TOPOLOGICAL RESOLUTION OF SINGULARITIES

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Abstract. In this note, we review the topological obstructions to resolving the singularities and show they all vanish for low-dimensional complex algebraic varieties.

Contents

1. Introduction

What do homology classes look like? From definition, a homology class is representable by a singular geometric cycle, which could have singularities as indicated by its name. It is natural to ask: can we find better representatives?

Definition 1.1. *Let X be a finite simplicial complex. We say the homology class $\sigma_n \in H_n(X)$ is representable by manifold if there exists a continuous map $f : M^n \to X$ such that $\sigma_n = f_*[M]$ where M is an oriented closed manifold.*

Exercise 1.2. *All 1-dimensional and 2-dimensional homology classes are representable by manifolds.*

Then Steenrod asked: *Can all homology classes be represented by manifolds*?

In 1954, René Thom [Thom54] answered this question both positively and negatively: yes, for mod 2 and rational homology; not necessarily for integral homology. Moreover, he found there are topological obstructions to *resolving the singularities* of a homology class and give an example where there are *non-resolvable* singularities.

However, ten years later, Hironaka [Hiro64] came along and showed, all complex algebraic varieties admit resolutions. The topological consequence of that is, all the obstructions discovered by Thom must vanish on *algebraic* homology classes, which is quite surprising.

In this note, we first discuss what those topological obstructions are, and then show that obstructions vanish for low-dimensional complex algebraic varieties.

2. Steenrod's problem and Thom's solution

A big step in Thom's approach to Steenrod's problem is to use duality to turn this geometric-homological problem into a algebraic-cohomological problem, which makes it easier to apply algebraic topology tools.

To start with, we embed X into an Euclidean space $\mathbb{R}^{n+q}$, and let N to be a small neighborhood of X with boundary ∂N.

Theorem 2.1 (Alexander-Lefschetz-Poincaré)**.** *$H_n(X) \simeq H^q(N, \partial N)$.*

We will refer to this duality as Alexander duality from now on.

Thom added a nice geometric insight into this duality theorem, and used that to characterize when a homology class is representable by manifold.

2.1. Mod 2 homology.

We now discuss Thom's characterization of representability by manifolds. To make life easier, we consider mod 2 homology first.

Let's assume $\sigma_n \in H_n(X; \mathbb{Z}/2)$ is representable by $f : M \to X$. By abusing the notation, we denote f followed by inclusion of X into N by f as well. Moreover, we can choose q big enough ($q > n$), so that f is homotopic to an embedding $M \hookrightarrow N$, and thus we can think of M as a submanifold in N, the homology class σ_n is induced by $M \hookrightarrow N$ followed by a deformation retract $N \to X$.

Then M has a tubular neighborhood in N, denoted by ν, which is diffeomorphic to the normal bundle of M in N, and induces a classifying map $g : M \to BO_q$ so that $(\nu, \partial\nu) = (g^*D(\gamma_q), g^*\partial D(\gamma_q))$ where $\gamma_q \to BO_q$ is the universal rank q real vector bundle, and $D(\gamma_q)$ is the unit disk bundle.

Theorem 2.2 (Universal Thom isomorphism)**.**

$$H^*(BO_q; \mathbb{Z}/2) \simeq H^{*+q}(D\gamma_q, \partial D\gamma_q; \mathbb{Z}/2)$$

by multiplying a universal Thom class $u_q \in H^q(D\gamma_q, \partial D\gamma_q; \mathbb{Z}/2)$.

Proof. We can think of BO_q as a finite dimensional manifold, the Grassmannian of q-dimensional real vector spaces in d-dimensional space for d big enough. Then we have Alexander duality $H^{*+q}(D\gamma_q, \partial D\gamma_q; \mathbb{Z}/2) \simeq H_{\dim BO_q - *}(BO_q; \mathbb{Z}/2)$, and Poincaré duality $H_{\dim BO_q - *}(BO_q; \mathbb{Z}/2) \simeq H^*(BO_q; \mathbb{Z}/2)$. The Thom class u_q is the Alexander dual of fundamental class of BO_q. □

So one can pull-back the Thom class to $H^q(N, \partial N; \mathbb{Z}/2)$ via $N/\partial N \to \nu/\partial\nu \to D\gamma_q/\partial D\gamma_q =: MO_q$.

Lemma 2.3 (Key lemma)**.** *The pull-back of the Thom class is Alexander dual to $\sigma_n \in H_n(X; \mathbb{Z}/2)$. Moreover, a homology class of X is representable by manifold if and only if its Alexander dual is the pull-back of Thom class by some continuous map $N/\partial N \to MO_q$.*

So the Steenrod's problem has become the following dual problem: *Are all cohomology classes of $N/\partial N$ pull-back of Thom class?*

Since cohomology functor $H^q(-; \mathbb{Z}/2) \simeq [-, K(\mathbb{Z}/2, q)]$, the Thom class u_q corresponds to a map $MO_q \to K(\mathbb{Z}/2, q)$, and the above problem now can be phrased as: *Does every map $N/\partial N \to K(\mathbb{Z}/2, q)$ lift to MO_q?* This is a homotopy lifting problem, and we can study it by obstruction theory.

Recall that the universal obstruction is the obstruction to constructing a section of the fibration $F \to E \to B$, and the obstructions are inductively defined by trying to construct sections on skeletons of the base. More precisely, if one has constructed a section over k-skeleton of B, then the next obstruction lives in $H^{k+1}(B; \pi_k(F))$. The obstructions to homotopy lifting problem are the pull-back of the universal obstructions.

Thom thus went on to study the universal obstructions, and he showed:

Proposition 2.4. *There exists a section over the $2q$-skeleton of $K(\mathbb{Z}/2, q)$ to the fibration $MO_q \to K(\mathbb{Z}/2, q)$. Therefore, the universal obstructions appear in H^{2q+*} for $* \geq 1$.*

Corollary 2.5. *All mod 2 homology classes can be represented by manifolds.*

Proof. Notice that the $H^*(N, \partial N)$ is zero for $* > q + n$, so as long as $q + n < 2q$, i.e. $q > n$, there's no obstructions to represent n-dimensional homology classes by manifolds. □

2.2. **Integral and rational homology.** Now we switch to integral coefficients, but this time we almost immediately get obstructions when we look at $MSO_q \to K(\mathbb{Z}, q)$: there are cohomology operations vanishing on the Thom class but do not vanish on the fundamental class of $K(\mathbb{Z}, q)$, βP^1 for instance, where P^1 is the mod 3 Steenrod first power and β is mod 3 Bockstein.

Example 2.6. *There is a homology class $x_1 * x_5 \in H_7(K(\mathbb{Z}/3 \times \mathbb{Z}/3, 1))$ whose Alexander dual is not a pull-back of Thom class, since βP^1 is non-zero on that class.*

However, the good news is all the obstructions are (odd-primary) torsion! This implies:

Theorem 2.7. *All rational homology classes can be represented by manifolds. In fact, there exists an odd number Od so that $Od \cdot H_*(X)$ is representable by manifold, where Od only depends on dimension of X.*

3. Vanishing of obstructions for complex algebraic varieties

Remark 3.1 (Warning)**.** *In this section, the symbols X, M etc. will have different meanings than the previous section.*

Let X be a (singular) complex algebraic variety of complex dimension n.

Theorem 3.2 (Hironaka)**.** *There exists a resolution of X, i.e. a smooth variety $\tilde{X}$ together with a morphism $\tilde{X} \to X$ which is a diffeomorphism away from singularity of X.*

Corollary 3.3. *The fundamental class of a complex algebraic variety is always representable by manifold. Therefore, all topological obstructions to resolving singularities vanish on $[X]$.*

Hironaka's argument uses heavy machinery from algebraic geometry and doesn't seem to use any algebraic topology. So one wonders, is there a topological explanation to vanishing of the obstructions?

3.1. **Obstructions at odd primes.** We can adapt Thom's analysis to our complex algebraic situation.

First we embed X into a smooth complex algebraic variety M of complex dimension $n+q$, and ask can the map induced by Alexander dual of $[X]$, i.e. a map $M/(M-X) \to K(\mathbb{Z}, 2q)$, be lifted to MU_q? The map $MU_q \to K(\mathbb{Z}, 2q)$ is induced by the complex Thom class is understood here.

To analyze the obstructions to this lifting problem, we need some relatively advanced algebraic topology tools.

We *localize the problem at an odd prime p*. By a theorem of Quillen [Qui69], after localization we can replace the fibration $MU_{2q} \to K(\mathbb{Z}, 2q)$ by its *p-local* version $BP_{2q} \to K(\mathbb{Z}_{(p)}, 2q)$.

It seems appropriate at this moment to list some facts about the space BP_{2q}.

(1) The space BP's are introduced by Brown and Peterson [BP66] so that the mod p stable cohomology is the mod p Steenrod algebra modulo the two sided ideal generated by Bockstein.
(2) The homotopy of these spaces are extremely nice.
$$\pi_*(BP_{2q}) \simeq s^{2q}\mathbb{Z}_{(p)}[v_1, v_2, \dots],$$
where v_i has degree $|v_i| = 2(p^i - 1)$ and s^{2q} means shift up degree by $2q$.
(3) Later Steve Wilson [Wil75] introduced "quotients" of these BP spaces, denoted by $BP\langle m\rangle_{2q}$ whose homotopy group is a quotient of BP_{2q}.
$$\pi_*(BP\langle m\rangle_{2q}) \simeq s^{2q}\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_m].$$
Note that $BP = BP\langle\infty\rangle$ and $BP\langle 0\rangle_{2q} = K(\mathbb{Z}_{(p)}, 2q)$.
(4) Moreover, these Wilson spaces fit into a sequence
$$BP = BP\langle\infty\rangle \to \cdots \to BP\langle m+1\rangle \to BP\langle m\rangle \to \cdots \to BP\langle 0\rangle = H\mathbb{Z}_{(p)}$$
(5) The fiber of $BP\langle m+1\rangle_{2q} \to BP\langle m\rangle_{2q}$ is precisely $BP\langle m+1\rangle_{2q+|v_{m+1}|}$.

Therefore, the lifting problem of the fibration $BP \to BP\langle 0\rangle$ can be thought of as an inductive lifting problem of the fibrations $BP\langle m+1\rangle \to BP\langle m\rangle$. This somewhat simplifies the description of the obstructions (even though still complicated) in the sense that we know the homotopy groups of the inductive fibers.

We now analyze the inductive obstructions. Suppose we have already lifted the map $M/(M-X) \to BP\langle 0\rangle_{2q} = K(\mathbb{Z}_{(p)}, 2q)$ to $BP\langle m\rangle_{2q}$, then the next obstructions appear in
$$H^{2q+*+1}(M, M-X; \pi_{2q+*}(BP\langle m+l\rangle_{2q+|v_{m+l}|}) \text{ for } * \geq |v_{m+1}|, l \geq 1.$$

Proposition 3.4. *Under the above assumption, if $2n \leq |v_{m+1}|$, then the fundamental class of X is representable by (stably almost complex) manifold.*

Proof. All obstruction appears in dimension $\geq 2q + |v_{m+1}| + 1$ but the cohomology of $M/(M-X)$ lives in dimension $\leq 2q + 2n$. □

3.2. **Lifting to $BP\langle 1\rangle$.** We now show, using mild algebraic geometry, that the map $M/(M-X) \to K(\mathbb{Z}_{(p)}, 2q)$ can always be lifted to $BP\langle 1\rangle_{2q}$.

For this we need the following well-known algebraic geometric lemma.

Lemma 3.5. *The sheaf $\mathcal{O}_X$ of holomorphic functions on X, treated as a coherent sheaf on M, admits a locally free resolution $\mathcal{E}^\bullet \to \mathcal{O}_X \to 0$.*

Note that locally free sheaves correspond to vector bundles, hence we get an element $[\mathcal{O}_X] := [\mathcal{E}^{od} - \mathcal{E}^{ev}] \in K(M)$. Moreover, since $\mathcal{O}_X$ is supported on X, the sequence $\mathcal{E}^\bullet \to 0$ is exact on $M - X$, hence $[\mathcal{O}_X]$ is in fact an element in $K(M, M-X) \simeq \tilde{K}(M/(M-X))$. This in turn gives a map $g : M/(M-X) \to BU$. Further, one notice that $M/(M-X)$ is $(2q-1)$-connected, hence g can be lifted to $(2q-1)$-connected cover of BU, denoted as $BU(2q, \infty)$.

Lemma 3.6 (Key lemma)**.** *The induced map $M/(M-X) \to BU(2q, \infty)$, up to homotopy, is independent of choice of resolutions, and the pull-back of the generator of $H^{2q}(BU(2q, \infty); \mathbb{Z})$ is Alexander dual to the fundamental class of X.*

Proof. The q-th Chern class of $[\mathcal{O}_X]$ is $(-1)^{q-1}(q-1)!$ times Alexander dual of $[X]$ by Hirzebruch-Riemann-Roch or by an explicit computation on an universal example where there is a Koszul resolution. Meanwhile, by Bott [BM58] the universal q-th Chern class is precisely divisible by $(q-1)!$ when pulled back to $BU(2q, \infty)$. □

Localized at p, $BU(2q, \infty)$ splits as a direct product of $BP\langle 1\rangle$'s. More precisely we have:

Theorem 3.7 (folk)**.**

$$BU(2q, \infty)_{(p)} = \prod_{i=0}^{p-2} BP\langle 1\rangle_{2q+i}.$$

Therefore $H^{2q}(BU(2q, \infty); \mathbb{Z}_{(p)}) \simeq H^{2q}(BP\langle 1\rangle_{2q}; \mathbb{Z}_{(p)}) \simeq \mathbb{Z}_{(p)}$.

Corollary 3.8. *The Alexander dual of the fundamental class of X can always be lifted to $BP\langle 1\rangle$. And thus all (odd primary) obstructions to resolving the singularities vanish when $\dim_{\mathbb{C}} X \leq 8$.*

Proof. Note that $8 = 3^2 - 1$, and apply proposition 3.4. □

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