intervals of affine invariant measure $\asymp 1$, and then write:

$$\operatorname{Var} X = \sum_{i=1}^{L} \operatorname{Var} X(\alpha_i, \alpha_{i+1}) + 2 \sum_{1 \le i < j \le L} \operatorname{Cov}(X(\alpha_i, \alpha_{i+1}), X(\alpha_j, \alpha_{j+1})) \tag{9.2}$$

Proposition 6.1 shows that the sum of variances is $\ll L$. Proposition 7.5, Lemma 7.6, and Proposition 6.1 imply that the sum of covariances is $\ll L$. Hence the right hand side is $\ll L$ as needed. ☐

10 Proof of Theorem 1.1

We need the following central limit theorem appearing in a survey article by Sunklodas [14]:

Theorem 10.1 (in English translation [13, p133 Theorem 10])**.** *Let* $X = \sum_{i=1}^{L} X_i$ *where* $X_1, \ldots, X_L$ *are random variables. Additionally suppose that:*

- $\mathbb{E}|X_i|^3 \le C_1$.
- $X_1, \ldots, X_L$ *are* α*-mixing with* $\alpha \le C_2 \exp(-\delta|i-j|)$.

for some $\delta > 0$ *and* $C_1, C_2 < \infty$*. Then there exists* $M < \infty$ *such that:*

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{X - \mathbb{E}X}{\sqrt{\operatorname{Var} X}} \le x \right) - \Phi(x) \right| \le M \frac{L(\log L)^2}{(\operatorname{Var} X)^{-3/2}} \tag{10.1}$$

We have everything necessary to apply Theorem 10.1 to N and A, except that our decomposition is "circular". Thus, for example, Theorem 10.1 shows immediately that $N(\alpha, \alpha+\pi)$ satisfies a central limit theorem for any α, but does not directly apply to give a central limit theorem for N. For completeness, we include the following proof, where we derive Theorem 1.1 just using Theorem 10.1 as a black box. The reader who is willing to believe the natural extension of Theorem 10.1 to our situation may want to omit it, as it is essentially just a straightforward calculation.

Proof of Theorem 1.1. Suppose K is given with $\mathrm{Area}(K) \gg 1$. Let X denote either N or A. In this proof $\delta > 0$ denotes some positive absolute constant, possibly different at each occurence.

The function $f(\alpha) = \mu_K([\alpha, \alpha+\pi])$ on $\mathbb{R}/2\pi$ satisfies $f(\alpha)+f(\alpha+\pi) = \mu_K(\mathbb{R}/2\pi)$. Thus by continuity we may find α such that $\mu_K([\alpha, \alpha+\pi]) = \mu_K([\alpha+\pi, \alpha+2\pi])$. Without loss of generality, we may assume $\mu_K([0,\pi]) = \mu_K([\pi, 2\pi])$. Set $L = \mu_K(\mathbb{R}/2\pi)$.

We let ℓ denote a quantity much smaller than L (we will eventually let ℓ equal some large multiple of $\log L$). We pick $\alpha_1, \beta_1, \alpha_2, \beta_2$ so that $\mu([0,\alpha_1]) = \mu([\beta_1, \pi]) = \mu([\pi, \alpha_2]) = \mu([\beta_2, 2\pi]) = \ell$. Then we set:

$$X_1 = X(\alpha_1, \beta_1) \tag{10.2}$$
$$X_2 = X(\alpha_2, \beta_2) \tag{10.3}$$

Observe that by partitioning $[\alpha_i, \beta_i]$ into intervals of affine invariant measure $\asymp 1$, we may apply Theorem 10.1 (appealing to Propositions 6.1 and 7.5). Thus remembering Proposition 8.1, we may write:

$$\left| P\left(\frac{X_i - \mathbb{E}[X_i]}{\sqrt{\operatorname{Var} X_i}} \le x \right) - \Phi(x) \right| \ll \frac{\log^2 L}{\sqrt{L}} \tag{10.4}$$

Let $\tilde{Y}$ equal the sum of independent copies of X_1 and X_2. Then (10.4) implies that:

$$\left| P\left(\frac{\tilde{Y} - \mathbb{E}[\tilde{Y}]}{\sqrt{\operatorname{Var} \tilde{Y}}} \le x \right) - \Phi(x) \right| \ll \frac{\log^2 L}{\sqrt{L}} \tag{10.5}$$

By Proposition 7.5, X_1 and X_2 are α-mixing with $\alpha \ll e^{-\delta \ell}$. Proposition 6.1 shows $\mathbb{E}[|X_i|^3]^{1/3} \ll L$. If we let $Y = X_1 + X_2$, then Lemma 7.7 implies that $|P(Y \le x) - P(\tilde{Y} \le x)| \ll e^{-\delta\ell}$. Lemma 7.6 implies:

$$\mathrm{Cov}(X_1, X_2) \ll L^2 e^{-\delta \ell} \tag{10.6}$$

Since $\operatorname{Var} Y = \operatorname{Var} \tilde{Y} + \mathrm{Cov}(X_1, X_2)$ and $\operatorname{Var} \tilde{Y} \asymp L$, we have $\operatorname{Var} Y = (1 + O(Le^{-\delta L})) \operatorname{Var} \tilde{Y}$. Hence $|\Phi(\sqrt{\operatorname{Var} Y} x) - \Phi(\sqrt{\operatorname{Var} \tilde{Y}} x)| \ll Le^{-\delta \ell}$. Hence we conclude that:

$$\begin{aligned} \left| P\left(\frac{Y - \mathbb{E}[Y]}{\sqrt{\operatorname{Var} Y}} \le x \right) - \Phi(x) \right| &\ll \frac{\log^2 L}{\sqrt{L}} + e^{-\delta\ell} + Le^{-\delta\ell} \\ &\ll \frac{\log^2 L}{\sqrt{L}} + Le^{-\delta\ell} \end{aligned} \tag{10.7}$$

Now the final part of our argument is to translate this into a statement about X. Let $E = X(\beta_2, \alpha_1) + X(\beta_1, \alpha_2)$. Thus by definition, we have $X = Y + E$. Using Proposition 6.1, it is evident that $\mathbb{E}[\exp(\delta \ell^{-1} E)] \ll 1$ for some absolute $\delta > 0$. From this, we conclude that $P(\exp(\delta\ell^{-1}E) \ge M) \ll M^{-1}$. Thus $P(E \ge M) \ll e^{-\delta M/\ell}$. Now we pick $M = \ell^2$, so that:

$$P(E \ge \ell^2) \ll e^{-\delta \ell} \tag{10.8}$$

Now examine (10.7), and consider what this says about $P\left(\frac{Y+E-\mathbb{E}[Y]}{\sqrt{\operatorname{Var} Y}} \leq x\right)$. We have $\sqrt{\operatorname{Var} Y} \asymp \sqrt{L}$, so (10.8) implies that $P(E/\sqrt{\operatorname{Var} Y} \notin [0, \ell^2/\sqrt{L}]) \ll e^{-\delta\ell}$. Thus:

$$\left|P\left(\frac{Y+E-\mathbb{E}[Y]}{\sqrt{\operatorname{Var} Y}} \leq x\right) - \Phi(x)\right| \\ \ll \left|P\left(\frac{Y-\mathbb{E}[Y]}{\sqrt{\operatorname{Var} Y}} \leq x\right) - \Phi(x)\right| + e^{-\delta\ell} + \frac{\ell^2}{\sqrt{L}} \\ \ll \frac{\log^2 L}{\sqrt{L}} + Le^{-\delta\ell} + \frac{\ell^2}{\sqrt{L}} \quad (10.9)$$

Now $\mathbb{E}[E] \asymp \ell$, so adding $\mathbb{E}[E]$ in the numerator adds at most $\ell/\sqrt{L}$ to the error. Hence:

$$\left|P\left(\frac{X-\mathbb{E}[X]}{\sqrt{\operatorname{Var} Y}} \leq x\right) - \Phi(x)\right| \ll \frac{\log^2 L}{\sqrt{L}} + Le^{-\delta\ell} + \frac{\ell^2}{\sqrt{L}} \quad (10.10)$$

Observe that $\operatorname{Var} E \ll \ell$, so $\operatorname{Var} X = \operatorname{Var} Y + 2\operatorname{Cov}(Y, E) + \operatorname{Var} E = \operatorname{Var} Y + O(\sqrt{L}\sqrt{\ell}) + O(\ell)$, so the relative error is $\ll \frac{\sqrt{\ell}}{\sqrt{L}} + \frac{\ell}{L}$. Thus we have:

$$\left|P\left(\frac{X-\mathbb{E}[X]}{\sqrt{\operatorname{Var} X}} \leq x\right) - \Phi(x)\right| \ll \frac{\log^2 L}{\sqrt{L}} + Le^{-\delta\ell} + \frac{\ell^2}{\sqrt{L}} + \frac{\sqrt{\ell}}{\sqrt{L}} + \frac{\ell}{L} \quad (10.11)$$

Taking ℓ to equal a sufficiently large multiple of $\log L$, we achieve the desired estimate. □

A Proof of Lemma 7.2

Lemmas A.1 and A.2 below combine easily to give Lemma 7.2. The proof of Lemma A.1 follows [7], where similar manipulations are performed.

Lemma A.1. *Let $[\theta_1, \theta_2]$ and $[\psi_1, \psi_2]$ be two disjoint intervals in $\mathbb{R}/2\pi$. Let $A \in \mathcal{F}_{[\theta_1,\theta_2]}$ and $B \in \mathcal{F}_{[\psi_1,\psi_2]}$. Then:*

$$|P(A \cap B) - P(A)P(B)| \\ \leq 2 \sum_{i,j\in\{1,2\}} \iint_{(p,q)\in R(\theta_i,\psi_j)} dP(W(\theta_i) = p)\, dP(W(\psi_j) = q) \quad (A.1)$$

where $R(\alpha, \beta)$ is the set of pairs $(p, q) \in K \times K$ such that it is impossible that $W(\alpha) = p$ and $W(\beta) = q$.

Proof. We have that $P(A \cap B)$ is given by:

$$\iiiint_{K^4} P(A \cap B | (W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2)) \\ dP((W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2)) \quad (A.2)$$

and $P(A)P(B)$ by:

$$\iint_{K^2} P(A|(W(\theta_1), W(\theta_2)) = (p_1, p_2))\, dP((W(\theta_1), W(\theta_2)) = (p_1, p_2)) \\ \times \iint_{K^2} P(B|(W(\psi_1), W(\psi_2)) = (q_1, q_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1, q_2)) \quad (A.3)$$

Now given $W(\theta_1)$, $W(\theta_2)$, $W(\psi_1)$, and $W(\psi_2)$, the events A and B are independent. In other words the two integrands above are equal (although the measures are not). Hence we conclude that:

$$|P(A \cap B) - P(A)P(B)| \leq \iiiint_{K^4} \\ \Big| dP((W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2)) \\ - dP((W(\theta_1), W(\theta_2)) = (p_1, p_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1, q_2)) \Big| \quad (A.4)$$

Now define the set:

$$R_{\theta_1,\theta_2,\psi_1,\psi_2} = \{(p_1, p_2, q_1, q_2) \in K^4 : \\ (W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2) \text{ is impossible}\} \quad (A.5)$$

We will calculate the right hand side of (A.4) by splitting up the integral as $I(R_{\theta_1,\theta_2,\psi_1,\psi_2}) + I({R_{\theta_1,\theta_2,\psi_1,\psi_2}}^{\complement})$ (that is, the integral over $R_{\theta_1,\theta_2,\psi_1,\psi_2}$ and the integral over its complement). Since the first measure in question $dP((W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2))$ is supported on ${R_{\theta_1,\theta_2,\psi_1,\psi_2}}^{\complement}$, we trivially have that:

$$\iiiint_{{R_{\theta_1,\theta_2,\psi_1,\psi_2}}^{\complement}} \Big[dP((W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2)) \\ - dP((W(\theta_1), W(\theta_2)) = (p_1, p_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1, q_2)) \Big] = \\ \iiiint_{R_{\theta_1,\theta_2,\psi_1,\psi_2}} dP((W(\theta_1), W(\theta_2)) = (p_1, p_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1, q_2)) \quad (A.6)$$

Now observe that on ${R_{\theta_1,\theta_2,\psi_1,\psi_2}}^{\complement}$, we have:

$$dP((W(\theta_1), W(\theta_2), W(\psi_1), W(\psi_2)) = (p_1, p_2, q_1, q_2)) \geq \\ dP((W(\theta_1), W(\theta_2)) = (p_1, p_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1, q_2)) \quad (A.7)$$

From this, it is clear that equation (A.6) is equivalent to:

$$I({R_{\theta_1,\theta_2,\psi_1,\psi_2}}^{\complement}) = I(R_{\theta_1,\theta_2,\psi_1,\psi_2}) \quad (A.8)$$

Thus the right hand side of (A.4) in fact equals $2I(R_{\theta_1,\theta_2,\psi_1,\psi_2})$. Hence we conclude that $|P(A \cap B) - P(A)P(B)|$ is bounded above by:

$$2 \iiiint_{R_{\theta_1,\theta_2,\psi_1,\psi_2}} dP((W(\theta_1), W(\theta_2)) = (p_1, p_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1, q_2)) \quad (A.9)$$

If we define $R(\theta,\psi) := \{(p,q) \in K^2 : (W(\theta), W(\psi)) = (p,q) \text{ is impossible}\}$, then:

$$R_{\theta_1,\theta_2,\psi_1,\psi_2} = \bigcup_{i,j\in\{1,2\}} R(\theta_i,\psi_j) \times K^2 \tag{A.10}$$

Thus we conclude that:

$$|P(A\cap B) - P(A)P(B)| \le 2 \sum_{i,j\in\{1,2\}} \iiiint_{R(\theta_i,\psi_j)\times K^2} dP((W(\theta_1), W(\theta_2)) = (p_1,p_2))\, dP((W(\psi_1), W(\psi_2)) = (q_1,q_2)) \tag{A.11}$$

Integrating out the undesired indices on the right hand side yields the correct result. ☐

Lemma A.2. *We have:*

$$\iint_{R_{\alpha,\beta}} dP(W(\alpha) = p)\, dP(W(\beta) = q) \le 2\int_K \exp(-A(p,\alpha)) \exp(-A(p,\beta))\, dp \tag{A.12}$$

Proof. This relies on the observation that $R_{\alpha,\beta} = \{(p,q) \in K^2 : q \notin H_{p,\beta}\} \cup \{(p,q) \in K^2 : p \notin H_{q,\alpha}\}$. Now recalling that $dP(W(\alpha) = p) = A(p,\alpha)\, dp$ (Lemma 3.4), we calculate:

$$\begin{aligned}\int_K \left[\int_{K-H_{p,\beta}} dP(W(\beta) = q)\right] dP(W(\alpha) = p) &= \int_K \exp(-A(p,\beta))\, dP(W(\alpha) = p) \\ &= \int_K \exp(-A(p,\beta)) \exp(-A(p,\alpha))\, dp \end{aligned} \tag{A.13}$$

Thus the result follows. ☐

References

[1] Problems from the AIM Workshop on Algorithmic Convex Geometry. *Available online at* http://www.aimath.org/WWN/convexgeometry/convexgeometry.pdf.

[2] I. Bárány and D. G. Larman. Convex bodies, economic cap coverings, random polytopes. *Mathematika*, 35(2):274–291, 1988.

[3] Imre Bárány and Matthias Reitzner. On the variance of random polytopes. *To appear in Advances in Mathematics. Available online at* http://www.renyi.hu/~barany/cikkek/var-aim.pdf.

[4] Imre Bárány and Matthias Reitzner. Poisson polytopes. *To appear in Annals of Probability. Available online at* http://www.renyi.hu/~barany/cikkek/poi-pol2.pdf.

[5] A. J. Cabo and P. Groeneboom. Limit theorems for functionals of convex hulls. *Probab. Theory Related Fields*, 100(1):31–55, 1994.

[6] H. Groemer. On the mean value of the volume of a random polytope in a convex set. *Arch. Math. (Basel)*, 25:86–90, 1974.

[7] Piet Groeneboom. Limit theorems for convex hulls. *Probab. Theory Related Fields*, 79(3):327–368, 1988.

[8] Tailen Hsing. On the asymptotic distribution of the area outside a random convex hull in a disk. *Ann. Appl. Probab.*, 4(2):478–493, 1994.

[9] John Pardon. Central limit theorems for uniform model random polygons. *Preprint.*

[10] Matthias Reitzner. Central limit theorems for random polytopes. *Probab. Theory Related Fields*, 133(4):483–507, 2005.

[11] A. Rényi and R. Sulanke. Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 2:75–84 (1963), 1963.

[12] A. Rényi and R. Sulanke. Über die konvexe Hülle von n zufällig gewählten Punkten. II. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 3:138–147 (1964), 1964.

[13] V. Statulevičius, editor. *Limit theorems of probability theory.* Springer-Verlag, Berlin, 2000. Translation of ıt Probability theory. 6 (Russian), Itogi Nauki i Tekhniki, Sovrem. Probl. Mat. Fund. Naprav., 81, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, 1991 [MR1157205 (92k:60001)], Translation edited by Yu. V. Prokhorov and V. Statulevičius.

[14] Ĭ. Sunklodas. Approximation of distributions of sums of weakly dependent random variables by the normal distribution. In *Probability theory, 6 (Russian)*, Itogi Nauki i Tekhniki, pages 140–199. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1991.

[15] Van Vu. Central limit theorems for random polytopes in a smooth convex set. *Adv. Math.*, 207(1):221–243, 2006.