Topological resolution of singularities

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Abstract

We review the topological obstructions to resolving singularities and show they all vanish for low dimensional complex algebraic varieties.

Contents

1 Introduction

What do homology classes look like? From its definition, a singular homology class is representable by a singular geometric cycle, which could carry singularities as indicated by its name. Can we find better representatives?

Definition 1.1. Let $\sigma_n \in H_n(X)$ be an n-dimensional singular homology class in X. We say σ_n is representable by manifolds if there exists a continuous map $f : M^n \to X$ from a closed oriented n-manifold into X so that $\sigma_n = f_*[M]$ where $[M] \in H_n(M)$ is the fundamental class of M.

Example 1.2. Homology classes up to dimension 6 are representable by manifolds.

Exercise 1.3. Show that all 1-dimensional and 2-dimensional homology classes are representable by manifolds.

Problem 1.4 (Steenrod)**.** Can all homology classes be represented by manifolds?

In 1954, René Thom [Tho54] answered Steenrod's problem both positively and negatively–yes, for mod 2 homology classes; but not true in general for integral homology classes. Moreover, he found there are topological obstructions to resolving the singularities of a homology class, and gave an example of a 7-dimensional homology class on which the obstructions do not vanish, thus showing it carries non-resolvable singularities.

However, ten years later, Hironaka [Hir64] came along and showed that all complex algebraic varieties admits resolutions. This in particular implies that all the obstructions to resolution of singularities discovered by Thom must vanish on *algebraic* homology classes.

This note aims to define Thom's obstructions, and show that they vanish on the fundamental classes of low dimensional complex algebraic varieties without referring to Hironaka's theorem.

2 Steenrod's problem and Thom's solution

A big step in Thom's approach to Steenrod's problem is to use duality to turn this homological-geometric problem into an cohomological-algebraic problem.

To start with, let X be a finite complex embedded into an Euclidean space $\mathbb{R}^{n+q}$, and let N be a small closed tubular neighborhood of X with boundary ∂N. Then we have the following well-known duality, which will be referred to as Alexander duality in this note.

Theorem 2.1 (Alexander-Lefschetz-Poincaré). $H_n(X) \cong H^q(N, \partial N)$.

Thom added a brilliant geometric insight into this duality, and used that to characterize when a homology is representable by manifolds. This has to do with the famous Thom class and Thom isomorphism.

2.2 Mod 2 homology

For simplicity, we shall for the moment surpress orientations and consider mod 2 homology classes. Now suppose $\sigma_n \in H_n(X;\mathbb{Z}_2)$ is represented by $f : M^n \to X$ from a (not necessarily orientable) closed manifold into X such that $\sigma_n = f_*[M]$. By abusing the notation, we denote f followed by the inclusion of X into N by f as well. Moreover, we can choose q large enough $(q > n)$ so that f is homotopic (via a small perturbation) to an smooth embedding $M \hookrightarrow N$. Thus we can think of M as a submanifold of N, and the homology class $\sigma_n \in H_n(X;\mathbb{Z}_2)$ is the push-forward of $[M] \in H_n(M;\mathbb{Z}_2)$ along $M \hookrightarrow N$ followed by a deformation retract $N \to X$.

Now M has a closed tubular neighborhood ν contained in N. Let $g : M \to BO_q$ be the map classifying the normal bundle of M in N, then we have a diffeomorphism of the pairs

$$(\nu, \partial\nu) \cong (g^* D\gamma_q, g^* \partial D\gamma_q)$$

where γ is the universal rank q vector bundle over BO_q and $D\gamma_q, \partial D\gamma_q$ are the unit closed disk, unit sphere bundles of γ respectively.

Theorem 2.3 (Universal Thom isomorphism)**.** Let u_q be the unique non-zero class in

$$H^q(D\gamma_q, \partial D\gamma_q; \mathbb{Z}_2) \cong \mathbb{Z}_2.$$

Then multiplication by u_q yields an isomorphism

$$H^*(BO_q; \mathbb{Z}_2) \cong H^{*+q}(D\gamma_q, \partial D\gamma_q; \mathbb{Z}_2).$$

Proof. Without loss of generality, we can think of BO_q as the Grassmannian of q-dimensional real vector spaces in a d-dimensional Euclidean space for d sufficiently large. Then by Alexander duality we have

$$H^{*+q}(D\gamma_q, \partial D\gamma_q; \mathbb{Z}_2) \cong H_{\dim BO_q - *}(BO_q; \mathbb{Z}_2).$$

On the other hand, by Poincaré duality we have

$$H_{\dim BO_q - *}(BO_q; \mathbb{Z}_2) \cong H^*(BO_q; \mathbb{Z}_2).$$

The class u_q, now known as the Thom class, is the Alexandar dual of the fundamental class of BO_q. ■

Now we can pull back the Thom class u_q to $H^q(N, \partial N; \mathbb{Z}_2)$ via

$$N/\partial N \xrightarrow[\text{a point}]{\text{map } N\backslash\nu \text{ to}} \nu/\partial\nu \xrightarrow{g} D\gamma_q/\partial D\gamma_q =: MO_q.$$

Lemma 2.4 (Key lemma)**.** The pull-back of the Thom class is Alexander dual to $\sigma_n \in H_n(X; \mathbb{Z}_2)$. Moreover, a homology class of X is representable by manifolds if and only if its Alexander dual is the pull-back of the Thom class by some continuous map $N/\partial N \to MO_q$.

We note here N depends on our choice of the embedding $X \subset \mathbb{R}^{n+q}$, and in fact one is allowed to vary the embedding in the above formulation, we suppress this point for simplicity.

So now the Steenrod's problem becomes the following dual problem:

Problem 2.5 (Steenrod's problem, dual version)**.** Are all cohomology classes of $N/\partial N$ pull-backs of the Thom class?

Notice that the cohomology functor $\widetilde{H}^q(-; \mathbb{Z}_2)$ is representable by the Eilenberg-MacLane space $K(\mathbb{Z}_2, q)$. in particular the Thom class $u_q \in \widetilde{H}^q(MO_q; \mathbb{Z}_2)$ corresponds to a (based) map

$$MO_q \to K(\mathbb{Z}_2, q).$$

Moreover, the above problem can be rephrased as: Can every map $N/\partial N \to K(\mathbb{Z}_2, q)$ be lifted to MO_q?

$$\begin{tikzcd} & MO_q \arrow[d, "\text{Thom class}"] \\ N/\partial N \arrow[r] \arrow[ur, dashed] & K(\mathbb{Z}_2, q) \end{tikzcd}$$

Now this lifting problem is a homotopy theoretical problem, and we can study it through obstruction theory.

Recall that the obstructions to constructing a section of the fiberation $F \to E \to B$ are inductively defined by trying to construct a section skeleton-by-skeleton. If a section has been constructed on the k-skeleton of B, then the next obstruction is a cohomology class in $H^{k+1}(B; \pi_k F)$. The obstructions to a lifting problem

$$\begin{tikzcd} & E \arrow[d] & F \arrow[l] \\ Y \arrow[r] \arrow[ur, dashed] & B & \end{tikzcd}$$

are similar. If a lifting has been constructed on the k-skeleton of Y, then the next obstruction is a cohomology class in $H^{k+1}(Y; \pi_k F)$.

Thom studied the universal obstructions to constructing a section to the map

$$MO_q \xrightarrow{\text{Thom class}} K(\mathbb{Z}_2, q).$$

And he showed

Proposition 2.6. There exists a section over the $2q$-skeleton of $K(\mathbb{Z}_2, q)$. Therefore the obstructions to our lifting problem appear in H^{2q+*} for $* \geq 1$.

Corollary 2.7. All mod 2 homology classes are representable by manifolds.

Proof. Notice that $H^*(N, \partial N) = 0$ for $* > q + n$ for simple dimension reason. So as long as $q + n < 2q$, i.e. $q > n$, there is no obstruction to finding a lifting. ■

2.8 Integral homology

Now we take orientations into account and consider integral homology classes. Similarly we must examine the map

$$MSO_q \xrightarrow{\text{Thom class}} K(\mathbb{Z}, q).$$

corresponding to the Thom class of the universal oriented vector bundle. But this time, we almost immediately meet obstructions–there are cohomology operations vanishing on the Thom class but not vanishing on the fundamental class of $K(\mathbb{Z}, q)$ (corresponding to the identity map), βP^1 for instance, where P^1 is the mod 3 Steenrod first power and β is the mod 3 Bockstein.

Example 2.9. There is a homology class $x_1 * x_5 \in H_7(K(\mathbb{Z}_3 \times \mathbb{Z}_3, 1); \mathbb{Z})$ whose Alexander dual is not a pull-back of the Thom class, for βP^1 does not vanish on it. The singularity type of this homology class is a surface times a cone over $\mathbb{CP}^2$. See [Sul71].

However, the good news is that all the obstructions are odd-primary torsion! This implies

Theorem 2.10. There exists an odd number Odd_n depending on n so that $Odd_n \cdot \sigma_n$ is representable by manifolds.

3 Vanishing of obstructions for complex algebraic varieties

We caution the reader that, in this section, the symbols X, M etc. will have different meanings than the previous section.

Let X be a compact (singular) complex algebraic variety of complex dimension n.

Theorem 3.1 (Hironaka)**.** There exists a resolution of X, that is a smooth variety $\widetilde{X}$ together with a morphism $\widetilde{X} \to X$ which is an isomorphism away from the singular locus of X.

Corollary 3.2. The fundamental class of a complex algebraic variety is always representable by manifolds. Therefore all Thom's obstructions discussed in the previous section vanish on $[X] \in H_{2n}(X; \mathbb{Z})$.

Hironaka's argument is quite involved, so one wonders if there is an easier argument showing Thom's obstructions vanish, which is (presumably much) weaker than Hironaka's theorem.

In this section, we combine Atiyah and Hirzebruch's argument in disproving integral Hodge conjecture [AH62] and some topological facts to show Thom's obstructions vanish on the fundamental classes of low dimensional complex algebraic varieties.

3.3 Complex version of Thom's obstructions

We adapt Thom's analysis to our complex algebraic situation by utilizing the complex structure. First we embed X into a smooth complex variety M of complex dimension $n+q$, and ask whether the map induced by the Alexander dual of $[X]$

$$M/(M-X) \to K(\mathbb{Z}, 2q)$$

can be lifted to MU_q along the map

$$MU_q \xrightarrow{\text{Thom class}} K(\mathbb{Z}, 2q)$$

corresponding to the Thom class of the universal complex rank q vector bundle.

To analyze the obstructions to this lifting problem, we need some algebraic topology tools. We *localize the problem at an odd prime p*, in effect we concentrate on obstructions of order powers of p. By a theorem of Quillen [Qui69], after localization the map $MU_q \to K(\mathbb{Z}, 2q)$ can be effectively replaced by a map

$$BP_{2q} \to K(\mathbb{Z}_{(p)}, 2q).$$

The following are some basic facts about the space BP_{2q}.

(i) The space BP's are introduced by Brown and Peterson [BP66] so that the mod p stable cohomology of BP is the mod p Steenrod algebra modulo the two sided ideal generated by Bockstein.

(ii) The homotopy groups of these spaces are extremely nice

$$\pi_*(BP_{2q}) = s^{2q}\mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

where v_i has degree $|v_i| = 2(p^i - 1)$ and s^{2q} means shift up degree by $2q$.

(iii) Later Wilson [Wil75] introduced "quotients" of these BP spaces, denoted by $BP\langle m\rangle_{2q}$, whose homotopy groups are quotients of those of BP

$$\pi_* BP\langle m\rangle_{2q} = s^{2q}\mathbb{Z}_{(p)}[v_1, \dots, v_m].$$

Note in particular $BP\langle 0\rangle_{2q} = K(\mathbb{Z}_{(p)}, 2q)$ and $BP\langle\infty\rangle_{2q} = BP_{2q}$.

(iv) Moreover, these Wilson spaces fit into a sequence

$$BP \to \cdots \to BP\langle m+1\rangle \to BP\langle m\rangle \to \cdots \to BP\langle 0\rangle.$$

(v) The fiber of $BP\langle m+1\rangle_{2q} \to BP\langle m\rangle_{2q}$ is exactly $BP\langle m+1\rangle_{2q+|v_{m+1}|}$.

Therefore, the lifting problem of the fiberation $BP \to BP\langle 0\rangle$ can be thought of as an inductive lifting problem of the fiberations

$$\begin{tikzcd} BP\langle m+1\rangle_{2q} \arrow[d] & BP\langle m+1\rangle_{2q+|v_{m+1}|} \arrow[l] \\ BP\langle m\rangle_{2q} & \end{tikzcd}$$

This somewhat simplifies the description of the obstructions (even though still complicated) in the sense that we know the homotopy groups of the inductive fibers.

Now let us analyze the inductive obstructions. Suppose we have already lifted the map

$$M/(M-X) \to BP\langle 0\rangle_{2q}$$

to $BP\langle m\rangle_{2q}$. Then all the following obstructions appear in

$$H^{2q+*+1}\left(M,M-X;\pi_{2q+*}(BP\langle m+l\rangle_{2q+|v_{m+l}|})\right)$$

for $*\geq|v_{m+1}|$ and $l\geq 1$.

Proposition 3.4. Under the above assumption, if $2n\leq|v_{m+1}|$, then the fundamental class of X is representable by (stably almost complex) manifolds.

Proof. All the following obstructions appear in dimensions $\geq 2q+|v_{m+1}|+1$, but the cohomology of $M/(M-X)$ live in dimensions $\leq 2q+2n$. ■

3.5 Lifting to $BP\langle 1\rangle$

We now show, by combining the above topological facts with Atiyah and Hirzebruch's argument in disproving integral Hodge conjecture [AH62], that the map $M/(M-X)\to BP\langle 0\rangle_{2q}$ can always be lifted to $BP\langle 1\rangle_{2q}$.

For this, we need the following well-known algebraic geometry lemma.

Lemma 3.6. The sheaf $\mathscr{O}_X$ of regular functions on X, when treated as a coherent sheaf on M, admits a locally free resolution of finite length

$$\mathscr{E}^\bullet\to\mathscr{O}_x\to 0.$$

Proof. Finitely generated modules over a regular Noetherian local ring admits a finite free resolution. ■

Since locally free sheaves correspond to vector bundles, we get an element

$$[\mathscr{O}_X]:=[\mathscr{E}^{even}]-[\mathscr{E}^{odd}]\in K(M).$$

Moreover, since $\mathscr{O}_X$ is supported on X, the sequence $\mathscr{E}^\bullet\to 0$ is exact on $M-X$, hence $[\mathscr{O}_X]$ lives in

$$K(M,M-X)\cong\widetilde{K}(M/(M-X)).$$

This in turn yields a map

$$M/(M-X)\xrightarrow{g}BU.$$

Furthermore, notice that $M/(M-X)$ is $(2q-1)$-connected, hence g can be lifted to the $(2q-1)$-connected cover $BU(2q,\infty)$ of BU.

Lemma 3.7 (Key lemma)**.** The induced map $M/(M-X)\to BU(2q,\infty)$, up to homotopy, is independent of our choice of the free resolution, and the pull-back of the generator of $H^{2q}(BU(2q,\infty);\mathbb{Z})=\mathbb{Z}$ is Alexander dual to the fundamental class of X.

Proof. The q-th Chern class of $[\mathscr{O}_X]$ is $(-1)^{q-1}(q-1)!$ times the Alexander dual of X. This can be proved by Hirzebruch-Riemann-Roch or by an explicit computation on an universal example where there is a Koszul resolution. Meanwhile by Bott [BM58], the universal q-th Chern class is precisely divisible by $(q-1)!$ when pullled back onto $BU(2q,\infty)$. ■

Localized at prime p, a folklore theorem (or perhaps due to Adams?) says the space $BU(2q,\infty)$ splits into a product of $BP\langle 1\rangle$'s.

Theorem 3.8.

$$BU(2q,\infty)_{(p)} \simeq \prod_{i=0}^{p-2} BP\langle 1\rangle_{2q+i}.$$

Therefore $H^{2q}(BU(2q,\infty)) = H^{2q}\left(BP\langle 1\rangle_{2q};\mathbb{Z}_{(p)}\right) = \mathbb{Z}_{(p)}$.

Corollary 3.9. The Alexander dual of the fundamental class of X can always be lifted to $BP\langle 1\rangle$.

Corollary 3.10. All odd primary topological obstructions to resolving singularities vanish when $\dim_{\mathbb{C}} \leq 8$.

Proof. Note that $8 = 3^2 - 1$, and apply Proposition 3.4. ■

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