Noncontractible periodic orbits in cotangent bundles and Floer homology

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Abstract

Let M be a closed connected Riemannian manifold and let α be a homotopy class of free loops in M. Then, for every compactly supported time-dependent Hamiltonian on the open unit disc cotangent bundle which is sufficiently large over the zero section, we prove existence of a 1-periodic orbit whose projection to M represents α. The proof shows that the Biran-Polterovich-Salamon capacity of the open unit disk cotangent bundle relative to the zero section is finite. If M is not simply connected, this leads to an existence result for noncontractible periodic orbits on level hypersurfaces corresponding to a dense set of values of any proper, bounded from below Hamiltonian on T^*M, whenever the levels enclose M. This implies a version of the Weinstein conjecture including multiplicities: existence of closed characteristics – one associated to each nontrivial α – on every contact type hypersurface in T^*M enclosing M.

1 Introduction and main results

Given a closed connected smooth manifold M, let $\pi : T^*M \to M$ be its cotangent bundle with canonical symplectic form $\Omega_{can} = -d\theta$, where θ is the Liouville form. Choosing in addition a Riemannian metric on M, let DT^*M denote the open unit disc bundle. A smooth function $H : S^1 \times DT^*M \to \mathbb{R}$ of compact support gives rise to the Hamiltonian vector field X_{H_t} on T^*M determined by $dH_t = \iota(X_{H_t})\Omega_{can}$. Here nondegeneracy of the symplectic form enters. Our aim is to detect 1-periodic orbits of X_{H_t} whose projection to M represents a given homotopy class α of free loops in M. We denote the set of these orbits by

$$\mathcal{P}(H;\alpha) := \{z \in C^\infty(S^1, T^*M) \mid \dot{z}(t) = X_{H_t}(z(t)), \forall t \in S^1, [\pi(z)] = \alpha\}. \quad (1)$$

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Here and throughout S^1 is identified with $\mathbb{R}/\mathbb{Z}$ and we think of H as a smooth, compactly supported function on $\mathbb{R} \times DT^*M$, $(t,z) \mapsto H(t,z) =: H_t(z)$, satisfying $H_{t+1} = H_t$. The associated symplectic action functional $\mathcal{A}_H$ is defined by (3). The *marked length spectrum* Λ_α is the set of lengths of all periodic geodesics in the Riemannian manifold M which represent α. It is closed by Lemma 3.3, hence its infimum is realized by a periodic geodesic in the class α:

$$\ell_\alpha := \inf \Lambda_\alpha \in \Lambda_\alpha. \tag{2}$$

Theorem A (Existence of 1-periodic orbits). *Let M be a closed connected smooth Riemannian manifold and DT^*M the associated open unit disc cotangent bundle with canonical symplectic form Ω_{can}. Then for every homotopy class α of free loops in M the following is true. Every compactly supported Hamiltonian $H \in C^\infty([0,1] \times DT^*M)$ with*

$$\sup_{[0,1]\times M} H =: -c \le -\ell_\alpha$$

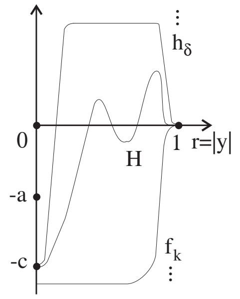
admits a 1-periodic orbit z whose projection to M represents α and $\mathcal{A}_H(z) \ge c$.

The existence problem for periodic orbits has a rich history (see [14], for example). One of the most prominent cornerstones was Floer's proof [7] of the Arnold conjecture for a certain class of symplectic manifolds by introducing Floer homology theory. Whereas periodic orbits detected in the past were typically contractible, only recently first steps have been taken in finding *noncontractible* ones, namely by Gatien-Lalonde [10] and by Biran-Polterovich-Salamon [2]. Both approaches are based on rather strong assumptions on the Riemannian manifold. For instance, Theorem A is proved in [2] in case that M is either the euclidean torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ or of negative sectional curvature. Generalizations of [10] are given in [16].

In the noncontractible case Theorem A is sharp in the sense that, firstly, the inequality cannot be improved. This follows from Remark 2.1 by perturbing and smoothing out the radial Hamiltonian on T^*M given by $(x,y) \mapsto \ell_\alpha(-1+|y|)$ for $|y| \in [0,1]$ and equal to zero whenever $|y| \ge 1$, in such a way that all slopes are strictly less then ℓ_α. Secondly, the zero section $\mathcal{O}_M = M$ in the inequality cannot be replaced by an arbitrary smooth section [2, Theorem C].

A main tool in [2] is existence of a natural homomorphism T from symplectic homology of DT^*M to relative symplectic homology of (DT^*M, M) which factors through Floer homology of H (see commuting triangle in Figure 1). The proof of Theorem A combines this and the main result of [19], namely that Floer homology of the Hamiltonian $T^*M \ni (x,y) \mapsto \frac{1}{2}|y|^2$ is represented by singular homology of the free loop space of M (see also [25]; after writing this manuscript a new proof appeared in [1]). The filtration provided by the symplectic action functional is another key ingredient.

A contribution of the present text is to compute symplectic homology $\underleftarrow{\mathrm{SH}}$ of DT^*M and relative symplectic homology $\underrightarrow{\mathrm{SH}}$ of (DT^*M, M) in terms of singular homology of the free loop space components $\mathcal{L}_\alpha M$ of M. This is the



$$\begin{tikzcd}
 & \underrightarrow{\mathrm{SH}}_*^{(a,\infty);c;\alpha} \arrow[r, "\simeq", "a\in(0,c]"'] & \mathrm{H}_*(\mathcal{L}_\alpha M) \\
\mathrm{HF}_*^{(a,\infty)}(H;\alpha) \arrow[ur, "\iota_H"] \arrow[r, phantom, "\circlearrowleft"] & {} \arrow[r, phantom, "\circlearrowleft"] & {} \\
 & \underleftarrow{\mathrm{SH}}_*^{(a,\infty);\alpha} \arrow[ul, "\pi_H"] \arrow[uu, "T"] \arrow[r, "\simeq", "a\in\mathbb{R}^+\setminus\Lambda_\alpha"'] & \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2} M) \arrow[uu, "{[\iota^\alpha_{a^2/2}]\neq 0 \text{ if } a\geq\ell_\alpha}"']
\end{tikzcd}$$

Figure 1: Idea of proof of Theorem A

content of Theorem 3.1 and it is illustrated by the rectangular block in Figure 1. Here ι_b^α is the natural inclusion $\mathcal{L}_\alpha^b M \hookrightarrow \mathcal{L}_\alpha M$ of a sublevel set of the classical action $\mathcal{S}_0$ (see Section 2.2; observe that its value for a periodic geodesic of length ℓ is $\ell^2/2$). Note that as the action window (a,∞) in the lower row gets bigger homology (eventually) gets smaller, because the increasing of the chain complex admits more cancellations. Throughout all homologies are with coefficients in $\mathbb{Z}_2$. To compute these symplectic homologies we construct families of functions f_k and h_δ as indicated in Figure 1. For appropriate k and δ they satisfy $f_k \leq H \leq h_\delta$. Together with the composition rule (18) for the induced monotone homomorphisms this leads to the commutative triangle in the figure. Now the homomorphism T is nonzero iff $a \in [\ell_\alpha, c]\setminus\Lambda_\alpha$. If $c \notin \Lambda_\alpha$ and c happens to be a regular value of the symplectic action, Theorem A follows: set $a = c$ and use that nontriviality of T implies nontriviality of Floer homology.

It is worthwhile to spell out a consequence of the commutative diagram in Figure 1 (see Theorem 3.1) in the *nondegenerate case*. Let the class α, the Hamiltonian H and the constant $c = c(H)$ be as in Theorem A. Then for every $a \in (\ell_\alpha, c] \setminus \Lambda_\alpha$ the following holds. If all 1-periodic orbits $z \in \mathcal{P}(H;\alpha)$ with $\mathcal{A}_H(z) > a$ are nondegenerate, then their number is at least

$$\operatorname{rank} \operatorname{im}[\iota^\alpha_{a^2/2}] =: \rho_\alpha(a).$$

The nondecreasing step function ρ_α takes values in the nonnegative integers and is continuous from above. Its points of discontinuity are elements of Λ_α. The minimum of the set on which ρ_α is nonzero equals ℓ_α.

A further consequence of Theorem 3.1 is finiteness of the *Biran-Polterovich-Salamon (BPS) capacity of DT^*M relative to M* (see Theorem 4.3)

$$\begin{aligned}
&c_{BPS}(DT^*M, M;\alpha) \\
&:= \inf\{c > 0 \mid \mathcal{P}(H;\alpha) \neq \emptyset \ \forall H \in C_0^\infty(S^1 \times DT^*M) : \sup_{S^1\times M} H \leq -c\} \\
&= \ell_\alpha.
\end{aligned}$$

In the terminology of [2] this means that *every* nontrivial α is symplectically essential (see Definition 4.4). Therefore we obtain the following two *multiplicity results* (both are proved in [2] for symplectically essential α).

Theorem B (Dense existence). *Let M be a closed connected smooth manifold and $H : T^*M \to \mathbb{R}$ be a smooth Hamiltonian which is proper and bounded from below. Suppose that the sublevel set $\{H < c\}$ contains M. Then for every nontrivial homotopy class α of free loops in T^*M there exists a dense subset $S_\alpha \subset (c, \infty)$ such that the following is true. For every $s \in S_\alpha$, the level set $\{H = s\}$ contains a periodic orbit $z = (x, y)$ of the Hamiltonian system (T^*M, Ω_{can}, H) which represents α and satisfies $\int_0^1 \langle y(t), \dot{x}(t)\rangle dt > 0$.*

Proof. Theorem 4.3 and [2, Theorem 3.4.1]. $\square$

Note that the period of the orbit in the previous theorem is not specified. Moreover, the theorem is not true in case $\alpha = 0$ as the example of the euclidean torus and the Hamiltonian $H(x, y) = \frac{1}{2}|y|^2$ shows.

Theorem C (Closed characteristics). *Let M be a closed connected smooth manifold and $W \subset T^*M$ be an open set with compact closure and smooth boundary $Q = \partial\overline{W}$ which is of contact type and encloses M. Let the characteristic line bundle $\mathcal{L}_Q$ be equipped with its canonical orientation. Then for every nontrivial homotopy class α of free loops in M the characteristic foliation of Q has a closed leaf $z \subset Q$ with $j_\#[z] = \alpha$. Here $j : Q \hookrightarrow T^*M \to M$ is the composition of inclusion and projection and $j_\#$ is the induced map on free homotopy.*

Proof. Theorem 4.3 and [2, Corollary 3.4.2]. $\square$

By definition, the hypersurface Q in the theorem is of *contact type* if there exists a smooth Liouville vector field Z (i.e. $\mathcal{L}_Z\Omega_{can} = \Omega_{can}$) which is defined on a neighbourhood U of Q in $\overline{W}$ and points outside $\overline{W}$ along Q. In this case Z gives rise to the contact form $\lambda_Q := (\iota(Z)\Omega_{can})\,|_{TQ}$ on Q with $d\lambda_Q = \Omega_{can}\,|_{TQ}$. The characteristic line bundle over Q is given by $\mathcal{L}_Q := \ker(\Omega_{can}\,|_{TQ})$. The associated foliation of Q is called *characteristic foliation* and its leaves *characteristics* . The Reeb vector field is a nonvanishing section of $\mathcal{L}_Q$ inducing the *canonical orientation*. This orients each leaf of the characteristic foliation.

Theorem C is related to the *Weinstein conjecture* [27]: given a symplectic manifold N and any compact hypersurface $Q \subset N$ of contact type, then the characteristic foliation of Q has a closed leaf. The conjecture was first proved by Viterbo [22] in case $N = \mathbb{R}^{2n}$. Later Hofer and Viterbo [13] proved it for cotangent bundles of closed manifolds under the additional hypothesis that Q encloses the zero section. The *multiplicity* result Theorem C refines their theorem in case that M is not simply connected. Viterbo informed us that the techniques of [13] should also provide multiplicities. In the unpublished part II of his PhD thesis Cieliebak proved Theorem C in the special case of fibrewise starshaped Q and *noncyclic* α ($\alpha^k \neq 0$ whenever $k \in \mathbb{Z} \setminus \{0\}$). In fact, in this case he proved [5, Part II Thm. 2] that the number of geometrically distinct closed characteristics on Q which represent α is at least

$$\left[\frac{\operatorname{cuplength}_{\mathbb{Z}_2}(\mathcal{L}_\alpha M)+1}{2}\right].$$

Here $[r]$ denotes the largest integer less or equal to r. Other recent multiplicity results [10, 2, 16] assume in addition existence of certain Riemannian metrics on M. In [23] Viterbo proved the Weinstein conjecture for cotangent bundles of closed manifolds with *finite* fundamental group (without the assumption that Q encloses the zero section). For further references concerning Weinstein conjecture and dense existence we refer to the survey [12].

This paper is organized as follows. In Section 2 we define action filtered Floer homology for a class of Hamiltonians which are radial and linear outside a compact subset of T^*M. We study Floer continuation and give a geometric criterion to decide whether a given 1-parameter family of Hamiltonians is action-regular (see Definition 2.6). In this case the associated continuation homomorphism is an isomorpism. Theorem 2.9 represents Floer homology of *convex* radial Hamiltonians in terms of singular homology of the free loop space of M. In Section 3 we recall the definition of symplectic [8, 6] and relative symplectic homology [2] and compute them in Theorem 3.1. In Section 4 we compute the capacity c_{BPS} and prove Theorem A.

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2 Floer homology

In Section 2.1 we define Floer homology for a class of Hamiltonians which outside of a compact subset of T^*M are radial and linear of nonnegative slope. In Section 2.2 we consider convex radial Hamiltonians and calculate their Floer homology in terms of singular homology of the free loop space of M.

Let ∇ be the Levi-Civita connection associated to the Riemannian metric on M. Throughout we use the isomorphism $T_{(x,y)}T^*M \to T_xM \oplus T_x^*M$ which takes the derivative $\dot{z}(t)$ of a curve $\mathbb{R} \to T^*M : t \mapsto z(t) = (x(t), y(t))$, where $y(t) \in T^*_{x(t)}M$, to the pair $(\dot{x}(t), \nabla_t y(t))$. Then the almost complex structure J_g on T^*M induced by the metric isomorphism $g : TM \to T^*M$ is represented by

$$J_g = \begin{pmatrix} 0 & -g^{-1} \\ g & 0 \end{pmatrix}.$$

It provides the Riemannian metric $G_g(\cdot,\cdot) := \Omega_{can}(\cdot, J_g\cdot)$ on T^*M. Define

$$\mathcal{L}_\alpha T^*M := \{z = (x,y) \mid x \in C^\infty(S^1, M),\, [x] = \alpha,\, y(t) \in T^*_{x(t)}M\}$$

where α is element of $\tilde{\pi}_1(M)$, the set of homotopy classes of free loops in M. Given $H \in C^\infty(S^1 \times T^*M)$, define the symplectic action $\mathcal{A}_H : \mathcal{L}_\alpha T^*M \to \mathbb{R}$ by

$$\mathcal{A}_H(x, y) := \int_0^1 \left(\langle y(t), \dot{x}(t)\rangle - H(t, x(t), y(t))\right) dt. \tag{3}$$

Its critical points are given by $\mathcal{P}(H;\alpha)$ as defined in (1). The set of critical values $\mathcal{A}_H(\mathcal{P}(H;\alpha))$ is called *action spectrum of H with respect to α*. It is denoted by $Spec(H;\alpha)$.

Remark 2.1 (Radial Hamiltonians). Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth symmetric function and define the autonomous Hamiltonian H^f on T^*M by $(x, y) \mapsto f(|y|)$. Throughout we simply denote H^f by f again. Its set of 1-periodic orbits

$$\mathcal{P}(f;\alpha) = \left\{(x, y) \in C^\infty(S^1, T^*M) \;\middle|\; \dot{x} = \tfrac{f'(|y|)}{|y|} y, \nabla_t y = 0, [x] = \alpha\right\}$$

equals the disjoint union of the two sets

$$\begin{aligned}\mathcal{P}^\pm(f;\alpha) :&= \big\{z = (x, y) \in C^\infty(S^1, T^*M) \;\big|\; x \text{ periodic geodesic}, \ell := |\dot{x}|,\\ &\qquad [x] = \alpha,\, y(t) = \pm\tfrac{r_z}{\ell}\dot{x}(t) \text{ where } r_z > 0 \text{ satisfies } f'(r_z) = \pm\ell\big\}.\end{aligned}$$

Note that $|y(t)| = r_z$ for all $t \in S^1$ and that $\ell \in \Lambda_\alpha$. Therefore we also call the elements of Λ_α *critical slopes*. For $z^\pm \in \mathcal{P}^\pm(f;\alpha)$ with $f'(r_{z^\pm}) = \pm\ell$ it holds

$$\mathcal{A}_f(z^\pm) = f'(r_{z^\pm})r_{z^\pm} - f(r_{z^\pm}) = \pm\ell r_{z^\pm} - f(r_{z^\pm}).$$

Given $a \in \mathbb{R}$, there are two methods to determine if the action of $z^+ \in \mathcal{P}^+(f;\alpha)$ is larger than a. (1) The inequality $\mathcal{A}_f(z^+) > a$ holds if and only if $f(r_{z^+})$ is located strictly below the line $r \mapsto -a + r\ell$. (2) This actually works for $z^\pm \in \mathcal{P}^\pm(f;\alpha)$: draw the tangent to the graph of f at the point $r_{z^\pm}$ which satisfies $f'(r_{z^\pm}) = \pm\ell$. This tangent $t(r) = \pm\ell r + b$ intersects the vertical coordinate axis at $b = -\mathcal{A}_f(z^\pm)$. Both methods are illustrated in Figure 2.

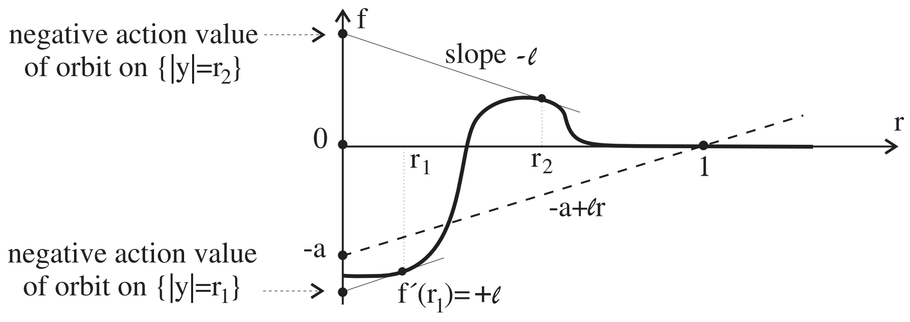

Figure 2: Action of 1-periodic orbits on $\{|y| = r\}$, where $f'(r) = \pm\ell \in \Lambda_\alpha$.

2.1 Definition for a class of admissible Hamiltonians

Consider the class of smooth time-1-periodic Hamiltonians on T^*M which are radial and linear of nonnegative slope outside some compact subset of the open radius-R-disk bundle $D_R T^*M$. For $R > 0$ define

$$\mathcal{K}_R := \{H \in C^\infty(S^1 \times T^*M) \mid \exists \lambda \geq 0 \; \exists c \in \mathbb{R} \text{ such that } H(t,x,y) = -c + \lambda\,|y| \text{ whenever } |y| \geq R\}.$$

Let $\mathcal{K}_R$ be equipped with the C^∞-topology on the closure of $S^1 \times D_{R+1}T^*M$. Now fix a homotopy class α of free loops in M and elements $a < b$ of $\mathbb{R} \cup \{\pm\infty\}$. In order to set up Floer homology filtered by the action interval (a,b) we need to impose the condition that a and b are not in the action spectrum, since this allows for small perturbations of the Hamiltonian without changing Floer homology. Moreover, it is necessary to ensure that the set of 1-periodic orbits with action in (a,b) is compact. This is not yet satisfied for all $H \in \mathcal{K}_R$. For instance, consider $H \in \mathcal{K}_R$ with $\lambda \in \Lambda_\alpha$ and $c \in [a,b]$. Such a Hamiltonian admits 1-periodic orbits of action c on all radius-r-sphere bundles $S_r T^*M$ with $r \geq R$. To exclude this we define the class of *admissible Hamiltonians* by

$$\mathcal{K}^{a,b}_{R;\alpha} := \{H \in \mathcal{K}_R \mid \{a,b\} \cap Spec(H;\alpha) = \emptyset \text{ and, if } H \equiv -c + \lambda\,|y| \text{ whenever } |y| \geq R, \text{ then } \lambda \notin \Lambda_\alpha \text{ or } c \notin [a,b]\}.$$

Next we would like to achieve nondegeneracy of all elements of the set

$$\mathcal{P}^{(a,b)}(H;\alpha) := \{z \in \mathcal{P}(H;\alpha) \mid \mathcal{A}_H(z) \in (a,b)\}.$$

Every 1-periodic orbit z corresponds to a fixed point $z_0 := z(0)$ of the time-1-map φ^H of the Hamiltonian flow and vice versa. Recall that z is called *nondegenerate* if 1 is not in the spectrum of $d\varphi^H(z_0) : T_{z_0}T^*M \to T_{z_0}T^*M$. It is well known that nondegeneracy can be achieved by perturbing H near the image of the 1-periodic orbit. A proof in a slightly different setting may be found in [26, Proof of Theorem 3.6]. This proof carries over to the situation at hand and allows for the following conclusion.

Remark 2.2. Fix $H \in \mathcal{K}^{a,b}_{R;\alpha}$ and an open neighborhood $U \subset S^1 \times D_R T^*M$ of the graphs of the elements of $\mathcal{P}^{(a,b)}(H;\alpha)$. Then there exists a neighborhood $\mathcal{U}$ of $0 \in C_0^\infty(U)$ and a subset $\mathcal{U}_{reg} \subset \mathcal{U}$ of the second category in the sense of Baire such that the following are true. If $W \in \mathcal{U}_{reg}$, then all elements of $\mathcal{P}^{(a,b)}(H+W;\alpha)$ are nondegenerate, their graphs lie in U and $H + W \in \mathcal{K}^{a,b}_{R;\alpha}$.

Choose $W \in \mathcal{U}_{reg}$. Then every element z of the discrete set $\mathcal{P}^{(a,b)}(H+W;\alpha)$ takes values in $D_R T^*M$ and this implies compactness, hence finiteness. As shown in [26] the Conley-Zehnder index $\mu_{CZ}(z)$ is well defined. Set $\sigma(z) := 0$, if $x^*TM \to S^1$ is trivial, and $\sigma(z) := 1$ otherwise, and define the integer

$$\mu(z) := -\mu_{CZ}(z) + \sigma(z). \tag{4}$$

This index provides the grading for the action filtered Floer chain groups

$$\mathrm{CF}_k^{(a,b)}(H+W;\alpha) := \bigoplus_{\substack{z\in\mathcal{P}^{(a,b)}(H+W;\alpha)\\ \mu(z)=k}} \mathbb{Z}_2 z, \tag{5}$$

for $k \in \mathbb{Z}$. Our convention is that a direct sum over the empty set is zero.

The following key proposition is used in its parameter and time independent form to define Floer's boundary operator. In this case it says, roughly speaking, that Floer cylinders with asymptotic boundary conditions located *inside* D_rT^*M cannot leave this set, whenever the Hamiltonian is radial outside.

Proposition 2.3 (C^0-bound for Floer cylinders). *Given $R, c \geq 0$, let $f \in C^\infty(\mathbb{R}\times S^1\times[R,\infty))$ be a smooth function such that $\partial_s f'_{s,t} \geq 0$ and $f''_{s,t} \geq -c$ (or $f''_{s,t} \leq c$) for every $(s,t)\in\mathbb{R}\times S^1$, where $f_{s,t}(r) := f(s,t,r)$. Assume further that $H \in C^\infty(\mathbb{R}\times S^1\times TM)$ satisfies*

$$|y| \geq R \qquad \Rightarrow \qquad H(s,t,x,y) = f_{s,t}(|y|), \tag{6}$$

that the pair $(u,v)\in C^\infty(\mathbb{R}\times S^1, TM)$ satisfies

$$\begin{pmatrix} \partial_s u - \nabla_t v \\ \nabla_s v + \partial_t u \end{pmatrix} - \nabla H(s,t,u,v) = 0, \tag{7}$$

and that there exists $T > 0$ such that

$$|s| \geq T \qquad \Rightarrow \qquad |v(s,\cdot)| \leq R. \tag{8}$$

Then $|v| \leq R$ on $\mathbb{R}\times S^1$.

Proof. Assume by contradiction that there exists $(s_*,t_*)\in(-T,T)\times S^1$ with

$$r_* := |v(s_*,t_*)| = \max_{[-T,T]\times S^1} |v| > R.$$

Let $a \in (R, r_*)$ be a regular value of $|v|$ and define the set

$$\Omega := \{(s,t)\in\mathbb{R}\times S^1 : \ |v(s,t)| \in [a,r_*]\} \subset (-T,T)\times S^1$$

with smooth boundary $\partial\Omega = |v(\cdot,\cdot)|^{-1}(a)$. Denote the connected component of Ω containing (s_*,t_*) by Ω_*. Assumptions (6) and (7) show that the pair (u,v) restricted to Ω_* satisfies the equations

$$\partial_s u - \nabla_t v = 0, \qquad \nabla_s v + \partial_t u - \frac{f'_{s,t}(|v|)}{|v|} v = 0.$$

Hence, setting $L := \partial_s^2 + \partial_t^2 - f''_{s,t}(|v|)\partial_s$ and applying ∇_t to the first and ∇_s to the second equation, we obtain on Ω_* the inequality

$$L\tfrac{1}{2}|v|^2 = |\partial_s u|^2 + |\nabla_s v|^2 + \left(\partial_s f'_{s,t}\right)(|v|)\cdot|v| \geq 0.$$

Now the proof, but not the statement, of the weak maximum principle [11, Theorem 3.1] carries over to our situation (we have a bound for $|s|$ but not for $|t|$ when viewed as a real variable): consider first the case $f''_{s,t} \geq -c$ and let $\varepsilon \in (0,1]$. Then on Ω_* we get

$$L(\varepsilon e^{-(c+1)s}) \geq \varepsilon(c+1)e^{-(c+1)s} \geq \varepsilon(c+1)e^{-(c+1)T} > 0.$$

In the first step we used $f''_{s,t} \geq -c$ and the second step uses $s \leq T$ on Ω_*. It follows that the function $\gamma_\varepsilon : \Omega_* \to \mathbb{R}$ given by

$$\gamma_\varepsilon(s,t) := |v(s,t)|^2 + \varepsilon e^{-(c+1)s}$$

satisfies $L\gamma_\varepsilon > 0$. Hence it cannot have any interior maximum: otherwise, if (s',t') was an interior maximum, then $\partial_s \gamma_\varepsilon(s',t') = 0$ and $(\partial_s^2 + \partial_t^2)\gamma_\varepsilon(s',t') \leq 0$, which contradicts $L\gamma_\varepsilon > 0$. Therefore $\|\gamma_\varepsilon\|_{L^\infty(\Omega_*)} = \|\gamma_\varepsilon\|_{L^\infty(\partial\Omega_*)}$ and this proves the third of the following equations

$$r_*^2 = \|v\|^2_{L^\infty(\Omega_*)} = \lim_{\varepsilon\to 0} \|\gamma_\varepsilon\|_{L^\infty(\Omega_*)} = \lim_{\varepsilon\to 0} \|\gamma_\varepsilon\|_{L^\infty(\partial\Omega_*)} = \|v\|^2_{L^\infty(\partial\Omega_*)} = a^2.$$

To obtain the second equation we used $s \geq -T$ in Ω_*, which implies $e^{-(c+1)s} \leq e^{(c+1)T}$, and therefore

$$\sup_{\Omega_*} |v|^2 \leq \sup_{\Omega_*} \left(|v|^2 + \varepsilon e^{-(c+1)s}\right) \leq \left(\sup_{\Omega_*} |v|^2\right) + \varepsilon e^{(c+1)T} \stackrel{\varepsilon\to 0}{\longrightarrow} \sup_{\Omega_*} |v|^2 .$$

Equation four follows similarly with Ω_* replaced by $\partial\Omega_*$. The conclusion $r_* = a$ contradicts the choice of a, hence the lemma is proved in case $f''_{s,t} \geq -c$. To prove the case $f''_{s,t} \leq c$ replace $\varepsilon e^{-(c+1)s}$ by $\varepsilon e^{(c+1)s}$. $\square$

Fix $z^\pm \in \mathcal{P}^{(a,b)}(H+W;\alpha)$ and consider the moduli space $\mathcal{M}(z^-, z^+; H + W, \alpha)$ of *connecting trajectories* which consists of smooth solutions $w : \mathbb{R}\times S^1 \to T^*M$ of Floer's elliptic partial differential equation

$$\partial_s w + J_g(w)\partial_t w - \nabla(H+W)(t,w) = 0 \tag{9}$$

subject to the asymptotic boundary conditions

$$\lim_{s\to\pm\infty} w(s,t) = z^\pm(t), \qquad \lim_{s\to\pm\infty} \partial_s w(s,t) = 0, \tag{10}$$

uniformly in $t \in S^1$. Every solution w of (9) and (10) satisfies the energy identity

$$E(w) := \|\partial_s w\|^2_{L^2(\mathbb{R}\times S^1)} = \mathcal{A}_{H+W}(z^-) - \mathcal{A}_{H+W}(z^+).$$

To define *Floer's boundary operator* the following observations are crucial.

(B1) All relevant Floer connecting trajectories stay inside the bounded set $D_R T^*M$: since all elements of $\mathcal{P}^{(a,b)}(H+W;\alpha)$ take values in $D_R T^*M$, Proposition 2.3 shows that $w(\mathbb{R}\times S^1) \subset D_R T^*M$ for all connecting trajectories $w \in \mathcal{M}(z^-, z^+; H+W, \alpha)$ and all $z^\pm \in \mathcal{P}^{(a,b)}(H+W;\alpha)$.

(B2) The moduli spaces $\mathcal{M}(z^-, z^+; H+W, \alpha)$ are compact with respect to C^∞-convergence on compact sets. This is a consequence of the energy identity, exactness of the canonical symplectic form Ω_{can}, and (B1).

(B3) Let D_w be the linear operator obtained by linearizing (9) at a solution w. By [9, Theorem 5.1 (ii)] surjectivity of D_w can be achieved by a small perturbation of $H^0 := H + W$ which vanishes up to second order along the asymptotic orbits $z^\pm$. More precisely, denote by $\mathcal{H} = \mathcal{H}(H^0)$ the subset of all $K \in C_0^\infty(S^1 \times D_R T^*M)$ which vanish up to second order on the elements of $\mathcal{P}^{(a,b)}(H^0; \alpha)$. Then there exists a subset $\mathcal{H}_{reg} = \mathcal{H}_{reg}(H^0)$ of the second category in the sense of Baire such that D_w is surjective for all $w \in \mathcal{M}(z^-, z^+; H^0 + K, \alpha)$, all $z^\pm \in \mathcal{P}^{(a,b)}(H^0 + K; \alpha)$ and all $K \in \mathcal{H}_{reg}$. Elements of $\mathcal{H}_{reg}$ are called *regular*. Moreover, for every sufficiently small $K \in \mathcal{H}$ we have $H^0 + K \in \mathcal{K}_{R;\alpha}^{a,b}$ and $\mathcal{P}^{(a,b)}(H^0 + K; \alpha) = \mathcal{P}^{(a,b)}(H^0; \alpha)$.

From now on we assume that H is regular. In this case the moduli spaces $\mathcal{M}(z^-, z^+; H, \alpha)$ are smooth manifolds of dimension $\mu_{CZ}(z^+) - \mu_{CZ}(z^-) = \mu(z^-) - \mu(z^+)$; see [20] and [18]. There is a free $\mathbb{R}$-action given by shifting the s-variable: $w(s,t) \mapsto w(s+\sigma, t)$. For $z^- \in \mathcal{P}^{(a,b)}(H; \alpha)$ with $\mu(z^-) = k$, define $\partial_k = \partial_k(H; \alpha) : \mathrm{CF}_k^{(a,b)}(H; \alpha) \to \mathrm{CF}_{k-1}^{(a,b)}(H; \alpha)$ by

$$\partial_k z^- := \sum_{\substack{z^+ \in \mathcal{P}^{(a,b)}(H;\alpha) \\ \mu(z^+) = k-1}} \#_2 \left(\frac{\mathcal{M}(z^-, z^+; H, \alpha)}{\mathbb{R}} \right) z^+.$$

Here $\#_2$ denotes the number of elements modulo two. We define $\partial_k z^- := 0$, whenever the sum is over the empty set. Since the index difference is one, the coefficients in the sum are finite. This uses the compactness property (B2) of the 1-dimensional moduli spaces. The identity $\partial_{*-1}\partial_* = 0$ follows as in Floer's original work [7] by analyzing the 2-dimensional moduli spaces. Here gluing of Floer cylinders and again (B2) enters. Action filtered Floer homology is defined by (we drop T^*M and the almost complex structure J_g in the notation, since both are fixed throughout)

$$\mathrm{HF}_k^{(a,b)}(H; \alpha) := \frac{\ker \partial_k}{\mathrm{im}\, \partial_{k+1}}, \qquad k \in \mathbb{Z}.$$

Continuation

We extend the definition of Floer homology from regular to *any* $H \in \mathcal{K}_{R;\alpha}^{a,b}$ and show that Floer homology does not change on connected components.

Lemma 2.4. *The subset $\mathcal{K}_{R;\alpha}^{a,b} \subset \mathcal{K}_R$ is open.*

Proof. The second condition in the definition of $\mathcal{K}_{R;\alpha}^{a,b}$, namely $\lambda \notin \Lambda_\alpha$ or $c \notin [a,b]$ is open, since Λ_α is closed by Lemma 3.3. The first condition $a, b \notin Spec(H; \alpha)$ is open, because its negation '$a \in Spec(H; \alpha)$ or $b \in Spec(H; \alpha)$'

is closed: Let (H_ν) be a sequence in $\mathcal{K}_R$ which C^∞-converges to $H \in \mathcal{K}_R$ and with $a \in Spec(H_\nu;\alpha)$ for every $\nu \in \mathbb{N}$. Hence there is a sequence of orbits $z_\nu \in \mathcal{P}(H_\nu;\alpha)$ of action a. This sequence admits a uniform C^2-bound, which implies existence of a C^1-convergent subsequence. It follows that the limit z is a 1-periodic orbit of H of action a and with $[\pi(z)] = \alpha$. □

Now fix $H \in \mathcal{K}^{a,b}_{R;\alpha}$ and choose a convex neighbourhood $\mathcal{W}$ of H *in* $\mathcal{K}^{a,b}_{R;\alpha}$ and regular Hamiltonians $H^0, H^1 \in \mathcal{W}$. Outside D_RT^*M these are of the form $H^i_t(x,y) = -c^i + \lambda^i\,|y|$, for $i = 0,1$. We may assume $\lambda^0 \le \lambda^1$, otherwise rename H^0 and H^1. Let $\beta : \mathbb{R} \to [0,1]$ be a smooth nondecreasing cutoff function which is zero on $(-\infty,-1]$ and one on $[1,\infty)$. Consider the smooth homotopy $\mathbb{R} \to \mathcal{W}$ given by

$$s \mapsto H_s := H^0 + \beta(s)(H^1 - H^0). \tag{11}$$

Observe that outside D_RT^*M it is of the form

$$f_s(|y|) = -c^0 + \lambda^0\,|y| + \beta(s)\left(c^0 - c^1 + |y|\,(\lambda^1 - \lambda^0)\right)$$

and satisfies $\partial_s f_s'(|y|) = \dot\beta(s)(\lambda^1 - \lambda^0) \ge 0$. For $z_i \in \mathcal{P}^{(a,b)}(H^i;\alpha)$, define the moduli space $\mathcal{M}(z_0,z_1;H_s,\alpha)$ of *continuation trajectories* to be the set of smooth solutions $w : \mathbb{R}\times S^1 \to T^*M$ of Floer's parameter-dependent equation

$$\partial_s w + J_g(w)\partial_t w - \nabla H_s(t,w) = 0 \tag{12}$$

which satisfy uniformly in $t \in S^1$ the asymptotic boundary conditions

$$\lim_{s\to-\infty} w(s,t) = z_0(t), \qquad \lim_{s\to+\infty} w(s,t) = z_1(t), \qquad \lim_{s\to\pm\infty} \partial_s w(s,t) = 0. \tag{13}$$

Every solution w of (12) and (13) satisfies the energy identity

$$E(w) = \mathcal{A}_{H^0}(z_0) - \mathcal{A}_{H^1}(z_1) - \int_{-\infty}^{\infty}\int_0^1 \left(\partial_s H_s\right)(t,w(s,t))\,dtds. \tag{14}$$

To define Floer's continuation chain map the following observations are crucial.

(C1) All continuation trajectories $w \in \mathcal{M}(z_0,z_1;H_s,\alpha)$ stay in D_RT^*M for all $z_k \in \mathcal{P}^{(a,b)}(H^k;\alpha)$ and $k = 0,1$: as observed in (B1) this is true for the elements of $\mathcal{P}^{(a,b)}(H^k;\alpha)$ with $k = 0,1$. Since $\partial_s f_s' \ge 0$, it follows that H_s satisfies the assumptions of Proposition 2.3 and this proves the claim.

(C2) The moduli spaces $\mathcal{M}(z_0,z_1;H_s,\alpha)$ are compact with respect to C^∞-convergence on compact sets. Again this is a consequence of the energy identity (14), exactness of the canonical symplectic form Ω_{can} and (C1).

(C3) A perturbation argument similar to (B3) shows that there is a subset of the second category in the sense of Baire of regular homotopies among all homotopies $\mathbb{R} \to \mathcal{W}$. A homotopy is called *regular* if the linear operator associated to (12) is surjective for all continuation trajectories w.

From now on assume that H_s is a regular homotopy. Then the moduli spaces $\mathcal{M}(z_0, z_1; H_s, \alpha)$ are smooth manifolds of dimension $\mu(z_0) - \mu(z_1)$. For $z_0 \in \mathcal{P}^{(a,b)}(H^0; \alpha)$ with $\mu(z_0) = k$ define *Floer's continuation map* by

$$\Phi^{10} z_0 = \Phi^{10}(H_s) z_0 := \sum_{\substack{z_1 \in \mathcal{P}^{(a,b)}(H^1;\alpha) \\ \mu(z_1)=k}} \#_2 \mathcal{M}(z_0, z_1; H_s, \alpha)\, z_1. \tag{15}$$

As shown in [2, Section 4.4] it is a chain map $\mathrm{CF}_*^{(a,b)}(H^0;\alpha) \to \mathrm{CF}_*^{(a,b)}(H^1;\alpha)$ whenever H^0 and H^1 are *sufficiently close* to $H \in \mathcal{K}_{R;\alpha}^{a,b}$ (here the energy identity (14) enters). It descends to a homomorphism $[\Phi^{10}]$ on homology which is independent of the choice of homotopy by standard arguments.

Lemma 2.5. *The homomorphism $[\Phi^{10}]$ is an isomorphism whenever H^0 and H^1 are* sufficiently close *to $H \in \mathcal{K}_{R;\alpha}^{a,b}$.*

Proof. A standard idea of proof is to consider the reverse homotopy H_{-s} from H^1 to H^0 and show that it induces the inverse map on homology. Unfortunately, this approach fails here, because H_{-s} leads to $\partial_s f'_{-s} \le 0$. Hence the proof of $(C1)$ fails. To overcome this problem we introduce an appropriate intermediate Hamiltonian H^2 (see Figure 3) and replace the homotopy $H_s =: H^{01}$ from H^0 to H^1 by two homotopies, namely by $H_s^{02} := H^0 + \beta(s)(H^2 - H^0)$ followed by $H_s^{21} := H^2 + \beta(s)(H^1 - H^2)$. The associated continuation maps Φ^{10}

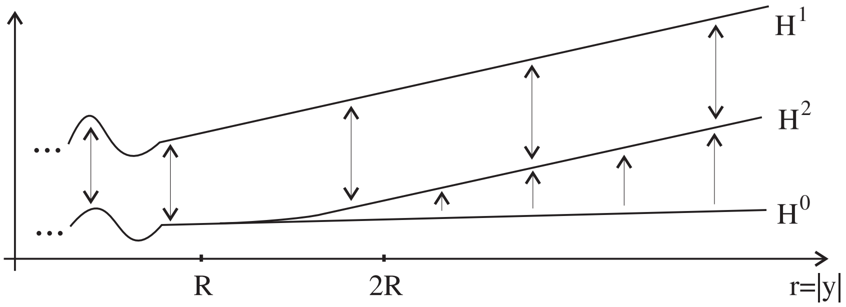

Figure 3: The Hamiltonians H^0, H^1, H^2 and homotopies.

and $\Phi^{12} \circ \Phi^{20}$ are chain homotopy equivalent by standard arguments (see [18, Section 3.4]), hence the induced maps on homology are equal. In other words, the following diagram commutes

$$\begin{tikzcd}
\mathrm{HF}_*^{(a,b)}(H^1;\alpha) & \\
\arrow[r, phantom, "\circlearrowright"] & \mathrm{HF}_*^{(a,b)}(H^2;\alpha)\,. \arrow[ul, "{[\Phi^{12}]}"', "\simeq"] \\
\mathrm{HF}_*^{(a,b)}(H^0;\alpha) \arrow[uu, "{[\Phi^{10}]}"] \arrow[ur, "{[\Phi^{20}]=[id]}"'] &
\end{tikzcd}$$

To conclude the proof of the lemma it suffices to show that (i) $\Phi^{20} = \mathrm{id}$ and (ii) Φ^{12} induces an isomorphism on homology. The Hamiltonian H^2 is defined as follows. On $D_R T^*M$ let it coincide with H^0 and on the complement extend it radially: starting at $r = R$ increase r and make a smooth left turn and continue, say outside $D_{2R}T^*M$, with the constant slope λ^1 of H^1 (see Figure 3).

(i) If H^0 and H^1 are sufficiently close, then the chain groups associated to H^0 and H^2 are identical: since $\mathbb{R} \setminus \Lambda_\alpha$ is open by Lemma 3.3, it follows $(\lambda_0, \lambda_1) \cap \Lambda_\alpha = \emptyset$. Hence the elements of $\mathcal{P}^{(a,b)}(H^0;\alpha)$ and $\mathcal{P}^{(a,b)}(H^2;\alpha)$ take values in $D_R T^*M$, but there H^0 and H^2 coincide. Proposition 2.3, shows that also all relevant continuation trajectories (12) take values in $D_R T^*M$, but here the homotopy from H^0 to H^2 is constant and so all trajectories are independent of s. This implies that the continuation map Φ^{20} is the identity.

(ii) The key fact is that outside $D_{2R}T^*M$ the homotopy H_s^{21} is of the form $f_s(|y|)$ with $\partial_s f_s' \equiv 0 \equiv \partial_s f_{-s}'$. Hence by Proposition 2.3 all continuation trajectories corresponding to H_s^{21} and also those corresponding to the reverse homotopy H_{-s}^{21} remain in $D_{2R}T^*M$. Therefore the standard argument of reversing the homotopy applies and shows $[\Phi(H_{-s}^{21})] = [\Phi(H_s^{21})]^{-1}$. □

Definition 2.6. (1) For $H \in \mathcal{K}_{R;\alpha}^{a,b}$, define $\mathrm{HF}_*^{(a,b)}(H;\alpha) := \mathrm{HF}_*^{(a,b)}(K;\alpha)$, where K is any regular Hamiltonian sufficiently close to H.
(2) A homotopy H_s taking values in a connected component of $\mathcal{K}_{R;\alpha}^{a,b}$ is called *action-regular*.

Definition 2.6 (1) is independent of the choice of K by Lemma 2.5. Moreover, given any action-regular homotopy H_s from H^0 to H^1, there exists $N \in \mathbb{N}_0$ sufficiently large such that with $s_k := \frac{k}{N}$ we have the composition of isomorphisms

$$[\Phi^{s_N s_{N-1}}] \circ \cdots \circ [\Phi^{s_1 s_0}] \; : \; \mathrm{HF}_*^{(a,b)}(H^0;\alpha) \to \mathrm{HF}_*^{(a,b)}(H^1;\alpha). \tag{16}$$

It is an open question, if the resulting isomorphism depends on the choice of the homotopy between H^0 and H^1; see [2, Remark 4.4.3]. If the homotopy is in addition monotone, an isomorphism can be defined directly in terms of the solutions of (12) and it is independent of the choice of the monotone action-regular homotopy. This will be discussed in the next section.

Monotone homotopies

Given $H^0, H^1 \in \mathcal{K}_{R;\alpha}^{a,b}$ with $H^0 \leq H^1$ pointwise, and a smooth homotopy H_s from H^0 to H^1 which satisfies $\partial_s H_s \geq 0$ everywhere and $H_s(t,x,y) = f_s(|y|)$ with $\partial_s f_s'(|y|) \geq 0$ whenever $|y| \geq R$. We call such a homotopy *monotone*. Any convex combination $H_s = (1-\beta(s))H^0 + \beta(s)H^1$ is a monotone homotopy. By a perturbation argument we may assume without loss of generality that the Hamiltonians H^0 and H^1 and the monotone homotopy H_s are regular. Let Floer's continuation map Φ^{10} be given by (15). Using the energy identity (14) one can show that it is a chain map from $\mathrm{CF}_*^{(a,b)}(H^0;\alpha)$ to $\mathrm{CF}_*^{(a,b)}(H^1;\alpha)$. The induced map on homology, called the *monotone homomorphism*, is denoted by

$$\sigma^{10} := [\Phi^{10}] : \mathrm{HF}_*^{(a,b)}(H^0;\alpha) \to \mathrm{HF}_*^{(a,b)}(H^1;\alpha). \tag{17}$$

Lemma 2.7 ([8, 6]). *The monotone homomorphism does not depend on the choice of the monotone homotopy used to define it and*

$$\sigma^{21} \circ \sigma^{10} = \sigma^{20}, \qquad \sigma^{00} = id, \tag{18}$$

whenever $H^0, H^1, H^2 \in \mathcal{K}^{a,b}_{R;\alpha}$ *satisfy* $H^0 \leq H^1 \leq H^2$.

Lemma 2.8 ([24]). *The monotone homomorphism associated to an action-regular monotone homotopy is an isomorphism.*

Proof. The isomorphisms in (16) are induced by monotone homotopies; now apply (18). □

Given $a \leq b \leq c$ and Hamiltonians $H^0 \leq H^1$ in $\mathcal{K}^{a,b}_{R;\alpha} \cap \mathcal{K}^{a,c}_{R;\alpha}$, we obtain the following commuting diagram whose rows are the short exact sequences associated to the natural inclusion ι^F and projection π^F (we temporarily suppress α in the notation)

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{CF}^{(a,b)}_*(H^0) \arrow[r, "\iota^F"] \arrow[d, "\Phi^{10}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{CF}^{(a,c)}_*(H^0) \arrow[r, "\pi^F"] \arrow[d, "\Phi^{10}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{CF}^{(b,c)}_*(H^0) \arrow[r] \arrow[d, "\Phi^{10}"] & 0 \\
0 \arrow[r] & \mathrm{CF}^{(a,b)}_*(H^1) \arrow[r, "\iota^F"] & \mathrm{CF}^{(a,c)}_*(H^1) \arrow[r, "\pi^F"] & \mathrm{CF}^{(b,c)}_*(H^1) \arrow[r] & 0
\end{tikzcd}.$$

The associated long exact sequences fit into the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{HF}^{(a,b)}_*(H^0) \arrow[r, "{[\iota^F]}"] \arrow[d, "\sigma^{10}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{HF}^{(a,c)}_*(H^0) \arrow[r, "{[\pi^F]}"] \arrow[d, "\sigma^{10}"] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{HF}^{(b,c)}_*(H^0) \arrow[r] \arrow[d, "\sigma^{10}"] & \cdots \\
\cdots \arrow[r] & \mathrm{HF}^{(a,b)}_*(H^1) \arrow[r, "{[\iota^F]}"] & \mathrm{HF}^{(a,c)}_*(H^1) \arrow[r, "{[\pi^F]}"] & \mathrm{HF}^{(b,c)}_*(H^1) \arrow[r] & \cdots
\end{tikzcd}. \tag{19}$$

Moreover, given $a \leq b \leq d$ and Hamiltonians $H^0 \in \mathcal{K}^{a,d}_{R;\alpha}$ and $H^1 \in \mathcal{K}^{a,d}_{R;\alpha} \cap \mathcal{K}^{a,b}_{R;\alpha}$ with $H^0 \leq H^1$, let H_s be a monotone homotopy from H^0 to H^1 such that the associated continuation map Φ^{10} maps $\mathcal{P}^{(a,d)}(H^0;\alpha)$ to $\mathcal{P}^{(a,b)}(H^1;\alpha)$. Then we obtain the homomorphism

$$\check{\sigma}^{10} = [\Phi^{10}] : \mathrm{HF}^{(a,d)}_*(H^0;\alpha) \to \mathrm{HF}^{(a,b)}_*(H^1;\alpha).$$

If in addition $c \in [b,d]$, $H^0 \in \mathcal{K}^{a,c}_{R;\alpha}$, and $\Phi^{10}(\mathcal{P}^{(a,c)}(H^0;\alpha)) \subset \mathcal{P}^{(a,b)}(H^1;\alpha)$, then the following diagram commutes (even on the chain level)

$$\begin{tikzcd}
\mathrm{HF}^{(a,d)}_*(H^0;\alpha) \arrow[r, "\sigma^{10}"] \arrow[dr, "\check{\sigma}^{10}"] & \mathrm{HF}^{(a,d)}_*(H^1;\alpha) \\
\mathrm{HF}^{(a,c)}_*(H^0;\alpha) \arrow[u, "{[\iota^F]}"] \arrow[r, "\check{\sigma}^{10}"] & \mathrm{HF}^{(a,b)}_*(H^1;\alpha) \arrow[u, "{[\iota^F]}"']
\end{tikzcd}. \tag{20}$$

2.2 Computation for convex radial Hamiltonians

We denote by $\mathcal{L}M = C^\infty(S^1, M)$ the free loop space of M and by $\mathcal{L}_\alpha M$ the connected component whose elements represent the free homotopy class α. For $V \in C^\infty(S^1 \times M)$ the *classical action functional* $\mathcal{S}_V$ on $\mathcal{L}M$ is given by

$$\mathcal{S}_V(x) := \int_0^1 \frac{1}{2}\left|\dot{x}(t)\right|^2 - V(t, x(t))\, dt. \tag{21}$$

For $a \in \mathbb{R}$ consider the sublevel set

$$\mathcal{L}_\alpha^a M := \{x \in \mathcal{L}_\alpha M \mid \mathcal{S}_0(x) \leq a\}$$

and denote its singular homology with $\mathbb{Z}_2$-coefficients by $\mathrm{H}_*(\mathcal{L}_\alpha^a M)$. Using simplified notation there are natural inclusions $\iota_a = \iota_a^\alpha : \mathcal{L}_\alpha^a M \hookrightarrow \mathcal{L}_\alpha M$ and $\iota_{ba} = \iota_{ba}^\alpha : \mathcal{L}_\alpha^a M \hookrightarrow \mathcal{L}_\alpha^b M$ whenever $b \geq a$.

Theorem 2.9 (Floer homology of convex radial Hamiltonians). *Fix a homotopy class α of free loops in M and a smooth symmetric function f with $f'' \geq 0$. If λ is a positive noncritical slope of f, i.e. $\lambda \in \mathbb{R}^+ \setminus \Lambda_\alpha$ and $f'(r) = \lambda$ for some $r > 0$, then there is a natural isomorphism*

$$\Psi^{f,\lambda} : \mathrm{HF}_*^{(-\infty, c_{f,\lambda})}(f;\alpha) \to \mathrm{H}_*(\mathcal{L}_\alpha^{\lambda^2/2} M), \qquad c_{f,\lambda} := rf'(r) - f(r).$$

If g is another such function, then there exists an isomorphism $[\Phi^{gf,\lambda}]$ such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{HF}_*^{(-\infty,c_{f,\lambda})}(f;\alpha) \arrow[rr, "{[\Phi^{gf,\lambda}]}", "\simeq"'] \arrow[dr, "\simeq", "\Psi^{f,\lambda}"'] & & \mathrm{HF}_*^{(-\infty,c_{g,\lambda})}(g;\alpha) \arrow[dl, "\Psi^{g,\lambda}", "\simeq"'] \\
& \mathrm{H}_*(\mathcal{L}_\alpha^{\lambda^2/2} M) &
\end{tikzcd} \quad . \tag{22}$$

If $\mu \in (0, \lambda] \setminus \Lambda_\alpha$ is another noncritical slope of f, then the following diagram commutes

$$\begin{tikzcd}
\mathrm{HF}_*^{(-\infty,c_{f,\mu})}(f;\alpha) \arrow[r, "{[\iota^F]}"] \arrow[d, "\Psi^{f,\mu}"', "\simeq"] & \mathrm{HF}_*^{(-\infty,c_{f,\lambda})}(f;\alpha) \arrow[d, "\Psi^{f,\lambda}", "\simeq"'] \\
\mathrm{H}_*(\mathcal{L}_\alpha^{\mu^2/2} M) \arrow[r, "{[\iota_{\frac{\lambda^2}{2}\frac{\mu^2}{2}}]}"] & \mathrm{H}_*(\mathcal{L}_\alpha^{\lambda^2/2} M)
\end{tikzcd} \quad . \tag{23}$$

The proof of the theorem is based on the main result in [19], namely existence of a natural isomorphism $\mathrm{HF}_*^{(-\infty,a)}(\frac{1}{2}|y|^2;\alpha) \simeq \mathrm{H}_*(\mathcal{L}_\alpha^a M)$ whenever a is a regular value of $\mathcal{S}_V$.

Proof. The main part of the proof is to establish the vertical homomorphisms in diagram (24), show that they are actually isomorphisms and prove that the diagram commutes.

$$
\begin{tikzcd}
\mathrm{HF}_*^{(-\infty,c_{f,\mu})}(f+F;\alpha) \arrow[r, "{[\iota^F]}"] \arrow[d] & \mathrm{HF}_*^{(-\infty,c_{f,\lambda})}(f+F;\alpha) \arrow[d, "{[\mathrm{id}]}"] \\
\mathrm{HF}_*^{(-\infty,\infty)}(f^{(\mu)}+\widetilde{F};\alpha) \arrow[r, "{[\iota^F]}", "{[\Phi]}"'] \arrow[d] & \mathrm{HF}_*^{(-\infty,\infty)}(f^{(\lambda)}+F;\alpha) \arrow[d, "{[\Phi]}"] \\
\mathrm{HF}_*^{(-\infty,\infty)}(f_0^{(\mu)}+\widetilde{W};\alpha) \arrow[r, "{[\Phi]}", "{[\iota^F]}"'] \arrow[d] & \mathrm{HF}_*^{(-\infty,\infty)}(f_0^{(\lambda)}+W;\alpha) \arrow[d, "{[\mathrm{id}]}"] \\
\mathrm{HF}_*^{(-\infty,\mu^2/2)}(f_0+W;\alpha) \arrow[r, "{[\iota^F]}"] \arrow[d] & \mathrm{HF}_*^{(-\infty,\lambda^2/2)}(f_0+W;\alpha) \arrow[d, "{[\Phi]}"] \\
\mathrm{HF}_*^{(-\infty,\mu^2/2)}(f_0,\mathcal{V};\alpha) \arrow[r, "{[\iota^F]}"] \arrow[d] & \mathrm{HF}_*^{(-\infty,\lambda^2/2)}(f_0,\mathcal{V};\alpha) \arrow[d, "{[19,\ \text{Section } 11]}"] \\
\mathrm{H}_*(\mathcal{L}_\alpha^{\mu^2/2}M) \arrow[r, "{[\iota]}"] & \mathrm{H}_*(\mathcal{L}_\alpha^{\lambda^2/2}M)
\end{tikzcd}
\tag{24}
$$

Here $f_0(|y|) := |y|^2/2$ and F and W are regular perturbations. The abstract perturbations $\mathcal{V}$ were introduced in [19] and are discussed further below.

Once the above is achieved we define $\Psi^{f,\lambda}$ to be the composition of the vertical maps on the right hand side of diagram (24). It follows that diagram (23) commutes. Definition of $\Phi^{gf,\lambda}$ is indicated in the following diagram

$$
\begin{tikzcd}
\mathrm{HF}_*^{(-\infty,c_{f,\lambda})}(f+F;\alpha) \arrow[r, "{[\Phi^{gf,\lambda}]:=}", "{[id_g{}^{-1}\circ\Phi\circ id_f]}"'] \arrow[d, "{[id_f]}"'] & \mathrm{HF}_*^{(-\infty,c_{g,\lambda})}(g+K;\alpha) \arrow[d, "{[id_g]}"] \\
\mathrm{HF}_*^{(-\infty,\infty)}(f^{(\lambda)}+F;\alpha) \arrow[r, "{[\Phi]}"] \arrow[d, "{[\Phi]}"'] & \mathrm{HF}_*^{(-\infty,\infty)}(g^{(\lambda)}+K;\alpha) \arrow[dl, "{[\Phi]}"] \\
\mathrm{HF}_*^{(-\infty,\infty)}(f_0^{(\lambda)}+W;\alpha) &
\end{tikzcd}
.
$$

Here Φ denotes Floer's continuation map. Since all Hamiltonians are outside a compact set radial of noncritical constant slope λ, the maps Φ induce isomorphisms on homology which do commute (e.g. see [18, Theorem 3.7]). Together with (24) this proves that diagram (22) commutes.

It remains to establish the vertical isomorphisms on the right hand side (thereby the ones on the left hand side) of diagram (24). Let $\tilde{f}^{(\lambda)}$ be given by following the graph of f until slope λ turns up for the first time, say at a point $R-1$, then continue linearly with slope λ. Since this function is only C^1, we approximate it by a smooth convex function $f^{(\lambda)}$ such that both functions coincide outside a small neighbourhood of $R-1$. Note that $R = R(f,\lambda)$.

ISOMORPHISM 1. By definition

$$\mathrm{HF}_*^{(-\infty,c_{f,\lambda})}(f;\alpha) := \mathrm{HF}_*^{(-\infty,c_{f,\lambda})}(f+F;\alpha),$$

where $F \in \mathcal{H}_{reg}(f)$ is a small perturbation of compact support in $S^1 \times D_R T^*M$ as in (B3). By (B1) all 1-periodic orbits of $f+F$ of action less than $c_{f,\lambda}$ and all their connecting trajectories take values in $D_R T^*M$. The same is true for the Hamiltonian $f^{(\lambda)}+F$ without any action restriction. But in $D_R T^*M$ both Hamiltonians coincide, hence both chain complexes are identical.

ISOMORPHISM 2. The second vertical map in (24) is induced by Floer's continuation map $\Phi(H_s)$ associated to any homotopy H_s arising from a convex combination of $f^{(\lambda)}$ and $f_0^{(\lambda)}$. Note that we are not restricted to monotone homotopies, since the action window is the whole real line. Outside a compact set both functions are of slope λ, hence in this domain we have $H_s(t,x,y) = f_s(|y|)$ with $\partial_s f_s' = 0$ and $\partial_s f_{-s}' = 0$. Therefore Proposition 2.3 applies to H_s and to H_{-s} and consequently $\Phi(H_s)$ and $\Phi(H_{-s})$ can be defined. Since any two convex combinations are homotopic through functions whose slope equals λ outside a compact set, the standard argument to prove independence of $[\Phi(H_s)]$ on H_s by studying homotopies of homotopies goes through. The composition rule for continuation maps with respect to concatenation of homotopies then implies $[\Phi(H_{-s})] = [\Phi(H_s)]^{-1}$.

ISOMORPHISM 3. The same argument as for isomorphism 1 applies.

ISOMORPHISM 4. The fourth vertical map is again Floer's continuation map. However, its definition and the proof that it is an isomorphism require new arguments. Floer homology as in line five of diagram (24) has been introduced in [19]. Here $f_0(|y|) = \frac{1}{2}|y|^2$ and the abstract perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ is required to satisfy five axioms (V0-V4) for which we refer the reader to [19, Section 2]. However, since we will use it below, let us recall the first axiom:

(V0) The perturbation $\mathcal{V}$ is continuous with respect to the C^0 topology on $\mathcal{L}M$. Moreover, there is a constant $C>0$ such that

$$\sup_{x\in\mathcal{L}M} |\mathcal{V}(x)| + \sup_{x\in\mathcal{L}M} \|\operatorname{grad}\mathcal{V}(x)\|_{L^\infty(S^1)} \le C.$$

This class of perturbations is rich enough such that the Morse-Smale transversality condition is satisfied generically. The special case $\mathcal{V}(x) = \int_0^1 V(t,x(t))\,dt$ where $V \in C^\infty(S^1 \times M)$ reproduces standard Floer theory associated to the Hamiltonian $H_V(t,x,y) = |y|^2/2 + V(t,x)$. The functionals $\mathcal{S}_\mathcal{V}$ and $\mathcal{A}_\mathcal{V}$ are obtained by replacing $\int_0^1 V(t,x(t))\,dt$ by $\mathcal{V}(x)$ in $\mathcal{S}_V$ and $\mathcal{A}_{H_V}$, respectively. The critical points of $\mathcal{S}_\mathcal{V}$ are given by

$$\mathcal{P}(\mathcal{V}) = \{x \in \mathcal{L}M \mid -\nabla_t\partial_t x - \operatorname{grad}\mathcal{V}(x) = 0\}.$$

They are identified with the critical points of $\mathcal{A}_{\mathcal{V}}$ by the map $x \mapsto (x, g(x)\partial_t x)$. Here grad $\mathcal{V}$ denotes the L^2 gradient of $\mathcal{V}$ which is defined by

$$\int_0^1 \langle \operatorname{grad} \mathcal{V}(u), \partial_s u \rangle dt := \frac{d}{ds}\mathcal{V}(u)$$

for every smooth path $\mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$. It is important to observe that on critical points both functionals coincide.

Let $\lambda' > \lambda$ be sufficiently small such that $(\lambda, \lambda') \cap \Lambda_\alpha = \emptyset$. Given any $(x, y) \in \mathcal{P}^{(-\infty, \lambda^2/2)}(\mathcal{V}; \alpha)$, we claim that (x, y) takes values in $D_{\lambda'}T^*M$ whenever $\mathcal{V}$ is sufficiently C^1-small. We recall the argument given in [26, App. A]. The first observation is that $\mathcal{S}_{\mathcal{V}}(x) < \lambda^2/2$ implies

$$\|\dot{x}\|_2^2 < \lambda^2 + 2\,\|\mathcal{V}\|_{L^\infty(\mathcal{L}M)}\,. \tag{25}$$

Secondly, use $-\nabla_t \dot{x} - \operatorname{grad} \mathcal{V}(x) = 0$ to get the pointwise identity

$$\frac{d}{d\tau}|\dot{x}(\tau)|^2 = -2\langle (\operatorname{grad} \mathcal{V}(x))(\tau), \dot{x}(\tau)\rangle.$$

Integrate this identity from t_0 to $t \in [t_0, t_0 + 1]$ to obtain

$$\begin{aligned} |\dot{x}(t_0)|^2 &= |\dot{x}(t)|^2 + 2\int_{t_0}^{t} \langle (\operatorname{grad} \mathcal{V}(x))(\tau), \dot{x}(\tau)\rangle \, d\tau \\ &\le |\dot{x}(t)|^2 + 2\int_{t_0}^{t_0+1} \varepsilon^{-1}\, |(\operatorname{grad} \mathcal{V}(x))(\tau)| \cdot \varepsilon\, |x(\tau)| \; d\tau \\ &\le |\dot{x}(t)|^2 + \varepsilon^{-2} \sup_{x\in\mathcal{L}M} \|\operatorname{grad} \mathcal{V}(x)\|_{L^\infty(S^1)}^2 + \varepsilon^2\,\|\dot{x}\|_2^2\,. \end{aligned}$$

Integrate this inequality over the interval $[0, 1]$ and take the supremum over $t_0 \in S^1$ to obtain for every $\varepsilon > 0$ the estimate

$$\|\dot{x}\|_\infty^2 \le (1 + \varepsilon^2)\,\|\dot{x}\|_2^2 + \varepsilon^{-2} \sup_{x\in\mathcal{L}M} \|\operatorname{grad} \mathcal{V}(x)\|_{L^\infty(S^1)}^2\,. \tag{26}$$

Combining (25) and (26) shows that we can achieve $\|y\|_\infty = \|\dot{x}\|_\infty \le \lambda'$ by choosing $\mathcal{V}$ sufficiently C^1-small and this proves the claim.

By (B1) the elements of $\mathcal{P}^{(-\infty, \lambda^2/2)}(f_0 + W; \alpha)$ also take values in $D_{\lambda'}T^*M$ whenever the regular Hamiltonian perturbation W is sufficiently close to zero. From now on assume that the abstract perturbation $\mathcal{V}$ is regular as well.

We aim to define the continuation homomorphism Φ by counting the solutions $w = (u, v) : \mathbb{R} \times S^1 \to TM$ of the parameter dependent Floer equation

$$\begin{aligned} \partial_s u - \nabla_t v &= \beta(s)\operatorname{grad} \mathcal{V}(u) + (1 - \beta(s))\nabla_1 W_t(u, v) \\ \nabla_s v + \partial_t u &= v + (1 - \beta(s))\nabla_2 W_t(u, v) \end{aligned} \tag{27}$$

for which the following limits exist uniformly in $t \in S^1$

$$\lim_{s\to\pm\infty} w(s, t) = z^\pm(t), \qquad \lim_{s\to\pm\infty} \partial_s w(s, t) = 0. \tag{28}$$

Here $z^- \in \mathcal{P}^{(-\infty,\lambda^2/2)}(f_0+W;\alpha)$ and $z^+ \in \mathcal{P}^{(-\infty,\lambda^2/2)}(f_0,\mathcal{V};\alpha)$ are supposed to have equal index, the cutoff function β is as in (11) and we identified TM and T^*M using the Riemannian metric on M. Moreover, the gradients $\nabla_1 W_t(u,v)$ and $\nabla_2 W_t(u,v)$ appearing in (27) are defined by the identity

$$\frac{d}{d\tau}W_t(x,\xi) =: \langle \nabla_1 W_t(x,\xi), \partial_s x\rangle + \langle \nabla_2 W_t(x,\xi), \nabla_s \xi\rangle,$$

for every path $\mathbb{R} \to TM : \tau \mapsto (x(\tau),\xi(\tau))$.

To define Φ we have to check if conditions (C1-C3) are satisfied. Here only the apriori bound in (C1) is nonstandard. Define $\mathcal{A}_s$ on $\mathcal{L}T^*M$ by

$$\mathcal{A}_s(x,y) := \int_0^1 \left(\langle y, \dot{x}\rangle - \frac{1}{2}|y|^2 - (1-\beta(s))W_t(x,y)\right) dt - \beta(s)\mathcal{V}(x)$$

and consider the identity

$$\frac{d}{ds}\mathcal{A}_s(u,v) = -\|\partial_s u\|^2_{L^2(S^1)} - \|\nabla_s v\|^2_{L^2(S^1)} + \beta'(s)\left(\int_0^1 W_t(u,v)\,dt - \mathcal{V}(u)\right)$$

for solutions (u,v) of (27). It implies the energy identity

$$E(u,v) = \mathcal{A}_W(z^-) - \mathcal{A}_{\mathcal{V}}(z^+) + \int_{-\infty}^{\infty} \beta'(s)\left(\int_0^1 W_t(u,v)\,dt - \mathcal{V}(u)\right) ds$$

for solutions (u,v) of (27) and (28). It is a consequence of this last two identities that Proposition 2.10 applies with $c_0 = {\lambda'}^2/2 + 2\|W\|_\infty + 2\sup_{x\in\mathcal{L}M}|\mathcal{V}(x)|$. It provides a uniform bound $R>0$ for the absolute value of the fibre component of all relevant continuation trajectories. Moreover, the argument in [2, Section 4.4]) shows that Φ is indeed a chain map with respect to the given action window whenever the perturbations $\mathcal{V}$ and W are sufficiently C^0-small. Here the energy identity enters again. The usual reverse homotopy argument then proves that $[\Phi]$ is an isomorphism.

Isomorphism 5. This is the main result of [19].

Commutativity. To show that diagram (24) commutes observe first that the vertical maps on the left hand side of (24) are the analogues of the ones on the right hand side. The perturbations on the left (marked by ˜) are restrictions of the ones on the right hand side. The first five horizontal maps can be viewed as being induced by inclusion of subcomplexes. Here convexity of the unperturbed Hamiltonians enters. Blocks one, three and four in diagram (24) commute already on the chain level. Horizontal maps two and three can alternatively be interpreted as being induced by continuation maps associated to monotone homotopies. Note that, due to the fact that orbits could enter from infinity during the homotopy, the induced maps on homology are in general not isomorphisms. However, continuation maps satisfy the obvious composition rule for catenations of homotopies (see [18, Lemmata 3.11 and 3.12]) and this implies commutativity of block two. Block five commutes by [19]. This concludes the proof of Theorem 2.9. □

Proposition 2.10 (Apriori C^0-bound). *Fix a constant $c_0 > 0$, a perturbation $\mathcal{V} : \mathcal{L}M \to \mathbb{R}$ which satisfies (V0) and [19, (V1)], and a perturbation $W \in C_0^\infty(\mathbb{R} \times S^1 \times D_\lambda T^*M)$ for some $\lambda > 0$. Then there is a constant $C > 0$ depending continuously on c_0, the C^2-norm of W and the constant appearing in (V0–V1) such that the following holds. If $(u, v) : \mathbb{R} \times S^1 \to T^*M$ solves (27) and*

$$E(u,v) \leq c_0, \qquad \sup_{s \in \mathbb{R}} \mathcal{A}_s(u(s,\cdot), v(s,\cdot)) \leq c_0$$

then $\|v\|_\infty \leq C$.

The proof of the Proposition follows line by line the proof of the apriori estimate [19, Thm. 5.1] in case $\varepsilon = 1$.

Remark 2.11. An apriori bound for s-independent Hamiltonians was obtained by Cieliebak [4, Theorem 5.4]. A different technique based on parabolic mean value inequalities was developed by Salamon and the present author in [19]. There is yet another approach towards C^0-bounds by Oancea [17] which requires $\partial_s H_s \geq 0$. Moreover, the assumptions of Proposition 2.3 fit into the framework of [17], which provides some uniform C^0-bound c. In contrast, Proposition 2.3 provides the specific value $c = r$, which is important for our purposes.

3 Symplectic and relative symplectic homology

3.1 Definition

To define symplectic homology [8, 6] of DT^*M and relative symplectic homology [2] of (DT^*M, M) we need to introduce inverse and direct limits, respectively. We shall discuss these notions only briefly and refer to [2, Section 4.6] for more details. The set $\mathcal{H} := C_0^\infty(S^1 \times DT^*M)$ is partially ordered by

$$H_0 \preceq H_1 \qquad :\Leftrightarrow \qquad H_0(t,z) \leq H_1(t,z) \quad \forall (t,z) \in S^1 \times DT^*M.$$

Fix $\alpha \in \tilde{\pi}_1(M)$, two numbers $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$ and let $\mathcal{H}_\alpha^{a,b}$ denote the set of $H \in \mathcal{H}$ such that $a, b \notin Spec(H; \alpha)$. The monotone homomorphisms $\sigma^{H_1 H_0}$ of Section 2 give rise to the *partially ordered system* (HF, σ) *of $\mathbb{Z}_2$-vector spaces over $\mathcal{H}_\alpha^{a,b}$*. By definition this means that HF assigns to each $H \in \mathcal{H}_\alpha^{a,b}$ the $\mathbb{Z}_2$-vector space ${\mathrm{HF}_*}^{(a,b)}(H; \alpha)$ and σ assigns to any two elements $H_0 \preceq H_1$ of $\mathcal{H}_\alpha^{a,b}$ the monotone homomorphism $\sigma^{H_1 H_0}$ satisfying the composition rule (18).

The partially ordered set $(\mathcal{H}_\alpha^{a,b}, \preceq)$ is *downward directed*: for all $H_1, H_2 \in \mathcal{H}_\alpha^{a,b}$ there exists $H_0 \in \mathcal{H}_\alpha^{a,b}$ such that $H_0 \preceq H_1$ and $H_0 \preceq H_2$. The functor (HF, σ) is an *inverse system of $\mathbb{Z}_2$-vector spaces over $\mathcal{H}_\alpha^{a,b}$*. Its *inverse limit* is called *symplectic homology of DT^*M with respect to α* and defined by

$$\underleftarrow{\mathrm{SH}}_*^{(a,b)}(DT^*M; \alpha) := \varprojlim_{H \in \mathcal{H}_\alpha^{a,b}} \mathrm{HF}_*^{(a,b)}(H; \alpha)$$

$$:= \left\{ \{a_H\}_{H \in \mathcal{H}_\alpha^{a,b}} \in \prod_{H \in \mathcal{H}_\alpha^{a,b}} \mathrm{HF}_*^{(a,b)}(H; \alpha) \;\middle|\; H_0 \preceq H_1 \Rightarrow \sigma^{H_1 H_0}(a_{H_0}) = a_{H_1} \right\}.$$

For $H \in \mathcal{H}_\alpha^{a,b}$ let

$$\pi_H : \underleftarrow{\mathrm{SH}}_*^{(a,b)}(DT^*M;\alpha) \to \mathrm{HF}_*^{(a,b)}(H;\alpha) \tag{29}$$

be the projection to the component corresponding to H. Then $\pi_{H_1} = \sigma^{H_1H_0} \circ \pi_{H_0}$, whenever $H_0 \preceq H_1$.

To define *relative* symplectic homology fix $c > 0$ and consider the subset

$$\mathcal{H}_{\alpha;c}^{a,b} := \left\{ H \in \mathcal{H}_\alpha^{a,b} \mid \sup_{S^1 \times M} H \leq -c \right\}.$$

This set is upward directed: for all $H_0, H_1 \in \mathcal{H}_{\alpha;c}^{a,b}$ there exists $H_2 \in \mathcal{H}_{\alpha;c}^{a,b}$ such that $H_0 \preceq H_2$ and $H_1 \preceq H_2$. The functor (HF, σ) is a *direct system of* $\mathbb{Z}_2$*-vector spaces over* $\mathcal{H}_{\alpha;c}^{a,b}$. Its *direct limit* is called *relative symplectic homology of the pair* (DT^*M, M) *with respect to* α *and the bound* c. It is defined as the quotient

$$\begin{aligned} \underrightarrow{\mathrm{SH}}_*^{(a,b);c}(DT^*M, M;\alpha) &:= \varinjlim_{H \in \mathcal{H}_{\alpha;c}^{a,b}} \mathrm{HF}_*^{(a,b)}(H;\alpha) \\ &:= \left\{ (H, a_H) \;\middle|\; H \in \mathcal{H}_{\alpha;c}^{a,b},\ a_H \in \mathrm{HF}_*^{(a,b)}(H;\alpha) \right\} \Big/ \sim \end{aligned}$$

where $(H_0, a_0) \sim (H_1, a_1)$ iff there exists $H_2 \in \mathcal{H}_{\alpha;c}^{a,b}$ such that $H_0 \preceq H_2$, $H_1 \preceq H_2$ and $\sigma^{H_2H_0}(a_0) = \sigma^{H_2H_1}(a_1)$. This is an equivalence relation, since $\mathcal{H}_{\alpha;c}^{a,b}$ is upward directed. The direct limit is a $\mathbb{Z}_2$-vector space with the operations

$$k[H_0, a_0] := [H_0, ka_0], \quad [H_0, a_0] + [H_1, a_1] := [H_2, \sigma^{20}(a_0) + \sigma^{21}(a_1)],$$

for all $k \in \mathbb{Z}_2$ and $H_2 \in \mathcal{H}_{\alpha;c}^{a,b}$ with $H_0 \preceq H_2$ and $H_1 \preceq H_2$. For $H \in \mathcal{H}_{\alpha;c}^{a,b}$ define the homomorphism

$$\iota_H : \mathrm{HF}_*^{(a,b)}(H;\alpha) \to \underrightarrow{\mathrm{SH}}_*^{(a,b);c}(DT^*M, M;\alpha), \qquad a_H \mapsto [H, a_H]. \tag{30}$$

It satisfies $\iota_{H_0} = \iota_{H_1} \circ \sigma^{10}$, whenever $H_0 \preceq H_1$.

The main feature of symplectic and relative symplectic homology for our purposes is the existence of a unique homomorphism between them which factors through Floer homology (see [2, Proposition 4.8.2]). This means that

$$\begin{tikzcd} & \mathrm{HF}_*^{(a,b)}(H;\alpha) \arrow[dr, "\iota_H"] & \\ \underleftarrow{\mathrm{SH}}_*^{(a,b)}(DT^*M;\alpha) \arrow[ur, "\pi_H"] \arrow[rr, "T_{\alpha;c}^{(a,b)}"] & & \underrightarrow{\mathrm{SH}}_*^{(a,b);c}(DT^*M, M;\alpha) \end{tikzcd} \tag{31}$$

commutes for every $H \in \mathcal{H}_{\alpha;c}^{a,b}$. Here the homomorphism is given by

$$T_{\alpha;c}^{(a,b)} : \{a_K\}_{K \in \mathcal{H}_\alpha^{a,b}} \mapsto [F, a_F], \tag{32}$$

where F is any element of $\mathcal{H}_{\alpha;c}^{a,b}$ (and $T_{\alpha;c}^{(a,b)}$ is independent of this choice).

3.2 Computation

Theorem 3.1. *Let α be a homotopy class of free loops in M, then*
(i) for every $a \in \mathbb{R} \setminus \Lambda_\alpha$ (with $a > 0$ if $\alpha = 0$), there is a natural isomorphism

$$\underleftarrow{\mathrm{SH}}_*^{(a,\infty)}(DT^*M;\alpha) \simeq \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2}M).$$

(ii) for all real numbers $a, c > 0$, there is a natural isomorphism

$$\underrightarrow{\mathrm{SH}}_*^{(a,\infty);c}(DT^*M, M;\alpha) \simeq \begin{cases} \mathrm{H}_*(\mathcal{L}_\alpha M), & \text{if } a \in (0,c], \\ 0, & \text{if } a > c. \end{cases}$$

(iii) for every $a \in (0,c] \setminus \Lambda_\alpha$, the following diagram commutes

$$\begin{tikzcd} \underrightarrow{\mathrm{SH}}_*^{(a,\infty);c}(DT^*M,M;\alpha) \arrow[r, "(ii)", "\simeq"'] & \mathrm{H}_*(\mathcal{L}_\alpha M) \\ \underleftarrow{\mathrm{SH}}_*^{(a,\infty)}(DT^*M;\alpha) \arrow[r, "(i)", "\simeq"'] \arrow[u, "T_{\alpha;c}^{(a,\infty)}"] & \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2}M) \arrow[u, "{[\iota_{a^2/2}]}"'] \end{tikzcd} .$$

Remark 3.2. 1) Assumption $a > 0$ in Theorem 3.1 (i) case $\alpha = 0$ avoids taking into account the trivial 1-periodic orbits near the boundary of DT^*M. Their existence is caused by the compact support condition for the Hamiltonians and their symplectic action is zero.

2) Assumption $a > 0$ in part (ii) is to avoid the elements of $\mathcal{P}^-(f;\alpha)$ (see Remark 2.1) whose symplectic action is negative.

3) The homomorphism T in (iii) is nonzero, if and only if $a \in [\ell_\alpha, c] \setminus \Lambda_\alpha$.

In the proof of Theorem 3.1 we use the following two lemmata.

Lemma 3.3. *Let $\alpha \in \tilde{\pi}_1(M)$. The marked length spectrum $\Lambda_\alpha \subset \mathbb{R}$ is a closed and nowhere dense subset. Equivalently, its complement is open and dense.*

Proof. The set Λ_α is closed: Let $\{\lambda_j\}_{j\in\mathbb{N}} \subset \Lambda_\alpha$ be such that $\lambda := \lim_{j\to\infty} \lambda_j$ exists. For each length λ_j choose a periodic geodesic x_j in the class α of length λ_j. Since $|\dot{x}_j(t)| = \lambda_j$, $\forall t \in S^1$, there exists a compact subset $K \subset TM$ such that $(q_j, v_j) := (x_j(0), \dot{x}_j(0)) \in K$, for every $j \in \mathbb{N}$. Passing to a subsequence, if necessary, we may assume without loss of generality that the limit

$$(q,v) := \lim_{j\to\infty}(q_j, v_j)$$

exists. Let $x : [0,1] \to M$ be the solution of the initial value problem $\nabla_t \dot{x} \equiv 0$ with $x(0) = q$ and $\dot{x}(0) = v$. It follows that x is periodic:

$$(x(0), \dot{x}(0)) = \lim_{j\to\infty}(x_j(0), \dot{x}_j(0)) = \lim_{j\to\infty}(x_j(1), \dot{x}_j(1)) = (x(1), \dot{x}(1)).$$

The last step uses the fact that the solution to a second order ordinary differential equation depends continuously on the initial values. It remains to show

$[x]=\alpha$: Since x_j converges to x uniformly in $t \in S^1$ and the injectivity radius of M is positive, it follows that x_j is homotopic to x, for all sufficiently large j.

The set Λ_α does not contain intervals: It suffices to prove this property for the set Σ_α of critical values of the classical action $\mathcal{S}_0 : \mathcal{L}_\alpha M \to \mathbb{R}$, since $\Lambda_\alpha = \{\sqrt{2\sigma} \mid \sigma \in \Sigma_\alpha\}$. Fix $a \in \mathbb{R}^+$ and consider the space $L^a_\alpha M$ of piecewise smooth maps $x : S^1 \to M$ such that $[x]=\alpha$ and $\mathcal{S}_0(x) \le a$. For some sufficiently large $N = N(a) \in \mathbb{N}$, there is a smooth finite dimensional manifold $P^N_\alpha M$ (the set of geodesic polygons; see [3]) such that the $\mathcal{S}_0$ restricted to $P^N_\alpha M$ is smooth and the critical points of the three functionals obtained by restricting $\mathcal{S}_0$ to each of the three domains

$$P^N_\alpha M \subset L^a_\alpha M \supset \mathcal{L}^a_\alpha M$$

coincide up to natural identification. Hence $\Sigma_\alpha \cap \{\mathcal{S}_0(x) \le a\}$ equals the set of critical values of $\mathcal{S}_0 : P^N_\alpha M \to \mathbb{R}$. The latter set is of measure zero by the Theorem of Sard [21].

The set Λ_α is nowhere dense in $\mathbb{R}$: By definition of *nowhere dense* we have to show that every nonempty open set $U \subset \mathbb{R}$ contains a nonempty open set V such that $V \cap \Lambda_\alpha = \emptyset$. Choose any such U and an open interval $I \subset \mathbb{R}$ contained in U. Since Λ_α does not contain intervals, we can find an element $s \in I$ such that $s \notin \Lambda_\alpha$. Because Λ_α is closed, it is possible to choose an open neighbourhood V of s in I which is disjoint from Λ_α. □

Lemma 3.4. *For every $\alpha \in \tilde{\pi}_1(M)$, there is a natural isomorphism*

$$\varinjlim_{a \in \mathbb{R}^+_0} \mathrm{H}_*(\mathcal{L}^a_\alpha M) \to \mathrm{H}_*(\mathcal{L}_\alpha M), \qquad [a, x_a] \mapsto [\iota_a](x_a).$$

Proof. Recall that ι_a denotes the natural inclusion $\mathcal{L}^a_\alpha M \hookrightarrow \mathcal{L}_\alpha M$ and the direct limit is defined by

$$\varinjlim_{a \in \mathbb{R}^+_0} \mathrm{H}_*(\mathcal{L}^a_\alpha M) := \{(a, x_a) \mid a \ge 0,\ x_a \in \mathrm{H}_*(\mathcal{L}^a_\alpha M)\}/\sim,$$

where $(a, x_a) \sim (b, x_b)$ iff there exists $c \ge \max\{a,b\}$ with $[\iota_{ca}](x_a) = [\iota_{cb}](x_b)$.

(i) well defined: Assume $(a, x_a) \sim (b, x_b)$, i.e. $[\iota_{ca}](x_a) = [\iota_{cb}](x_b)$ for some $c \ge \max\{a,b\}$. Apply $[\iota_c]$ to both sides to obtain $[\iota_a](x_a) = [\iota_b](x_b)$.

(ii) surjective: Represent $x \in \mathrm{H}_*(\mathcal{L}_\alpha M)$ by a cycle $f(C)$. Here C is the total space of a closed simplicial complex and $f : C \to \mathcal{L}_\alpha M$ is a continuous map. Let $\mathcal{L}_\alpha M$ be equipped with the $W^{1,2}$-metric, then $\mathcal{S}_0$ given by (21) is continuous (see [15, section 5.4]). Since $f(C)$ is compact and $\mathcal{S}_0$ is continuous, we obtain the real number $a := 1 + \max_{f(C)} \mathcal{S}_0$. Now $f(C) \subset \mathcal{L}^a_\alpha M$ shows $x_a := [f(C)] \in \mathrm{H}_*(\mathcal{L}^a_\alpha M)$.

(iii) injective: Assume $[\iota_a](x_a) = 0 \in \mathrm{H}_*(\mathcal{L}_\alpha M)$. We need to show that $[a, x_a]$ is the zero element of the direct limit, i.e. that there exists $b \ge a$ such that $[\iota_{ba}](x_a) = 0 \in \mathrm{H}_*(\mathcal{L}^b_\alpha M)$. Choose a cycle $f_a : C \to \mathcal{L}^a_\alpha M$ representing x_a. By assumption there is a cycle $F : W \to \mathcal{L}_\alpha M$, where W is the total space of a simplicial complex with $\partial W = C$ and F is a continuous map which coincides with f_a on $\partial W = C$. Define $b := 1+\max_{F(W)} \mathcal{S}_0 \geq a$ and $F_b := F : W \to \mathcal{L}^b_\alpha M$. Then $F_b(p) = F(p) = f_a(p) = (\iota_{ba} \circ f_a)(p)$, for every $p \in \partial W = C$. Hence $[\iota_{ba}](x_a) = [(\iota_{ba} \circ f_a)(C)] = [F_b(C)] = 0$. □

Proof of Theorem 3.1. Fix a homotopy class α of free loops in M. We prove Theorem 3.1 in five steps. Part (i) is equivalent to

Step 1. (a) *For every $a \in \mathbb{R}^+ \setminus \Lambda_\alpha$, there is a natural isomorphism*

$$\underleftarrow{\mathrm{SH}}_*^{(a,\infty)}(DT^*M;\alpha) \simeq \mathrm{H}_*(\mathcal{L}^{a^2/2}_\alpha M).$$

(b) *If $\alpha \neq 0$ and $a \leq 0$, then $\underleftarrow{\mathrm{SH}}_*^{(a,\infty)}(DT^*M;\alpha) = 0$.*

Proof. (a) Fix $a \in \mathbb{R}^+ \setminus \lambda_\alpha$ and consider the infimum of all geodesic lengths strictly greater than a, namely

$$\ell_+ := \inf\left(\Lambda_\alpha \cap (a,\infty)\right). \tag{33}$$

We use the convention that $\inf \emptyset = \infty$. It follows $a < \ell_+ \in \Lambda_\alpha \cup \{\infty\}$ by Lemma 3.3.

Consider the sequence of linear functions T_k (of increasing slopes) through the points $(r_{k1}, -a)$ and $(r_{k2}, 0)$, where

$$\begin{aligned} r_{k1} = r_{k,1;a,\ell_+} &:= 1 - \frac{1}{k} + \frac{3}{16k}\left(1 - \frac{a}{\ell_+}\right), \\ r_{k2} = r_{k,2;a,\ell_+} &:= 1 - \frac{3}{16k}\left(1 - \frac{a}{\ell_+}\right), \end{aligned}$$

and

$$T_k(r) = T_{k;a,\ell_+}(r) := ak\frac{r - 1 + \frac{3}{16k}\left(1-\frac{a}{\ell_+}\right)}{1 - \frac{3}{8}\left(1 - \frac{a}{\ell_+}\right)}, \qquad k \in \mathbb{N}.$$

Note that $0 < r_{k1} < r_{k2} < 1$, for all $k \in \mathbb{N}$, and that both numbers converge to $+1$ as $k \to \infty$. Moreover, the slope m_1 of T_1 satisfies

$$a < m_1 = \frac{a}{1 - \frac{3}{8}\left(1 - \frac{a}{\ell_+}\right)} < \ell_+. \tag{34}$$

Define the sequence of piecewise linear functions (see Figure 4)

$$\hat{f}_k = \hat{f}_{k;a,\ell_+}(r) := \begin{cases} -A_k := T_k(r_{k1}/2), & \text{if } r \in [0, r_{k1}/2], \\ T_k(r), & \text{if } r \in [r_{k1}/2, r_{k2}], \\ 0, & \text{if } r \geq r_{k2}. \end{cases} \tag{35}$$

We obtain a sequence of smooth functions $f_k = f_{k;a,\ell_+}$ by smoothing out $\hat{f}_k$ nearby $r_{k1}/2$ and r_{k2} subject to the following conditions. The new functions

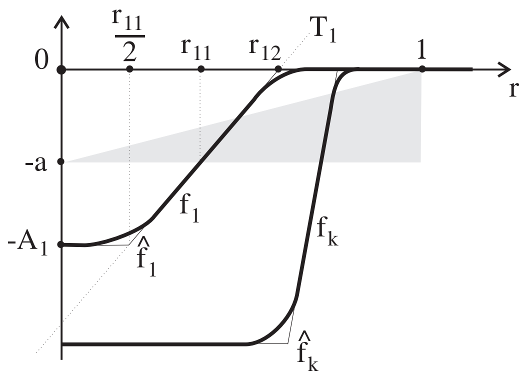

Figure 4: The functions f_k.

f_k coincide with the old functions $\hat{f}_k$ away from small neighbourhoods of $r_{k1}/2$ and r_{k2}. In particular, these neighbourhoods are chosen sufficiently small such that graph f_k = graph $\hat{f}_k$ in the region which lies below the line $r \mapsto -a + ar$ and above the line $r \mapsto -a$ (this region is grey shaded in Figure 4). Moreover, the derivatives are required to satisfy $f_k' \geq 0$ everywhere, $f_k'' \geq 0$ near $r_{k1}/2$, $f_k'' \leq 0$ near r_{k2} and $f_k'' = 0$ elsewhere.

We will prove that every function f_k gives rise to a natural isomorphism

$$\eta^{f_k,a} : \mathrm{HF}_*^{(a,\infty)}(f_k;\alpha) \to \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2}M), \qquad \forall k \in \mathbb{N}. \tag{36}$$

First we need to check that $a \notin Spec(f_k;\alpha)$: The graph of f_k admits a tangent through the point $(0,-a)$, but the slope of this tangent lies in the interval (a,ℓ_+) (see Figure 5) and is therefore noncritical.

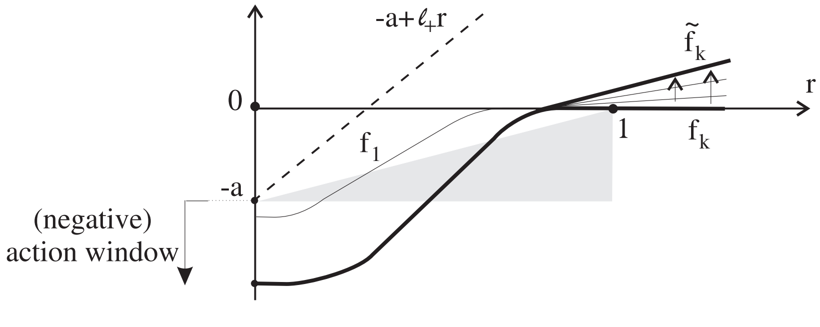

Figure 5: An action-regular monotone homotopy between f_k and $\tilde{f}_k$.

Let $\tilde{f}_k$ be the function obtained by following the graph of f_k until it takes on slope a for the second time (near r_{k2}, say at a point p), then continue linearly with slope a. The resulting function is only of class C^1 at p. Local smoothing near p yields a smooth function $\tilde{f}_k$ (see Figure 5). Figure 5 also indicates a monotone action-regular homotopy which induces the monotone isomorphism

$$\sigma^{\tilde{f}_k f_k} : \mathrm{HF}_*^{(a,\infty)}(f_k; \alpha) \to \mathrm{HF}_*^{(a,\infty)}(\tilde{f}_k; \alpha). \tag{37}$$

The homotopy is action-regular, because all points which do not remain constant during the homotopy intersect the vertical coordinate axis strictly above the point $-a$. Therefore all 1-periodic orbits arising (if any) are of action strictly less than a (see Remark 2.1; method 2).

Next consider the function ${f_k}^{(a)}$ obtained by following the graph of $\tilde{f}_k$ (likewise f_k) until it takes on slope a for the first time (say at a point q), then continue linearly with slope a (see Figure 6). Again smoothing near q yields ${f_k}^{(a)}$. Figure 6 shows a monotone homotopy between ${f_k}^{(a)}$ and $\tilde{f}_k$ which is also

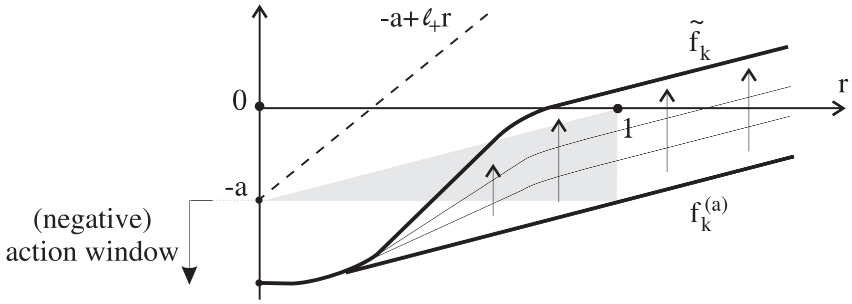

Figure 6: An action-regular monotone homotopy between ${f_k}^{(a)}$ and $\tilde{f}_k$.

action-regular: all tangents to members of the homotopy which pass through the point $(0, -a)$ lie above the ray $\mathbb{R}^+ \ni r \mapsto -a + ar$ and strictly below $-a + \ell_+ r$. Hence their slopes lie in the interval $[a, \ell_+)$ and so are noncritical. Consequently the homotopy induces a monotone isomorphism with inverse

$$\left(\sigma^{\tilde{f}_k {f_k}^{(a)}}\right)^{-1} : \mathrm{HF}_*^{(a,\infty)}(\tilde{f}_k; \alpha) \to \mathrm{HF}_*^{(a,\infty)}({f_k}^{(a)}; \alpha). \tag{38}$$

Since ${f_k}^{(a)}$ does not admit 1-periodic orbits of action less than a, the map $[\pi^F]$ in (19) in case $(a, b, c) = (-\infty, a, \infty)$ is an isomorphism with inverse

$$[\pi^F]^{-1} : \mathrm{HF}_*^{(a,\infty)}({f_k}^{(a)}; \alpha) \to \mathrm{HF}_*^{(-\infty,\infty)}({f_k}^{(a)}; \alpha). \tag{39}$$

Similarly, there is the isomorphism

$$[\iota^F]^{-1} : \mathrm{HF}_*^{(-\infty,\infty)}({f_k}^{(a)}; \alpha) \to \mathrm{HF}_*^{(-\infty,C)}({f_k}^{(a)}; \alpha), \tag{40}$$

where the constant $C = C({f_k}^{(a)}, a)$ is defined in Theorem 2.9. Here we use the assumption $a \notin \Lambda_\alpha$. Theorem 2.9 provides the isomorphism

$$\Psi^{{f_k}^{(a)}, a} : \mathrm{HF}_*^{(-\infty,C)}({f_k}^{(a)}; \alpha) \to \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2} M). \tag{41}$$

Composing (37-41) gives the desired isomorphisms (36).

To conclude the proof of part (a) of Step 1 we show that $\{f_k\}_{k\in\mathbb{N}}$ is a downward exhausting sequence (see [2, Section 4.7]) for the inverse limit defining symplectic homology. There are two properties to be checked. Firstly, given any $H \in \mathcal{H}^{a,\infty}_\alpha$, there exists f_k such that $H(t,x,y) \geq f_k(|y|)$, for every $(t,x,y) \in S^1 \times DT^*M$. This can be achieved by choosing $k \in \mathbb{N}$ sufficiently large. Secondly, for every $k \in \mathbb{N}$, the map

$$\sigma^{f_k f_{k+1}} : \mathrm{HF}^{(a,\infty)}_*(f_{k+1};\alpha) \to \mathrm{HF}^{(a,\infty)}_*(f_k;\alpha)$$

induced by a monotone homotopy between f_{k+1} and f_k is an isomorphism. To see this think of k as a continuous parameter in $[1,\infty)$ instead of a positive integer. Consider the family $\{f_r\}_{r\in[1,\infty)}$ defined analogous to $\{f_k\}_{k\in\mathbb{N}}$. Then the homotopy $\{f_{k-s}\}_{s\in[-1,0]}$ from f_{k+1} to f_k is clearly monotone and in addition action-regular: similar arguments as above show that all tangents to members of the homotopy which pass through the point $(0,-a)$ are located between the lines $-a+ar$ and $-a+\ell_+ r$. Hence their slope is in the interval $[a,\ell_+)$ which does not contain critical slopes. Consequently the boundary $+a$ of the action window remains disjoint from the action spectrum throughout the homotopy. This proves property two and therefore that the sequence is exhausting. Hence the homomorphism (29), given in the situation at hand by

$$\pi_{f_k} : \underleftarrow{\mathrm{SH}}^{(a,\infty)}_*(DT^*M;\alpha) \to \mathrm{HF}^{(a,\infty)}_*(f_k;\alpha), \tag{42}$$

is an isomorphism by [2, Lemma 4.7.1 (ii)]. This and (36) prove Step 1 (a).

(b) We follow the line of argument in [2, Pf. of Thm. 5.1.2 Step 1]. Fix $a \leq 0$ and $\alpha \neq 0$. Then the smallest length bigger than a is given by $\ell_\alpha > 0 \geq a$, where ℓ_α is defined by (2). Consider the *new* family $f_k = f_{k;-a,\ell_\alpha}$ as defined subsequent to (35). It is easy to see that all points of slope $\lambda \in \Lambda_\alpha$ on the graph of f_k (which is nonpositive on $\mathbb{R}^+$), are located strictly below the line $r \mapsto -a+\ell_\alpha r$ (which is strictly positive on $\mathbb{R}^+$ due to $-a \geq 0$ and $\ell_\alpha > 0$). Therefore they are located strictly below the line $r \mapsto -a+\lambda r$. Hence, by method 1 of Remark 2.1, all 1-periodic orbits (representing $\alpha \neq 0$) are of action strictly bigger than a. This shows that the action window (a,∞) is admissible and $Spec(f_k;\alpha) \subset (a,\infty)$. Consider the upper horizontal row in diagram (19) in case $(a,b,c) = (-\infty,a,\infty)$ and $H^0 = f_k$. It follows $[\iota^F] = 0$ and

$$[\pi^F] : \mathrm{HF}^{(a,\infty)}_*(f_k;\alpha) \to \mathrm{HF}^{(-\infty,\infty)}_*(f_k;\alpha)$$

is an isomorphism. The target is independent of the Hamiltonian, by Floer continuation, and therefore zero: every sufficiently C^2-small compactly supported Hamiltonian admits only *contractible* 1-periodic orbits. The same exhausting sequence argument as in part (a) proves (b) and therefore Step 1. □

Fix $a,c > 0$. Step 2 and Step 3 of the proof rely on a family of functions $h_\delta \in \mathcal{H}^{a,\infty}_{\alpha;c}$ defined as follows. By Lemma 3.3 there is a real number δ with

$$\delta \in \left(0,\frac{a}{c}\right), \qquad \mu_\delta := \frac{a}{\delta} - (c-a) \notin \Lambda_\alpha. \tag{43}$$

It follows $\mu_\delta > 0$. Using the conventions $\sup\emptyset = 0$ and $\inf\emptyset = \infty$ we define

$$\ell_- := \sup\left((0,\mu_\delta)\cap\Lambda_\alpha\right), \quad \ell_+ := \inf\left((\mu_\delta,\infty)\cap\Lambda_\alpha\right), \quad \ell_-' := \frac{\mu_\delta+\ell_-}{2}. \tag{44}$$

Since $\mu_\delta \notin \Lambda_\alpha$, Lemma 3.3 shows

$$0 \le \ell_- < \ell_-' < \mu_\delta < \ell_+ \le +\infty, \qquad (\ell_-,\ell_+)\cap\Lambda_\alpha = \emptyset.$$

The assumption $\delta < a/c$ ensures that the lines $-c + (a/\delta)r$ and $-a + \mu_\delta r$ intersect strictly above the r-axis. As a first approximation for h_δ consider the piecewise linear curve in $\mathbb{R}^2$ obtained by starting at the point $(0,-c)$ with slope a/δ. Upon meeting the horizontal line through $(0, \max\{0, \ell_-' - a\})$, follow this horizontal line to the right until it intersects the vertical line through $(1,0)$. Now go straight down to the point $(1,0)$ and follow the horizontal coordinate axis to $+\infty$. Smooth out this piecewise linear curve near its corners such that the result is the graph of a smooth function h_δ with $h_\delta'(0) = 0 = h_\delta'(1)$.

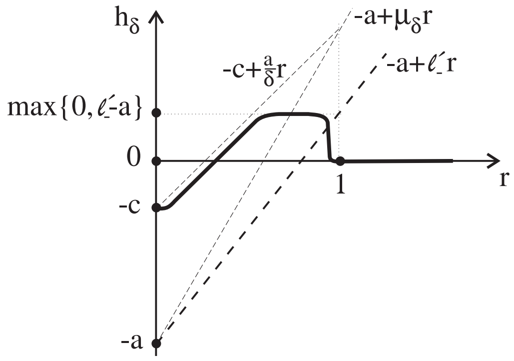

Figure 7: The function h_δ in case $c < a$.

Step 2. *If $a > c > 0$, then $\underrightarrow{\mathrm{SH}}_*^{(a,\infty);c}(DT^*M, M;\alpha) = 0$.*

Proof. Let δ be as in (43) and h_δ the associated function (see Figure 7), then

$$\mathrm{HF}_*^{(a,\infty)}(h_\delta;\alpha) = 0. \tag{45}$$

This is because the action of all 1-periodic orbits is strictly less than a: all tangents to the graph of h_δ intersect the vertical coordinate axis strictly above $-a$ (see Remark 2.1 method 2). By Lemma 3.3 there exists a strictly decreasing real sequence $\delta_k \to 0$ such that each δ_k satisfies (43) and we claim that $\{h_k := h_{\delta_k}\}_{k\in\mathbb{N}}$ is an upward exhausting sequence for the direct limit: by [2, Section 4.7] there are two properties to be checked. Firstly, given any $H \in \mathcal{H}^{a,\infty}_{\alpha;c}$, there exists $h_k \ge H$. This is clearly true. Secondly, the monotone homomorphism

$$\sigma^{h_{k+1},h_k} : \mathrm{HF}_*^{(a,\infty)}(h_k;\alpha) \to \mathrm{HF}_*^{(a,\infty)}(h_{k+1};\alpha),$$

is an isomorphism for every $k \in \mathbb{N}$: both Floer homologies are zero by (45). This proves the claim. Hence, by [2, Lemma 4.7.1 (i)], the homomorphism in (30)

$$\iota_{h_k} : \mathrm{HF}_*^{(a,\infty)}(h_k; \alpha) \to \underrightarrow{\mathrm{SH}}_*^{(a,\infty);c}(DT^*M, M; \alpha), \qquad x_{h_k} \mapsto [h_k, x_{h_k}],$$

is an isomorphism for every $k \in \mathbb{N}$ and (45) completes the proof of Step 2. $\square$

STEP 3. *Given $0 < a \leq c$ and δ satisfying (43), there is a natural isomorphism*

$$\eta^{h_\delta, a} : \mathrm{HF}_*^{(a,\infty)}(h_\delta; \alpha) \to \mathrm{H}_*(\mathcal{L}_\alpha^{\mu_\delta^2/2} M), \qquad \mu_\delta := \frac{a}{\delta} - (c-a). \tag{46}$$

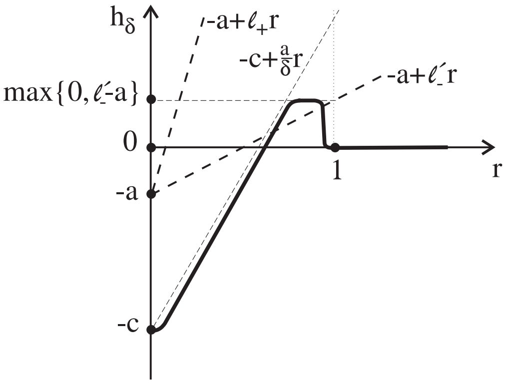

Figure 8: The function h_δ in case $c \geq a$.

Proof. The idea is to deform h_δ (see Figure 8) by action-regular monotone homotopies to a convex function, so that we can apply Theorem 2.9. Let $0 \leq \ell_- < \ell'_- < \mu_\delta < \ell_+ \leq +\infty$ be given by (44). First we need to check that the action window (a, ∞) is indeed admissible for h_δ: it suffices to check positive critical slopes only. They appear in two clusters, one of which is located near the point $(0, -c)$. Their tangents hit the vertical coordinate axis below $-c < -a$. So they correspond to action values strictly bigger than a. The other cluster is located in the region between the lines $-a + \ell_- r$ and $-a + \ell_+ r$. The slope of any such tangent passing through the point $(0, -a)$ lies in the interval (ℓ_-, ℓ_+) and is therefore noncritical. Here we use the assumption $\mu_\delta \notin \Lambda_\alpha$, which implies $\ell_- < \mu_\delta < \ell_+$ and $(\ell_-, \ell_+) \cap \Lambda_\alpha = \emptyset$.

Next we define the graph of a new function $\tilde{h}_\delta$ by initially following the graph of h_δ. When h_δ makes its first right turn, keep going with constant slope a/δ until you are about to hit the line $-a + \mu_\delta r$. Perform a smooth right turn and follow that line closely and linearly as indicated in Figure 9. The figure also shows a monotone homotopy between h_δ and $\tilde{h}_\delta$ which is action-regular: we claim that no 'critical slope tangents' to members of the homotopy

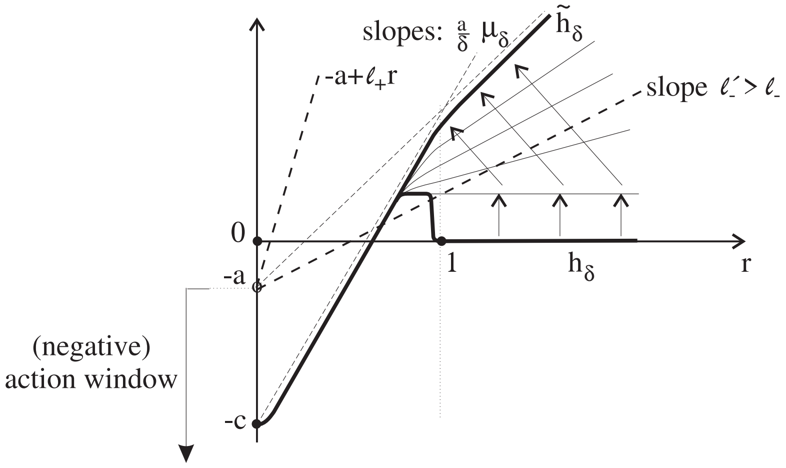

Figure 9: An action-regular monotone homotopy between h_δ and $\tilde{h}_\delta$.

pass through the point $(0,-a)$, thereby we exclude that periodic orbits enter or leave the action window (a,∞). All points on members of the homotopy whose tangents pass through the point $(0,-a)$ lie strictly between the lines $-a+\ell_- r$ and $-a+\ell_+ r$ (see Figure 9). Hence these slopes are noncritical and this proves the claim. Therefore we obtain the monotone *isomorphism*

$$\sigma^{\tilde{h}_\delta h_\delta} : \mathrm{HF}_*^{(a,\infty)}(h_\delta;\alpha) \to \mathrm{HF}_*^{(a,\infty)}(\tilde{h}_\delta;\alpha). \tag{47}$$

Define the function $h_\delta^{(\mu_\delta)}$ by following $\tilde{h}_\delta$ until slope μ_δ shows up for the first time. Then make a smooth turn and continue linearly with slope μ_δ as indicated in Figure 10.
The monotone homotopy shown is action-regular: all tangents to all members of the homotopy hit the vertical coordinate axis strictly below the point $-a$. Hence we obtain an associated monotone isomorphism with inverse

$$\left(\sigma^{\tilde{h}_\delta h_\delta^{(\mu_\delta)}}\right)^{-1} : \mathrm{HF}_*^{(a,\infty)}(\tilde{h}_\delta;\alpha) \to \mathrm{HF}_*^{(a,\infty)}(h_\delta^{(\mu_\delta)};\alpha). \tag{48}$$

Since $h_\delta^{(\mu_\delta)}$ does not admit 1-periodic orbits of action less than a, the map $[\pi^F]$ in (19) with $(a,b,c,)$ given by $(-\infty,a,\infty)$ is an isomorphism with inverse

$$[\pi^F]^{-1} : \mathrm{HF}_*^{(a,\infty)}(h_\delta^{(\mu_\delta)};\alpha) \to \mathrm{HF}_*^{(-\infty,\infty)}(h_\delta^{(\mu_\delta)};\alpha). \tag{49}$$

Similarly, there is the isomorphism

$$[\iota^F]^{-1} : \mathrm{HF}_*^{(-\infty,\infty)}(h_\delta^{(\mu_\delta)};\alpha) \to \mathrm{HF}_*^{(-\infty,C)}(h_\delta^{(\mu_\delta)};\alpha), \tag{50}$$

where the constant $C = C(h_\delta^{(\mu_\delta)},\mu_\delta)$ is defined in Theorem 2.9. This theorem applies, since $h_\delta^{(\mu_\delta)}$ is convex and $\mu_\delta \notin \Lambda_\alpha$, and it provides the isomorphism

$$\Psi^{h_\delta^{(\mu_\delta)},\mu_\delta} : \mathrm{HF}_*^{(-\infty,C)}(h_\delta^{(\mu_\delta)};\alpha) \to \mathrm{H}_*(\mathcal{L}_\alpha^{{\mu_\delta}^2/2}M). \tag{51}$$

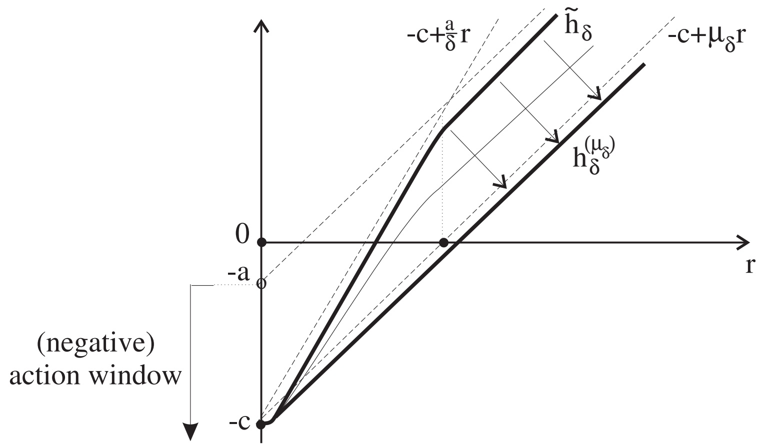

Figure 10: An action-regular monotone homotopy between $h_\delta^{(\mu_\delta)}$ and $\tilde{h}_\delta$.

The proof of Step 3 concludes by composing (47-51). ◻

STEP 4. *If* $0 < a \le c$, *then* $\underrightarrow{\mathrm{SH}}_*^{(a,\infty);c}(DT^*M, M;\alpha) \simeq \mathrm{H}_*(\mathcal{L}_\alpha M)$ *naturally.*

Proof. Consider the set

$$\Delta := \{\delta \in (0, a/c) \mid (43) \text{ holds}\}$$

with partial order defined by $\delta_0 \preccurlyeq \delta_1$ iff $\delta_0 \ge \delta_1$. Observe that $(\Delta, \preccurlyeq)$ is upward directed and that $\delta_0 \ge \delta_1$ is equivalent to $\mu_{\delta_0} \le \mu_{\delta_1}$ and to $h_{\delta_0} \le h_{\delta_1}$. Define

$$\mu_\Delta := \{\mu_\delta \mid \delta \in \Delta\}, \qquad \mathcal{H}_\Delta := \{h_\delta \mid \delta \in \Delta\}.$$

Hence we can identify the upward directed sets $(\Delta, \preccurlyeq)$, $(\mu_\Delta, \le)$ and $(\mathcal{H}_\Delta, \le)$. Take the direct limit of both sides of (46) with respect to $(\Delta, \preccurlyeq)$ to obtain

$$\varinjlim_{h_\delta \in \mathcal{H}_\Delta} \mathrm{HF}_*^{(a,\infty)}(h_\delta;\alpha) \simeq \varinjlim_{\mu \in \mu_\Delta} \mathrm{H}_*(\mathcal{L}_\alpha^{\mu^2/2} M).$$

By Lemma 3.4 the right hand side is naturally isomorphic to $\mathrm{H}_*(\mathcal{L}_\alpha M)$. Concerning the left hand side, there is the natural inclusion homomorphism

$$\varinjlim_{h_\delta \in \mathcal{H}_\Delta} \mathrm{HF}_*^{(a,\infty)}(h_\delta;\alpha) \to \varinjlim_{H \in \mathcal{H}_{\alpha;c}^{a,\infty}} \mathrm{HF}_*^{(a,\infty)}(H;\alpha), \quad [h_\delta, x_{h_\delta}] \mapsto [h_\delta, x_{h_\delta}]. \tag{52}$$

Using (18) and the fact that $(\mathcal{H}_\Delta, \le)$ is upward directed, it is easy to see that this map is well defined, injective and surjective. ◻

Steps 2 and 4 prove part (ii) of Theorem 3.1.

STEP 5. *We prove part* (iii) *of Theorem 3.1.*

Proof. Throughout we use the notation introduced in Steps 1-4. Fix $0 < a \le c$ and choose $\delta \in \Delta$ sufficiently small such that

(a) $\mu_\delta > a$.

Then choose $k \in \mathbb{N}$ sufficiently large such that

(b) $f_k \in \mathcal{H}^{a,\infty}_{\alpha;c}$

(c) $f_k \le h_\delta$.

Let η^{f_k} and η^{h_δ} be given by (36) and (46), respectively, and assume that the diagram

$$\begin{tikzcd} \mathrm{HF}_*^{(a,\infty)}(f_k;\alpha) \arrow[r, "\sigma^{h_\delta f_k}"] \arrow[d, "\eta^{f_k}"', "\simeq"] & \mathrm{HF}_*^{(a,\infty)}(h_\delta;\alpha) \arrow[d, "\simeq"', "\eta^{h_\delta}"] \\ \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2}M) \arrow[r, "{[\iota_{\frac{\mu_\delta^2}{2}\frac{a^2}{2}}]}"] & \mathrm{H}_*(\mathcal{L}_\alpha^{\mu_\delta^2/2}M) \end{tikzcd} \tag{53}$$

commutes. We claim that in this case the following diagram, whose existence uses (b), commutes too

$$\begin{tikzcd} \underleftarrow{\mathrm{SH}}_*^{(a,\infty)}(DT^*M;\alpha) \arrow[d, "\pi_{f_k}"', "\simeq"] \arrow[dr, "T^{(a,\infty)}_{\alpha;c}"] & \\ \mathrm{HF}_*^{(a,\infty)}(f_k;\alpha) \arrow[r, "\iota_{f_k}"] \arrow[d, "\eta^{f_k}"', "\simeq"] & \underrightarrow{\mathrm{SH}}_*^{(a,\infty);c}(DT^*M,M;\alpha) \arrow[d, "\simeq"', "\eta"] \\ \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2}M) \arrow[r, "I_{\frac{a^2}{2}}"] & \varinjlim_{\mu\in\mu_\Delta} \mathrm{H}_*(\mathcal{L}_\alpha^{\mu^2/2}M) \simeq \mathrm{H}_*(\mathcal{L}_\alpha M) \end{tikzcd} \quad . \tag{54}$$

This implies Step 5. To prove the claim observe that the upper triangle commutes by (31). The map $I_{a^2/2}$ is the inclusion homomorphism of direct limits defined analogous to (30) and

$$\eta([h_\delta, x_{h_\delta}]) := \left[\frac{\mu_\delta^2}{2}, \eta^{h_\delta,a}(x_{h_\delta})\right].$$

Here we use (52). Commutativity of the lower rectangular block follows from the identity

$$I_{\frac{a^2}{2}}(\eta^{f_k}x_{f_k}) = \left[\frac{\mu_\delta^2}{2}, [\iota_{\frac{\mu_\delta^2}{2}\frac{a^2}{2}}](\eta^{f_k}x_{f_k})\right],$$

which uses (a), the identity

$$\eta\circ\iota_{f_k}(x_{f_k}) = \eta([f_k, x_{f_k}]) = \eta([h_\delta, \sigma^{h_\delta f_k}x_{f_k}]) = \left[\frac{\mu_\delta^2}{2}, \eta^{h_\delta}(\sigma^{h_\delta f_k}x_{f_k})\right],$$

which uses (c), and the commutative diagram (53). This proves the claim.

It remains to show that diagram (53) commutes. Simplifying notation let us rewrite (53) as follows

$$
\begin{tikzcd}
\mathrm{HF}_*^{(a,\infty)}(f_k) \arrow[r, "\sigma"] \arrow[d, "\sigma"', "(37)"] & \mathrm{HF}_*^{(a,\infty)}(h_\delta) \arrow[d, "(47)"', "\sigma"] & \\
\mathrm{HF}_*^{(a,\infty)}(\tilde{f}_k) \arrow[r, "\sigma"] & \mathrm{HF}_*^{(a,\infty)}(\tilde{h}_\delta) & \\
\mathrm{HF}_*^{(a,\infty)}(f_k^{(a)}) \arrow[r, "\sigma"] \arrow[u, "\sigma", "(38)"'] & \mathrm{HF}_*^{(a,\infty)}(h_\delta^{(\mu_\delta)}) \arrow[u, "(48)", "\sigma"'] & \\
\mathrm{HF}_*^{(-\infty,\infty)}(f_k^{(a)}) \arrow[r, "\sigma"] \arrow[u, "{[\pi^F]}", "(39)"'] \arrow[dr, "\check{\sigma}"] & \mathrm{HF}_*^{(-\infty,\infty)}(h_\delta^{(\mu_\delta)}) \arrow[u, "(49)", "{[\pi^F]}"'] & \\
\mathrm{HF}_*^{(-\infty,c_{f,a})}(f_k^{(a)}) \arrow[r, "{\check{\sigma}=[\Phi^{hf,a}]}", "\simeq"'] \arrow[u, "{[\iota^F]}", "(40)"'] \arrow[dr, "{\Psi^{f,a},(41)}"'] & \mathrm{HF}_*^{(-\infty,c_{h,a})}(h_\delta^{(\mu_\delta)}) \arrow[r, "{[\iota^F]}"] \arrow[u, "{[\iota^F]}"] \arrow[d, "\simeq"', "\Psi^{h,a}"] & \mathrm{HF}_*^{(-\infty,c_{h,\mu})}(h_\delta^{(\mu_\delta)}) \arrow[ul, "{[\iota^F],(50)}"'] \arrow[d, "(51)"', "\Psi^{h,\mu_\delta}"] \\
& \mathrm{H}_*(\mathcal{L}_\alpha^{a^2/2}M) \arrow[r, "{[\iota]}"] & \mathrm{H}_*(\mathcal{L}_\alpha^{\mu_\delta^2/2}M)
\end{tikzcd}
.
$$

The first two rectangular blocks of the diagram commute due to property (18) of the monotone homomorphisms. The third block commutes by (19). The fourth block, which consists of two triangles, commutes by (20) with $(a, b_1, b_2, c) = (-\infty, c_{h,a}, c_{f,a}, \infty)$. The triangle to its right commutes already on the chain level. The lowest rectangular block commutes by (23) in Theorem 2.9 and the triangle to its left by (22). This concludes the proof of Step 5 and Theorem 3.1. $\square$

4 Relative BPS-capacity

We compute the BPS-capacity of DT^*M relative to M and show that every nontrivial element α of $\tilde{\pi}_1(M)$, the set of homotopy classes of free loops in M, is symplectically essential. Note that $\tilde{\pi}_1(DT^*M) \simeq \tilde{\pi}_1(M)$. For $c > 0$, set $\mathcal{H}_c := \{H \in C_0^\infty(S^1 \times DT^*M) \mid \sup_{S^1 \times M} H \le -c\}$. In the following definition we use the conventions $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

Definition 4.1 ([2, Sections 3.2 and 4.9]). Let $\alpha \in \tilde{\pi}_1(M)$ and $a \in \mathbb{R}$. The *Biran-Polterovich-Salamon capacities of DT^*M relative M* are defined by

$$
\begin{aligned}
c_{BPS}(DT^*M, M; \alpha, a) &:= \inf\{c > 0 \mid \forall H \in \mathcal{H}_c \ \exists z \in \mathcal{P}(H; \alpha) \colon \mathcal{A}_H(z) \ge a\} \\
c_{BPS}(DT^*M, M; \alpha) &:= \inf\{c > 0 \mid \text{ every } H \in \mathcal{H}_c \text{ admits } z \in \mathcal{P}(H; \alpha)\}.
\end{aligned}
$$

For $c > 0$, set

$$A_c(DT^*M, M; \alpha) := \{a \in \mathbb{R}\ (a > 0 \text{ if } \alpha = 0) \mid T^{(a,\infty)}_{\alpha;c} \neq 0\},$$

then the *homological* relative BPS-capacity is defined by

$$\hat{c}_{BPS}(DT^*M, M; \alpha, a) := \inf\{c > 0 \mid \sup A_c(DT^*M, M; \alpha) > a\}.$$

The significance of the homological capacity $\hat{c}_{BPS}$ lies in the fact that it is accessible to computation and bounds c_{BPS} from above [2, Prop. 4.9.1].

Proposition 4.2. *$\hat{c}_{BPS}(DT^*M, M; \alpha, a) = \max\{\ell_\alpha, a\}$, $\forall \alpha \in \tilde{\pi}_1(M)$, $\forall a \in \mathbb{R}$.*

Proof. Theorem 3.1 shows that

$$\sup A_c(DT^*M, M; \alpha) = \sup[\ell_\alpha, c] = \begin{cases} c, & \text{if } c \geq \ell_\alpha, \\ -\infty, & \text{else.} \end{cases}$$

Hence $\hat{c}_{BPS}(DT^*M, M; \alpha, a) = \inf\{c > 0 \mid c > a \text{ and } c \geq \ell_\alpha\} = \max\{a, \ell_\alpha\}$. □

Theorem 4.3 (BPS-capacity of DT^*M relative M). *Let M be a closed connected smooth Riemannian manifold. Then for every homotopy class α of free loops in M and every $a \in \mathbb{R}$ the relative BPS-capacity is finite and given by*

$$c_{BPS}(DT^*M, M; \alpha, a) = \max\{\ell_\alpha, a\}, \qquad c_{BPS}(DT^*M, M; \alpha) = \ell_\alpha.$$

Proof. The proof of [2, Proof of Theorem 3.2.1] carries over almost literally. In case $\alpha = 0$ every $H \in \mathcal{H}$ admits constant 1-periodic orbits of action zero, because it is zero near the boundary of DT^*M. Therefore $c_{BPS}(DT^*M, M; 0, a) = 0$ whenever $a \leq 0$. The remaining part of proof proceeds like [2, Proof of Theorem 3.2.1 in case $\alpha \neq 0$]. Just replace ℓ by ℓ_α, Theorem 5.1.1 and Theorem 5.1.2 by Proposition 4.2, and let

$$m := \begin{cases} \max\{\ell_\alpha, a\}, & \text{in case } \alpha \neq 0 \text{ and } a \in \mathbb{R}, \\ a, & \text{in case } \alpha = 0 \text{ and } a > 0. \end{cases}$$

□

Definition 4.4 ([2, Section 3.4]). Let M be a closed connected smooth manifold. A nontrivial homotopy class α of free loops in M is called *symplectically essential* if there exists an open set $W \subset T^*M$ containing the zero section such that $c_{BPS}(W, M; \iota_\# \alpha)$ is finite. Here $\iota_\# : \tilde{\pi}_1(M) \to \tilde{\pi}_1(W)$ is the map induced by the inclusion $\iota : M \hookrightarrow W$.

Corollary 4.5. *Every nontrivial homotopy class α of free loops in a closed connected smooth Riemannian manifold M is symplectically essential.*

Proof. Let $W := DT^*M$, then $\iota_\#$ is an isomorphism. Apply Theorem 4.3. □

Proof of Theorem A. By [2, Proof of Theorem B], we may assume without loss of generality that H is periodic, i.e. $H \in C_0^\infty(S^1 \times DT^*M)$. By Theorem 4.3 we have $c_{BPS}(DT^*M, M; \alpha, c) = \max\{\ell_\alpha, c\} = c$ and this implies that the set

$$\{C > 0 \mid \forall H \in \mathcal{H}_C \; \exists z \in \mathcal{P}(H; \alpha) : \mathcal{A}_H(z) \geq c\}$$

is nonempty. On the other hand, by [2, Proposition 3.3.4] this set is either empty or has a minimum. This proves Theorem A. $\square$

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