

Morse functions associated to triangulations of closed 2-dimensional manifolds

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December 20, 1995

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Abstract Given a smooth triangulation (Σ, h) of a 2-dimensional, closed, smooth manifold M , we construct a smooth Morse function on M , such that its critical points of Morse index k , $k \in \{0, 1, 2\}$, coincide with the images under h of the barycenters of the k -faces of the simplices of Σ . There is a metric on M such that the unstable manifolds (with respect to the negative gradient flow) of the critical points of this Morse function coincide essentially with the images under h of the k -faces of the simplices of Σ . The coincidence may fail near the critical points of Morse index 0.

A critical point of Morse index 2 has exactly one connecting orbit to each of its three surrounding (with respect to the triangulation) critical points p_i of Morse index 1. Each of the latter ones has exactly one connecting orbit to each of the two critical points of Morse index 0 lying on the boundary of the (image under h of the) 1-face containing p_i .

Our result will lead to a canonical chain map between the chain complexes of the Morse homology of M $HM_*(M; \mathbb{Z}_2)$ and the simplicial homology of Σ $H_*^{\text{simp}}(\Sigma; \mathbb{Z}_2)$. This chain map is even an isomorphism, thus it yields an isomorphism on the level of homology. The isomorphism $H_*^{\text{simp}}(\Sigma; \mathbb{Z}_2) \simeq H_*^{\text{sing}}(M; \mathbb{Z}_2)$ is well known.

We expect the methods and arguments used in this paper to be generalizable to the n -dimensional case.

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Acknowledgement The author would like to thank Nico Frieze, Markus Klein, Andreas Knauf and Ruedi Seiler for valuable discussions and comments.

1. Local constructions

Consider the 2-simplex Δ^2 in $\mathbb{R}^2$

$$\Delta^2 := \langle v^0, v^1, v^2 \rangle := \left\{ \sum_{i=0}^2 \lambda_i v^i \mid \sum_{i=0}^2 \lambda_i = 1, \lambda_i \geq 0, i = 0, 1, 2 \right\},$$

where $v^0 := (-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$, $v^1 := (\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and $v^2 := (0, 1)$ are the vertices of Δ^2 . By $\langle \cdot, \cdot \rangle$ we denote the euclidean inner product on $\mathbb{R}^2$; let $m \in \mathbb{N}$

$$\text{dist}(x, \Delta^2) := \inf_{y \in \Delta^2} \text{dist}(x, y) := \inf_{y \in \Delta^2} \langle x - y, x - y \rangle^{1/2},$$

$$U_{1/m}(\Delta^2) := \{x \in \mathbb{R}^2 \mid \text{dist}(x, \Delta^2) < 1/m\}.$$

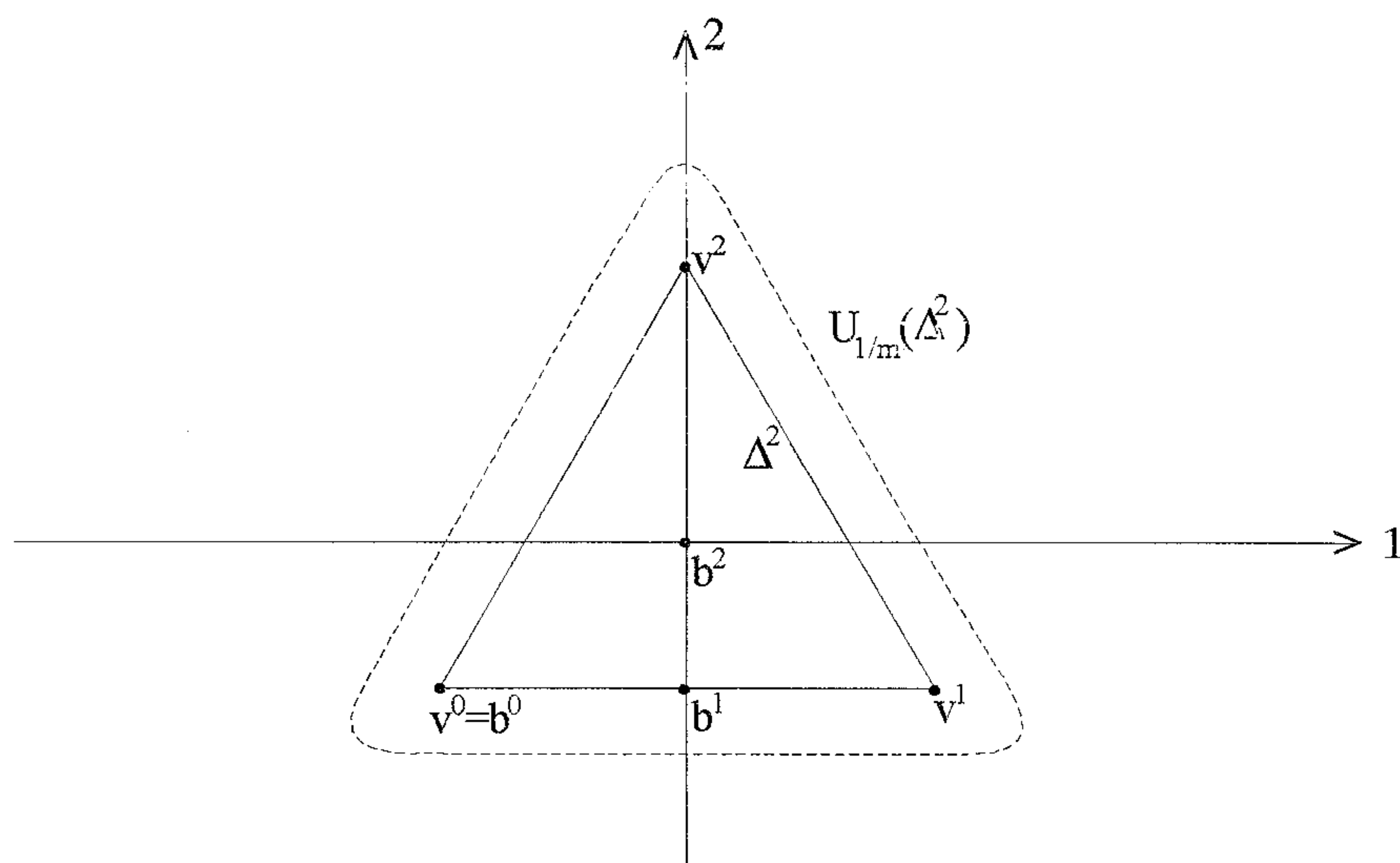


figure 1 b^0, b^1 and b^2 denote the barycenters of the corresponding faces of Δ^2

In the first part 'THE MORSE FUNCTION d_m ON Δ^2 ' we are going to construct a smooth function $d_m: U_{1/m}(\Delta^2) \rightarrow \mathbb{R}$, such that its critical points are nondegenerate and the set of critical points of Morse index k (denoted by $\text{Crit}_k d_m$) coincides with the set of barycenters of the k -dimensional faces of Δ^2 .

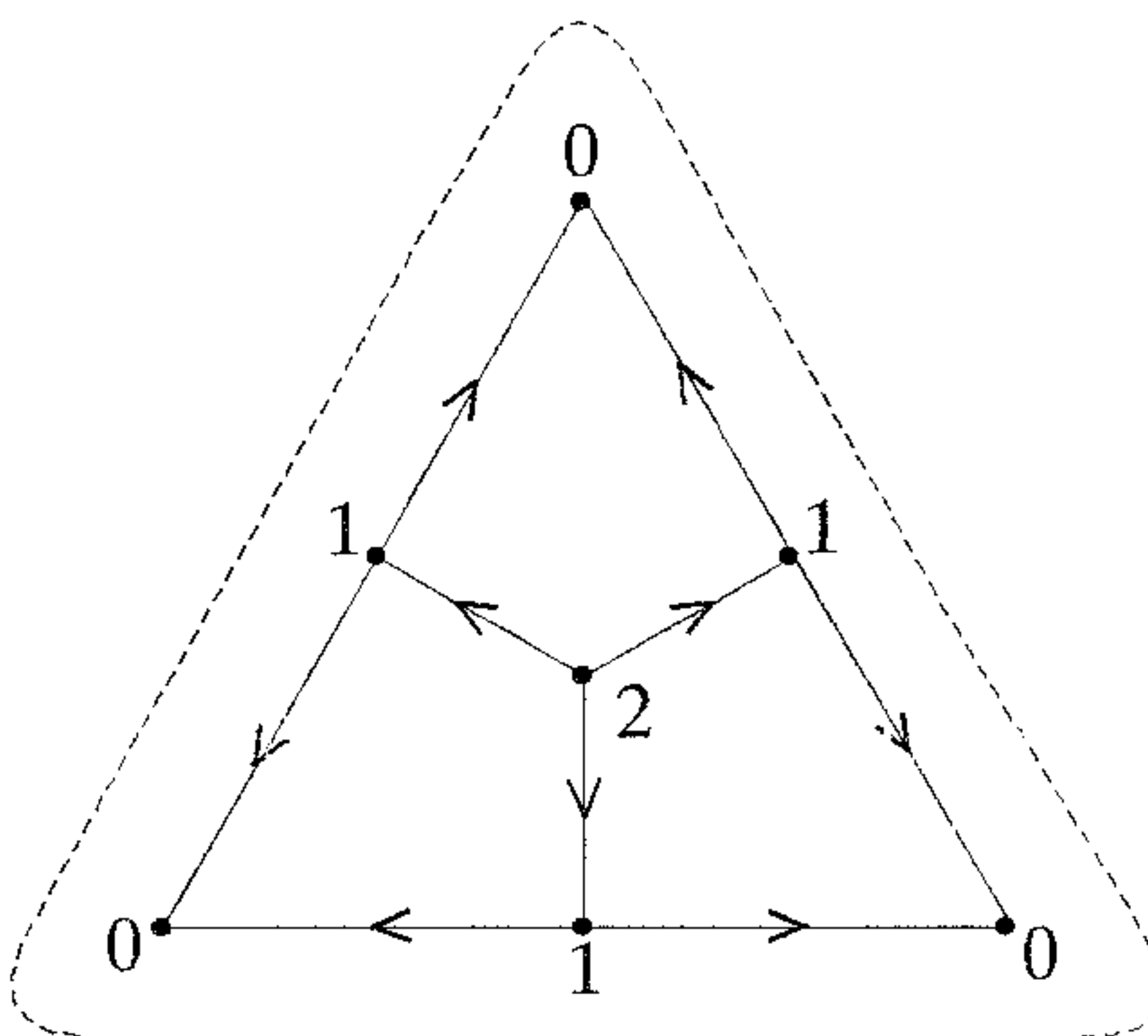


figure 2 Connecting orbits between critical points of index difference 1

The vector field $-\nabla d_m$ on $U_{1/m}(\Delta^2)$, where the gradient is taken with respect to $\langle \cdot, \cdot \rangle$, generates the negative gradient flow $\phi_t: U_{1/m}(\Delta^2) \rightarrow U_{1/m}(\Delta^2)$. The images of its connecting orbits between two critical points of index difference 1 are precisely the (open) straight line segments between the corresponding critical points. This is shown in figure 2, where the numbers indicate the Morse indices.

1.1 THE MORSE FUNCTION d_m ON Δ^2

We start with the continuous function

$$\begin{aligned} d: U_{2/m}(\Delta^2) &\rightarrow \mathbb{R} \\ x &\mapsto \text{dist}(x, \{v^0, v^1, v^2\}) \end{aligned}$$

We observe that $d(x) = \min_{y \in \{v^0, v^1, v^2\}} \|x - y\|^2$ is piecewise given by smooth functions $d_i(x) = \|x - v^i\|^2$ where $x \in A_i$ and $i = 0, 1, 2$. The open subsets A_0, A_1 and A_2 of $U_{2/m}(\Delta^2)$ are as indicated in figure 3.

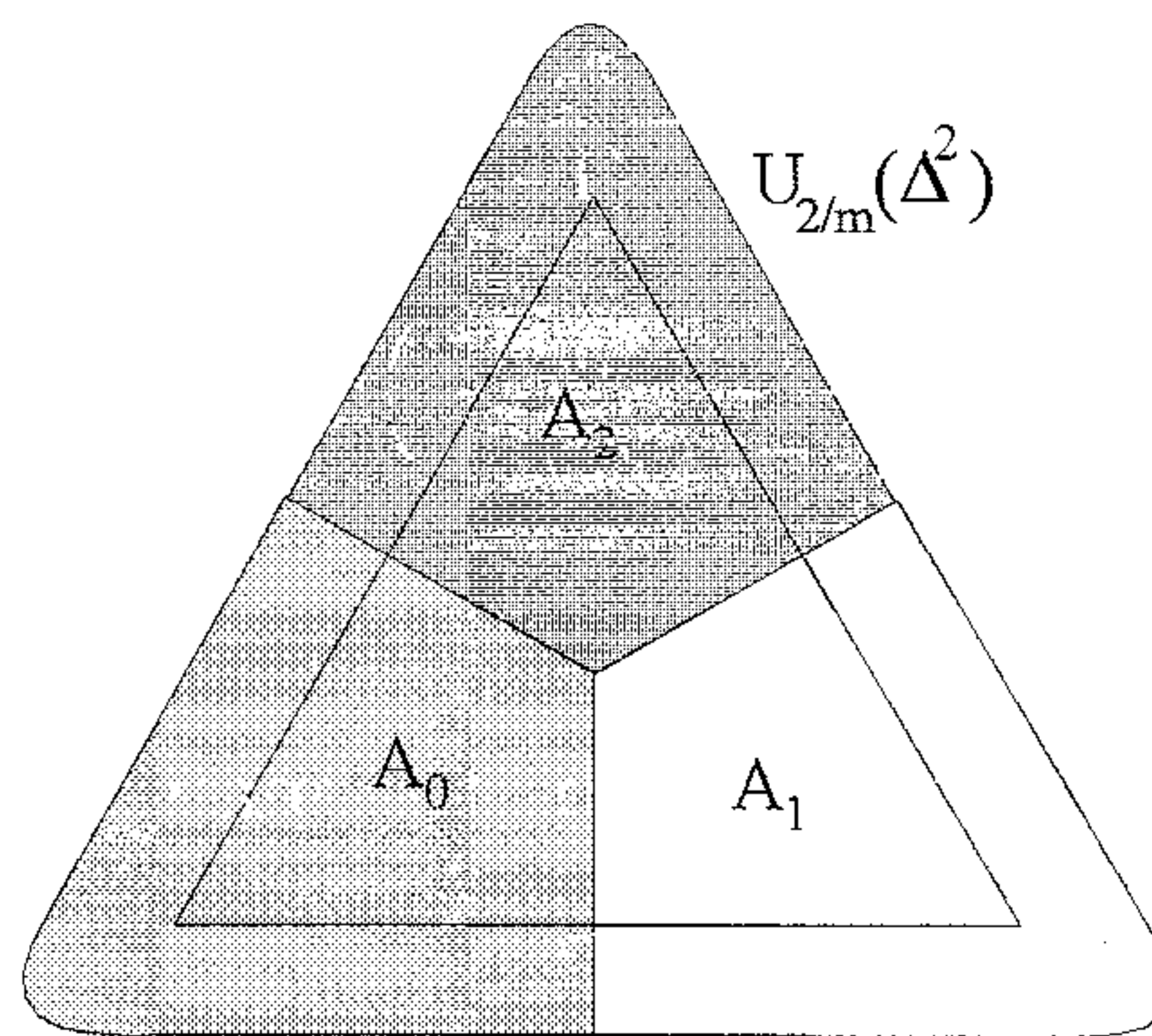


figure 3

The compactness of $\overline{U_{2/m}(\Delta^2)}$ implies

$$L := \max_{i \in \{0,1,2\}} \sup_{x \in A_i} \|Dd(x)\|_{\mathcal{L}(\mathbb{R}^2)} < \infty.$$

L is a Lipschitz constant with respect to d , i.e.

$$\forall x, y \in U_{2/m}(\Delta^2) \quad |d(x) - d(y)| \leq L \cdot \|x - y\|.$$

Consider now for $m \in \mathbb{N}$ the **standard convolution kernel**

$$\begin{aligned} J_m: B_{1/m}(0) &\rightarrow \mathbb{R} \\ x &\mapsto m^2 \cdot j(mx) := \frac{m^2}{I} \exp\left(\frac{-1}{1 - m^2 x^2}\right) \end{aligned}$$

J_m is smooth and vanishes with all derivatives on $\partial B_{1/m}(0)$. The normalizing factor I is

given by

$$I := \int_{B_{1/m}(0) \subset \mathbb{R}^2} m^2 \cdot e^{\frac{-1}{1-m^2x^2}} dx = \int_{B_1(0) \subset \mathbb{R}^2} e^{\frac{-1}{1-y^2}} dy = 2\pi \int_0^1 e^{\frac{-1}{1-r^2}} r \cdot dr < \infty ,$$

where the last equality follows by transformation to polar coordinates.

Proposition 1 $\forall m \in \mathbb{N}$ we have

$$i) \int_{B_{1/m}(0)} J_m(x) dx = 1$$

$$ii) DJ_m(x) = \left(\frac{d}{dx_1} J_m(x), \frac{d}{dx_2} J_m(x) \right) = - \frac{2m^2}{(1-m^2x^2)^2} J_m(x) (x_1, x_2)$$

$$iii) D^2 J_m(x) = ((J_{ij}))_{i,j \in \{1,2\}} \text{ where } J_{ij} = \frac{d}{dx_j} \frac{d}{dx_i} J_m(x) \text{ and}$$

$$J_{11} = J_m(x) \cdot \left(\frac{-2m^2 + 2m^4x^2}{(1-m^2x^2)^3} + \frac{8m^6x_1^2x^2 - 4m^4x_1^2}{(1-m^2x^2)^4} \right)$$

$$J_{22} = J_m(x) \cdot \left(\frac{-2m^2 + 2m^4x^2}{(1-m^2x^2)^3} + \frac{8m^6x_2^2x^2 - 4m^4x_2^2}{(1-m^2x^2)^4} \right)$$

$$J_{12} = J_{21} = J_m(x) \cdot x_1x_2 \cdot \left(\frac{-4m^4 + 8m^6x^2}{(1-m^2x^2)^4} \right) .$$

We define the **convolution of d by J_m** to be the map

$$\begin{aligned} J_m * d : U_{1/m}(\Delta^2) &\rightarrow \mathbb{R} \\ x &\mapsto \int_{B_{1/m}(0)} J_m(y) d(x-y) dy . \end{aligned}$$

For fixed $x \in U_{1/m}(\Delta^2)$ we change variables $z = x - y$ and get

$$d_m(x) := (J_m * d)(x) = \int_{B_{1/m}(x)} J_m(x-z) d(z) dz .$$

According to [Hi, Se. 2.2] we have

Lemma 2 Let J_m be as above and $f \in C^0(U_{2/m}(\Delta^2), \mathbb{R})$.

a) $J_m * f$ is C^∞ and for any $k \in \mathbb{N}$ and $x \in U_{1/m}(\Delta^2)$

$$D^k(J_m * f)(x)(\xi_1, \dots, \xi_k) = \int_{B_{1/m}(x)} D^k J_m(x-z)(\xi_1, \dots, \xi_k) \cdot f(z) dz .$$

b) If f is C^k , then $D^k(J_m * f) = J_m * D^k f$.

c) Suppose $f \in C^r(U_{2/m}(\Delta^2), \mathbb{R})$, $0 \leq r \leq \infty$. Let $K \subset \overline{U_{1/m}(\Delta^2)}$ be compact, then

$$\forall \epsilon > 0 \quad \exists m_0 \in \mathbb{N} \quad \forall m \geq m_0 \quad \|J_m * f - f\|_{r,K} < \epsilon \quad .$$

In our case c) implies that $J_m * d \rightarrow d$ uniformly on compact subsets of $U_{1/m}(\Delta^2)$ as $m \rightarrow \infty$. We are now going to prove a result analogous to b) in the case that d is Lipschitz continuous.

Lemma 3 Let J_m and d be as above and $x \in U_{1/m}(\Delta^2)$, then

$$D(J_m * d)(x) \xi = \int_{B_{1/m}(x) \setminus Z} J_m(x-z) \cdot Dd(z) \xi \, dz \quad ,$$

where $\xi \in \mathbb{R}^2$ and Z is the set of measure zero on which d is not continuously differentiable.

PROOF We know by a) that $J_r * d$ is smooth and by c) that $J_r * d \rightarrow d$ pointwise as $r \rightarrow \infty$. Therefore $D(J_r * d) \rightarrow Dd$ as $r \rightarrow \infty$ pointwise almost everywhere. Clearly $Dd \in L^1(U_{1/m}(\Delta^2), \mathbb{R}^2)$ and $D(J_r * d)$ is dominated by $2L$ for r sufficiently large. Now we can apply the dominated convergence theorem [Ad, Thm. 1.45]

$$\begin{aligned} (*) \quad D \int_{B_{1/m}(x)} J_m(x-z) \cdot (J_r * d)(z) \, dz &\stackrel{b)}{=} \int_{B_{1/m}(x)} J_m(x-z) \cdot D(J_r * d)(z) \, dz \\ &\rightarrow \int_{B_{1/m}(x) \setminus Z} J_m(x-z) \cdot Dd(z) \, dz \quad \text{as } r \rightarrow \infty. \end{aligned}$$

On the other hand

$$\begin{aligned} (**) \quad D \int_{B_{1/m}(x)} J_m(x-z) \cdot (J_r * d)(z) \, dz &\stackrel{a)}{=} \int_{B_{1/m}(x)} DJ_m(x-z) (J_r * d)(z) \, dz \\ &\rightarrow \int_{B_{1/m}(x)} DJ_m(x-z) d(z) \, dz \quad \text{as } r \rightarrow \infty. \end{aligned}$$

In the last step we again applied the dominated convergence theorem using $J_r * d \rightarrow d$ uniformly, hence pointwise, and the compactness of $\overline{U_{2/m}(\Delta^2)}$ giving a constant dominating $J_r * d$ for large r .

By (*) and (**) we get

$$D \int_{B_{1/m}(x)} J_m(x-z) d(z) \, dz = \int_{B_{1/m}(x)} DJ_m(x-z) d(z) \, dz = \int_{B_{1/m}(x) \setminus Z} J_m(x-z) \cdot Dd(z) \, dz \quad .$$

□

Theorem 4 $\exists m_0 \in \mathbb{N} \setminus \{1\} \quad \forall m \geq m_0$

i) The barycenters of the k -faces of Δ^2 are critical points of d_m .

ii) They are nondegenerate. The Hessians of d_m at b^0, b^1, b^2 are given in the coordinates (z_1, z_2) by

$$D^2 d_m(b^0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad D^2 d_m(b^1) = \begin{bmatrix} 2 - 2\sqrt{3}c_m & 0 \\ 0 & 2 \end{bmatrix}, \quad D^2 d_m(b^2) = \begin{bmatrix} 2 - 2\sqrt{3}c_m & 0 \\ 0 & 2 - 3c_m \end{bmatrix},$$

where $c_m \rightarrow \infty$ as $m \rightarrow \infty$ and $2 - 3c_m < 0$ for $m \geq m_0$.

The Hessians at the other barycenters can be transformed to one of the above normal forms by an affine change of coordinates.

PROOF i) At any barycenter b of a k -face we can find two linearly independent symmetry axes ξ, η for d . We will see that they are also symmetry axes for d_m , implying that $Dd(b)\xi^\perp = 0 = Dd(b)\eta^\perp$, where $\xi^\perp$ ($\eta^\perp$) is one of the two vectors of norm 1 orthogonal to ξ (η).

Let ξ be a symmetry axis of d through b . We choose orthonormal coordinates (y_1, y_2) such that the y_2 -axis is parallel to ξ and the y_1 -axis orthogonal to ξ . In this coordinates we have

$$\begin{aligned} d(y_1, y_2) &= d(-y_1, y_2) \quad , \quad (y_1, y_2) \in U_{2/m}(\Delta^2) \\ d_m(y_1, y_2) &= \int_{B_{1/m}(y_1, y_2)} J_m((y_1, y_2) - (z_1, z_2)) d(z_1, z_2) dz_1 dz_2 \quad , \quad x_1 := -z_1, x_2 := z_2 \\ &\quad , \quad \left| \frac{\partial(x_1, x_2)}{\partial(z_1, z_2)} \right| = 1 \\ &= \int_{B_{1/m}(-y_1, y_2)} J_m(y_1 + x_1, y_2 - x_2) d(x_1, x_2) dx_1 dx_2 \\ &= \int_{B_{1/m}(-y_1, y_2)} J_m(-y_1 - x_1, y_2 - x_2) d(x_1, x_2) dx_1 dx_2 \quad , \quad \text{sym. of } J_m \text{ in 1st comp.} \\ &= d_m(-y_1, y_2) \end{aligned}$$

$$d_m(y_1, 0) = d_m(0, 0) + Dd_m(0, 0)(y_1, 0) + O(y_1^2)$$

||

$$d_m(-y_1, 0) = d_m(0, 0) + Dd_m(0, 0)(-y_1, 0) + O(y_1^2)$$

$$\Rightarrow Dd_m(0, 0)(1, 0) = Dd_m(0, 0)(-1, 0) = -Dd_m(0, 0)(1, 0)$$

$$\Rightarrow 0 = Dd_m(0, 0)(1, 0) = Dd_m(b)\xi^\perp.$$

Analogously we can show that $Dd_m(b)\eta^\perp = 0$. But $\xi^\perp$ and $\eta^\perp$ are linearly independent and therefore $Dd_m(b) \equiv 0$.

ii) The critical point b^0 : In $B_{1/m}(b^0)$ $d(z)$ is given by $d_0(z) = \|z - v^0\|^2$, thus it is smooth and we can apply Lemma 2 b) to get for $i = 1, 2$

$$\begin{aligned}
D^2(J_m * d)(b^0)(e_i, e_i) &= \int_{B_{1/m}(b^0)} J_m(b^0 - z) \cdot D^2 d_0(z)(e_i, e_i) dz \\
&= \int_{B_{1/m}(b^0)} J_m(b^0 - z) \cdot \underbrace{\frac{d}{dz_i} 2(z_i - v_i^0)}_{=2} dz \\
&= 2 \quad , \text{ by Prop. 1 i) } .
\end{aligned}$$

$$D^2(J_m * d)(b^0)(e_1, e_2) = \int_{B_{1/m}(b^0)} J_m(b^0 - z) \cdot \underbrace{\frac{d}{dz_2} 2(z_1 - v_1^0)}_{=0} dz = 0 .$$

The critical point b^1 : In $B_{1/m}(b^1)$ $d(z)$ is piecewise smooth and given by

$$d_i(z) = \|z - v^i\|^2 \quad , \quad z \in B_i \quad , \quad i = 0, 1$$

$$B_0 := \{y \in B_{1/m}(b^1) \mid y_1 < b_1^1\} \quad , \quad B_1 := \{y \in B_{1/m}(b^1) \mid y_1 > b_1^1\} \quad .$$

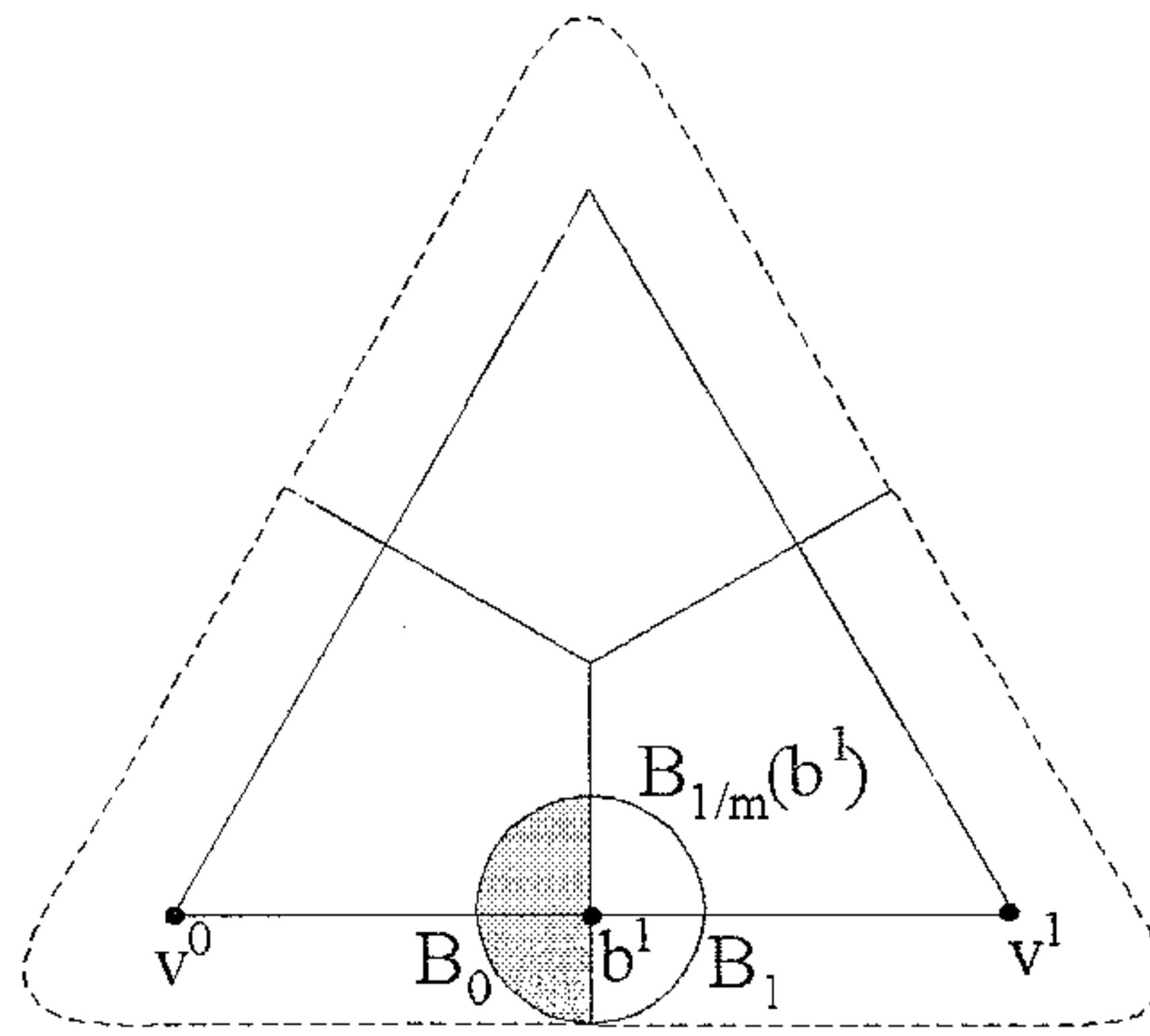


figure 4

$$\begin{aligned}
D^2(J_m * d)(b^1)(e_1, e_2) &= \int_{B_{1/m}(b^1)} D^2 J_m(b^1 - z)(e_1, e_2) \cdot d(z) dz \quad , \text{ Lemma 2 a)} \\
&= \int_{B_{1/m}(b^1)} \partial_1 \partial_2 J_m(b^1 - z) \cdot d(z) dz = \sum_{i=0}^1 \int_{B_i} \partial_1 \partial_2 J_m(b^1 - z) \cdot d_i(z) dz \\
&= \sum_{i=0}^1 \int_{B_i} \left(\partial_1 (\partial_2 J_m(b^1 - z) \cdot d_i(z)) - \partial_2 J_m(b^1 - z) \cdot \partial_1 d_i(z) \right) dz \\
&= \underbrace{\int_{B_0 \cup B_1} \partial_1 (\partial_2 J_m(b^1 - z) \cdot d(z)) dz}_{=0 \text{ (integrate w.r.t. } z_1)} - \sum_{i=0}^1 \int_{B_i} \partial_2 J_m(b^1 - z) \cdot \partial_1 d_i(z) dz
\end{aligned}$$

$$\begin{aligned}
& \stackrel{\text{part. integr.}}{=} \stackrel{\text{w.r.t. } z_2}{=} - \left(\int_{-\frac{1}{m}}^0 [J_m(b^1 - z) \, 2(z_1 - v_1^0)]_{z_2 = -\sqrt{\cdot}}^{+\sqrt{\cdot}} dz_1 - \int_{B_0} J_m(b^1 - z) \cdot \overbrace{\partial_2 \partial_1 d_0(z)}^{=0} dz \right) \\
& - \left(\int_0^{\frac{1}{m}} [J_m(b^1 - z) \, 2 \overbrace{(z_1 - v_1^1)}^{=-(z_1 - v_1^0)}]_{z_2 = -\sqrt{\cdot}}^{+\sqrt{\cdot}} dz_1 - \int_{B_1} J_m(b^1 - z) \cdot \overbrace{\partial_2 \partial_1 d_1(z)}^{=0} dz \right) \\
& = 0 \quad .
\end{aligned}$$

$$D^2(J_m * d)(b^1)(e_2, e_2) \stackrel{\text{as before}}{=} \dots = - \sum_{i=0}^1 \int_{B_i} \partial_2 J_m(b^1 - z) \cdot \partial_2 d_i(z) dz$$

$$\begin{aligned}
& \stackrel{\text{part. integr.}}{=} \stackrel{\text{w.r.t. } z_2}{=} - \left(\int_{-\frac{1}{m}}^0 [J_m(b^1 - z) \, 2(z_1 - v_1^0)]_{z_2 = -\sqrt{\cdot}}^{+\sqrt{\cdot}} dz_1 - \int_{B_0} J_m(b^1 - z) \cdot \overbrace{\partial_2 \partial_2 d_0(z)}^{=2} dz \right) \\
& - \left(\int_0^{\frac{1}{m}} [J_m(b^1 - z) \, 2 \overbrace{(z_1 - v_1^1)}^{=-(z_1 - v_1^0)}]_{z_2 = -\sqrt{\cdot}}^{+\sqrt{\cdot}} dz_1 - \int_{B_1} J_m(b^1 - z) \cdot \overbrace{\partial_2 \partial_2 d_1(z)}^{=2} dz \right) \\
& = 2 \int_{B_0 \cup B_1} J_m(b^1 - z) dz \stackrel{\text{Prop.1}}{=} 2 \int_{B_{1/m}(b^1)} J_m(b^1 - z) dz \stackrel{i)}{=} 2 \quad .
\end{aligned}$$

$$D^2(J_m * d)(b^1)(e_1, e_1) \stackrel{\text{as before}}{=} \dots = - \sum_{i=0}^1 \int_{B_i} \partial_1 J_m(b^1 - z) \cdot \partial_1 d_i(z) dz$$

$$\begin{aligned}
& \stackrel{\text{part. integr.}}{=} \stackrel{\text{w.r.t. } z_1}{=} - \left(\int_{-\frac{1}{2}-\frac{1}{m}}^{-\frac{1}{2}+\frac{1}{m}} [J_m(b^1 - z) \, 2(z_1 - v_1^0)]_{z_1 = -\sqrt{\cdot}}^0 dz_2 - \int_{B_0} J_m(b^1 - z) \cdot \overbrace{\partial_1 \partial_1 d_0(z)}^{=2} dz \right) \\
& - \left(\int_{-\frac{1}{2}-\frac{1}{m}}^{-\frac{1}{2}+\frac{1}{m}} [J_m(b^1 - z) \, 2(z_1 - v_1^1)]_{z_1 = 0}^{+\sqrt{\cdot}} dz_2 - \int_{B_1} J_m(b^1 - z) \cdot \overbrace{\partial_1 \partial_1 d_1(z)}^{=2} dz \right) \\
& = 2 - \int_{-\frac{1}{2}-\frac{1}{m}}^{-\frac{1}{2}+\frac{1}{m}} \left(J_m(b^1 - (0, z_2)) \cdot \underbrace{(-2v_1^0)}_{=2v_1^1} - J_m(b^1 - (0, z_2)) \cdot (-2v_1^1) \right) dz_2 \\
& \stackrel{s := z_2 + \frac{1}{2}}{=} 2 - 4v_1^1 \int_{-\frac{1}{m}}^{\frac{1}{m}} J_m(0, -s) ds =: 2 - 4v_1^1 c_m \quad ,
\end{aligned}$$

$$c_m = \int_{-\frac{1}{m}}^{\frac{1}{m}} \frac{m^2}{I} \cdot e^{\frac{-1}{1-m^2s^2}} ds \stackrel{u:=ms}{=} \frac{m}{I} \int_{\underbrace{-1}_{=\text{const} \in (0,\infty)}}^1 e^{\frac{-1}{1-u^2}} du \rightarrow \infty \quad \text{as } m \rightarrow \infty.$$

Recall that $v_1^1 = \frac{1}{2}\sqrt{3}$, therefore $\exists m_0 \in \mathbb{N} \quad \forall m \geq m_0 \quad D^2(J_m * d)(b^1)(e_1, e_1) < 0$.

The critical point b^2 : In $B_{1/m}(b^2)$ $d(z)$ is piecewise smooth and given by

$$d_i(z) = \|z - v^i\|^2, \quad z \in B_i, \quad i = 0, 1, 2,$$

where B_0, B_1, B_2 are the open subsets of $B_{1/m}(b^2)$ indicated in figure 5.

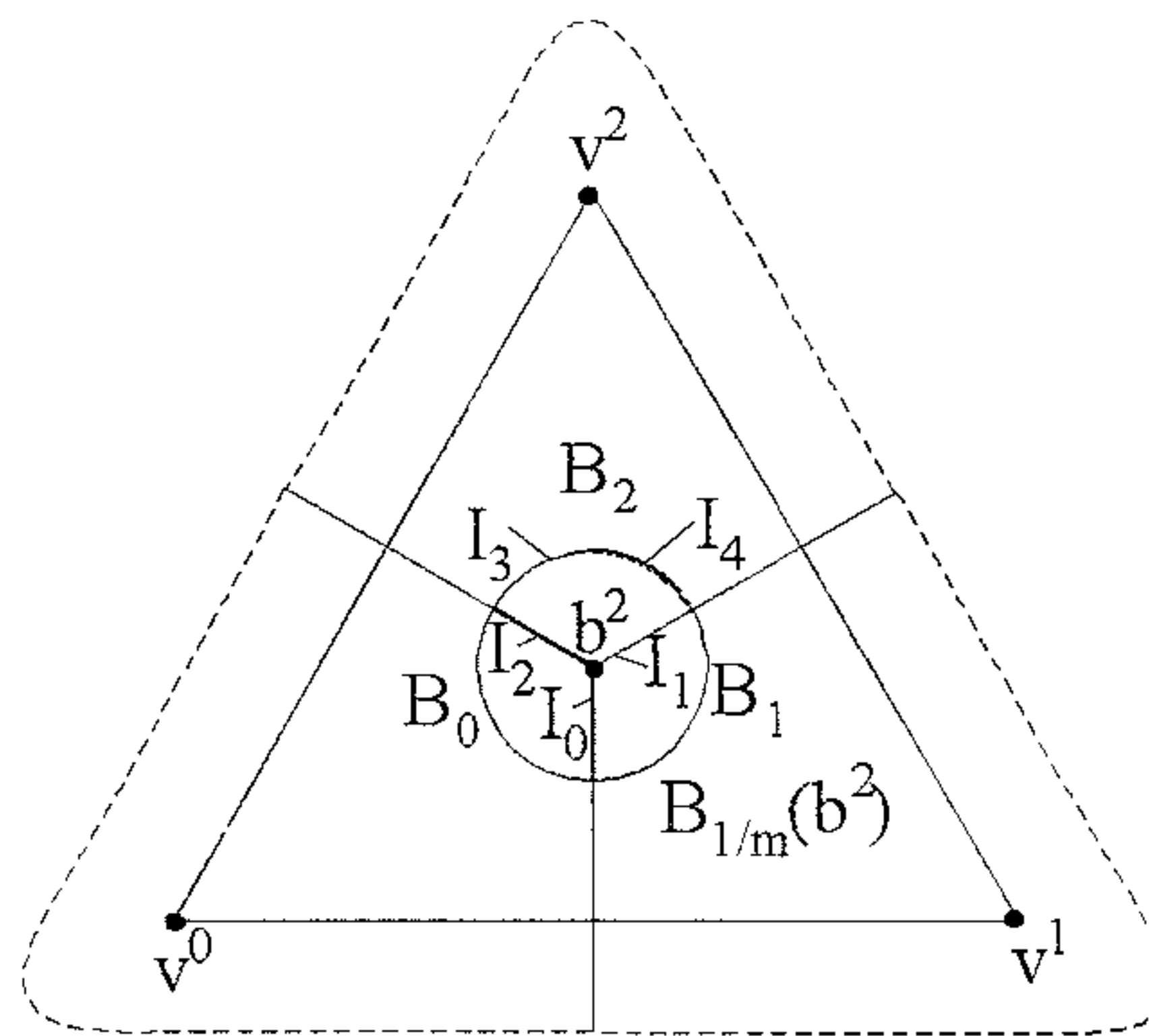


figure 5

$$\begin{aligned} D^2(J_m * d)(b^2)(e_1, e_2) &= \int_{B_{1/m}(b^2)} D^2 J_m(b^2 - z)(e_1, e_2) \cdot d(z) dz && , \text{ Lemma 2 a)} \\ &= \int_{B_{1/m}(b^2)} \partial_1 \partial_2 J_m(b^2 - z) \cdot d(z) dz = \sum_{i=0}^2 \int_{B_i} \partial_1 \partial_2 J_m(b^2 - z) \cdot d_i(z) dz \\ &= \sum_{i=0}^2 \int_{B_i} \left(\partial_1 (\partial_2 J_m(b^2 - z) \cdot d_i(z)) - \partial_2 J_m(b^2 - z) \cdot \partial_1 d_i(z) \right) dz \\ &= \int_{B_0 \cup B_1 \cup B_2} \overbrace{\partial_1 (\partial_2 J_m(b^2 - z) \cdot d(z))}^{= 0 \text{ (integrate w.r.t. } z_1)} dz - \int_{B_0 \cup B_1 \cup B_2} \overbrace{\partial_2 J_m(b^2 - z)}^{\text{antisym. in } z_2} \cdot \underbrace{\partial_1 d_i(z)}_{\text{indep. of } z_2} dz \\ &= 0 \quad . \end{aligned}$$

As indicated in figure 5, by I_i , $i \in \{0, 1, 2\}$, we denote the part of the straight line inside $B_{1/m}(b^2)$. By I_3, I_4 we denote the part of the circle arc as indicated. In what follows we use this notation to indicate the boundary of the domain of integration.

$$D^2(J_m * d)(b^2)(e_1, e_1) \stackrel{\text{as before}}{=} \dots = - \sum_{i=0}^2 \int_{B_i} \partial_1 J_m(b^2 - z) \cdot \partial_1 d_i(z) dz$$

$$\begin{aligned}
& \stackrel{\text{part. integr.}}{=} \stackrel{\text{w.r.t. } z_1}{=} - \left(\int_{-\frac{1}{m}}^{I_2} [J_m(b^2 - z) \cdot 2(z_1 - v_1^0)]_{z_1 = -\sqrt{-}}^{I_0 \cup I_2} dz_2 - \int_{B_0} J_m(b^2 - z) \cdot \overbrace{\partial_1 \partial_1 d_0(z)}^{=2} dz \right) \\
& - \left(\int_{-\frac{1}{m}}^{I_1} [J_m(b^2 - z) \cdot 2(z_1 - v_1^1)]_{z_1 = I_0 \cup I_1}^{+\sqrt{-}} dz_2 - \int_{B_1} J_m(b^2 - z) \cdot \overbrace{\partial_1 \partial_1 d_1(z)}^{=2} dz \right) \\
& - \left(\int_{I_1 \cup I_2}^{\frac{1}{m}} [J_m(b^2 - z) \cdot 2(z_1 - v_1^2)]_{z_1 = I_2 \cup I_3}^{I_1 \cup I_4} dz_2 - \int_{B_2} J_m(b^2 - z) \cdot \overbrace{\partial_1 \partial_1 d_2(z)}^{=2} dz \right) \\
& = 2 - \int_{I_2} J_m(b^2 - z) \cdot 2(v_1^2 - v_1^0) - \int_{I_1} J_m(b^2 - z) \cdot 2(v_1^1 - v_1^2) - \int_{I_0} J_m(b^2 - z) \cdot 2(v_1^1 - v_1^0) \\
& = 2 - 2\sqrt{3} \int_{-\frac{1}{m}}^{\frac{1}{m}} J_m(b^2 - (0, z_2)) dz_2 = 2 - 2\sqrt{3} c_m \rightarrow -\infty \text{ as } m \rightarrow \infty.
\end{aligned}$$

$$D^2(J_m * d)(b^2)(e_2, e_2) \stackrel{\text{as before}}{=} \dots = - \sum_{i=0}^2 \int_{B_i} \partial_2 J_m(b^2 - z) \cdot \partial_2 d_i(z) dz$$

$$\begin{aligned}
& \stackrel{\text{part. integr.}}{=} \stackrel{\text{w.r.t. } z_2}{=} - \left(\int_{-\sqrt{-}}^{I_0 \cup I_2} [J_m(b^2 - z) \cdot 2(z_2 - v_2^0)]_{z_2 = -\sqrt{-}}^{I_2} dz_1 - \int_{B_0} J_m(b^2 - z) \cdot \overbrace{\partial_2 \partial_2 d_0(z)}^{=2} dz \right) \\
& - \left(\int_{I_0 \cup I_1}^{\sqrt{-}} [J_m(b^2 - z) \cdot 2(z_2 - v_2^1)]_{z_2 = -\sqrt{-}}^{I_1} dz_1 - \int_{B_1} J_m(b^2 - z) \cdot \overbrace{\partial_2 \partial_2 d_1(z)}^{=2} dz \right) \\
& - \left(\int_{I_2 \cup I_3}^{I_1 \cup I_4} [J_m(b^2 - z) \cdot 2(z_2 - v_2^2)]_{z_2 = I_1 \cup I_2}^{+\sqrt{-}} dz_1 - \int_{B_2} J_m(b^2 - z) \cdot \overbrace{\partial_2 \partial_2 d_2(z)}^{=2} dz \right) \\
& = 2 - \int_{I_2} J_m(b^2 - z) \cdot 2(\overbrace{v_2^2 - v_2^0}^{=3/2}) - \int_{I_1} J_m(b^2 - z) \cdot 2(\overbrace{v_2^2 - v_2^1}^{=3/2}) \\
& = 2 - 3 \int_{-\frac{1}{m}}^{\frac{1}{m}} J_m(b^2 - (0, z_2)) dz_2 = 2 - 3 c_m \rightarrow -\infty \text{ as } m \rightarrow \infty.
\end{aligned}$$

□

Lemma 5 Let $m_0 \in \mathbb{N}$ be as in Theorem 4 and $m \geq m_0$, then the barycenters of the k -faces ($k = 0, 1, 2$) of Δ^2 are exactly the critical points of d_m in $U_{1/m}(\Delta^2)$.

Corollary 6 For any $m \geq m_0$, m_0 as in Theorem 4, d_m is a Morse function on $U_{1/m}(\Delta^2)$.

PROOF (of Lemma 5) Fix $m \in \mathbb{N}$ such that $m \geq m_0$. Let A_0, A_1, A_2 be the open subsets of $U_{2/m}(\Delta^2)$ indicated in figure 3. Because of the nondegeneracy of the barycenters as critical points of $d_m: U_{1/m}(\Delta^2) \rightarrow \mathbb{R}$, there exists $\delta > 0$, such that in the δ -neighborhood $U_\delta(b)$ of each barycenter b there is no other critical point of d_m . Therefore it suffices to show that

$$\forall x \in \left(U_{1/m}(\Delta^2) \setminus \bigcup_{b \in \{\text{barycenters}\}} U_\delta(b) \right) : D(J_m * d)(x) \neq 0 .$$

Because of the symmetry of the function d_m , we can in addition assume without loss of generality that $x \in A_0$. There are three cases to be analyzed:

Case I $x \in \left((U_{1/m}(\Delta^2) \cap A_0) \setminus \bigcup_{b \in \{\text{barycenters}\}} U_\delta(b) \right)$ and $U_{1/m}(x) \subset A_0$.

Denote by g^1 (g^2) the straight line through v^0 , which is parallel to the x_1 - (x_2 -) coordinate axis.

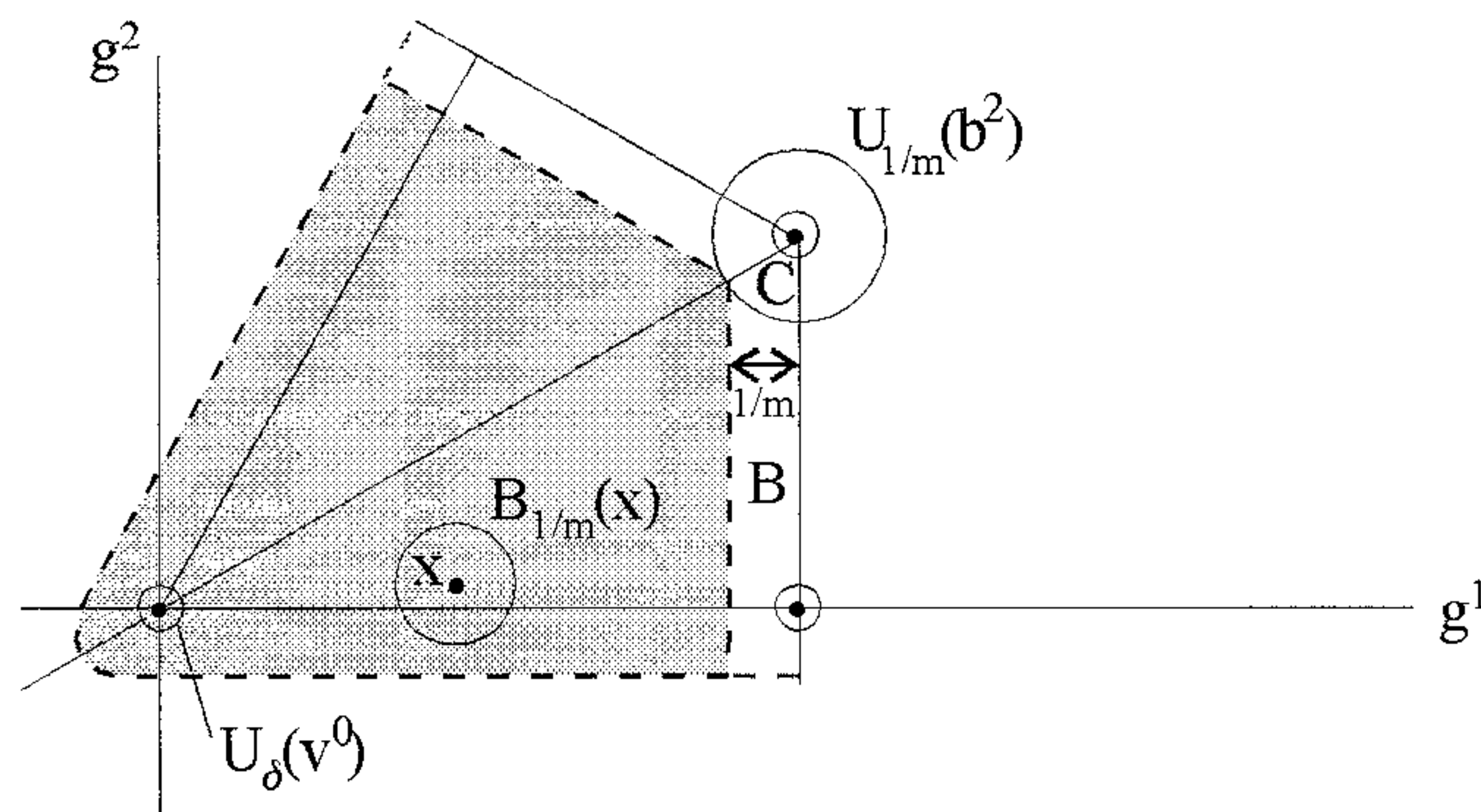


figure 6

(*) If $\text{dist}(x, g^1) < \text{dist}(x, g^2)$ then $i := 1$ otherwise $i := 2$.

Assume $i = 1$, then

$$\begin{aligned} D(J_m * d)(x) e_1 &= \int_{B_{1/m}(x)} J_m(x - z) \cdot Dd_0(z) e_1 \, dz \\ &= \int_{B_{1/m}(x)} J_m(x - z) \cdot 2(z_1 - v_1^0) \, dz . \end{aligned}$$

Assume furthermore $x_1 > -\frac{1}{2}\sqrt{3}$ and note that (in our case $i = 1$) there exists $c > 0$ such that $|x_1 + \frac{1}{2}\sqrt{3}| > c > 0$. In the case $B_{1/m}(x) \cap g^2 = \emptyset$ we are done, because $2(z_1 + \frac{1}{2}\sqrt{3}) > 0$ and therefore

$$D(J_m * d)(x) e_1 > 0 \quad .$$

In the case $B_{1/m}(x) \cap g^2 \neq \emptyset$ (still assuming $x_1 + \frac{1}{2}\sqrt{3} > c > 0$) we can divide $B_{1/m}(x)$ into 3 subsets S_0, S_1, S_2 (up to sets of measure zero) as follows

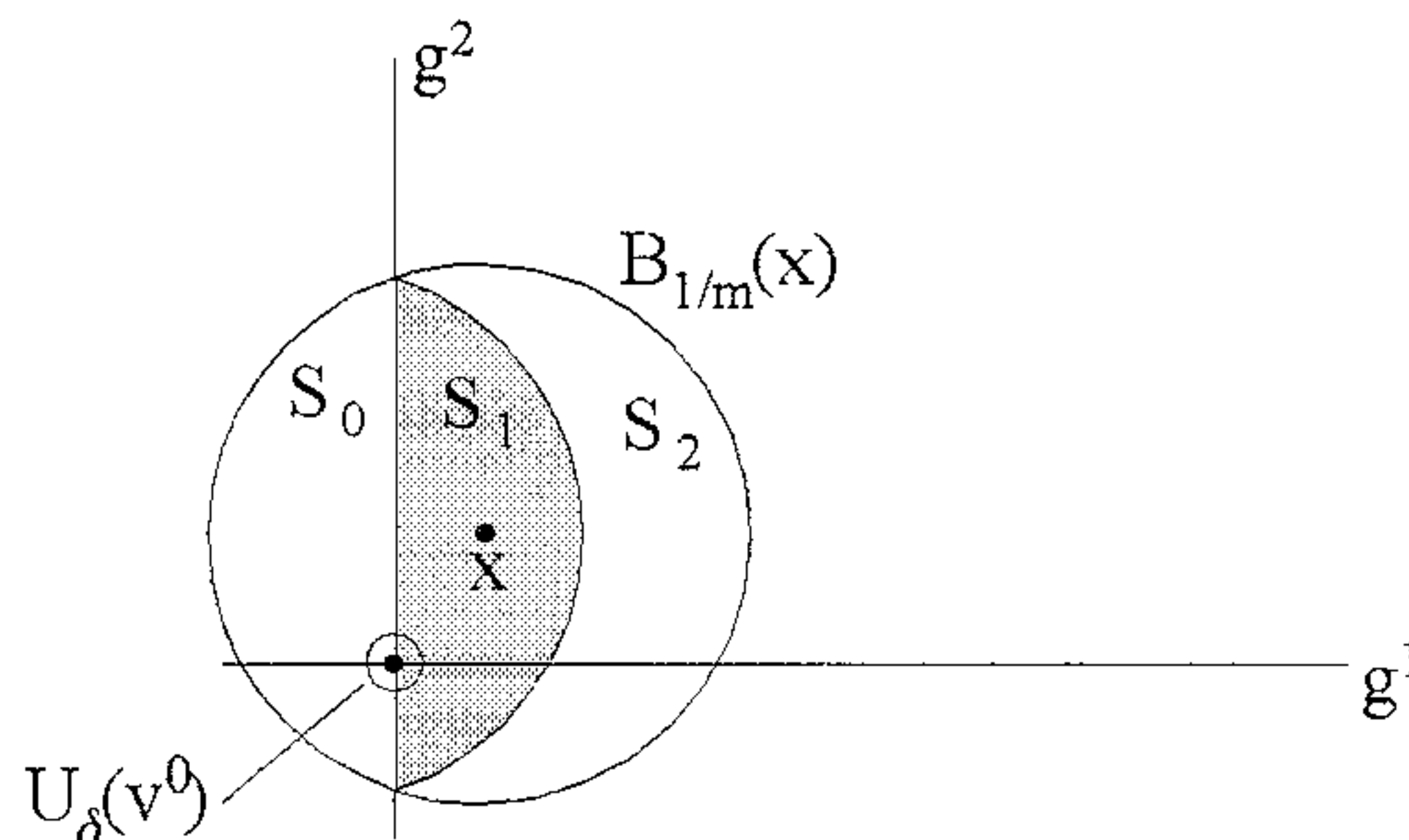


figure 7

$$S_0 := B_{1/m}(x) \cap \{(z_1, z_2) \mid z_1 < -\frac{1}{2}\sqrt{3}\}$$

$$S_1 := \{(-2 \cdot \frac{1}{2}\sqrt{3} - z_1, z_2) \mid (z_1, z_2) \in S_0\} \text{ (mirror image of } S_0 \text{ w.r.t. } g^2)$$

$$S_2 := B_{1/m}(x) \setminus (\overline{S_0} \cup S_1)$$

$$D(J_m * d)(x) e_1 = \underbrace{\left(\int_{S_0} + \int_{S_1} + \int_{S_2} \right)}_{\substack{> 0 \\ \text{by the antisym. of } f(z_1) \\ \text{and the properties of } J_m}} J_m(x - z) \overbrace{2(z_1 + \frac{1}{2}\sqrt{3})}^{=: f(z_1)} dz > 0 \quad .$$

The case $i = 2$ is proven analogously.

Case II $x \in B$ (as in figure 6, but g^2 is now defined to be the x_2 -coordinate axis)

Assume $i = 2$, see (*).

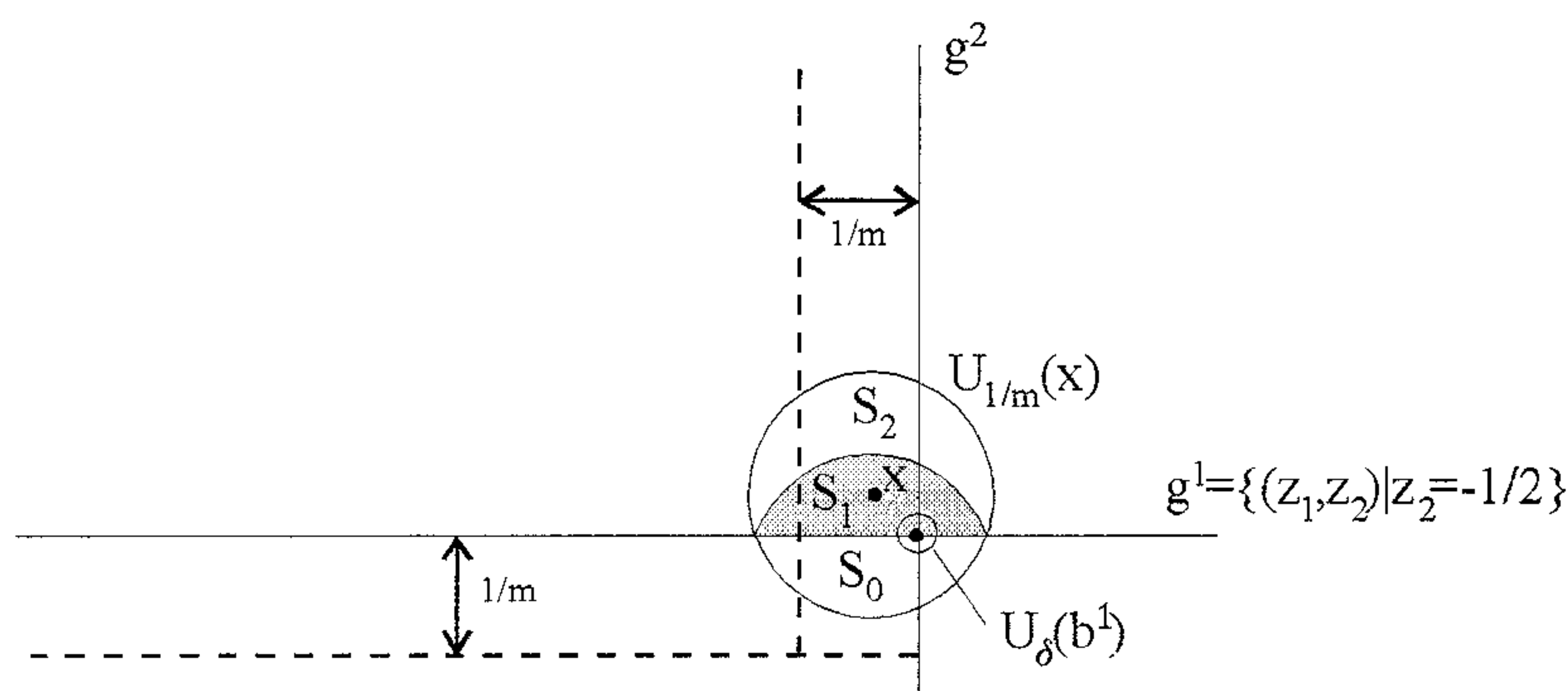


figure 8

S_0, S_1, S_2 are defined in analogy to Case I.

$$D(J_m * d)(x) e_2 = \left(\underbrace{\int_{S_0} + \int_{S_1}}_{\geq 0} + \int_{S_2} \right) J_m(x-z) \overbrace{2(z_2 + \frac{1}{2})}^{=: f(z_2)} dz > 0 .$$

by the antisym. of $f(z_2)$
and the properties of J_m

The case $i = 1$ works analogous.

Case III $x \in C$ (as in figure 6)

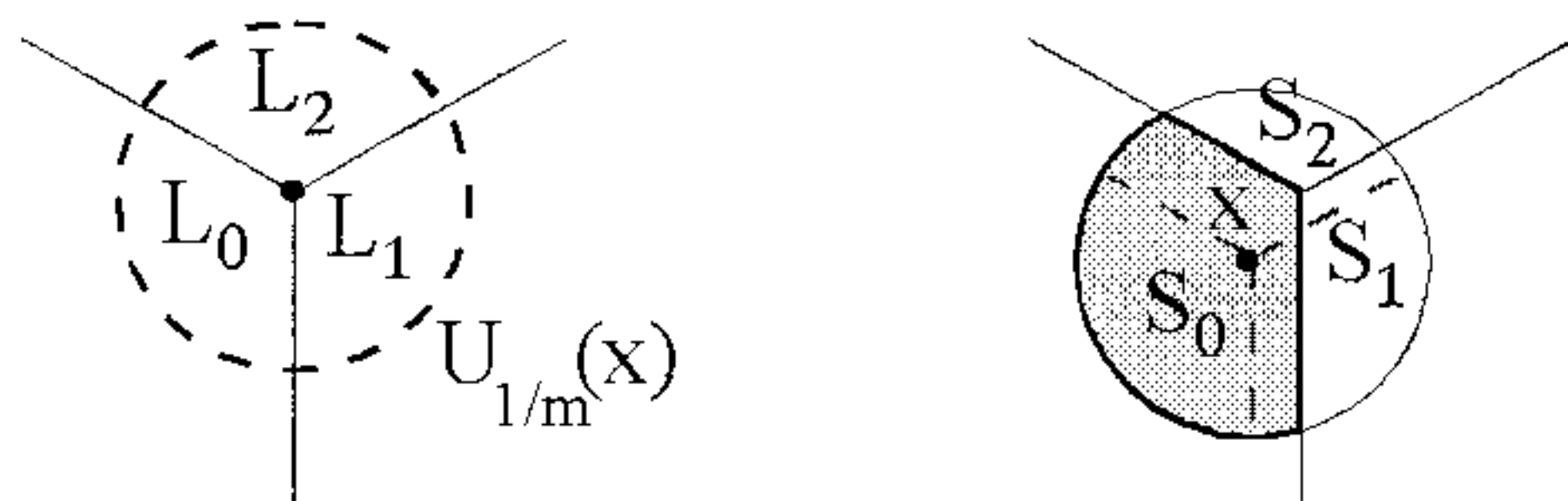


figure 9

a) $x = b^2$

b) $x \in C$

$$\begin{aligned} D(J_m * d)(x) e_2 &= \int_{B_{1/m}(x)} J_m(x-z) \cdot \frac{d}{dz_2} d(z) dz \\ &= \underbrace{\int_{S_0 \cup S_1} J_m(x-z) 2(z_2 + \frac{1}{2}) dz}_{> \int_{L_0 \cup L_1} J_m(b^2 - z) 2(z_2 + \frac{1}{2}) dz} + \underbrace{\int_{S_2} J_m(x-z) \overbrace{2(z_2 - 1)}^{=: f(z_2) < 0, m \geq 2} dz}_{> \int_{L_2} J_m(b^2 - z) 2(z_2 - 1) dz} \\ &> D(J_m * d)(b^2) e_2 = 0 \quad \text{by Theorem 4 i).} \end{aligned}$$

□

Theorem 7 Fix an integer $m \geq m_0$, m_0 as in Theorem 4. The connecting orbits with respect to the negative gradient flow of the Morse function d_m (w.r.t. the euclidean metric on $U_{1/m}(\Delta^2) \subset \mathbb{R}^2$) between critical points of index difference 1 are precisely given by the open straight line segments between the corresponding critical points (see figure 2, where the numbers indicate the Morse indices).

PROOF By the symmetry of d_m it suffices to consider the connecting orbit between

- i) b^0 and b^1 , denoted by $\mathcal{O}(b^0, b^1)$,
- ii) b^1 and b^2 , denoted by $\mathcal{O}(b^1, b^2)$.

We denote the gradient of d_m w.r.t. the euclidean metric $\langle \cdot, \cdot \rangle$ by ∇d_m .

ad i) For $p \in \mathcal{O}(b^0, b^1) \setminus \{b^0, b^1\}$ we have to show that $\nabla d_m(p) = (a, 0)$, where $a < 0$.

$$\begin{aligned} \langle \nabla d_m(p), (0, b) \rangle &= Dd_m(p)(0, b) = b \cdot \frac{d}{dz_2} d_m(p) \\ &= b \cdot \int_{B_{1/m}(p)} J_m(p-z) \cdot 2(z_2 + \frac{1}{2}) dz = 0 \quad , \quad \forall (b, 0), \end{aligned}$$

because $p = (p_1, -\frac{1}{2})$ implies the antisymmetry of the integrand in z_2 . This, together with $\nabla d_m(p) \neq 0$ (because the critical points are precisely the barycenters, see Lemma 5) implies $\nabla d_m(p) = (a, 0)$. Because d_m is decreasing along trajectories of the negative gradient flow and $d_m(b^0) < d_m(b^1)$ we conclude $a < 0$.

ad ii) analogous. $\square$

Corollary 8 $-\nabla d_m$ (as in Theorem 7) is a **Morse-Smale vector field**, i.e. the stable and unstable manifolds of any two critical points of the Morse function d_m intersect transversally.

PROOF As we are in the case of dimension 2, the only property we have to check is that there don't exist connecting orbits between critical points of Morse index 1. This is excluded by Theorem 7 (the flow is known qualitatively, because the connecting orbits between critical points of index difference 1 are known).

Another argument is the following: d_m takes the same value on all three critical points of Morse index 1. On the other hand one can show that d_m is strictly decreasing along trajectories of the negative gradient flow, therefore a connecting orbit between two of the three points can't exist. $\square$

1.2 MODIFICATION OF THE TRIANGULATION

Assume we have (see figure 10)

- i) a triangle $\Delta^2 \subset \mathbb{R}^2$ and let U be an open neighborhood of Δ^2
- ii) a diffeomorphism $\eta: U \rightarrow V := \eta(U) \subset M^2$, M^2 is a 2-dimensional manifold
- iii) a vector field X on U with a singularity p in the barycenter of one of the 1-faces Δ^1 , such that $DX(p)$ has two real eigenvalues $\lambda_1 > 0 > \lambda_2$.

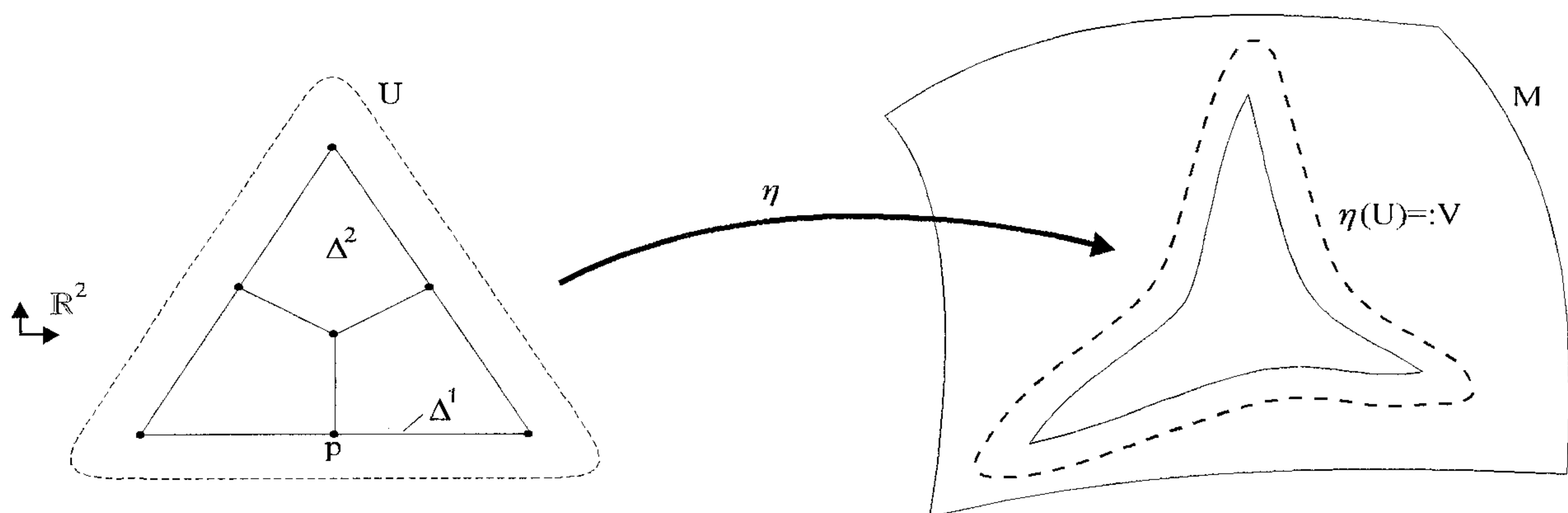


figure 10

By $\hat{\xi}_1, \hat{\xi}_2$ we denote the eigendirections associated to the eigenvalues λ_1, λ_2 of $DX(p)$. Without loss of generality let $p = 0$ and $\hat{\xi}_1 \in T_0 \Delta^1$, see figure 11.

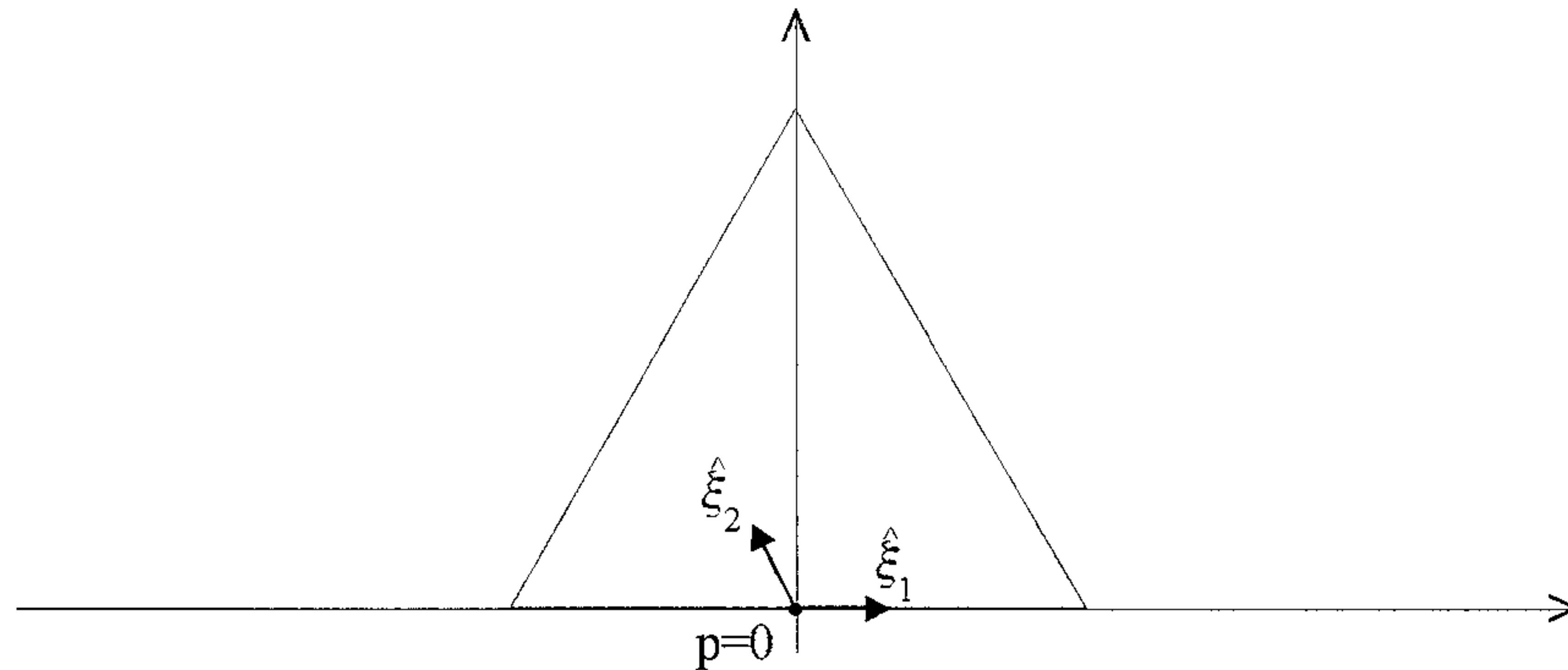


figure 11

Proposition 9 *The push-forward vector field η_*X on $\eta(U) \subset M$ has a singularity at $q := \eta(0)$, $D(\eta_*X)(q)$ has the same eigenvalues $\lambda_1 > 0 > \lambda_2$ as $DX(0)$ and the corresponding eigenvectors are $\xi_i := D\eta|_0 \circ \hat{\xi}_i$, $i = 1, 2$.*

PROOF By the definition of the push-forward of a vector field we get

$$(\eta_*X)(q) = D\eta|_{\eta^{-1}(q)} \circ X|_{\eta^{-1}(q)} = D\eta|_0 \circ X|_0 = D\eta|_0(0) = 0 \quad .$$

Now we have to calculate the linearization of η_*X at the singularity q . Consider the functions

$$\begin{aligned} r : U &\mapsto U \times U & , U \subset \mathbb{R}^2 \text{ as before} \\ x &\mapsto (x, x) \\ s : U &\mapsto TV & , V := \eta(U) \text{ as before} \\ (x, y) &\mapsto D\eta|_x \circ X|_y \end{aligned}$$

(identify $T_x U, T_y U$ with the ambient $\mathbb{R}^2$ and ignore the base point $\eta(x) \in V$). The linearizations are ($v, w \in \mathbb{R}^2$)

$$\begin{aligned} Dr|_x \circ v &= (Id_{2 \times 2}, Id_{2 \times 2})^t v = (v, v)^t \\ Ds|_{(x,y)} \circ (v, w) &= (D_1 s|_{(x,y)}, D_2 s|_{(x,y)})(v, w)^t & , D_{1(2)} \text{ derivative w.r.t. } x(y) \\ &= (D_1 (D\eta|_x \circ X|_y), D_2 (D\eta|_x \circ X|_y))(v, w)^t \\ &= (D^2 \eta|_x (X|_y, \cdot), D\eta|_x \circ DX|_y(\cdot))(v, w)^t \\ &= D^2 \eta|_x (X|_y, v) + D\eta|_x \circ DX|_y(w) \quad . \end{aligned}$$

Using

$$(\eta_*X)(q) = D\eta|_{\eta^{-1}(q)} \circ X|_{\eta^{-1}(q)}$$

we get $\forall u \in T_q M$

$$D(\eta_*X)|_q u = D(s \circ r \circ \eta^{-1})|_q u$$

$$\begin{aligned}
&= (Ds|_{(\eta^{-1}q, \eta^{-1}q)} \circ Dr|_{\eta^{-1}q} \circ D\eta^{-1}|_q) u \\
&= (D^2\eta|_{\eta^{-1}q}(X|_{\eta^{-1}q}, \cdot), D\eta|_{\eta^{-1}q} \circ DX|_{\eta^{-1}q}(\cdot)) (D\eta^{-1}|_q u, D\eta^{-1}|_q u)^t,
\end{aligned}$$

i.e.

$$D(\eta_*X)|_q u = D^2\eta|_{\eta^{-1}q}(X|_{\eta^{-1}q}, D\eta^{-1}|_q u) + D\eta|_{\eta^{-1}q} \circ DX|_{\eta^{-1}q} \circ D\eta^{-1}|_q u.$$

In our special case of q being a singularity of η_*X , i.e. $\eta^{-1}q = 0$ being a singularity of X , the above formula simplifies

$$D(\eta_*X)|_q u = D\eta|_0 \circ DX|_0 (D\eta^{-1}|_q u).$$

Now assume $DX(0)\hat{\xi}_i = \lambda_i\hat{\xi}_i$, $i = 1, 2$, then

$$\begin{aligned}
D(\eta_*X)|_q \hat{\xi}_i &= D(\eta_*X)|_q \circ D\eta|_0 \hat{\xi}_i \\
&= D\eta|_0 \circ DX|_0 (D\eta^{-1}|_q \circ D\eta|_0 \hat{\xi}_i) \\
&= D\eta|_0 \circ DX|_0 \hat{\xi}_i = \lambda_i \cdot D\eta|_0 \hat{\xi}_i = \lambda_i \cdot \hat{\xi}_i. \quad \square
\end{aligned}$$

Note that $\{\xi_1, \xi_2\}$ is a basis of T_qM , therefore any $u \in T_qM$ can be written in the form $u = u^1\xi_1 + u^2\xi_2$.

Theorem 10 For any $u \in T_qM$ with $u^2 > 0$ and for any neighborhood $\tilde{V} \subset V$ of q there exists a diffeomorphism $\psi: U \rightarrow V$, such that

- i) ψ and η coincide on $U \setminus \overline{\eta^{-1}(\tilde{V})}$,
- ii) $D\psi|_0 \hat{\xi}_2 = u$,
- iii) $\psi(x) = \eta(x)$, $\forall x \in \Delta^1$ ($= 1$ -face containing 0).

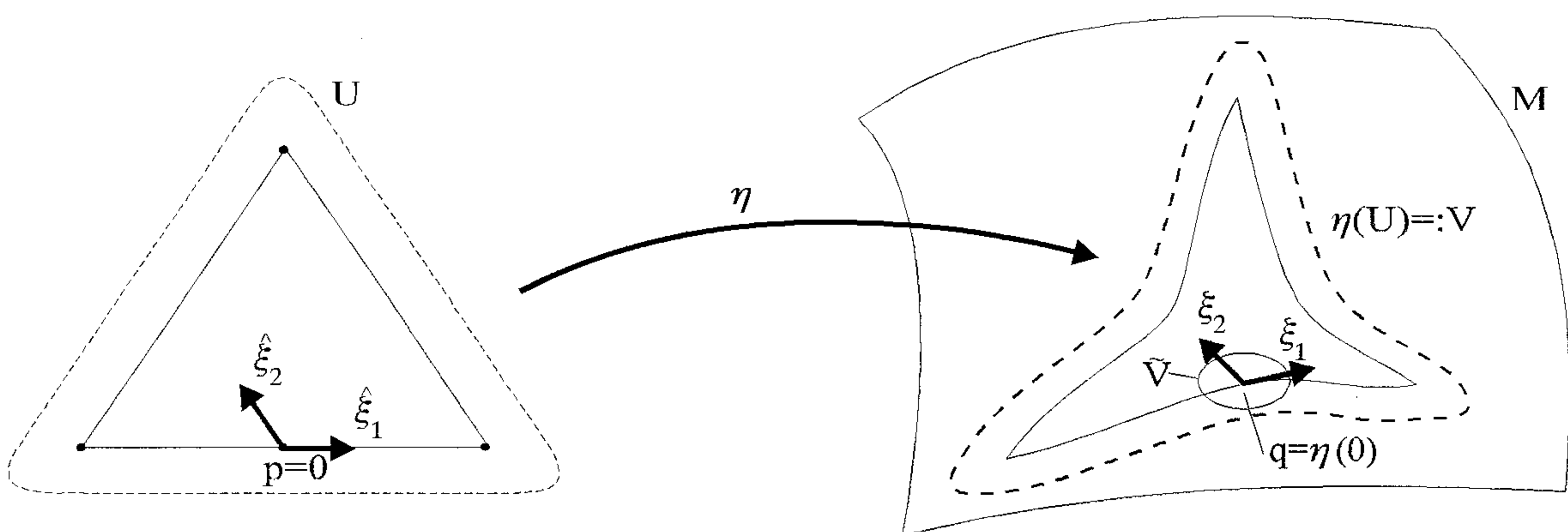


figure 12

The theorem tells us that we can i) alter our diffeomorphism η locally at 0 in such a way, that ii) u is eigenvector of $D(\psi_*X)(q)$ with corresponding eigenvalue λ_2 :

$$D(\psi_*X)|_q u = D(\psi_*X)|_q \circ D\psi|_0 \hat{\xi}_2 = D\psi|_0 \circ DX|_0 \circ D\psi^{-1}|_q \circ D\psi|_0 \hat{\xi}_2$$

$$= D\psi|_0 \circ DX|_0 \hat{\xi}_2 = \lambda \cdot D\psi|_0 \hat{\xi}_2 = \lambda_2 \cdot u \quad .$$

Statement iii) implies that ξ_1 is eigenvector of $D(\psi_*X)|_q$ with corresponding eigenvalue λ_1 : Define $c(t) = t \cdot \hat{\xi}_1$ for $t \in (-\epsilon, \epsilon)$ with $\epsilon > 0$ sufficiently small such that $c(t) \in \Delta^1$. Then $c(0) = 0$, $c'(0) = \hat{\xi}_1$ and

$$\begin{aligned} D\psi|_0 \hat{\xi}_1 &= \frac{d}{dt} (\psi \circ c(t))|_{t=0} = \frac{d}{dt} (\eta \circ c(t))|_{t=0} \quad , \quad c(t) \in \Delta^1 \\ &= D\eta|_0 \hat{\xi}_1 = \xi_1 \quad . \end{aligned}$$

The proof now continues as before.

PROOF (of Theorem 10) Our proof is based on the proof of an isotopy extension theorem in [Hi, Thm. 8.1.4]. We identify T_0U with $\mathbb{R}^2$. Let $\tilde{U} \subset U$ be an open neighborhood of 0, such that $\eta(\tilde{U}) \subset \tilde{V}$; let $A \subset \tilde{U}$ be a compact neighborhood of 0. We define $\hat{u} := D\eta^{-1}|_{\eta(0)} u$ and denote by $P_t \in Gl(2, \mathbb{R})$, $t \in [0, 1]$, the map determined by

$$\begin{aligned} \hat{\xi}_1 &\mapsto \hat{\xi}_1 \\ \hat{\xi}_2 &\mapsto \hat{\xi}_2 + t(\hat{u} - \hat{\xi}_2) \quad . \end{aligned}$$

Therefore $P_0 = Id$ and P_1 maps $\hat{\xi}_2$ to $\hat{u}$ (and $\hat{\xi}_1$ to $\hat{\xi}_1$). We are in the following situation

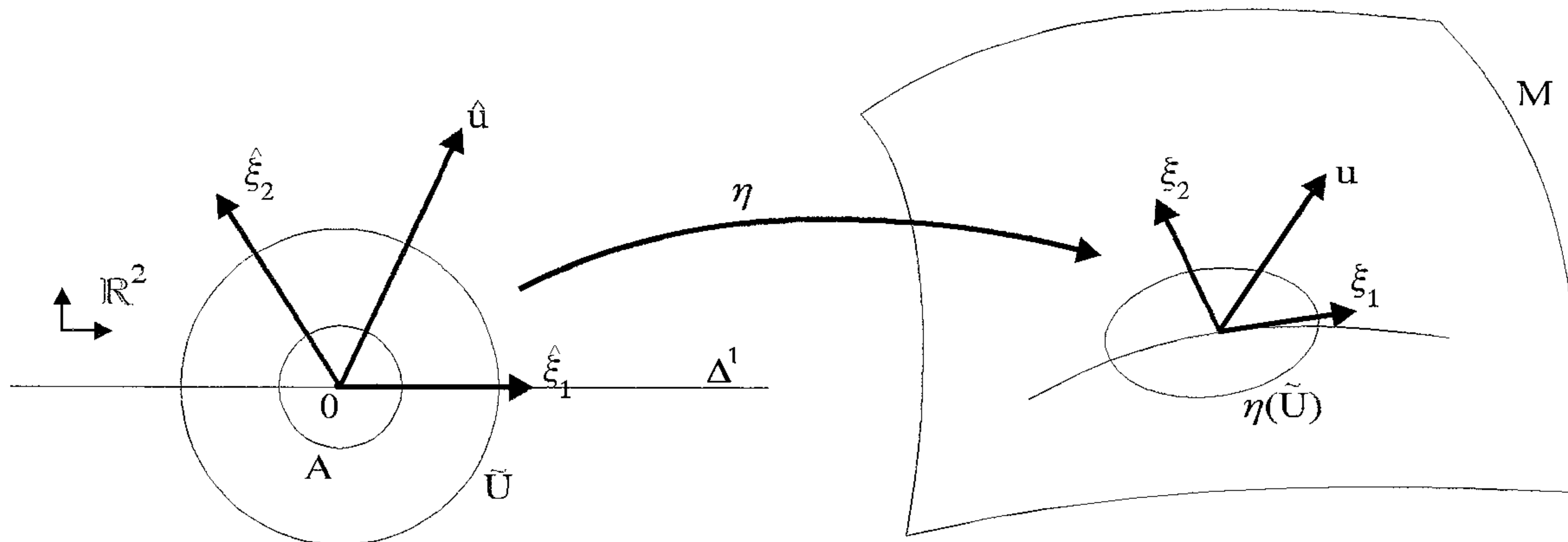


figure 13

Denote by I the unit interval $[0, 1]$. We now define an **isotopy** F of $\tilde{U}$ in U , i.e. a map $F: \tilde{U} \times I \rightarrow U$ such that for each $t \in I$ the map

$$F_t: \tilde{U} \rightarrow U \quad , \quad x \mapsto F_t(x) = F(x, t) \quad ,$$

is an embedding and F_0 is the inclusion; F is required to be smooth in t .

The map F defined by

$$\begin{aligned} F: \tilde{U} \times I &\rightarrow U \\ (x, t) &\mapsto P_t(x) \end{aligned}$$

is in fact an isotopy: It is for each $t \in I$ an embedding because $P_t \in Gl(2, \mathbb{R})$. Furthermore $F_0 = P_0 = Id|_{\tilde{U}}$ and F is smooth in t , because P_t is smooth in t . The

continuity of P_t together with the compactness of I imply for $\tilde{U}$ sufficiently small: $P_t(\tilde{U}) \subset U$, $\forall t \in I$, and therefore $\eta \circ P_t(\tilde{U}) \subset \tilde{V}$.

Note that for any $t \in I$ F_t is the identity on Δ^1 : Let $x \in \Delta^1$, i.e. $x = \lambda \hat{\xi}_1$ with $\lambda \in \mathbb{R}$

$$F_t(x) = P_t(\lambda \hat{\xi}_1) = \lambda P_t \hat{\xi}_1 = \lambda \hat{\xi}_1 = x \quad .$$

Our goal is to extend the isotopy F to all of U in such a way that its support is compact (and is therefore contained in the open set U). Recall that the support **supp** F of an isotopy $F: U \times I \rightarrow U$ is the closure of $\{x \in U \mid F(x, t) \neq F(x, 0) \text{ for some } t \in I\}$. The **track** of the isotopy F is defined to be the map

$$\begin{aligned} \hat{F} : \tilde{U} \times I &\rightarrow U \times I \\ (x, t) &\mapsto (F(x, t), t) \quad . \end{aligned}$$

$\hat{F}$ is in fact an embedding, because

$$\det D\hat{F}(x, t) = \det \begin{bmatrix} DF_t(x) & \partial_t F(x, t) \\ 0_{1 \times 2} & 1 \end{bmatrix} = \det \begin{bmatrix} P_t & \partial_t P_t(x) \\ 0_{1 \times 2} & 1 \end{bmatrix} = 1 \cdot \det P_t \neq 0 \quad ,$$

i.e. $D\hat{F}(x, t)$ has (maximal) rank 3 at any point (x, t) .

- $\hat{F}(\tilde{U} \times I)$ is open in $U \times I$: For any $y_0 \in \hat{F}(\tilde{U} \times I)$ we have to show that a whole neighborhood of y_0 belongs to $\hat{F}(\tilde{U} \times I)$. Let $(x_0, t_0) \in \tilde{U} \times I$ be such that $\hat{F}(x_0, t_0) = y_0$. Now because $D\hat{F}(x_0, t_0)$ has maximal rank, the inverse function theorem tells us that $\hat{F}$ is locally at (x_0, t_0) a diffeomorphism, so an open neighborhood of (x_0, t_0) is mapped to an open neighborhood of y_0 .

- The compactness of $A \times I$ and the continuity of $\hat{F}$ imply that $\hat{F}(A \times I) \subset U \times I$ is compact [Jä, Kap. 1 §8]: Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of $\hat{F}(A \times I)$, then $A \times I = \hat{F}^{-1}(U_{\lambda_1}) \cup \dots \cup \hat{F}^{-1}(U_{\lambda_r})$ for some finite (compactness of $A \times I$!) choice $\{U_{\lambda_1}, \dots, U_{\lambda_r}\}$. Obviously $\hat{F}(A \times I) = U_{\lambda_1} \cup \dots \cup U_{\lambda_r}$, so $\hat{F}(A \times I)$ is compact. (Recall that a topological space X is called **compact**, iff any open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ has a finite subcover $\{U_{\lambda_i}\}_{i=1}^N$, and a function is continuous, iff preimages of open sets are open sets).

- Construction of a vector field Y on $\hat{F}(\tilde{U} \times I)$: $\hat{F}$ being an embedding implies that $\hat{F}$ is a diffeomorphism onto its image, i.e. any point $(y_0, t_0) \in \hat{F}(\tilde{U} \times I)$ belongs to a unique arc

$$\begin{aligned} \gamma : I &\rightarrow \hat{F}(\tilde{U} \times I) \\ t &\mapsto (F_t(x_0), t) \quad , \end{aligned}$$

where $x_0 := F_{t_0}^{-1} y_0$, i.e. $\gamma(0) = (x_0, 0)$ and $\gamma(t_0) = (y_0, t_0)$. Consider the vector field on $\hat{F}(\tilde{U} \times I)$

$$(*) \quad \frac{d}{dt} \gamma(t) = \left(\frac{d}{dt} F_t x_0, 1 \right) =: \left(Y(F_t x_0, t_0), 1 \right) \in T_{F_t x_0} F_t \tilde{U} \times T_t I \quad .$$

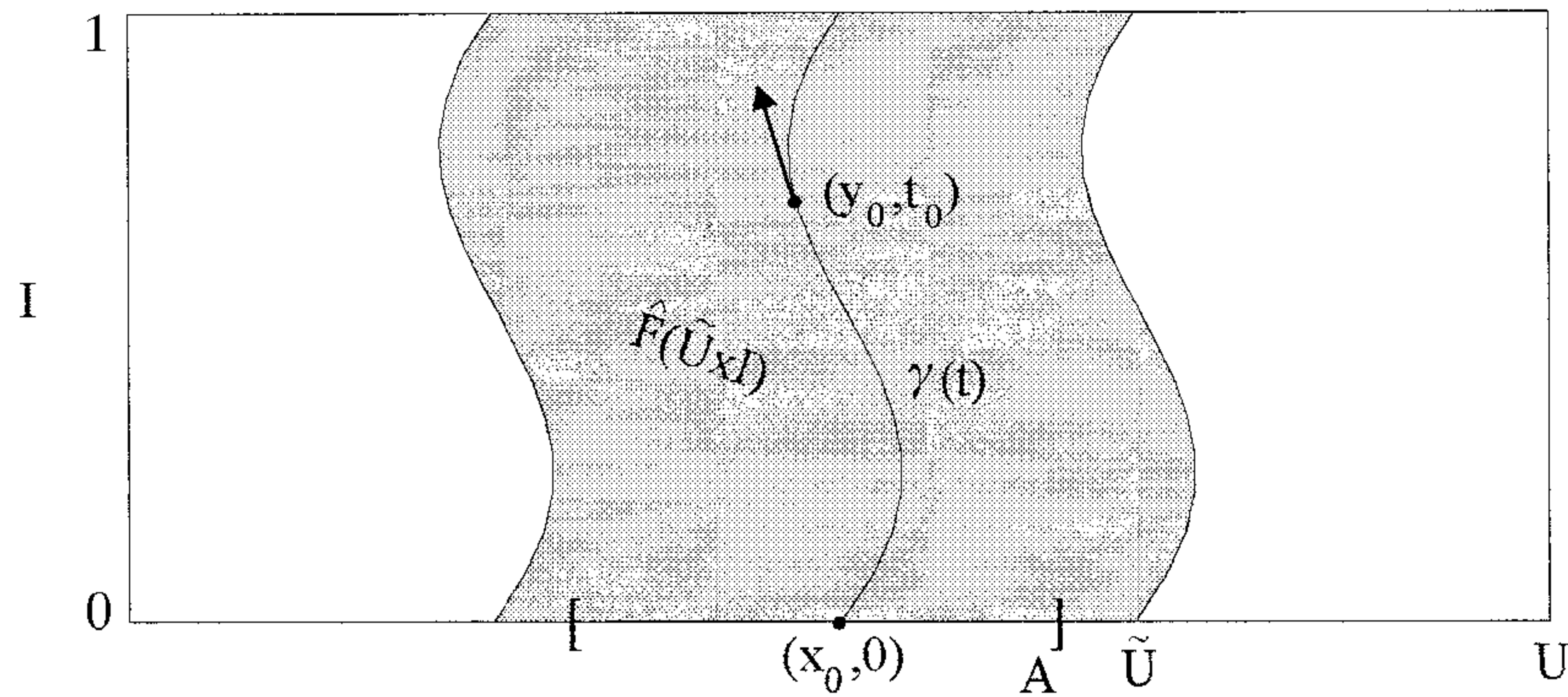


figure 14

• Construction of a vector field on $U \times I$: Pick a cutoff function $\rho \in C^\infty(U \times I, [0, 1])$, such that

$$\rho(y_0, t_0) = \begin{cases} 1 & , (y_0, t_0) \in \hat{F}(A \times I) \\ 0 & , (y_0, t_0) \notin \hat{F}(\tilde{U} \times I) \end{cases} .$$

Here we need the fact that $\hat{F}(\tilde{U} \times I)$ is open in $U \times I$ and contains the compact set $\hat{F}(A \times I)$. The existence of ρ now follows by a standard 'partition of unity' argument. Clearly $\text{supp } \rho \subset \text{cl } \hat{F}(\tilde{U} \times I)$ and is therefore compact.

Now we define the vector field on $U \times I$

$$Z(y_0, t_0) = (X(y_0, t_0), 1) = \begin{cases} (\rho(y_0, t_0) \cdot Y(y_0, t_0), 1) & , (y_0, t_0) \in \hat{F}(\tilde{U} \times I) \\ (0, 1) & , (y_0, t_0) \notin \hat{F}(\tilde{U} \times I) \end{cases} .$$

This vector field generates a flow $\psi_t = \phi_t \oplus \varphi_t$ on $U \times I$ by

$$\frac{d}{dt} \psi_t(y_0, t_0) = Z \circ \psi_t(y_0, t_0) \quad , \quad \psi_0 = \text{id} \mid_{U \times I} ,$$

which is equivalent to

$$(**) \quad \begin{cases} \frac{d}{dt} \phi_t y_0 = X(\phi_t y_0, t_0) & , \phi_0 = \text{id} \mid_U , \\ \frac{d}{dt} \varphi_t t_0 = 1 & , \varphi_0 = \text{id} \mid_I . \end{cases}$$

Now we set the initial value $(y_0, t_0) = (x, 0) \in U \times I$. The second equation can be integrated easily: $\varphi_t 0 = t + 0$.

X can be regarded as a time-dependent vector field on U with compact support. The first equation can therefore be integrated to yield the 1-parameter group ϕ_t of diffeomorphisms of U with $\phi_0 = \text{id} \mid_U$ [Hi, Thm. 8.1.1 & 8.1.2].

$$\begin{aligned} \phi : U \times I &\rightarrow U \\ (x, t) &\mapsto \phi_t x \end{aligned}$$

is an isotopy of U in U (called a **diffeotopy**), such that

a) it coincides with F on A ,

b) ϕ_t is the identity on $U \setminus \overline{\eta^{-1}(\tilde{V})}$, $\forall t \in I$,

c) ϕ_t is the identity on Δ^1 , $\forall t \in I$.

ad a) Let $(x, 0) \in A \times I$, then $\rho(x, 0) = \rho(F_0 x, 0) = 1$ because $(F_0 x, 0) \in \hat{F}(A \times I)$. Equation $(**)$ now reads

$$\frac{d}{dt} \phi_t x = Y(\phi_t x, 0) \quad , \quad \phi_0 x = x \quad ,$$

which coincides with the equation $(*)$ for F_t . The uniqueness of the solution of an initial value problem of an ODE now tells us $\phi_t = F_t$ on A .

ad b) $\text{supp } \rho \subset \overline{\hat{F}(\tilde{U} \times I)} = \text{cl} \left(\bigcup_{t \in I} (F_t \tilde{U}, t) \right)$

$$= \text{cl} \left(\bigcup_{t \in I} (P_t \tilde{U}, t) \right) \subset \text{cl} \left(\bigcup_{t \in I} (\eta^{-1} \tilde{V}, t) \right) = (\text{cl } \eta^{-1} \tilde{V}) \times I \quad .$$

Note that above we assumed $\tilde{U}$ to be sufficiently small for the inclusion to hold. Let $(y_0, t_0) \in (U \setminus \text{cl } \eta^{-1} \tilde{V}) \times I$, i.e. $(y_0, t_0) \notin \text{supp } \rho$. We get

$$\frac{d}{dt} \phi_t y_0 = X(y_0, t_0) = 0$$

and therefore $\phi_t = \text{id}$ on $U \setminus \text{cl } \eta^{-1} \tilde{V}$.

ad c) Pick $(x, 0) \in \Delta^1 \times 0 \subset U \times I$, then

$$Y(F_t x, 0) = \frac{d}{dt} F_t x = \frac{d}{dt} x = 0 \quad ,$$

i.e. F_t solves

$$\frac{d}{dt} F_t x = 0 = \rho(F_t x, 0) \cdot Y(F_t x, 0) \quad ,$$

which is exactly the equation $(**)$ for ϕ_t with $(y_0, t_0) = (x, 0) \in \Delta^1 \times I$. We get

$$\phi_t|_{\Delta^1} = F_t|_{\Delta^1} = \text{id}|_{\Delta^1} \quad .$$

• The diffeomorphism ψ : We define $\psi := \eta \circ \phi_1: U \rightarrow V$.

i) $\psi = \eta$ on $U \setminus \text{cl } \eta^{-1} \tilde{V}$: By b) $\phi_1 = \text{id}$ on $U \setminus \text{cl } \eta^{-1} \tilde{V}$.

ii) $D\psi|_0 \hat{\xi}_2 = D\eta|_{\phi_1 0} \circ D\phi_1|_0 \hat{\xi}_2 = D\eta|_0 \circ DP_1|_0 \hat{\xi}_2 = D\eta|_0 \circ P_1 \hat{\xi}_2$
 $= D\eta|_0 \hat{u} = D\eta|_0 \circ D\eta^{-1}|_{\eta(0)} u = u \quad .$

Note that $P_1 0 = 0$. By a) we have on A : $\phi_1 = F_1 = P_1$. Furthermore P_1 is the restriction of a linear map, so $DP_1|_0 = P_1$.

iii) Let $x \in \Delta^1$, then $\psi(x) = \eta \circ \phi_1(x) = \eta(x)$, by c). □

2. Global patchwork

Let us start by recalling some notions from algebraic topology; hereby we follow the exposition of [Th, Se. 3.2 & 3.3].

Definition 11 *i) An n -simplex σ^n is the convex hull of $n+1$ affinely independent points $v^0, \dots, v^n$ (in some affine space). The convex hull of some subset of $\{v^0, \dots, v^n\}$ consisting of k elements is a k -face, or k -dimensional subsimplex, of σ^n .*

*ii) A **simplicial complex** is a locally finite collection Σ of simplices (in some affine space), satisfying the following two conditions: Any face of a simplex in Σ is also in Σ , and the intersection of two simplices in Σ is either empty or a face of both.*

*iii) The union of all simplices in Σ is called the **polyhedron of Σ** , and denoted by $|\Sigma|$.*

*iv) A **triangulation** (Σ, h) of a topological space X is a simplicial complex Σ together with a homeomorphism $h: |\Sigma| \rightarrow X$.*

*v) A **smooth triangulation** (Σ, h) of a differentiable manifold N is a simplicial complex Σ together with a homeomorphism $h: |\Sigma| \rightarrow N$, such that h restricted to any top-dimensional, closed simplex is a diffeomorphic embedding. (A map defined on a closed set S is said to be differentiable, if it is the restriction of a differentiable map defined on some open neighborhood of S).*

The next theorem goes back to Cairns [Ca] and may be found for example in [Th, Thm. 3.3.1].

Theorem 12 *Every differentiable manifold admits a smooth triangulation.*

The simplicial complex Σ of a smooth triangulation (Σ, h) of our closed 2-manifold M consists of finitely many, let's say N 2-simplices $\tilde{\sigma}_k^2$, $k \in \{1, \dots, N\}$. For any $\tilde{\sigma}_k^2$ there exists an open neighborhood $\tilde{V}_k$ (in the affine space spanned by $\tilde{\sigma}_k^2$), such that $h_k: \tilde{V}_k \rightarrow V_k := h_k(\tilde{V}_k) \subset M$ is a diffeomorphism (by def. 11 v)). Define $\sigma_k^2 := h_k(\tilde{\sigma}_k^2)$.

By $A_{k,m}$, $k \in \{1, \dots, N\}$, we denote the restriction to $U_{1/m}(\Delta^2)$ of the affine map from $\mathbb{R}^2$ to the affine space spanned by $\tilde{\sigma}_k^2$, which is uniquely determined by

$$\begin{aligned} v^0 &\mapsto \tilde{v}^0 \\ v^1 - v^0 &\mapsto \tilde{v}^1 - \tilde{v}^0 \\ v^2 - v^0 &\mapsto \tilde{v}^2 - \tilde{v}^0 \end{aligned} \quad .$$

We observe that there exists $m_1 \in \mathbb{N}$, $m_1 > m_0$, such that for any $m \geq m_1$ $A_{k,m}(U_{1/m}(\Delta^2)) \subset \tilde{V}_k$ for any $k \in \{1, \dots, N\}$. In what follows we assume $m \geq m_1$. The

situation so far is indicated in figure 15.

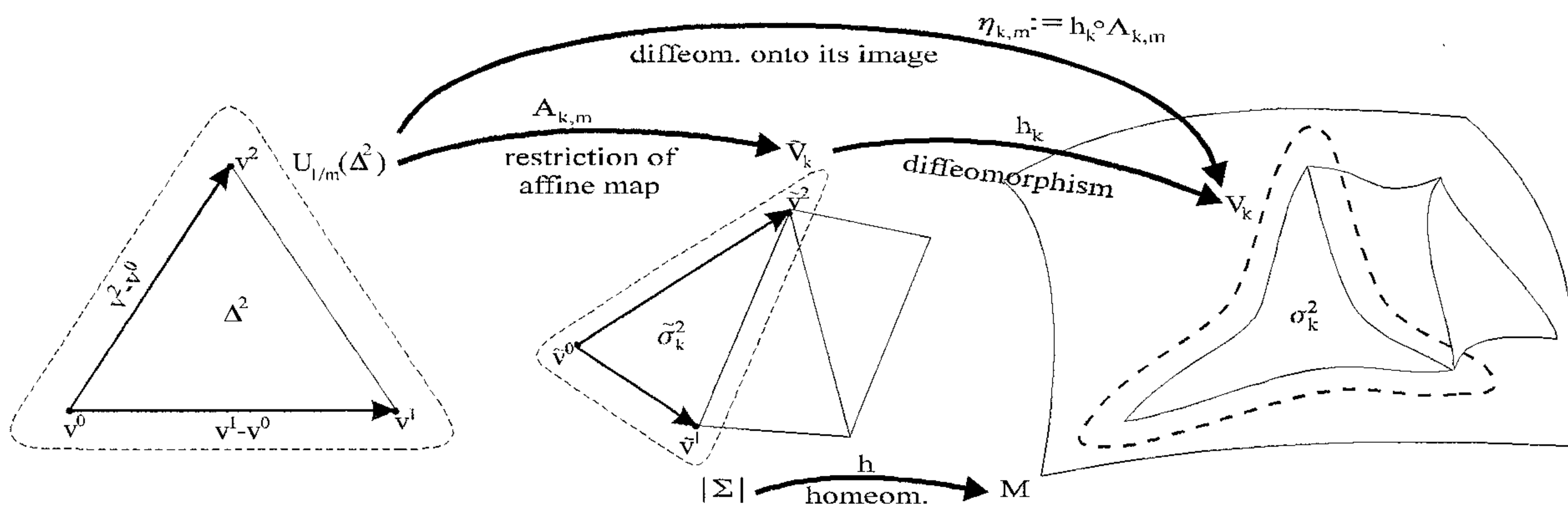


figure 15

The map

$$\eta_{\mathbf{k},\mathbf{m}} := h_{\mathbf{k}} \circ A_{\mathbf{k},\mathbf{m}} : U_{1/\mathbf{m}}(\Delta^2) \rightarrow U_{\mathbf{k},\mathbf{m}} := \eta_{\mathbf{k},\mathbf{m}}(U_{1/\mathbf{m}}(\Delta^2)) \subset M$$

is a diffeomorphism onto its image $U_{\mathbf{k},\mathbf{m}}$. Note that $\{U_{\mathbf{k},\mathbf{m}}\}_{\mathbf{k}=1}^N$ is an **open cover of M** . By $\{\lambda_{\mathbf{k},\mathbf{m}}\}_{\mathbf{k}=1}^N$ we denote a **smooth partition of unity subordinate to this open cover**, i.e.

- i) $\lambda_{\mathbf{k},\mathbf{m}} \in C^\infty(M, [0, 1])$ and $\text{supp } \lambda_{\mathbf{k},\mathbf{m}} \subset U_{\mathbf{k},\mathbf{m}} \ \forall k \in \{1, \dots, N\}$,
- ii) $\{\text{supp } \lambda_{\mathbf{k},\mathbf{m}}\}_{\mathbf{k}=1}^N$ is locally finite,
- iii) $\sum_{\mathbf{k}=1}^N \lambda_{\mathbf{k},\mathbf{m}}(x) = 1 \ \forall x \in M$.

By [Hi, Ch. 2, Thm. 2.1] such a family of maps does indeed exist in our case. In what follows we fix $m \in \mathbb{N}$, $m \geq m_1$, and **omit therefore the index m in our notation**.

Via push-forward of the vector field $-\nabla d$ on $U(\Delta^2) \subset \mathbb{R}^2$ by the diffeomorphism $\eta_{\mathbf{k}}$, we get smooth vector fields on $U_{\mathbf{k}} \subset M$

$$Y_{\mathbf{k}} := (\eta_{\mathbf{k}})_*(-\nabla d) \quad , \quad k \in \{1, \dots, N\} \quad .$$

Summation over k and multiplication by $\lambda_{\mathbf{k}}$ yields a smooth vector field on M depending on the partition of unity $\{\lambda_{\mathbf{k}}\}_{\mathbf{k}=1}^N$ (short notation $\{\lambda\}$)

$$X_{\{\lambda\}} := \sum_{\mathbf{k}=1}^N \lambda_{\mathbf{k}} \cdot Y_{\mathbf{k}} \quad .$$

Observe that the image p of a barycenter b of Δ^2 under $\eta_{\mathbf{k}}$, i.e. $p := \eta_{\mathbf{k}}(b)$, is a hyperbolic singularity of $X_{\{\lambda\}}$, i.e. $X_{\{\lambda\}}(p) = 0$, and $DX_{\{\lambda\}}(p)$ has real nonzero eigenvalues:

• **Case $b = b^2$:** If necessary, we shrink the support of $\lambda_{\mathbf{k}}$, $k \in \{1, \dots, N\}$, such that $\text{supp } \lambda_{\mathbf{k}}$ is a sufficiently small (closed) neighborhood of $\sigma_{\mathbf{k}}^2$ in order to guarantee that $p^2 := \eta_{\mathbf{k}}(b^2) \notin \text{supp } \lambda_{\mathbf{i}}$, $i \neq k$, i.e. p^2 is contained in the support of exactly one $\lambda_{\mathbf{k}}$. Now

apply Proposition 9 to see that the eigenvalues of $DX_{\{\lambda\}}(p^2)$ are given by $\{2\sqrt{3}c_m - 2, 3c_m - 2\}$ (cf. thm. 4), therefore p^2 is a source.

• Case $b = b^1$: After shrinking (as above) the support of λ_k , $k \in \{1, \dots, N\}$, we may assume that $p^1 := \eta_i(b^1) \in \text{supp } \lambda_i \cap \text{supp } \lambda_j$, $i \neq j$, and $p^1 \notin \text{supp } \lambda_k$, $k \in \{1, \dots, N\} \setminus \{i, j\}$. Now apply Proposition 9 to $(\eta_i)_*(-\nabla d)$ and modify η_j according to Theorem 10; apply again Proposition 9. This shows that p^1 is a hyperbolic singularity of $X_{\{\lambda\}}$; the eigenvalues of $DX_{\{\lambda\}}(p^1)$ are given by -2 and $2\sqrt{3}c_m - 2$ (c.f. Theorem 4), therefore p^1 is a saddle.

• Case $b = b^0$: In this case $p^0 := \eta_k(b^0)$ may be contained in the supports of various λ_k 's. However, p^0 is a singularity of $X_{\{\lambda\}}$, because either it is a singularity of $(\eta_k)_*(-\nabla d)$ or it is not contained in $\text{supp } \lambda_k$, i.e. in any case $X_{\{\lambda\}}(p^0) = 0$.

$$\begin{aligned} DX_{\{\lambda\}}(p^0) &= \sum_{k=1}^N \left(\underbrace{D\lambda_k(p^0) \cdot ((\eta_k)_*(-\nabla d))(p^0)}_{\text{at least one of these two factors is zero}} + \lambda_k(p^0) \cdot D((\eta_k)_*(-\nabla d))(p^0) \right) \\ &= \sum_{k=1}^N \lambda_k(p^0) \cdot D((\eta_k)_*(-\nabla d))(p^0) . \end{aligned}$$

The last expression is a sum of linear maps from $T_{p^0}M$ to itself with two strictly negative eigenvalues each. At least one of the nonnegative coefficients is nonzero, therefore the sum is a linear map with two strictly negative eigenvalues. We see that p^0 is a sink.

Now it remains to check, that our procedure of summing up the known vector fields $Y_k = (\eta_k)_*(-\nabla d)$ via partition of unity, doesn't create new singularities or even limit cycles. This is not the case if $\text{supp } \lambda_k$ is a sufficiently small neighborhood of σ_k^2 .

The next two lemmata state the existence of neighborhoods around saddles and sinks (independent of the choice of partition of unity), which contain no other singularities.

Lemma 13 *Let p^1 be one of the saddles of $X_{\{\lambda\}}$ as above (i.e. $p^1 = \eta_k(b^1)$ for some k), then there exists a neighborhood $W(p^1)$, such that $W(p^1) \cap \{q \in M \mid X_{\{\lambda\}}(q) = 0\} = \{p^1\}$ for any partition of unity.*

PROOF As this is a local problem, we may localize around p^1 using a chart of M . Until the end of the proof we assume our objects to be located in $\mathbb{R}^2$ (without changing notation!).

We may assume as above (c.f. "Case $b = b^1$ ") that $p^1 \in \text{supp } \lambda_i \cap \text{supp } \lambda_j$, $i \neq j$, and $p^1 \notin \text{supp } \lambda_k$, $k \in \{1, \dots, N\} \setminus \{i, j\}$. Let $U(p^1)$ be a sufficiently small neighborhood of p^1 such that $U(p^1) \subset U_i \cap U_j$ and $U(p^1) \cap U_k = \emptyset \quad \forall k \in \{1, \dots, N\} \setminus \{i, j\}$. Recall that

$Y_s := (\eta_s)_*(-\nabla d)$ is defined on U_s and $\text{supp } \lambda_s \subset U_s$, $s \in \{i, j\}$.

Applying Taylors theorem [Hö, formula (1.1.7)] to Y_s we get

$$(*) \quad Y_s(p^1 + v) = \underbrace{Y_s(p^1)}_{=0} + DY_s(p^1) v + R_{2,s}(v) \quad , \quad s \in \{i, j\} ,$$

where

$$R_{2,s}(v) = \int_0^1 D^2 Y_s(p^1 + tv)(v, v) \cdot (1-t) dt = O(\|v\|^2) \quad \text{as } v \rightarrow 0 .$$

We observe that

$$r_{2,s}(v)(\cdot) = \int_0^1 D^2 Y_s(p^1 + tv)(v, \cdot) \cdot (1-t) dt = o(1) \quad \text{as } v \rightarrow 0 ,$$

because

$$\frac{\|r_{2,s}(v)\|}{1} \rightarrow \left\| \int_0^1 D^2 Y_s(p^1)(0, \cdot) \cdot (1-t) dt \right\| = 0 \quad \text{as } v \rightarrow 0 .$$

Therefore

$$(**) \quad \forall \epsilon > 0 \quad \exists \delta > 0 : v \in B_\delta(p^1) \Rightarrow \|r_{2,s}(v)\| < \epsilon \quad \forall s \in \{i, j\} .$$

Note that in the basis of eigenvectors of $D := DY_i(p^1)$ ($= DY_j(p^1)$ after a modification according to Theorem 10) we have, by Proposition 9,

$$D = \begin{bmatrix} 2\sqrt{3}c_m - 2 & 0 \\ 0 & -2 \end{bmatrix} , \quad \text{cf. Theorem 4 .}$$

As $D \in Gl(2, \mathbb{R})$ and $Gl(2, \mathbb{R})$ is open in $Mat(2, \mathbb{R})$ there exists $\epsilon > 0$ such that $C \in Mat(2, \mathbb{R})$ with $\|D - C\| < \epsilon$ implies $C \in Gl(2, \mathbb{R})$. Let $v \in B_\delta(p^1)$

$$\begin{aligned} X_{\{\lambda\}}(p^1 + v) &= \sum_{k=1}^N \lambda_k(p^1 + v) \cdot Y_k(p^1 + v) \\ &= \lambda_i(p^1 + v) \cdot Y_i(p^1 + v) + \lambda_j(p^1 + v) \cdot Y_j(p^1 + v) \\ &\stackrel{(*)}{=} \lambda_i(p^1 + v) \cdot (DY_i(p^1)v + r_{2,i}(v)v) + \lambda_j(p^1 + v) \cdot (DY_j(p^1)v + r_{2,j}(v)v) \\ &= Dv + \lambda_i(p^1 + v) \cdot r_{2,i}(v)v + \lambda_j(p^1 + v) \cdot r_{2,j}(v)v \\ &= (D + \lambda_i(p^1 + v) \cdot r_{2,i}(v) + \lambda_j(p^1 + v) \cdot r_{2,j}(v))v =: Cv \end{aligned}$$

then, because of

$$\begin{aligned} \|D - C\| &= \left\| -\lambda_i(p^1 + v) \cdot r_{2,i}(v) - \lambda_j(p^1 + v) \cdot r_{2,j}(v) \right\| \\ &\leq \left\| \lambda_i(p^1 + v) \cdot r_{2,i}(v) \right\| + \left\| \lambda_j(p^1 + v) \cdot r_{2,j}(v) \right\| \\ &= \lambda_i(p^1 + v) \cdot \underbrace{\|r_{2,i}(v)\|}_{< \epsilon} + \lambda_j(p^1 + v) \cdot \underbrace{\|r_{2,j}(v)\|}_{< \epsilon} < \epsilon , \end{aligned}$$

we see that $C \in Gl(2, \mathbb{R})$. Hence $X_{\{\lambda\}}(p^1 + v) = 0$ implies $v = 0 \quad \forall v \in B_\delta(p^1)$, independent

of the chosen partition of unity. $\square$

Lemma 14 *Let p^0 be one of the sinks of $X_{\{\lambda\}}$ (i.e. $p^0 = \eta_k(b^0)$ for some k), then there exists a neighborhood $W(p^0)$, such that for any partition of unity $\{\lambda\}$*

i) $W(p^0) \cap \{q \in M \mid X_{\{\lambda\}}(q) = 0\} = \{p^0\}$

ii) *the ω -limit set for any $q \in W(p^0)$ is $\{p^0\}$, i.e. $\forall q \in W(p^0) \ \omega(q) = \{p^0\}$. In other words any $q \in W(p^0)$ is taken by the flow associated to $X_{\{\lambda\}}$ in forward time to $\{p^0\}$.*

PROOF **ad i)** In fact this proof is a generalization of the proof of Lemma 13. We are going to use the same notation, we also assume all objects to be located in $\mathbb{R}^2$ (without changing the global notation!).

Let $\Lambda := \{s \in \{1, \dots, N\} \mid p^0 \in U_s\}$, i.e. Λ is the set of indices such that p^0 is in the domain of $Y_s := (\eta_s)_*(-\nabla d)$. Now choose $\delta_0 > 0$ sufficiently small, such that $B_{\delta_0}(p^0) \subset U_s \ \forall s \in \Lambda$. Pick $v \in B_{\delta_0}(p^0)$, then

$$\begin{aligned} X_{\{\lambda\}}(p^0 + v) &= \sum_{k=1}^N \lambda_k(p^0 + v) \cdot Y_k(p^0 + v) \\ &= \sum_{s \in \Lambda} \lambda_s(p^0 + v) \cdot Y_s(p^0 + v) \quad , \ p^0 \notin U_k \Rightarrow p^0 \notin \text{supp } \lambda_k \\ &\stackrel{(*)}{=} \left(\sum_{s \in \Lambda} \lambda_s(p^0 + v) \cdot DY_s(p^0) + \sum_{s \in \Lambda} \lambda_s(p^0 + v) \cdot r_{2,s}(v) \right) v \\ &=: \left(L_{\{\lambda\},v} + \sum_{s \in \Lambda} \lambda_s(p^0 + v) \cdot r_{2,s}(v) \right) v =: C_v v \quad . \end{aligned}$$

Observe that $\mathcal{N}_v := \{L_{\{\lambda\},v} \mid \{\lambda\} \text{ is a partition of unity subordinate to } \{U_k\}\}$ is a compact subset of $Gl(2, \mathbb{R})$. This follows because $\mathcal{N}_v$ is the set of linear combinations of a finite number $|\Lambda|$ of bijective linear maps (with strictly positive eigenvalues), where the coefficients are allowed to vary in a closed, hence compact subset of $[0, 1]^{|\Lambda|}$. The openness of $Gl(2, \mathbb{R})$ in $Mat(2, \mathbb{R})$ together with the compactness of $\mathcal{N}_v$ imply

$$\exists \epsilon_v > 0 \quad \forall C \in Mat(2, \mathbb{R}) \quad \forall L_{\{\lambda\},v} \in \mathcal{N}_v : \quad \|L_{\{\lambda\},v} - C\| < \epsilon_v \Rightarrow C \in Gl(2, \mathbb{R}) \quad .$$

Define

$$\epsilon := \inf_{v \in B_{\delta_0}(p^0)} \epsilon_v \quad ,$$

then $\epsilon > 0$ by compactness of $\overline{B_{\delta_0}(p^0)}$. Now $r_{2,s}(v) = o(1)$ implies that there exists $\delta_1 > 0$, such that $\|r_{2,s}(v)\| < \epsilon \quad \forall s \in \Lambda \quad \forall v \in B_{\delta_1}(p^0)$. We may assume without loss of generality $\delta_1 < \delta_0$. With C_v as defined above and $v \in B_{\delta_1}(p^0)$

$$\|L_{\{\lambda\},v} - C_v\| = \left\| - \sum_{s \in \Lambda} \lambda_s(p^0 + v) \cdot r_{2,s}(v) \right\|$$

$$\leq \sum_{s \in \Lambda} \lambda_s(p^0 + v) \cdot \underbrace{\|r_{2,s}(v)\|}_{< \epsilon \text{ as } v \in B_{\delta_1}(p^0)} < \epsilon$$

implies $C_v \in Gl(2, \mathbb{R})$. Hence $X_{\{\lambda\}}(p^0 + v) = 0$ implies $v = 0 \quad \forall v \in B_{\delta_1}(p^0)$, independent of the chosen partition of unity.

ad ii) We continue using the same notation as above and assuming all objects to be located in $\mathbb{R}^2$ (working in local coordinates without changing the global notation). We may assume $p^0 = 0$. By Theorem 4 ii) and Proposition 9 the linear map $DY_s(0): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ has an eigenvalue -2 of geometric multiplicity 2, $\forall s \in \Lambda$. The set of eigenvectors is an ellipse centered at 0 (w.r.t. the euclidean metric of $\mathbb{R}^2$). The associated quadratic forms

$$\begin{aligned} Q_s : \mathbb{R}^2 &\rightarrow \mathbb{R} & , s \in \Lambda \\ v &\mapsto \langle v, DY_s(0) v \rangle \end{aligned}$$

are therefore negative definite, i.e.

$$\exists \epsilon_1 < 0 \quad \forall s \in \Lambda \quad \forall v \in \mathbb{R}^2 : \|v\| = 1 \Rightarrow Q_s(v) = \langle v, DY_s(0) v \rangle < \epsilon_1 .$$

Now define $\epsilon_2 := -\epsilon_1/2 > 0$ and pick $\delta_2 > 0$, such that $\|r_{2,s}(v)\| < \epsilon_2 \quad \forall v \in B_{\delta_2}(0) \quad \forall s \in \Lambda$ (recall Taylors theorem (*) and its consequence (**) in the proof of Lemma 13; we use the same notation here). We may assume without loss of generality that $\delta_2 < \delta_1$, where δ_1 is as in the proof of part i); therefore $p^0 = 0$ is the only singularity of $X_{\{\lambda\}}$ in $B_{\delta_2}(0)$.

We claim that the smooth function

$$\begin{aligned} f : B_{\delta_2}(0) &\rightarrow \mathbb{R} \\ v &\mapsto \frac{1}{2} |v|^2 \end{aligned}$$

is a Lyapunov function for $X_{\{\lambda\}}|_{B_{\delta_2}(0)}$, i.e. we have to show that f is strictly decreasing in direction $X_{\{\lambda\}}$ at any point $v \in B_{\delta_2}(0)$ which is not a singularity of $X_{\{\lambda\}}$, i.e. $v \neq 0$.

Using $Df(v) = 2 \cdot (1/2) \cdot |v| \cdot (v^t/|v|) = v^t$, we get with $v \in B_{\delta_2}(0) \setminus \{0\}$ and using Taylors expansion of Y_s as mentioned above

$$\begin{aligned} (X_{\{\lambda\}}f)(v) &= Df(v) X_{\{\lambda\}}(v) \\ &= \sum_{s \in \Lambda} \left(\lambda_s(v) \cdot \underbrace{Df(v)}_{=v^t} \circ DY_s(0)v + \lambda_s(v) \cdot \underbrace{Df(v)}_{=v^t} \circ r_{2,s}(v)v \right) \\ &= \sum_{s \in \Lambda} \left(\lambda_s(v) \cdot |v| \cdot \underbrace{\langle DY_s(0) \frac{v}{|v|}, \frac{v}{|v|} \rangle}_{< \epsilon_1 < 0} + \lambda_s(v) \cdot \langle v, r_{2,s}(v) \rangle \right) \\ &= |v|^2 \epsilon_1 + \sum_{s \in \Lambda} \lambda_s(v) \cdot |v|^2 \cdot \underbrace{\|r_{2,s}(v)\|}_{< \epsilon_2 = -\epsilon_1/2} \\ &< |v|^2 (\epsilon_1 - \epsilon_1/2) = \frac{1}{2} |v|^2 \epsilon_1 < 0 . \end{aligned}$$

A standard argument in the theory of dynamical systems shows that any $q \in B_{\delta_2}(0)$ is taken (in forward time) to p^0 by the flow associated to $X_{\{\lambda\}}$ (one shows: 1) $\omega(q) \neq \emptyset$ 2) $\omega(q) \subset \{p^0\}$). Now we define $W(p^0) := B_{\delta_2}(p^0 = 0)$. $\square$

So far we are in the following situation: Around our sinks, saddles and sources p^k of $X_{\{\lambda\}}$ there are neighborhoods $W(p^k)$ which contain no other singularities beside p^k and are independent of the choice of the partition of unity $\{\lambda\}$.

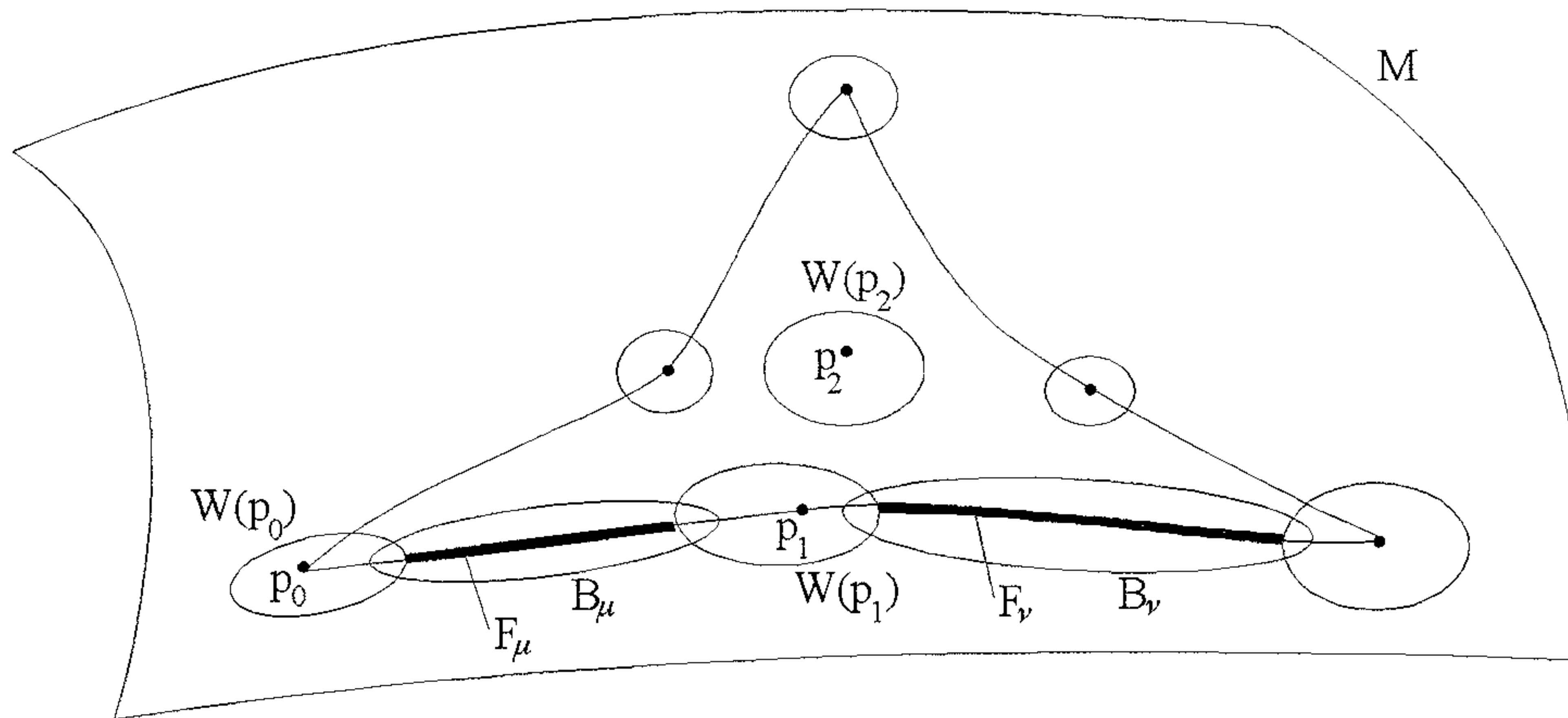


figure 16

On a set F_μ as indicated in figure 16 (a subset of the image under h of a 1-face connecting $W(p^0)$ and $W(p^1)$) there will be only two summands contributing to $X_{\{\lambda\}}$ (eventually only after shrinking $\{\lambda\}$), let's say $\lambda_1 \cdot Y_1$ and $\lambda_2 \cdot Y_2$ (recall $Y_i := (\eta_i)_*(-\nabla d)$). Moreover Y_1 and Y_2 coincide on F_μ and are not zero, therefore there exists a neighborhood B_μ of F_μ such that Y_1 and Y_2 are not antiparallel in B_μ . This implies that $X_{\{\lambda\}}|_{B_\mu} = (\lambda_1 Y_1 + \lambda_2 Y_2)|_{B_\mu}$ cannot be zero on B_μ . Apply this arguments to all sets of type F_μ as indicated in figure 16.

Now shrink the supports of the partition of unity, until all overlaps of supports of λ_k are inside the union U of the F_μ 's and $W(p^k)$'s. By construction inside U there are no other singularities except the p^k 's and outside U $X_{\{\lambda\}}$ is given by a single summand $(\eta_k)_*(-\nabla d)$, so there is exactly one singularity in the interior of every 2-simplex σ_k^2 , namely the source p^2 .

So far we have proven that our procedure of patching together the vectorfields Y_s on U_s , $s \in \{1, \dots, N\}$, using a partition of unity subordinated to the open cover $\{U_s\}_{s=1}^N$ of M does not create additional singularities of $X_{\{\lambda\}} = \sum_{k=1}^N \lambda_k Y_k$ beside the sources, saddles and sinks coming from the ones located at the barycenters of the k -faces, $k \in \{2, 1, 0\}$, of our model simplex $\Delta^2 \subset \mathbb{R}^2$.

It remains to be proven that the structure of the connecting orbits between any

two singularities of $X_{\{\lambda\}}$ is qualitatively as indicated in figure 2, in fact we will see that it is as in figure 17.

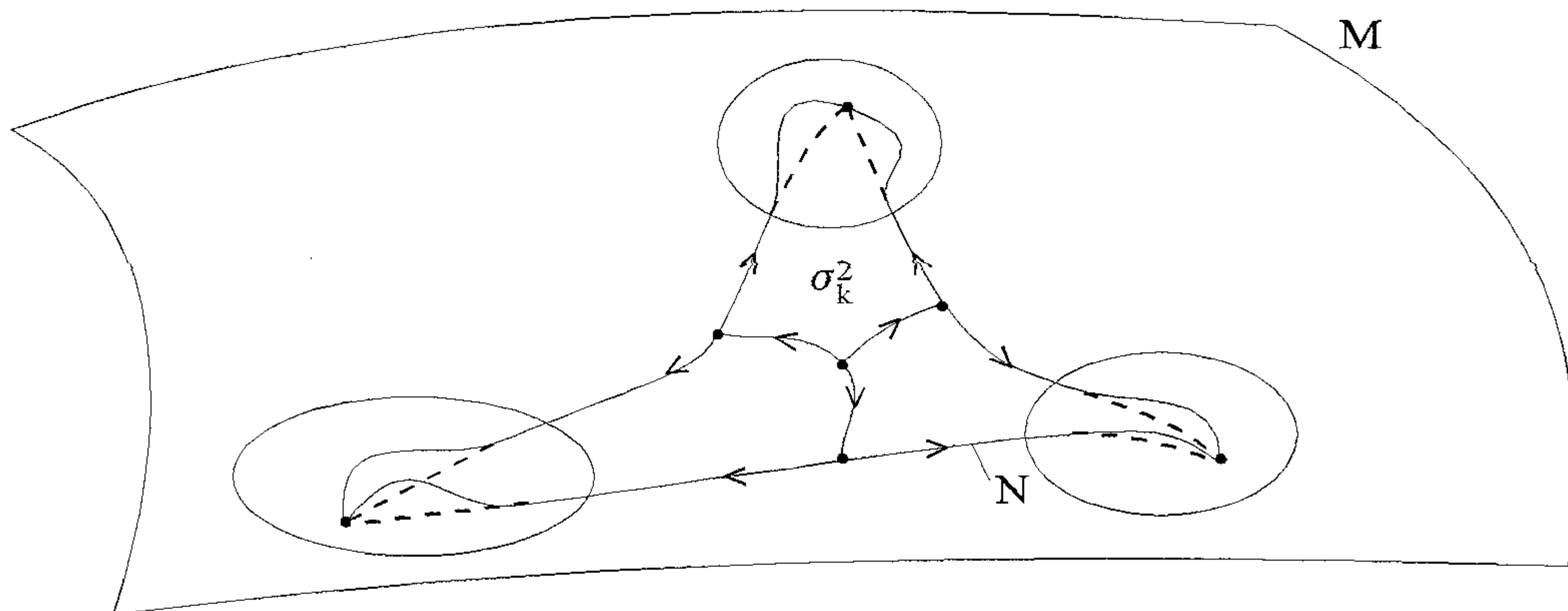


figure 17

Follow the flow generated by $X_{\{\lambda\}}$ along the unstable manifold of a saddle p^1 . The orbit runs along the 1-face until it reaches $W(p^0)$, once in $W(p^0)$ it will run into the sink p^0 by Lemma 14 ii). In doing so its orbit may leave the 1-face as indicated in figure 17.

Now the union of the saddles and sinks (belonging to one 2-simplex σ_k^2) and its connecting orbits is a compact flow invariant set N . The stable manifold of any saddle intersects N transversally, therefore the α -limit of any point in this stable manifold has to be inside N , but the only possibility is the source p^2 .

A priori this α -limit set could also be a limit cycle S . Now S would have to lie in a region where at least two λ_k 's are supported, i.e. near N . But then S had to run through $W(p^0)$ and by Lemma 14 ii) in a sink p^0 , which is a contradiction.

Finally we have to integrate $X_{\{\lambda\}}$ to be the negative gradient of a Morse function $f \in C^\infty(M, \mathbb{R})$. Using a smooth version of the (topological) Grobman-Hartman theorem, we see that $X_{\{\lambda\}}$ is locally around a singularity p_k conjugate by a diffeomorphism to $DX_{\{\lambda\}}(p)$, providing simple local coordinates near p_k .

Theorem 15 (Grobman-Hartman) [St, Thm 1] Let $\tilde{U}(0)$ be a neighborhood of 0 in $\mathbb{R}^n$ and $X: \tilde{U}(0) \rightarrow \mathbb{R}^n$ a C^∞ -diffeomorphism onto its image. Assume the eigenvalues $\lambda_1, \dots, \lambda_n$ of $DX(0)$ satisfy the conditions $\lambda_i \neq \lambda_1^{k_1} \dots \lambda_n^{k_n}$ for all i and $k_1, \dots, k_n \in \mathbb{N}_0$ with $k_1 + \dots + k_n \geq 2$, then X is conjugate to $DX(0)$ near 0; i.e. there are open neighborhoods $V(0)$ and $U(0) \subset \tilde{U}(0)$ of 0 and a C^∞ -diffeomorphism $h: V(0) \rightarrow U(0)$, $0 \mapsto 0$, such that

$$h \circ DX(0) = X \circ h \quad .$$

To apply the Grobman-Hartman theorem we first choose any local coordinates (φ, W) near p_k and apply Theorem 15 to $X_{\{\lambda\}}$ expressed in these coordinates. One may have to

perturb $X_{\{\lambda\}}$ slightly in a neighborhood of p_k to meet the nonresonance condition in Theorem 15. However, after a sufficiently small perturbation the singularity will still be hyperbolic.

With respect to the basis of $T_{p_k}M$ made up of the eigenvectors ξ_1, ξ_2 of $DX_{\{\lambda\}}(p_k)$ we get

$$DX_{\{\lambda\}}(p_k) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad a, b \in \mathbb{R} \setminus \{0\}.$$

In the mentioned coordinates we define $F_k: T_{p_k}M \supset V(0) \rightarrow \mathbb{R}$, $(x, y) \mapsto -\frac{a}{2}x^2 - \frac{b}{2}y^2$, and use the metric G_k on $V(0) \subset T_{p_k}M$ obtained by regarding ξ_1, ξ_2 as being orthogonal to get

$$\nabla_{G_k} F_k(x, y) = DX_{\{\lambda\}}(p_k) (x, y)^t.$$

Now we are ready to apply

Theorem 16 [Sm, Thm. B] *Let X be a smooth vector field on M with*

(1) *For any singularity β of X there exists a neighborhood N_β of β , a smooth function f_β and a Riemannian metric g_β , such that $\nabla_{g_\beta} f_\beta = X$ on N_β . Moreover, β is a nondegenerate critical point of f_β . Let $\beta_1, \dots, \beta_m$ denote the singularities of X .*

(2) *For any $x \in M$: $\alpha(x) \cup \omega(x) \subset \bigcup_{i=1}^m \beta_i$ (where α and ω are the limit sets generated by the flow associated to X).*

(3) *$W^u(\beta_i)$ and $W^s(\beta_j)$ intersect transversally for any $i, j \in \{1, \dots, m\}$.*

If (1)-(3) hold, then there exists a smooth function f on M with the following properties

(a) *The critical points of f coincide with the singularities of X and vice versa. For any critical point β of f there exists a neighborhood U_β of β , such that $f|_{U_\beta}$ coincides with the function $f_\beta|_{U_\beta}$ of (1) up to a constant $c_\beta \in \mathbb{R}$.*

(b) *If $X(p) \neq 0$ then X intersects the level set $f^{-1}(f(p))$ transversally at p .*

(c) *Let $\beta \in M$ be a critical point of f , then $f(\beta) = \text{Ind}_f(\beta)$.*

Note that (3) holds in our case, because we know that there are no saddle-saddle connections, since we know qualitatively the orbit structure as is exposed in figure 17. (2) is also satisfied, because we can rule out limit cycles: A limit cycle S has index $+1$ or -1 , because a unit tangent vector to S winds once around the origin (counter- or clockwise) when transported once around S . Now a result in differential topology tells us that the interior of S contains at least one singularity of the vector field; together with the known orbit structure between saddles and sinks, a limit cycle S could exist only around a source p^2 . Near p^2 we have a gradient vector field, therefore S had to be

located near the 1-faces; but this is excluded as above by property ii) of $W(p^0)$ in Lemma 14.

One can patch together the metrics G_k near the singularities of $X_{\{\lambda\}}$ to get

Lemma 17 [We, Prop. 2.1.12] *There exists a smooth Riemannian metric $\hat{g}$ on M , such that $\nabla_{\hat{g}}f = X_{\{\lambda\}}$, where f is as in Theorem 16.*

PROOF We use the notation of Theorem 16. Moreover, let $\{N_i\}_{i \in \Lambda}$ be a set of open subsets of M^2 , such that

$$(*) \quad \{N_i\}_{i \in \Lambda} \cup \{N_{\beta_1}, \dots, N_{\beta_m}\}$$

is an open cover of M^2 and

$$\left(\bigcup_{i \in \Lambda} N_i \right) \cap \{\beta_1, \dots, \beta_m\} = \emptyset \quad .$$

By compactness of M we may assume $|\Lambda| < \infty$. Let $\{\rho_i\}_{i \in \Lambda \cup \{\beta_1, \dots, \beta_m\}}$ be a partition of unity subordinated to the open cover $(*)$.

If we already found Riemannian metrics g_i on V_i , $\forall i \in \Lambda$, such that

$$(**) \quad \nabla_{g_i}f = X_{\{\lambda\}} \quad \text{on} \quad V_i \quad ,$$

then defining

$$\hat{g}(\xi, \eta) := \sum_{i \in \Lambda \cup \{\beta_1, \dots, \beta_m\}} \rho_i(q) \cdot g_i(\xi, \eta) \quad , \quad q \in M, \xi, \eta \in T_q M \quad ,$$

we were done. Symmetry, positive definiteness and nondegeneracy of $\hat{g}$ follow by the fact, that the summands have these properties. We have to show $X_{\{\lambda\}} = \nabla_{\hat{g}}f$: Let $q \in M$, $\xi \in T_q M$

$$\begin{aligned} \hat{g}(X_{\{\lambda\}}, \xi) &= \sum_{i \in \Lambda \cup \{\beta_1, \dots, \beta_m\}} \rho_i(q) \cdot g_i(X_{\{\lambda\}}, \xi) && , \text{ by def.} \\ &= \sum_i \rho_i(q) \cdot g_i(\nabla_{g_i}f, \xi) && , (**) \\ &= \sum_i \rho_i(q) \cdot df(\xi) = df(\xi) \cdot \sum_i \rho_i(q) = df(\xi) \quad , \end{aligned}$$

which is equivalent to $X_{\{\lambda\}} = \nabla_{\hat{g}}f$.

It remains to construct the metrics g_i on V_i . Let $q \in V_i$, then on the level set $f^{-1}(f(q))$ there exists a frame $\{e\}$ in a small neighborhood $W_i = f^{-1}(f(q)) \cap V_i$ of q (for V_i sufficiently small). By Theorem 16 (c) $X_{\{\lambda\}}$ is transversal to level sets, therefore

$$\{e(q), X_{\{\lambda\}}(q)\} \quad , \quad q \in W_i \quad ,$$

is a basis of $T_q M$. We can extend this frame to a small neighborhood of W_i in M using the linearization $D\phi_t$ of the time- t -map ϕ_t associated to $X_{\{\lambda\}}$:

$$\{D\phi_t|_q \circ e(q), D\phi_t|_q(q)\} \quad , \quad q \in W_i \quad , \quad t \in (-\epsilon, \epsilon) \quad , \quad \epsilon > 0 \text{ small} \quad ,$$

is a frame of V_i , eventually after shrinking V_i .

We define the metric g_i on V_i as follows: Let $q \in V_i$, $e = e(q)$, $X_{\{\lambda\}} = X_{\{\lambda\}}(q)$

$$\begin{aligned} g_i(e, e) &= 1 , \\ g_i(X_{\{\lambda\}}, e) &= df(e) , \\ g_i(X_{\{\lambda\}}, X_{\{\lambda\}}) &= df(X_{\{\lambda\}}) . \end{aligned}$$

In fact g_i satisfies $X_{\{\lambda\}} = \nabla_{g_i} f$ on V_i : Let $q \in V_i$ and $\xi \in T_q M$

$$\begin{aligned} g_i(X_{\{\lambda\}}, \xi) &= g_i(X_{\{\lambda\}}, \xi_1 e + \xi_2 X_{\{\lambda\}}) \\ &= \xi_1 g_i(X_{\{\lambda\}}, e) + \xi_2 g_i(X_{\{\lambda\}}, X_{\{\lambda\}}) \\ &= \xi_1 df(e) + \xi_2 df(X_{\{\lambda\}}) \\ &= df(\xi_1 e + \xi_2 X_{\{\lambda\}}) \\ &= df(\xi) . \end{aligned}$$

□

3. Application to Morse homology

Let f be a smooth Morse function on the smooth, closed Riemannian manifold (M, g) of dimension 2. We assume the **Morse-Smale condition** to hold, i.e. stable and unstable manifolds of any two critical points of f (determined by the flow associated to $-\nabla f$) intersect transversally. $\text{Crit}_k f$ denotes the set of critical points of f of Morse index k ($\#$ of negative eigenvalues of the Hessian of f at a critical point). One can show that $\#\text{Crit}_k f$ is finite for any $k \in \mathbb{Z}$.

The Morse chain complex with coefficients in $\mathbb{Z}_2$ is defined as follows: The **chain group** $CM_k(M, f, g; \mathbb{Z}_2)$ is defined to be the free, abelian module generated over $\mathbb{Z}_2$ by $\text{Crit}_k f$, i.e.

$$CM_k(M, f, g; \mathbb{Z}_2) = \bigoplus_{x \in \text{Crit}_k f} \mathbb{Z}_2 x \quad , \quad CM_k(M, f, g; \mathbb{Z}_2) = 0 \text{ if } \text{Crit}_k f = \emptyset \quad ,$$

$$CM_*(M, f, g; \mathbb{Z}_2) = \bigoplus_{k \in \mathbb{Z}} CM_k(M, f, g; \mathbb{Z}_2) \quad .$$

For $x \in \text{Crit}_k f$, $y \in \text{Crit}_{k-1} f$ we denote by $n_2(x, y)$ the number (mod 2) of orbits (of the negative gradient flow of f) connecting x to y (there are finitely many such orbits if one factors out the $\mathbb{R}$ -action on an orbit given by translations in time). Now we can define a **boundary operator**

$$\begin{aligned} \partial_k : CM_k(M, f, g; \mathbb{Z}_2) &\rightarrow CM_{k-1}(M, f, g; \mathbb{Z}_2) \\ x &\mapsto \sum_{y \in \text{Crit}_{k-1} f} n_2(x, y) y \quad . \end{aligned}$$

Theorem A $\partial_{k-1} \circ \partial_k = 0 \quad \forall k \in \mathbb{Z}$.

Definition The Morse homology groups with coefficients in $\mathbb{Z}_2$ are defined to be

$$HM_k(M, f, g; \mathbb{Z}_2) := \frac{\text{Ker } \partial_k}{\text{Im } \partial_{k+1}} \quad .$$

Theorem B $HM_k(M, f^0, g^0; \mathbb{Z}_2) \simeq HM_k(M, f^1, g^1; \mathbb{Z}_2) \quad \forall k \in \mathbb{Z}$.

By Theorem B we may denote the Morse homology groups by $HM_k(M; \mathbb{Z}_2)$.

Theorem C $HM_k(M; \mathbb{Z}_2) \simeq H_k^{\text{sing}}(M; \mathbb{Z}_2) \quad \forall k \in \mathbb{Z}$.

Theorems A, B and C are well-known, see e.g. [Po, Sa, Sch, We2]. A detailed proof of Theorem A, using elementary differential topology, may be found in [We1]. An immediate consequence of our previous work combined with Theorem B is a simple proof of Theorem C.

PROOF (of Theorem C) Given a triangulation (h, K) of M it is well-known that

$$H_*^{\text{sing}}(M; \mathbb{Z}_2) \simeq H_*^{\text{sing}}(|K|; \mathbb{Z}_2) \simeq H_*^{\text{simp}}(K; \mathbb{Z}_2) \quad ,$$

so it remains to show

$$HM_*(M; \mathbb{Z}_2) \simeq H_*^{\text{simp}}(K; \mathbb{Z}_2) \quad .$$

By our construction of a Morse function associated to the triangulation (h, K) we have a 1:1 correspondence between the chain groups

$$\begin{aligned} \chi_k : CM_k(M, f, g; \mathbb{Z}_2) &\rightarrow C_k^{\text{simp}}(K; \mathbb{Z}_2) \\ x &\mapsto k\text{-face whose barycenter is } x . \end{aligned}$$

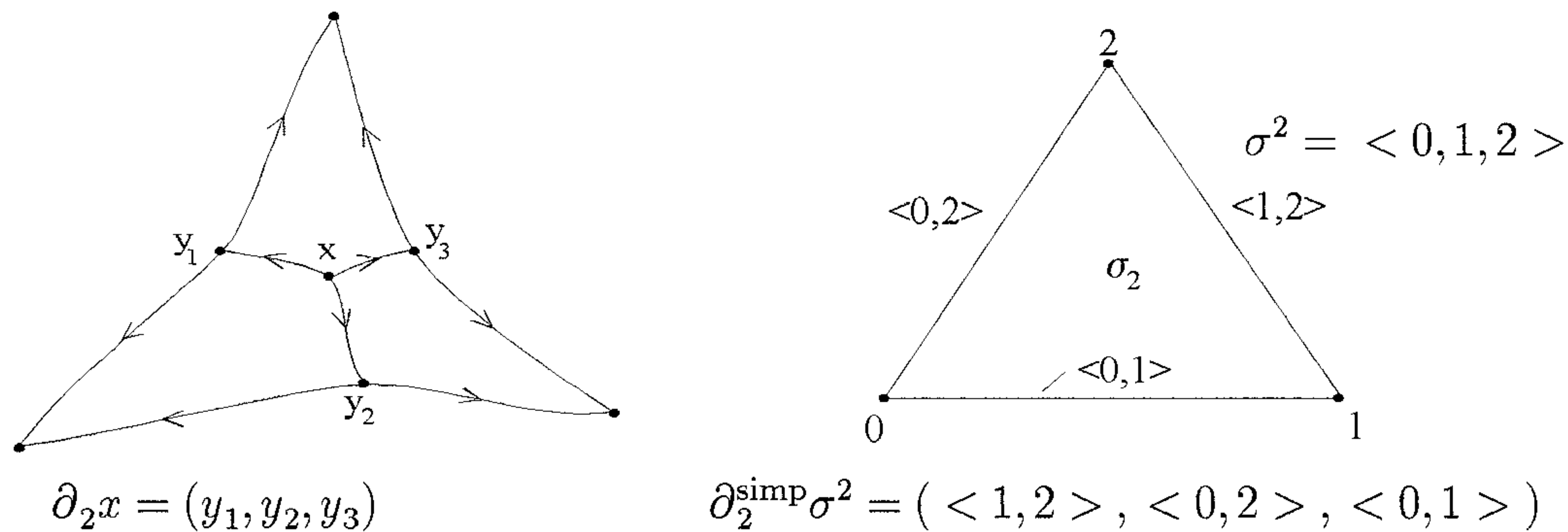


figure 18

The structure of the connecting orbits (figure 18) shows that the boundary operator ∂_k of the Morse complex acts on $x \in \text{Crit}_k f$ in the same way as the boundary operator ∂_k^{simp} of the simplicial complex K does on $\chi_k(x)$, i.e. the following diagram commutes

$$\begin{array}{ccc} CM_k(M, f, g; \mathbb{Z}_2) & \xrightarrow{\chi_k} & C_k^{\text{simp}}(K; \mathbb{Z}_2) \\ \partial_k \downarrow & & \downarrow \partial_k^{\text{simp}} \\ CM_{k-1}(M, f, g; \mathbb{Z}_2) & \xrightarrow{\chi_{k-1}} & C_{k-1}^{\text{simp}}(K; \mathbb{Z}_2) \quad . \end{array}$$

This means that χ_k is a chain map. As it is an isomorphism already on chain level, the induced map in homology is an isomorphism, too. $\square$

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