TOPOLOGY OF $Sp(2,\mathbb{R})$ AND THE CONLEY-ZEHNDER INDEX

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ABSTRACT. The purpose of this note is to provide immediate insight in the topology of the symplectic linear group to non-experts by simply visualizing it via modern computer graphics. We use an explicit homeomorphism of $Sp(2,\mathbb{R})$ onto the interior of the solid 2-torus. In this picture the (generalized) Conley-Zehnder index of a path in $Sp(2,\mathbb{R})$ may be easily identified with the intersection number of the path and the Maslov cycle.

CONTENTS

1. INTRODUCTION

The symplectic linear group arises naturally in the study of linear Hamiltonian equations (cf. [Ar1])

$$\dot{\zeta}(t) = -J_0 S\zeta(t)\ ,\ J_0 = \begin{pmatrix} 0 & -1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}$$

where S is the Hessian of the Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. The solution ζ with initial condition $\zeta(0) = \zeta_0$ is given by

$$\zeta(t) = e^{-tJ_0 S}\zeta_0 \stackrel{def}{=} A(t)\zeta_0$$

and $A(t)$ is an element of $Sp(2n,\mathbb{R}) = \{A \in Mat(2n,\mathbb{R}) \mid A^T J_0 A = J_0\}$ for any value of t. In the case $n = 1$ the symplectic linear group coincides with $Sl(2,\mathbb{R})$, the special linear group, whose elements have determinant 1 and so are area-preserving.

In 1984 Conley and Zehnder introduced an index for continuous paths in $Sp(2n,\mathbb{R})$ which start at the identity and end at a matrix whose spectrum

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does not contain 1. This index became an important ingredient in the construction of Floer homology for symplectic manifolds [Fl], [RS1], which lead to a proof of the *Arnold conjecture* in considerable generality (the number of fixed points of an exact symplectic diffeomorphism on a symplectic manifold can be estimated below by the sum of its Betti numbers provided that the fixed points are nondegenerate, [Ar2]).

This article is organized as follows: Our first claim is to show that $Sp(2,\mathbb{R})$ is homeomorphic to the interior of the full 2-torus $S^1 \times D^2$, where D^2 denotes the open unit disc. The explicit homeomorphism was taken from an article of Gelfand and Lidskii [GL], 1958. Visualizing particular subsets of $Sp(2,\mathbb{R})$ in figure 4, we discuss the notion of an eigenvalue of the first and second kind. This is the content of section 2.

In section 3 we briefly recall the definition of the Conley-Zehnder index. We interpret this index in the case $n = 1$ as intersection number of the path with the Maslov cycle $\mathcal{C}_+$, which is an algebraic subvariety of codimension one. It is shown in figure 6. Note that only paths starting at the identity and ending outside $\mathcal{C}_+$ are allowed so far. We then introduce a generalized Conley-Zehnder index, where we drop the condition on the endpoint of the path. A simple example is discussed (figure 7). Finally we visualize and shortly discuss another Maslov-type index, the Robbin-Salamon index, which may be interpreted as intersection number of an arbitrary path with the Maslov cycle Sp_1 (figure 8), where endpoints only contribute half.

2. Topology of $Sp(2,\mathbb{R})$

Proposition 2.1. (**Polar decomposition**) *Let $Y \in Sp(2,\mathbb{R})$, then there exists a unique $R \in Sp(2,\mathbb{R}) \cap SO(2,\mathbb{R})$ and a unique $S \in Sp(2,\mathbb{R})$, positive definite and symmetric, such that $Y = SR$.*

Proof. YY^T is clearly positive definite, symmetric and symplectic. The first two properties also hold for $S := (YY^T)^{1/2}$ (functional calculus). S is symplectic by Lemma 2.19 in [MS] which says that any real power of a positive definite, symmetric and symplectic matrix is itself symplectic. Now we define $R = S^{-1}Y$; R is symplectic and also orthogonal: Let $x, y \in \mathbb{R}^2$ and denote by $< \cdot, \cdot >$ the euclidean inner product on $\mathbb{R}^2$, then

$$< Rx, Ry > = < S^{-1}Yx, S^{-1}Yy > = < x, Y^T {S^{-1}}^T S^{-1} Yy > = < x, y > \ .$$

The last equation holds, because ${S^{-1}}^T = S^{-1}$, so

$${S^{-1}}^T S^{-1} = S^{-1}S^{-1} = (YY^T)^{-1/2}(YY^T)^{-1/2} = (YY^T)^{-1} \ .$$

Finally $det\, R = (det\, S)^{-1} \cdot det\, Y = +1$ as $S, Y \in Sp(2,\mathbb{R}) = Sl(2,\mathbb{R})$. □

Hence we may write any $Y \in Sp(2,\mathbb{R})$ in the form

$$Y = \begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \ , \tag{1}$$

where $\psi \in \mathbb{R}/2\pi\mathbb{Z}$ and

$$s_{11}s_{22} - {s_{12}}^2 = 1 \ . \tag{2}$$

Lemma 2.2. $s_{11}, s_{22} > 0$ *and* $s_{11} \cdot s_{22} \geq 1$.

Proof. The last statement is a simple consequence of (2). (2) also implies that either $s_{11} > 0$ and $s_{22} > 0$ or $s_{11} < 0$ and $s_{22} < 0$. Assume the second case, then the positive definiteness of S leads to a contradiction: $Tr\ S = s_{11} + s_{22} > 0$. ☐

Now (2) implies that s_{22} is uniquely determined once $(s_{11}, s_{12}) \in \mathbb{R}^+ \times \mathbb{R}$ has been chosen:

$$s_{22}(s_{11}, s_{12}) = \frac{1 + (s_{12})^2}{s_{11}}.$$

Hence the set of possible parameters is given by

$$\mathcal{M} = \mathbb{R}^+ \times \mathbb{R},$$

which is diffeomorphic to the open unit disc in $\mathbb{R}^2$. Recall the hyperbolic trigonometric functions

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}) \ , \ \cosh x = \tfrac{1}{2}(e^x + e^{-x}) \ , \ \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

whose qualitative behaviour is shown in figure 1. They satisfy the relation

$$\cosh^2 x - \sinh^2 x = 1 \ . \tag{3}$$

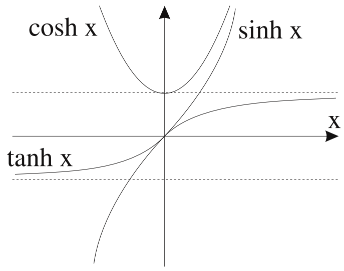

FIGURE 1. Hyperbolic trigonometric functions

We reparametrize $\mathcal{M}$ in the following way ([GL], §9)

$$\begin{aligned} \mathcal{S} \ : \ (0,\infty) \times \mathbb{R}/2\pi\mathbb{Z} \ \sqcup \ (0,0) &\to \mathcal{M} \\ (\tau, \sigma) &\mapsto (s_{11}(\tau,\sigma), s_{12}(\tau,\sigma)) \ , \end{aligned}$$

where

$$\begin{aligned} &i) \ \ s_{11}(\tau,\sigma) = \cosh\tau + \sinh\tau\cos\sigma \\ &ii) \ \ s_{12}(\tau,\sigma) = \sinh\tau\sin\sigma \ . \end{aligned} \tag{4}$$

The parameter $s_{22}(\tau,\sigma)$ is then given by

$$\begin{aligned} iii)\ \ s_{22}(\tau,\sigma) &= \frac{1+(s_{12})^2}{s_{11}} = \frac{1+\sinh^2\tau\sin^2\sigma+\sinh^2\tau-\sinh^2\tau}{\cosh\tau+\sinh\tau\cos\sigma} \\ &= \frac{\cosh^2\tau-\sinh^2\tau(1-\sin^2\sigma)}{\cosh\tau+\sinh\tau\cos\sigma} = \cosh\tau-\sinh\tau\cos\sigma. \end{aligned}$$

Proposition 2.3. *The map $\mathcal{S}$ is a homeomorphism. On $(0,\infty)\times\mathbb{R}/2\pi\mathbb{Z}$ it is a diffeomorphism onto $\mathcal{M}\setminus(1,0)$.*

Proof. Smoothness of $\mathcal{S}$ follows from the one of the hyperbolic trigonometric functions. $\mathcal{S}$ is injective: Assume $\mathcal{S}(\tau,\sigma)=\mathcal{S}(\tau',\sigma')$, then

$$\begin{aligned} &(4i)+iii) \Rightarrow 2\cosh\tau' = 2\cosh\tau \Rightarrow \tau'=\tau\ , \\ &(4i)-iii) \Rightarrow 2\sinh\tau\cos\sigma' \overset{\tau'=\tau}{=} 2\sinh\tau\cos\sigma \Rightarrow \cos\sigma'=\cos\sigma\ , \\ &(4ii) \Rightarrow \sin\sigma'=\sin\sigma\ , \text{hence } \sigma'=\sigma\ . \end{aligned}$$

$\mathcal{S}$ is surjective: Addition and subtraction of s_{11}, $s_{22}(s_{11},s_{12})$ leads to a smooth inverse on $\mathbb{R}^+\times\mathbb{R}\setminus(0,1)$

$$\mathcal{S}^{-1}(s_{11},s_{12}) = (\tau(s_{11},s_{12}),\sigma(s_{11},s_{12}))\ ,$$

where

$$\begin{aligned} \tau(s_{11},s_{12}) &= \operatorname{arc\,cosh}\frac{s_{11}+s_{22}(s_{11},s_{12})}{2} \\ &= \operatorname{arc\,cosh}\frac{(s_{11})^2+1+(s_{12})^2}{2s_{11}}\ , \end{aligned}$$

$$\begin{aligned} \sigma(s_{11},s_{12}) &= (\mathit{sign}\ s_{12})\ \arccos\left(\frac{s_{11}-s_{22}(s_{11},s_{12})}{2\sinh\circ\operatorname{arc\,cosh}\frac{s_{11}+s_{22}(s_{11},s_{12})}{2}}\right) \\ &= (\mathit{sign}\ s_{12})\ \arccos\left(\frac{(s_{11})^2-1-(s_{12})^2}{\sqrt{\left((s_{11})^2-1-(s_{12})^2\right)^2-4(s_{11})^2}}\right). \end{aligned} \tag{5}$$

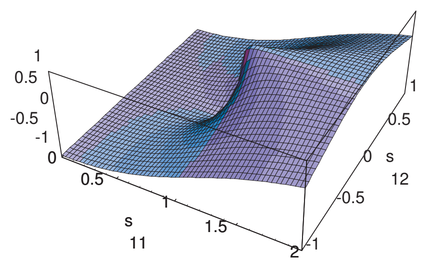

FIGURE 2. The argument of arccos in (5)

Note that we used the identity $\operatorname{arc\,cosh} x = \operatorname{arc\,sinh}\sqrt{x^2-1}$ for $x \geq 1$ and the convention $sign\ 0 = +1$. The argument of arccos in (5) is a smooth function on $\mathbb{R}^+ \times \mathbb{R} \setminus (1,0)$ which is not continuous at $(1,0)$. A numerical *Mathematica*[1]-plot of it is shown in figure 2. It is identically $+1$ on $(0,1)\times 0$ and identically -1 on $(1,\infty)\times 0$. This corresponds to $\sigma = \pi$ and $\sigma = 0$, respectively. Therefore multiplication by $sign\ s_{12}$ is well-defined and the full range $[-\pi,\pi]/\{\pi,-\pi\}$ of σ is covered. Finally we define $\mathcal{S}^{-1}(1,0) = (0,0)$. □

Rescale τ by setting $r(\tau) = \tanh^2\tau$, $\tau \in [0,\infty)$, i.e. $r \in [0,1)$. We interpret the parameters

$$(\psi, r, \sigma) \in \mathbb{R}/2\pi\mathbb{Z} \times [(0,1)\times \mathbb{R}/2\pi\mathbb{Z} \sqcup (0,0)]$$

as coordinates of the open solid 2-torus (figure 3).

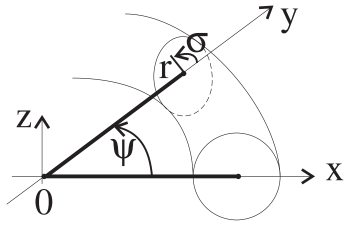

FIGURE 3. Coordinates on the full 2-torus

In this coordinates the matrices

$$E = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ,\ \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\ ,\ W_- = \begin{pmatrix}2 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\ ,\ \pm J_0 = \pm\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

correspond to

$$(0,0,0)\ ,\ (\pi,0,0)\ ,\ (0,\frac{3}{5},0)\ ,\ (\frac{(3)\pi}{2},0,0)$$

as follows from their polar decompositions

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\ ,$$
$$\begin{pmatrix}2 & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ ,\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \mp 1\\ \pm 1 & 0\end{pmatrix}\ .$$

The condition for $Y \in Sp(2,\mathbb{R}) = Sl(2,\mathbb{R})$ to have non-real eigenvalues is equivalent to $|Tr\, Y| < 2$ as

$$0 = \det(Y - \lambda E) = \det\begin{pmatrix}a-\lambda & b\\ c & d-\lambda\end{pmatrix} = \lambda^2 - \lambda\cdot Tr\, Y + 1$$
$$\Leftrightarrow\ \lambda_{1,2} = \frac{Tr\, Y \pm \sqrt{(Tr\, Y)^2 - 4}}{2}\ .$$

[1] *Mathematica* is a registered trademark of Wolfram Research, Inc.

Using the polar decomposition (1) of Y we get

$$|Tr\,Y| = |(s_{11}+s_{22})\cos\psi| = 2|\cos\psi|\cosh\tau$$
$$= 2|\cos\psi|\cosh\circ\text{arc}\tanh\sqrt{r}\;.$$

Lemma 2.4. *Let Y be as in (1), then $|Tr\,Y| = 2$ is equivalent to $r=\sin^2\psi$.*

Proof. $|Tr\,Y| = 2$ implies $\psi \notin \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ and is equivalent to

$$\pm\text{arc}\cosh\frac{1}{|\cos\psi|} = \text{arc}\tanh\sqrt{r} \Leftrightarrow \tanh^2\circ\pm\text{arc}\cosh\frac{1}{|\cos\psi|} = r$$
$$\Leftrightarrow \frac{\cosh^2\circ\pm\text{arc}\cosh\frac{1}{|\cos\psi|}-1}{\cos^{-2}\psi} = r \Leftrightarrow 1-\cos^2\psi = r \Leftrightarrow \sin^2\psi = r.$$

In the third equivalence we used $\tanh x = \sinh x/\cosh x$ and (3). □

As the eigenvalues of $Y\in Sp(2,\mathbb{R})$ come as pairs (λ,λ^{-1}),we observe that

$$Spec\,Y = \{+1\} \Leftrightarrow Tr\,Y = 2 \Leftrightarrow \sin^2\psi = r\;,\psi\in(-\pi/2,\pi/2)\;,$$
$$Spec\,Y = \{-1\} \Leftrightarrow Tr\,Y = -2 \Leftrightarrow \sin^2\psi = r\;,\psi\in(\pi/2,3\pi/2)\;.$$

We define

$$\mathcal{C}_\pm = \{Y\in Sp(2,\mathbb{R})|Spec\,Y = \{\pm1\}\}\;.$$

Remark 2.5. Our results so far are visualized in figure 4, which has been created using *Mathematica*[1] and *Geomview*[2]. Note that figure 4 is not a sketch and it is not obtained by numerical approximation, but represents exactly the set $|Tr\,Y| = 2$ in the coordinates (ψ, r, σ). The same holds for figure 6 in the case $Tr\,Y = 2$.

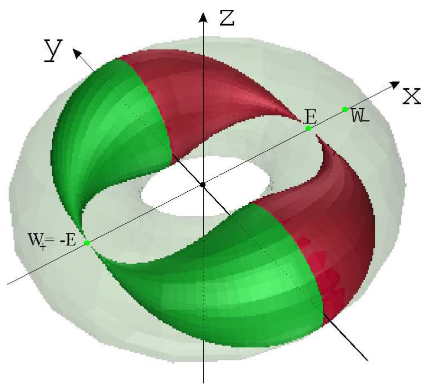

FIGURE 4. $\mathcal{C}_-$ and $\mathcal{C}_+$

The sets $\mathcal{C}_+\setminus\{E\}$ resp. $\mathcal{C}_-\setminus\{-E\}$ are smooth 2-dimensional surfaces; they are indicated in figure 4 red (dark) resp. green (light). The region enclosed by them consists of two connected components – corresponding to

[2]written at the Geometry Center, University of Minnesota

the matrices with non-real eigenvalues. The outside region consists also of two connected components. The one having E in its closure corresponds to the matrices with positive real pairs of eigenvalues (β,β^{-1}), $\beta \in \mathbb{R}^+ \setminus \{1\}$, the other corresponds to the ones having negative real pairs. Note that the set of all possible eigenvalues of elements of $Sp(2,\mathbb{R})$ is the union of $S^1 \subset \mathbb{C}$ with the real line minus zero (figure 5).

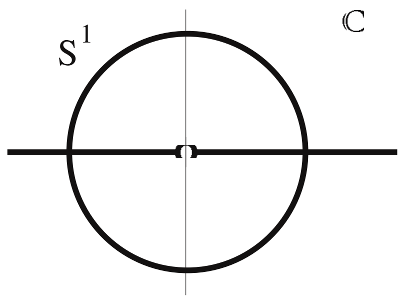

FIGURE 5. Spectrum of $Sp(2,\mathbb{R})$

At first sight one might be tempted to relate the two parts of S^1 lying in the upper resp. lower half plane $H^\pm \subset \mathbb{C}$ with the two connected components corresponding to matrices with non-real eigenvalues. This intuition points the right direction; the relation, however, is more subtle.

Definition 2.6. Let $Y \in Sp(2,\mathbb{R})$ with eigenvalues $(\lambda, \lambda^{-1} = \bar{\lambda})$ in $S^1 \setminus \{\pm 1\}$ and eigenvectors $\xi_\lambda, \xi_{\bar{\lambda}}$. We call λ an **eigenvalue of the first kind**, if

$$Im\, \omega_0(\bar{\xi_\lambda}, \xi_\lambda) > 0 \;,$$

and an **eigenvalue of the second kind** otherwise.

This definition does not depend on the choice of the eigenvector nor on the choice which eigenvalue was called λ resp. $\bar{\lambda}$. ω_0 denotes the standard symplectic form on $\mathbb{R}^2$; in coordinates (x,y) it is given by $\omega_0 = dx \wedge dy$, in terms of the euclidean inner product $< \cdot, \cdot >$ we have

$$\omega_0(\cdot,\cdot) = < \cdot, -J_0 \cdot > \;, \text{ where } J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \;.$$

As $\bar{\xi_\lambda}$ is eigenvector of $\bar{\lambda}$, ξ_λ and $\bar{\xi_\lambda}$ are linearly independent, hence in view of the nondegeneracy of ω_0 we have $\omega_0(\bar{\xi_\lambda}, \xi_\lambda) \neq 0$. Moreover this value is purely imaginary:

$$\begin{aligned} \overline{\omega_0(\bar{\xi_\lambda}, \xi_\lambda)} &= < \bar{\bar{\xi_\lambda}}, \overline{-J_0} \bar{\xi_\lambda} > \\ &= < J_0 \xi_\lambda, \bar{\xi_\lambda} > = -\omega_0(\bar{\xi_\lambda}, \xi_\lambda) \;. \end{aligned}$$

We used the fact that ${J_0}^T = -J_0$. Now the eigenvalues of J_0 are $\lambda = i$ and $\bar{\lambda} = -i$ with corresponding eigenvectors $\xi_\lambda = (i, 1)$ and $\xi_{\bar{\lambda}} = (-i, 1)$. As $\omega_0(\bar{\xi_\lambda}, \xi_\lambda) = -2i$, we conclude that i is eigenvalue of the second kind and hence $-i$ of the first kind. A continuity argument allows us to conclude that the matrices with non-real eigenvalues which are in the same connected component as J_0, have their eigenvalues of the first kind in $S^1 \cap H^-$. Analogously for the connected component containing $-J_0$.

The notion of eigenvalues of the first and the second kind will become more important in the case of $Sp(2n, \mathbb{R})$, $n \geq 2$, where the eigenvalues come in quadrupels $(\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1})$. Two pairs of eigenvalues on S^1 can meet and leave S^1, if and only if eigenvalues of different kind meet.

3. The Conley-Zehnder index

In this section we recall the definition of the Conley-Zehnder index μ_{CZ} for paths in $Sp(2n, \mathbb{R})$ starting at the identity and ending at a matrix without eigenvalue 1. This index was introduced by Conley and Zehnder in 1984 [CZ]. We state the results about the topology of $Sp(2n, \mathbb{R})$ needed to show its well-definedness. We are not going to prove these results in full generality; instead we restrict to $Sp(2, \mathbb{R})$ and use the explicit homeomorphism from section 2 to visualize them in this case. We shall see that μ_{CZ} may be equivalently defined as intersection number of the path with the Maslov cycle $\mathcal{C}_+$. Define

$$Sp^*_\pm = \{A \in Sp(2n, \mathbb{R}) \mid \det(A - E) \gtrless 0\} \ , \ Sp^* = Sp^*_+ \cup Sp^*_- \ ,$$
$$\mathcal{P} = \{\gamma : [0, 1] \to Sp(2n, \mathbb{R}) \mid \gamma \text{ continuous} \ , \ \gamma(0) = E \ , \ \gamma(1) \in Sp^*\} \ .$$

Lemma 3.1. *([CZ], Lemma 1.7) Sp^*_+ and Sp^*_- are connected and any loop in them is contractible in $Sp(2n, \mathbb{R})$. Moreover, $W_+ = -E$ lies in Sp^*_+ and $W_- = diag\,(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1)$ in Sp^*_-.*

The **Conley-Zehnder index**

$$\mu_{\mathbf{CZ}} : \mathcal{P} \to \mathbb{Z}$$

is defined as follows: Pick $\gamma \in \mathcal{P}$ and associate to it a path $u = F_2 \circ F_1(\gamma) : [0, 1] \to U(n, \mathbb{C})$, where

$$F_1 : C^0\left([0, 1], Sp(2n, \mathbb{R})\right) \to C^0\left([0, 1], Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})\right)$$
$$\gamma \mapsto (\gamma\gamma^T)^{-1/2}\gamma = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$
$$F_2 : C^0\left([0, 1], Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})\right) \to C^0\left([0, 1], U(n, \mathbb{C})\right)$$
$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \mapsto \qquad X + iY =: u \ .$$

Now choose $\alpha \in C^0([0, 1], \mathbb{R})$ such that

$$\det u(t) = e^{i\alpha(t)} \ .$$

Of course α is only determined modulo 2π. However,

$$\Delta(\gamma) := \frac{\alpha(1) - \alpha(0)}{\pi}$$

is well-defined. Note that Δ is additive: $\Delta(\gamma_1 \circ \gamma_2) = \Delta(\gamma_1) + \Delta(\gamma_2)$, $\Delta(\gamma^{-1}) = -\Delta(\gamma)$. As by assumption $\gamma(1) \in Sp^*$, we can choose a continuous extension $\tilde{\gamma} : [0,1] \to Sp^*$ such that $\tilde{\gamma}(0) = \gamma(1)$ and $\tilde{\gamma}(1)$ equals either W_+ or W_-, depending on whether $\tilde{\gamma}(0)$ is in Sp^*_+ or in Sp^*_-. Then define

$$\mu_{CZ}(\gamma) = \Delta(\tilde{\gamma} \circ \gamma) ,$$

where $\tilde{\gamma} \circ \gamma$ means: follow first γ then $\tilde{\gamma}$. Of course one has to show

$$\begin{aligned} &i) \text{ independence of the choice of the extension } \tilde{\gamma} , \\ &ii) \ \Delta(\tilde{\gamma} \circ \gamma) \in \mathbb{Z} . \end{aligned} \tag{6}$$

Now we restrict to the case $n = 2$ and represent the results of section 2 via figure 6 (cf. remark 2.5).

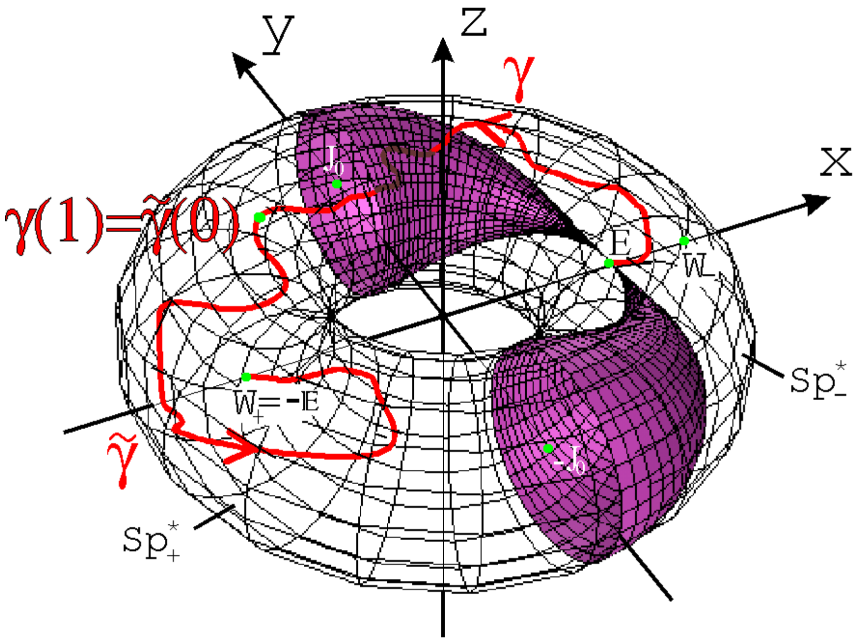

FIGURE 6. The Maslov cycle $\mathcal{C}_+$ and a path of Conley-Zehnder index $+1$

$\mathcal{C}_+ = Sp(2,\mathbb{R}) \setminus Sp^*$ is called **Maslov cycle**. After removing the point E, it is a smooth 2-manifold. In our case $u(t) \in U(1,\mathbb{C}) \simeq SO(2,\mathbb{R}) \simeq S^1$ corresponds to the matrix $R(t)$ in the polar-decomposition of Lemma 2.1, hence

$$\det u(t) = e^{i\psi(t)} ,$$

where $\psi(t)$ is one of our torus coordinates (cf. figure 3). As $\gamma(0) = E$, we have $\psi(0) = 2k\pi$, for $k \in \mathbb{Z}$. If $\tilde{\gamma}(1) = -E$ it follows $\psi(1) = (2l+1)\pi$ with $l \in \mathbb{Z}$, hence $\Delta(\tilde{\gamma} \circ \gamma) = 2l + 1 - 2k \in \mathbb{Z}$; if $\tilde{\gamma}(1) = E$ then $\psi(1) = 2l\pi$ with $l \in \mathbb{Z}$, hence $\Delta(\tilde{\gamma} \circ \gamma) = 2l - 2k \in \mathbb{Z}$. This proves (6ii). Independence of the choice of the extension $\tilde{\gamma}$, follows from the fact that another extension $\tilde{\tilde{\gamma}}$ is homotopic to $\tilde{\gamma}$ (by Lemma 3.1 or figure 6),hence $\tilde{\gamma} \circ \tilde{\tilde{\gamma}}^{-1}$ is contractible and we have $0 = \Delta(\tilde{\gamma} \circ \tilde{\tilde{\gamma}}^{-1}) = \Delta(\tilde{\gamma}) - \Delta(\tilde{\tilde{\gamma}})$. The first equality follows from the fact that if γ is a loop, $\Delta(\gamma)$ is the degree of the map $det \circ F_2 \circ F_1 \circ \gamma : S^1 \to S^1$, and therefore is a homotopy invariant. So if the loop is contractible the degree is zero.

Similarly it follows that μ_{CZ} descends to the equivalence classes $\hat{\mathcal{P}}$

$$\hat{\mu}_{CZ} : \hat{\mathcal{P}} = \mathcal{P}/\sim \to \mathbb{Z} ,$$

where $\gamma_0 \sim \gamma_1 \Leftrightarrow \exists F : [0,1] \times [0,1] \to Sp(2,\mathbb{R})$ continuous, such that $F(0,t) = \gamma_0(t)$, $F(1,t) = \gamma_1(t)$, $F(s,0) = E$, $F(s,1) \in Sp^*$.

If we assign appropriate orientations to the two connected components of $\mathcal{C}_+ \setminus \{E\}$, we can interpret μ_{CZ} as intersection number of γ with $\mathcal{C}_+$. We have to be careful, however, in which direction our path starts off at $\gamma(0) = E$:

1. If $\exists \delta > 0$ such that $\gamma((0,\delta)) \subset Sp^*_+$, then assign to $\gamma(0)$ the intersection number $+1$ if the angle $\psi(t)$ increases and -1 if it decreases.
2. If $\exists \delta > 0$ such that $\gamma((0,\delta)) \subset Sp^*_-$, then assign to $\gamma(0)$ the intersection number 0.
3. If $\exists \delta > 0$ such that $\gamma((0,\delta)) \subset \mathcal{C}_+$, then perturb γ slightly (with fixed end points) to end up in one of the former cases.

For a general continuous path $\gamma : [0,1] \to Sp(2,\mathbb{R})$ with $\gamma(0) = E$ we may define a **generalized Conley-Zehnder index**

$$\mu^{\mathbf{g}}_{\mathbf{CZ}} : C^0\left(([0,1],0),(Sp(2,\mathbb{R}),E)\right) \to \tfrac{1}{2} \cdot \mathbb{Z}$$

by setting $\mu^g_{CZ}(\gamma) = \mu_{CZ}(\gamma)$, if $\gamma \in \mathcal{P}$. If $\gamma(1) \in \mathcal{C}_+ \setminus \{E\}$, then let $\mu^g_{CZ}(\gamma)$ be the intersection number of γ with $\mathcal{C}_+$, where the endpoint $\gamma(1)$ contributes half (i.e. $\pm\frac{1}{2}$). If $\gamma(1) = E$, then assign to the endpoint one of the integers $-1, 0, +1$ as above ($(0,\delta)$ must be replaced by $(1-\delta, 1)$).

Example

Consider the path $A : [0,1] \to Sp(2,\mathbb{R})$

$$A(t) = \begin{pmatrix} 1 & \frac{t}{4\pi^2} \\ 0 & 1 \end{pmatrix}$$

which arises from linearizing the hamiltonian flow along a 1-periodic solution x of $\dot{x}(t) = X_H(x(t))$, where X_H is the hamiltonian vector field on the symplectic manifold $T^*S^1 \cong (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$ equipped with its canonical symplectic structure $dq \wedge dp$. $H = p^2/4\pi^2$ is the Hamiltonian of a free particle on S^1, cf. [We] section 4. As $Spec\,A(t) = \{+1\}$ for any $t \in [0,1]$, we observe that $A(t) \in \mathcal{C}_+\ \forall t \in [0,1]$. A numerical *Mathematica*[1]-plot of the path A is

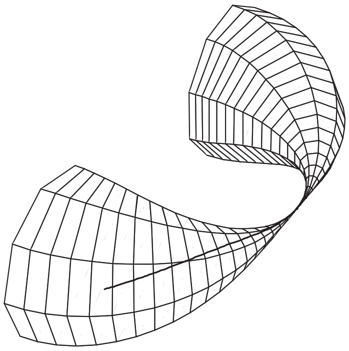

FIGURE 7. The path $A(\cdot)$

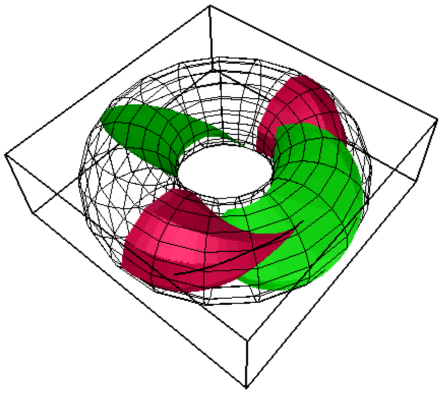

FIGURE 8. Sp_1 and $\mathcal{C}_+$

shown in figure 7. Here we extended the domain for t to $[0, 200]$ in order to scale the image of A to a viewable size.

To compute the generalized Conley-Zehnder index of A we have to perturb A keeping the endpoints fixed. Two particularly simple perturbations of A are given by paths $A^{\pm}$ with the same endpoints and $A^{\pm}((0,1)) \subset SP^*_{\pm}$. It turns out

$$\mu^g_{CZ}(A) = \mu^g_{CZ}(A^{\pm}) = -1/2.$$

To see this note that in case of A^- the initial point contributes 0 and the endpoint $1/2 \cdot$ *intersection number* $= -1/2$. In case of A^+ the initial point contributes -1 and the endpoint $1/2 \cdot$ *intersection number* $= 1/2$.

Remark

As is pointed out by Robbin and Salamon in ([RS2], se.4) there is another way of constructing a Maslov-type index for arbitrary paths $\Psi : [a,b] \to Sp(2n,\mathbb{R})$, the Robbin-Salamon index $\mu_{RS}(\Psi)$. For $n = 1$ it is constructed as follows: Let $V = 0 \times \mathbb{R}$ and define

$$Sp_k = \{M \in Sp(2,\mathbb{R}) \mid dim(MV \cap V) = k\}\ ,\ k = 0, 1.$$

Setting

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we get $M \in Sp_1$ iff $b = 0$ and $M \in Sp_0$ iff $b \neq 0$. Sp_1 is a submanifold of $Sp(2,\mathbb{R})$ of codimension one and $\mu_{RS}(\Psi)$ may be viewed as intersection number of the path with the Maslov cycle Sp_1. Sp_1 is sketched in figure 8 via numerical approximation using *Mathematica*[1]. We see that Sp_1 has 2 connected components. In figure 8 the path of the former example is indicated. After suitably orienting Sp_1 and counting intersections at endpoints only half, we observe that this path has Robbin-Salamon index $-1/2$. Alternatively we compute the index via the intersection form as introduced in [RS2] and get

$$\Gamma(A, t = 0)y = -y^2/4\pi^2\ ,\ y \in \mathbb{R}.$$

The intersection form Γ has signature -1, hence the index is $-1/2$.

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