QUATERNIONIC CLIFFORD MODULES, SPIN^h MANIFOLDS AND SYMPLECTIC K-THEORY

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Abstract. Through a systematic study of quaternionic Clifford algebras and their modules, we extend some of the fundamental algebraic and topological results related to spin and spin^c manifolds to their quaternionic counterpart–spin^h manifolds. On the algebraic side, we obtain an Atiyah-Bott-Shapiro type isomorphism relating quaternionic modules over the Clifford algebras to symplectic K-theory of a point. On the topological side, we define a natural transformation from spin^h cobordism theory to symplectic K-theory, which in particular assigns to each spin^h manifold an integer or mod 2 valued cobordism invariant. These invariants can be expressed in terms of differential geometrical data using a quaternionic version of the index theorem. We also offer a complete description of the cohomology of the stable spin^h group, which, combined with the aforementioned invariants, allows us to show the spin^h cobordism groups are non-zero in dimensions $5, 6$ mod 8 and determine low dimensional groups with explicit generators.

Contents

Introduction

This note grew out of the observation that the notion of spin^h manifold is closely related to symplectic K-theory, analogous to the way spin and spin^c manifolds are related to real and complex K-theories respectively.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
Spin	Spin^c	Spin^h
KO	KU	KSp

The algebraic and topological theories related to spin and spin^c manifolds are well-developed and successfully applied to solve geometric problems, such as Gromov-Lawson theory and Seiberg-Witten theory, just to name a few. Compared to its real and complex siblings, the theory of spin^h manifolds appears to be less discussed in the past but has been attracting more attentions in geometry and physics in recent decades.

We plan to explain the above analogy and set up some foundation stones for studying spin^h manifolds. We focus mainly on the algebraic and topological sides, however index theory (Dirac operators etc.) will be used along the way.

Our major goal is to define and study a natural transformation from spin^h cobordism theory to symplectic K-theory. In particular, we shall obtain cobordism invariants for spin^h manifolds with values in $\mathbb{Z}$ and $\mathbb{Z}_2$. These invariants will be expressed in terms differential-geometrical data. It will then be clear that they are analogous to those for spin manifolds which obstruct the existence of positive scalar curvature metrics. To define this natural transformation, we must construct Thom classes in symplectic K-theory (in an appropriate sense) for spin^h vector bundles, which of course should arise from representations of the spin^h group. Since the spin^h group is contained in the quaternionic Clifford algebra (Definition 1.1), the desired representations will be obtained through the study of the quaternionic Clifford algebras and their modules. This will be treated in §1.1 − §1.3. Main results therein is a fully classification of the quaternionic Clifford algebras and their modules. The structure among these modules is best revealed by an Atiyah-Bott-Shapiro type isomorphism (Theorem 1.26) relating quaternionic Clifford modules to symplectic K-theory. We also present necessary algebraic discussions in preparation for studying the indices of Dirac operators on spin^h manifolds, this will be the content of §1.4 and §1.5.

In Section 2, we present a topological discussion for spin^h vector bundles. Using the representations obtained in Section 1, Thom classes are constructed for spin^h vector bundles in appropriate K-theories. Readers familiar with cobordism theories should see immediately the Thom classes constructed therein give rise to the desired natural transformation from spin^h cobordism to symplectic K-theory. We however suppress this point until a better differential-geometrical understanding of the previously mentioned cobordism invariants of spin^h manifolds is obtained. The integer valued invariants are integrations of certain characteristic class that is analogous to the $\hat{A}$-class for spin manifolds. A Riemann-Roch theorem for spin^h maps (Theorem 2.23) is developed in order to pick out such characteristic class for spin^h manifolds. The integrality of the corresponding characteristic number is thus an easy consequence of Bott's theory. To complete the understanding of characteristic classes for spin^h vector bundles and to provide an input for analyzing spin^h cobordism groups in the future, we offer at the end of Section 2 a complete description of the cohomology of (the classifying space of) the stable spin^h group. Somewhat surprisingly, all but one of the Wu classes for spin^h vector bundles in degrees power of 2 admit lifts in integral cohomology.

In Section 3, we employ index theory to study invariants for spin^h manifolds. Due to the quaternionic nature of the problem, the principle symbol of the elliptic operator we concern lands in the Quaternionic K-theory which is a quaternionic analog of Atiyah's Real K-theory defined on the category of spaces with involution. The corresponding algebraic results needed to define topological index are contained in §1.5. A quaternionic version of index theorem for families must be used to deal with the more refined $\mathbb{Z}_2$-valued invariants and to identify the topological index with the analytic index. In order to compute the analytic index, results in §1.4 are used. These considerations are all parallel to the spin case. The invariants extracted from this index theoretical discussion are as expected (Theorem 3.9): the integer valued ones are indices of certain Dirac operators, and coincide with the characteristic numbers obtained in Section 2; the mod 2 invariants are parities of the dimensions of appropriately defined harmonic spinors. The fact that these geometrically defined invariants are spin^h cobordism invariants is a (non-trivial) consequence of the index theory. Furthermore these invariants are identified with the evaluation at a point of the aforementioned natural transformation from spin^h cobordism to symplectic K-theory. Near the end of Section 3, we use all the information gathered to study spin^h cobordism groups. A non-trivial fact we reveal, without doing any heavy homotopy-theoretical calculation, is that the spin^h cobordism group is always nonzero in degrees $5, 6$ mod 8. We also obtain and describe explicit generators for low dimensional spin^h cobordism groups. A complete determination (especially the torsion) should be interesting but looks to be very difficult since the spin^h cobordism theory is not multiplicative.

Finally we will indicate how the natural transformation relating spin^h cobordism to symplectic K-theory can be applied to study real vector bundles over an arbitrary compact (possibly with boundary) manifold. A detailed discussion towards this direction will appear elsewhere. Roughly by considering mappings from spin^h manifolds into the given compact manifold, one can attach numerical invariants (with $\mathbb{Z}$- and $\mathbb{Z}_2$- values) to the real vector bundle in question, and one expects these numerical invariants to completely determine the bundle up to stable equivalence. A quick thought on odd primary torsion in real K-group shows these invariants are unfortunately not enough, the way to fix it is to include mappings from spin^h manifolds with boudary. For this, we need the index theory for spin^h manifolds with boudary discussed in §3.6.

We should point out, many aspects herein have appeared in math and physics literature. [AM21] contains a nice survey on where the spin^h structures appeared in the literature. Nagase [Nag95] discussed Dirac operators on spin^h manifolds and found a Lichnerowicz-Weitzenböck type formula. The index of such a Dirac operator is known to Mayer [May65], and also discussed in Bär [Bär99]. The natural transformation we shall construct is studied by Freed and Hopkins [FH21].

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1. Quaternionic Clifford modules

1.1. **Quaternionic Clifford algebras.** We will write $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively for the real, complex and quaternion number-fields. If $\mathbb{K}$ is any one of these fields, $\mathbb{K}(n)$ will be the full $n \times n$ matrix algebra over $\mathbb{K}$. The following identities are well-known [ABS64]:

$$\begin{aligned} &\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{K} \cong \mathbb{K}(n), \mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) \cong \mathbb{R}(nm) \text{ for all } n, m \\ &\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \\ &\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(2) \\ &\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4) \end{aligned} \tag{1}$$

Let Cl_n be the real Clifford algebra associated to $\mathbb{R}^n$ with quadratic form given by square of the Euclidean norm. If $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$ then Cl_n is the universal associative $\mathbb{R}$-algebra generated by a unit and the symbols $e_1, \ldots, e_n$ subject to the relations $e_i^2 = -1$; $e_ie_j + e_je_i = 0, i \neq j$. As a $\mathbb{R}$-vector space, Cl_n is of dimension 2^n with a basis given by

$$\{e_{i_1} e_{i_2} \cdots e_{i_k} | i_1 < i_2 < \cdots < i_k, 0 \leq k \leq n\}.$$

The complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ is the associative $\mathbb{C}$-algebra $\mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$. The real and complex Clifford algebras are "periodic":

$$\begin{aligned} \mathrm{Cl}_{n+8} &\cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_8 \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{R}(16) \\ \mathbb{C}\mathrm{l}_{n+2} &\cong \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}_2 \cong \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2) \end{aligned} \tag{2}$$

Definition 1.1. We define the (n-th) quaternionic Clifford algebra $\mathrm{Cl}_{n,\mathbb{H}}$ to be the associative $\mathbb{R}$-algebra $\mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H}$ and define the (n-th) complexified quaternionic Clifford algebra $\mathbb{C}\mathrm{l}_{n,\mathbb{H}}$ to be the associative $\mathbb{C}$-algebra $\mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 1.2. For $n \geq 0$, there are isomorphisms of associative $\mathbb{R}$-algebras

$$\mathrm{Cl}_{n+4} \cong \mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{R}(2), \quad \mathrm{Cl}_{n+4,\mathbb{H}} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{R}(8);$$

and isomorphisms of associative $\mathbb{C}$-algebras

$$\mathbb{C}\mathrm{l}_{n,\mathbb{H}} \cong \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2).$$

Proof. The complex case directly follows from definition and (1):

$$\mathbb{C}\mathrm{l}_{n,\mathbb{H}} = \mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}(2) \cong \mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2).$$

In the real case, we shall first prove $\mathrm{Cl}_{n+4} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H}(2)$, or equivalently $\mathrm{Cl}_{n+4} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_4$ since $\mathbb{H}(2) \cong \mathrm{Cl}_4$. Let $e_1, e_2, \dots, e_{n+4}$ be an orthonormal basis of $\mathbb{R}^{n+4}$. Let $e'_1, \dots, e'_n$ and $e''_1, \dots, e''_4$ denote standard generators of Cl_n and Cl_4 respectively. Define a linear map $f : \mathbb{R}^{n+4} \to \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_4$ by

$$f(e_i) = \begin{cases} 1 \otimes e''_i & \text{for } 1 \le i \le 4 \\ e'_{i-4} \otimes e''_1 e''_2 e''_3 e''_4 & \text{for } 5 \le i \le n+4 \end{cases} \tag{3}$$

It is straightforward to check that $f(e_i)^2 = -1$ and $f(e_i)f(e_j) + f(e_j)f(e_i) = 0$ for $i \neq j$. Therefore f extends to an algebra morphism $\mathrm{Cl}_{n+4} \to \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_4$. Now notice f maps onto a set of generators and the two algebras in question have the same dimension, we conclude $\mathrm{Cl}_{n+4} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_4$. Combining this isomorphism with (1) we get:

$$\mathrm{Cl}_{n+4} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(2) \cong \mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{R}(2),$$
$$\mathrm{Cl}_{n+4,\mathbb{H}} = \mathrm{Cl}_{n+4} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H}(2) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{R}(8).$$

∎

Remark 1.3. In particular, Cl_{n+4} is a matrix algebra over $\mathrm{Cl}_{n,\mathbb{H}}$, thus Cl_{n+4} is *Morita equivalent* to $\mathrm{Cl}_{n,\mathbb{H}}$. This means, $V \mapsto V \otimes_{\mathbb{R}} \mathbb{R}^2$ is an equivalence between the category of (left) $\mathbb{R}$-modules of $\mathrm{Cl}_{n,\mathbb{H}}$ and that of Cl_{n+4}, where $V \otimes_{\mathbb{R}} \mathbb{R}^2$ is realized as a Cl_{n+4}-module through the isomorphism $\mathrm{Cl}_{n+4} \cong \mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{R}(2)$ and $\mathbb{R}(2)$ acts on $\mathbb{R}^2$ by left matrix multiplication. The same comments apply to $\mathrm{Cl}_{n+4,\mathbb{H}}$ and Cl_n as well.

Corollary 1.4. There are "periodicity" isomorphisms

$$\mathrm{Cl}_{n+8,\mathbb{H}} \cong \mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathrm{Cl}_8, \quad \mathbb{C}\mathrm{l}_{n+2,\mathbb{H}} \cong \mathbb{C}\mathrm{l}_{n,\mathbb{H}} \otimes_{\mathbb{C}} \mathbb{C}\mathrm{l}_2 .$$

Proof. These isomorphisms are obtained by applying Proposition 1.2 twice. ∎

Using the classification of the real Clifford algebras and Proposition 1.2, all the quaternionic Clifford algebras $\mathrm{Cl}_{n,\mathbb{H}}$ and their complexifiations $\mathbb{C}\mathrm{l}_{n,\mathbb{H}}$ can be easily deduced from the following table.

TABLE 1. Clifford algebras

n	Cl_n	$\mathbb{C}\mathrm{l}_n$	$\mathrm{Cl}_{n,\mathbb{H}}$	$\mathbb{C}\mathrm{l}_{n,\mathbb{H}}$
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}(2)$
1	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$
2	$\mathbb{H}$	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$
3	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$
5	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$
6	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{C}(16) \oplus \mathbb{C}(16)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$

The real Clifford algebra Cl_n admits a canonical automorphism of order 2 extended from the antipodal map $\mathbb{R}^n \to \mathbb{R}^n, e \mapsto -e$. The eigenspace decomposition of this automorphism defines a $\mathbb{Z}_2$-grading

$$\mathrm{Cl}_n = \mathrm{Cl}_n^0 \oplus \mathrm{Cl}_n^1$$

where Cl_n^α is the eigenspace of eigenvalue $(-1)^\alpha$ for $\alpha = 0, 1$. As a $\mathbb{R}$-vector space Cl_n^0 (resp. Cl_n^1) is spanned by products of even (resp. odd) numbers of e_i's. The quaternionic Clifford algebra $\mathrm{Cl}_{n,\mathbb{H}}$ inherits a natural $\mathbb{Z}_2$-grading by setting

$$\mathrm{Cl}_{n,\mathbb{H}}^\alpha = \mathrm{Cl}_n^\alpha \otimes_{\mathbb{R}} \mathbb{H} \quad (\alpha = 0, 1).$$

Remark 1.5. Note that the even part (i.e. the degree 0 part) of each quaternionic Clifford algebra forms a subalgebra. It is known that $\mathbb{R}^n \to \mathrm{Cl}^0_{n+1}$, $e_j \mapsto e_j e_{n+1}$ extends to an isomorphism $\mathrm{Cl}_n \cong \mathrm{Cl}^0_{n+1}$. Consequently this induces an isomorphism $\mathrm{Cl}_{n,\mathbb{H}} \cong \mathrm{Cl}^0_{n+1,\mathbb{H}}$.

Recall for $\mathbb{Z}_2$-graded vector spaces $V = V^0 \oplus V^1$ and $W = W^0 \oplus W^1$, their $\mathbb{Z}_2$-graded tensor product $V \hat{\otimes} W = (V\hat{\otimes}W)^0 \oplus (V\hat{\otimes}W)^1$ is defined to be

$$(V\hat{\otimes}W)^0 = (V^0 \otimes W^0) \oplus (V^1 \otimes W^1), \quad (V\hat{\otimes}W)^1 = (V^1 \otimes W^0) \oplus (V^0 \otimes W^1).$$

If further V and W are $\mathbb{Z}_2$-graded (associative) algebras, then $V\hat{\otimes}W$ is made into a $\mathbb{Z}_2$-graded algebra with the usual Koszul rule: $(v\hat{\otimes}w)\cdot(v'\hat{\otimes}w') = (-1)^{\deg w \deg v'} vv'\hat{\otimes}ww'$. The following lemma highlights the importance of the $\mathbb{Z}_2$-gradings on Clifford algebras.

Lemma 1.6 (see [LM89, Prop. 1.5])**.** Let e_i, e'_i, e''_i be standard orthonormal base of $\mathbb{R}^{m+n}, \mathbb{R}^m, \mathbb{R}^n$ respectively. Then the linear map $\mathbb{R}^{m+n} \to \mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_n$ given by

$$e_i \mapsto \begin{cases} e'_i\hat{\otimes}1 & \text{for } i \le m \\ 1\hat{\otimes}e''_{i-m} & \text{for } i > m \end{cases}$$

extends to an isomorphism of $\mathbb{Z}_2$-graded algebras

$$\mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_n \cong \mathrm{Cl}_{m+n}$$

Similarly $\mathbb{C}\mathrm{l}_m \hat{\otimes}_{\mathbb{C}} \mathbb{C}\mathrm{l}_n \cong \mathbb{C}\mathrm{l}_{m+n}$. ∎

Proposition 1.7. For all $m, n \ge 0$ there are isomorphisms of $\mathbb{Z}_2$-graded algebras

(i) $\mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_{n,\mathbb{H}} \cong \mathrm{Cl}_{m+n,\mathbb{H}}$;
(ii) $\mathrm{Cl}_{m,\mathbb{H}} \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_{n,\mathbb{H}} \cong \mathrm{Cl}_{m+n} \otimes_{\mathbb{R}} \mathbb{R}(4)$.

In particular, Cl_{m+n} is Morita equivalent to $\mathrm{Cl}_{m,\mathbb{H}} \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_{n,\mathbb{H}}$ as $\mathbb{Z}_2$-graded algebras. Here the $\mathbb{Z}_2$-grading on $\mathrm{Cl}_{m+n} \otimes_{\mathbb{R}} \mathbb{R}(4)$ is given by $(\mathrm{Cl}_{m+n} \otimes_{\mathbb{R}} \mathbb{R}(4))^\alpha = \mathrm{Cl}^\alpha_{m+n} \otimes_{\mathbb{R}} \mathbb{R}(4)$ for $\alpha = 0, 1$.

By Morita equivalence of $\mathbb{Z}_2$-graded algebras, we mean the Morita equivalence functors between the categories of modules preserve the $\mathbb{Z}_2$-gradings: the equivalence functor takes $\mathbb{Z}_2$-graded modules to $\mathbb{Z}_2$-graded modules.

Proof. (i) follows from Lemma 1.6 by tensoring with $\mathbb{H}$. For (ii), recall the $\mathbb{Z}_2$-grading on $\mathrm{Cl}_{n,\mathbb{H}}$ is given by $\mathrm{Cl}^\alpha_{n,\mathbb{H}} = \mathrm{Cl}^\alpha_n \otimes_{\mathbb{R}} \mathbb{H}$ for $\alpha = 0, 1$. Therefore as $\mathbb{Z}_2$-graded algebras

$$\mathrm{Cl}_{m,\mathbb{H}} \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_{n,\mathbb{H}} \cong (\mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_n) \otimes_{\mathbb{R}} (\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}) \cong \mathrm{Cl}_{n+m} \otimes_{\mathbb{R}} \mathbb{R}(4).$$

Since $\mathbb{R}(4)$ does not contribute to the $\mathbb{Z}_2$-grading of $\mathrm{Cl}_{n+m} \otimes_{\mathbb{R}} \mathbb{R}(4)$, the Morita equivalence for Cl_{n+m} and $\mathrm{Cl}_{n+m} \otimes \mathbb{R}(4)$ is a $\mathbb{Z}_2$-graded one. ∎

1.2. **Quaternionic Clifford modules.**

Definition 1.8. An $\mathbb{H}$-module of Cl_n is a pair (V, ϕ) consisting of a left $\mathbb{H}$-module V, i.e. a quaternionic vector space, together with a morphism of associative $\mathbb{R}$-algebras $\phi : \mathrm{Cl}_n \to \mathrm{End}_{\mathbb{H}}(V)$ where $\mathrm{End}_{\mathbb{H}}(V)$ is the associative $\mathbb{R}$-algebra consisting of quaternionic linear operators on V.

Lemma 1.9. The category of $\mathbb{H}$-modules of Cl_n is isomorphic to the category of $\mathbb{R}$-modules of $\mathrm{Cl}_{n,\mathbb{H}}$.

Proof. Given an $\mathbb{H}$-module (V, ϕ) of Cl_n, we construct a $\mathbb{R}$-module of $\mathrm{Cl}_{n,\mathbb{H}}$ as follows. Let $V_{\mathbb{R}}$ be the underlying $\mathbb{R}$-vector space of V and we define a $\mathbb{R}$-linear action of $\mathrm{Cl}_{n,\mathbb{H}}$ on $V_{\mathbb{R}}$ given on simple elements of form $a \otimes z \in \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H} = \mathrm{Cl}_{n,\mathbb{H}}$ by

$$(a \otimes z) \cdot v = \phi(a)(z \cdot v).$$

On the other hand, given a $\mathbb{R}$-module W of $\mathrm{Cl}_{n,\mathbb{H}}$, we may equip it with a left $\mathbb{H}$-module structure through the action of $1 \otimes \mathbb{H} \subset \mathrm{Cl}_{n,\mathbb{H}}$. Denote this left $\mathbb{H}$-module by $W_{\mathbb{H}}$. Then since $\mathrm{Cl}_n \otimes 1$ and $1 \otimes \mathbb{H}$ commute within $\mathrm{Cl}_{n,\mathbb{H}}$, the action of $\mathrm{Cl}_n \otimes 1 \subset \mathrm{Cl}_{n,\mathbb{H}}$ on W becomes an $\mathbb{H}$-linear action on $W_{\mathbb{H}}$. Thus $W_{\mathbb{H}}$ is an $\mathbb{H}$-module of Cl_n.

The functors $V \mapsto V_{\mathbb{R}}$ and $W \mapsto W_{\mathbb{H}}$ establish the desired isomorphism of categories. ∎

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, let $\mathfrak{M}_n^{\mathbb{K}}$ (resp. $\hat{\mathfrak{M}}_n^{\mathbb{K}}$) denote the Grothendieck group of equivalence classes of ungraded (resp. $\mathbb{Z}_2$-graded) finite dimensional $\mathbb{K}$-modules of Cl_n with respect to direct sum. Since Cl_n is semi-simple (from Table 1), $\mathfrak{M}_n^{\mathbb{K}}$ (resp. $\hat{\mathfrak{M}}_n^{\mathbb{K}}$) is a free abelian group generated by inequivalent irreducible ungraded (reps. $\mathbb{Z}_2$-graded) $\mathbb{K}$-modules.

The ungraded and $\mathbb{Z}_2$-graded modules are related as follows. Given an ungraded $\mathbb{K}$-module V of Cl_n and a $\mathbb{Z}_2$-graded $\mathbb{K}$-module $W = W^0 \oplus W^1$ of Cl_{n+1}, by identifying $\mathrm{Cl}_n \cong \mathrm{Cl}_{n+1}^0$ as mentioned in Remark 1.5, the functors

$$V \mapsto V \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n+1} = (V \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n+1}^0) \oplus (V \otimes_{\mathrm{Cl}_n} \mathrm{Cl}_{n+1}^1), \quad W \mapsto W^0$$

are inverses to each other, implying the category of ungraded $\mathbb{K}$-modules of Cl_n is isomorphic to the category of $\mathbb{Z}_2$-graded $\mathbb{K}$-modules of Cl_{n+1}. Therefore we have:

Lemma 1.10. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $n \geq 0$, there are isomorphisms $\mathfrak{M}_n^{\mathbb{K}} \cong \hat{\mathfrak{M}}_{n+1}^{\mathbb{K}}$. ■

Remark 1.11. In the special case $n = 0$, $\mathrm{Cl}_0 = \mathbb{R}$ is concentrated in degree 0. So Cl_0 has two inequivalent irreducible $\mathbb{Z}_2$-graded $\mathbb{K}$-modules, each of which is of dimension one, concentrated in degree 0 and 1 respectively. Hence $\hat{\mathfrak{M}}_0^{\mathbb{K}} \cong \mathbb{Z} + \mathbb{Z}$.

For the purpose of building Atiyah-Bott-Shapiro isomorphism, let us consider the isometric embedding $i : \mathbb{R}^n \to \mathbb{R}^{n+1}, e \mapsto (e, 0)$. This induces an inclusion $i_* : \mathrm{Cl}_n \to \mathrm{Cl}_{n+1}$ of $\mathbb{Z}_2$-graded algebras, which in turn yields restriction homomorphisms

$$i^* : \hat{\mathfrak{M}}_{n+1}^{\mathbb{K}} \to \hat{\mathfrak{M}}_n^{\mathbb{K}}, \quad i^* : \mathfrak{M}_{n+1}^{\mathbb{K}} \to \mathfrak{M}_n^{\mathbb{K}}.$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, we denote the cokernel of i^* by $\hat{\mathfrak{N}}_n^{\mathbb{K}} := \hat{\mathfrak{M}}_n^{\mathbb{K}} / i^* \hat{\mathfrak{M}}_{n+1}^{\mathbb{K}}$.

Proposition 1.12. For $n \geq 0$, there are isomorphisms

(i) $\mathfrak{M}_{n+4}^{\mathbb{R}} \cong \mathfrak{M}_n^{\mathbb{H}}$ and $\mathfrak{M}_{n+4}^{\mathbb{H}} \cong \mathfrak{M}_n^{\mathbb{R}}$
(ii) $\hat{\mathfrak{M}}_{n+4}^{\mathbb{R}} \cong \hat{\mathfrak{M}}_n^{\mathbb{H}}$ and $\hat{\mathfrak{M}}_{n+4}^{\mathbb{H}} \cong \hat{\mathfrak{M}}_n^{\mathbb{R}}$
(iii) $\hat{\mathfrak{N}}_n^{\mathbb{H}} \cong \hat{\mathfrak{N}}_{n+4}^{\mathbb{R}}$ and $\hat{\mathfrak{N}}_n^{\mathbb{R}} \cong \hat{\mathfrak{N}}_{n+4}^{\mathbb{H}}$.

Proof. (i) follows from the Morita equivalences built in Proposition 1.2. For $n \geq 1$, (ii) follows from (i) by applying Lemma 1.10. And (iii) follows from (ii) by first observing the following square commutes

$$\begin{tikzcd} \hat{\mathfrak{M}}_n^{\mathbb{K}} \arrow[d, "\cong"] & \hat{\mathfrak{M}}_{n+1}^{\mathbb{K}} \arrow[l, "i^*"'] \arrow[d, "\cong"] \\ \mathfrak{M}_{n-1}^{\mathbb{K}} & \mathfrak{M}_n^{\mathbb{K}} \arrow[l, "i^*"'] \end{tikzcd}$$

where $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$ (also true for $\mathbb{K} = \mathbb{C}$). As such we have $\hat{\mathfrak{N}}_n^{\mathbb{K}} \cong \mathfrak{M}_{n-1}^{\mathbb{K}} / i^* \mathfrak{M}_n^{\mathbb{K}}$. Second, we note the linear map f in the proof of Proposition 1.2 is compatible with the isometric embedding i, that is, the following diagram commutes

$$\begin{tikzcd} \mathbb{R}^{n+3} \arrow[r, "i"] \arrow[d, "f"] & \mathbb{R}^{n+4} \arrow[d, "f"] \\ \mathrm{Cl}_{n-1} \otimes_{\mathbb{R}} \mathbb{H}(2) \arrow[r, "i_* \otimes 1"] & \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H}(2) \end{tikzcd}$$

Therefore the isomorphisms in (i) commute with i^*, this completes the proof of (ii) and (iii) for $n \geq 1$. In the special case $n = 0$, one can easily verify (ii) and (iii) hold using the classification in Table 1. ■

Using Lemma 1.10 and Proposition 1.12, we can determine all the groups $\hat{\mathfrak{M}}_n^{\mathbb{H}}$, $\hat{\mathfrak{N}}_n^{\mathbb{H}}$ from the knowledge of $\hat{\mathfrak{M}}_n^{\mathbb{R}}$, $\hat{\mathfrak{N}}_n^{\mathbb{R}}$.

TABLE 2. real and quaternionic Clifford modules

n	0	1	2	3	4	5	6	7	8
$\hat{\mathfrak{M}}_n^{\mathbb{R}}$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$
$\hat{\mathfrak{M}}_n^{\mathbb{H}}$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$
$\hat{\mathfrak{N}}_n^{\mathbb{R}}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\hat{\mathfrak{N}}_n^{\mathbb{H}}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

We shall now describe the generators of $\hat{\mathfrak{M}}_n^{\mathbb{R}}$ and $\hat{\mathfrak{M}}_n^{\mathbb{H}}$ more concretely. From Table 1, whenever $n \not\equiv 3 \mod 4$, Cl_n and $\mathrm{Cl}_{n,\mathbb{H}}$ are of form $\mathbb{K}(N)$. It is well known that $\mathbb{K}(N)$ has a unique (equivalence class of) irreducible $\mathbb{R}$-module, given by the left matrix multiplication on $\mathbb{K}^N$. In view of Lemma 1.10, we have described generators of $\hat{\mathfrak{M}}_n^{\mathbb{R}}$ and $\hat{\mathfrak{M}}_n^{\mathbb{H}}$ for all $n \not\equiv 0 \mod 4$.

When $n \equiv 0 \bmod 4$, the following lemma is the key to our analysis in these dimensions.

Lemma 1.13 (see [LM89, Prop. 3.3])**.** Let $\omega_n = e_1 e_2 \cdots e_n$ be the (oriented) volume element of Cl_n. Then

(i) $\omega_n^2 = (-1)^{n(n+1)/2}$.
(ii) $e\omega_n = (-1)^{n-1}\omega_n e$ for all $e \in \mathbb{R}^n$.

■

Let $\mathrm{Cl}_4 = \mathbb{H}(2)$ act on $\mathbb{H}^2$ by left matrix multiplication. Then since $(\omega_4)^2 = 1$, $\mathbb{H}^2$ splits into a direct sum of ± 1 eigenspaces $(1 \pm \omega_4)\mathbb{H}^2$ of ω_4, denoted by $\mathbb{H}_\pm$. Since $e\omega_4 = -\omega_4 e$, multiplication by any $e \in \mathbb{R}^4 - 0$ yields an isomorphism of real vector spaces $\mathbb{H}_+ \cong \mathbb{H}_-$. So each of $\mathbb{H}_\pm$ is of real dimension 4. Further, since ω_4 commutes with Cl_4^0, $\mathbb{H}_\pm$ are invariant under the action of $\mathrm{Cl}_4^0 \cong \mathrm{Cl}_3$. Thus we may treat $\mathbb{H}_\pm$ as Cl_3-modules. Notice now ω_3 is in the center of Cl_3 and the action of ω_3 on $\mathbb{H}_\pm$ is through ω_4, we conclude $\mathbb{H}_\pm$ are inequivalent as Cl_3-modules. The two inequivalent $\mathbb{Z}_2$-graded $\mathbb{R}$-modules of Cl_4 corresponding to the two inequivalent Cl_3-modules $\mathbb{H}_\pm$, denoted by $\Delta_{4,\mathbb{R}}^\pm$, are tautologous: the underlying vector spaces of $\Delta_{4,\mathbb{R}}^\pm$ are both simply $\mathbb{H}^2$, with $\mathbb{Z}_2$-gradings given by

$$\Delta_{4,\mathbb{R}}^{\pm,0} = \mathbb{H}_\pm, \Delta_{4,\mathbb{R}}^{\pm,1} = \mathbb{H}_\mp.$$

As $\mathbb{H}^2$ is an irreducible ungraded Cl_4-module, $\Delta_{4,\mathbb{R}}^\pm$ are irreducible $\mathbb{Z}_2$-graded Cl_4-modules. It follows $\mathbb{H}_\pm$ are irreducible ungraded $\mathbb{R}$-modules of Cl_3.

Observe that $\mathbb{H}^2$ carries a natural right $\mathbb{H}$-multiplication which commutes with the left matrix multiplication from Cl_4. Since $\mathbb{H}^{op} \cong \mathbb{H}$ by conjugation, left and right modules of $\mathbb{H}$ are no different. We can thus view $\mathbb{H}^2$ as a left $\mathbb{H}$-module and therefore an $\mathbb{H}$-module of Cl_4. Equipped with this $\mathbb{H}$-module structure, $\Delta_{4,\mathbb{R}}^\pm$ are enhanced into two inequivalent irreducible $\mathbb{Z}_2$-graded $\mathbb{H}$-modules of Cl_4, denoted by $\Delta_{4,\mathbb{H}}^\pm$.

Similarly by considering the eigenspace decomposition of the volume element ω_8 through the matrix multiplication of $\mathrm{Cl}_8 = \mathbb{R}(16)$ on $\mathbb{R}^{16}$, we obtain two inequivalent irreducible $\mathbb{Z}_2$-graded $\mathbb{R}$-modules $\Delta_{8,\mathbb{R}}^\pm$ with

$$\Delta_{8,\mathbb{R}}^{\pm,0} = \mathbb{R}_\pm^{16}, \Delta_{8,\mathbb{R}}^{\pm,1} = \mathbb{R}_\mp^{16}.$$

The two inequivalent irreducible $\mathbb{Z}_2$-graded $\mathbb{H}$-modules of Cl_8, denoted by $\Delta_{8,\mathbb{H}}^\pm$, can be obtained by considering the eigenspace decomposition of the volume element $\omega_8 \otimes 1$ through the matrix multiplication of $\mathrm{Cl}_{8,\mathbb{H}} = \mathbb{H}(16)$ on $\mathbb{H}^{16}$ as before. It is not hard to see

$$\Delta_{8,\mathbb{H}}^\pm = \Delta_{8,\mathbb{R}}^\pm \otimes_\mathbb{R} \mathbb{H}.$$

Using periodicity, we now have a complete description of irreducible $\mathbb{Z}_2$-graded (and ungraded) $\mathbb{R}$- and $\mathbb{H}$-modules for the Clifford algebras.

Definition 1.14. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, let $\Delta_{n,\mathbb{K}}$ denote the unique (up to equivalence) irreducible $\mathbb{Z}_2$-graded $\mathbb{K}$-module of Cl_n for $n \not\equiv 0 \bmod 4$. For $n \equiv 0 \bmod 4$, let $\Delta_{n,\mathbb{K}}^\pm$ denote the two inequivalent irreducible $\mathbb{Z}_2$-graded $\mathbb{K}$-module of Cl_n, so that ω_n acts on $\Delta_{n,\mathbb{K}}^{\pm,0}$ by ± 1. We call these modules the fundamental $\mathbb{Z}_2$-graded $\mathbb{K}$-modules of the Clifford algebras.

Next we consider complex modules of the quaternionic Clifford algebras. First of all, since $\mathbb{C}$ is commutative, $\mathbb{C}$-modules of $\mathrm{Cl}_{n,\mathbb{H}}$ is no different from $\mathbb{C}$-modules of $\mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\mathrm{l}_{n,\mathbb{H}}$. Second, from Proposition 1.2, $\mathbb{C}\mathrm{l}_{n,\mathbb{H}}$ is isomorphic to $\mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2)$ as associative $\mathbb{C}$-algebras. This isomorphism can be enhanced into a $\mathbb{Z}_2$-graded one if we grade $\mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2)$ by

$$(\mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2))^\alpha = \mathbb{C}\mathrm{l}_n^\alpha \otimes_{\mathbb{C}} \mathbb{C}(2) \quad (\alpha = 0, 1).$$

This implies $\mathbb{C}\mathrm{l}_{n,\mathbb{H}}$ is Morita equivalent to $\mathbb{C}\mathrm{l}_n$ as $\mathbb{Z}_2$-graded $\mathbb{C}$-algebras. As such, if V is a $\mathbb{Z}_2$-graded $\mathbb{C}$-module of $\mathbb{C}\mathrm{l}_n$, then $V \otimes_{\mathbb{C}} \mathbb{C}^2$ is a $\mathbb{Z}_2$-graded $\mathbb{C}$-module of $\mathbb{C}\mathrm{l}_n \otimes_{\mathbb{C}} \mathbb{C}(2) \cong \mathbb{C}\mathrm{l}_{n,\mathbb{H}}$, with the natural $\mathrm{Cl}_{n,\mathbb{H}}$-action induced from the $\mathbb{C}\mathrm{l}_n$-module structure of V and the left matrix multiplication of $\mathbb{C}(2)$ on $\mathbb{C}^2$; conversely every $\mathbb{Z}_2$-graded $\mathbb{C}$-module of $\mathbb{C}\mathrm{l}_{n,\mathbb{H}}$, up to equivalence, arises in this way.

Therefore we denote the Grothendieck group of finite dimensional ungraded (resp. $\mathbb{Z}_2$-graded) $\mathbb{C}$-modules of $\mathrm{Cl}_{n,\mathbb{H}}$ by $\mathfrak{M}_n^{\mathbb{C}^2}$ (resp. $\hat{\mathfrak{M}}_n^{\mathbb{C}^2}$). Similar to the real case, for each $n \geq 0$ we have $\mathfrak{M}_n^{\mathbb{C}^2} \cong \hat{\mathfrak{M}}_{n+1}^{\mathbb{C}^2}$, and $- \otimes_{\mathbb{C}} \mathbb{C}^2$ induces a group isomorphism

$$\hat{\mathfrak{M}}_n^{\mathbb{C}} \xrightarrow{\cong} \hat{\mathfrak{M}}_n^{\mathbb{C}^2}$$

which in turn yields a group isomorphism

$$\hat{\mathfrak{N}}_n^{\mathbb{C}} \xrightarrow{\cong} \hat{\mathfrak{N}}_n^{\mathbb{C}^2}$$

where $\hat{\mathfrak{N}}_n^{\mathbb{C}^2} := \hat{\mathfrak{M}}_n^{\mathbb{C}^2} / i^* \hat{\mathfrak{M}}_{n+1}^{\mathbb{C}^2}$.

Explicit generators of $\hat{\mathfrak{M}}_n^{\mathbb{C}}$, and consequently generators of $\hat{\mathfrak{M}}_n^{\mathbb{C}^2}$, can be constructed in a similar way as did in real case. For instance, consider the left matrix multiplication of $\mathbb{C}\mathrm{l}_{2n} = \mathbb{C}(2n)$ on $\mathbb{C}^{2n}$ and observe the *complex volume element* $\omega_{2n}^{\mathbb{C}} = (\sqrt{-1})^n \omega_{2n}$ satisfies $(\omega_{2n}^{\mathbb{C}})^2 = 1$, we obtain $\mathbb{Z}_2$-graded $\mathbb{C}$-modules $\Delta_{2n,\mathbb{C}}^{\pm}$ for $\mathbb{C}\mathrm{l}_{2n}$ by

$$\Delta_{2n,\mathbb{C}}^{\pm,0} = (1 \pm \omega_{2n}^{\mathbb{C}}) \cdot \mathbb{C}^{2n}, \quad \Delta_{2n,\mathbb{C}}^{\pm,1} = (1 \mp \omega_{2n}^{\mathbb{C}}) \cdot \mathbb{C}^{2n}.$$

Consequently we have $\mathbb{Z}_2$-graded $\mathbb{C}$-modules $\Delta_{2n,\mathbb{C}^2}^{\pm} = \Delta_{2n,\mathbb{C}}^{\pm} \otimes_{\mathbb{C}} \mathbb{C}^2$ for $\mathbb{C}\mathrm{l}_{2n,\mathbb{H}}$. We remark that if $n \equiv 4 \pmod 8$ then $\omega_n^{\mathbb{C}} = -\omega_n$ and if $n \equiv 0 \pmod 8$ then $\omega_n^{\mathbb{C}} = \omega_n$.

We now explain how $\mathbb{C}$-modules are related to $\mathbb{R}$- and $\mathbb{H}$-modules.

Definition 1.15. If $\mathbb{K} \subset \mathbb{L}$ are two of the number-fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$. We will let $\varepsilon_{\mathbb{K}}^{\mathbb{L}}$ denote the field extension morphism induced by $- \otimes_{\mathbb{K}} \mathbb{L}$. We let $\rho_{\mathbb{L}}^{\mathbb{K}}$ denote the forgetful morphism induced by taking the underlying $\mathbb{K}$-vector space of an $\mathbb{L}$-vector space. Whenever it is clear in the context, we will simply write ε and ρ for $\varepsilon_{\mathbb{K}}^{\mathbb{L}}$ and $\rho_{\mathbb{L}}^{\mathbb{K}}$ respectively.

Proposition 1.16. (i) The composition $\mathfrak{M}_n^{\mathbb{C}} \xrightarrow[\cong]{-\otimes \mathbb{C}^2} \mathfrak{M}_n^{\mathbb{C}^2} \xrightarrow{\rho_{\mathbb{C}}^{\mathbb{R}}} \mathfrak{M}_n^{\mathbb{H}} \xrightarrow{\rho_{\mathbb{H}}^{\mathbb{C}}} \mathfrak{M}_n^{\mathbb{C}}$ is multiplication by 2.

(ii) The composition $\mathfrak{M}_n^{\mathbb{H}} \xrightarrow{\varepsilon_{\mathbb{R}}^{\mathbb{C}}} \mathfrak{M}_n^{\mathbb{C}^2} \xrightarrow{\rho_{\mathbb{C}}^{\mathbb{R}}} \mathfrak{M}_n^{\mathbb{H}}$ is multiplication by 2. Moreover,
   (a) if $n \equiv 2, 3, 4 \bmod 8$, then $\varepsilon_{\mathbb{R}}^{\mathbb{C}} : \mathfrak{M}_n^{\mathbb{H}} \to \mathfrak{M}_n^{\mathbb{C}^2}$ is an isomorphism;
   (b) if $n \equiv 6, 7, 8 \bmod 8$, then $\rho_{\mathbb{C}}^{\mathbb{R}} : \mathfrak{M}_n^{\mathbb{C}^2} \to \mathfrak{M}_n^{\mathbb{H}}$ is an isomorphism.

Proof. The first two assertions are obvious. For (ii)(a) and (ii)(b), by periodicity we may assume $n \leq 8$, then (ii)(a) and (ii)(b) follow case by case from Table 1 by dimension counts. For instance, for $n = 3$ there are two inequivalent irreducible $\mathbb{R}$-modules $\Delta_{4,\mathbb{H}}^{\pm,0}$ of $\mathrm{Cl}_{3,\mathbb{H}}$, each of which is of real dimension 4. Then each of $\varepsilon_{\mathbb{R}}^{\mathbb{C}}(\Delta_{4,\mathbb{H}}^{\pm,0})$ is of complex dimension 4, distinguished by the action of the volume element. Hence they must be the two inequivalent irreducible complex modules of $\mathbb{C}\mathrm{l}_{3,\mathbb{H}} = \mathbb{C}(4) + \mathbb{C}(4)$; indeed $\varepsilon_{\mathbb{R}}^{\mathbb{C}}(\Delta_{4,\mathbb{H}}^{\pm,0}) = \Delta_{4,\mathbb{C}}^{\mp,0} \otimes_{\mathbb{C}} \mathbb{C}^2$. The other cases are similar. ∎

Corollary 1.17. (i) The composition $\hat{\mathfrak{N}}_n^{\mathbb{C}} \xrightarrow[\cong]{-\otimes \mathbb{C}^2} \hat{\mathfrak{N}}_n^{\mathbb{C}^2} \xrightarrow{\rho_{\mathbb{C}}^{\mathbb{R}}} \hat{\mathfrak{N}}_n^{\mathbb{H}} \xrightarrow{\rho_{\mathbb{H}}^{\mathbb{C}}} \hat{\mathfrak{N}}_n^{\mathbb{C}}$ is multiplication by 2.

(ii) The composition $\hat{\mathfrak{N}}_n^{\mathbb{H}} \xrightarrow{\varepsilon_{\mathbb{R}}^{\mathbb{C}}} \hat{\mathfrak{N}}_n^{\mathbb{C}^2} \xrightarrow{\rho_{\mathbb{C}}^{\mathbb{R}}} \hat{\mathfrak{N}}_n^{\mathbb{H}}$ is multiplication by 2. Moreover,
   (a) if $n \equiv 4 \bmod 8$, then $\varepsilon_{\mathbb{R}}^{\mathbb{C}} : \hat{\mathfrak{N}}_n^{\mathbb{H}} \to \hat{\mathfrak{N}}_n^{\mathbb{C}^2}$ is an isomorphism;
   (b) if $n \equiv 0 \bmod 8$, then $\rho_{\mathbb{C}}^{\mathbb{R}} : \hat{\mathfrak{N}}_n^{\mathbb{C}^2} \to \hat{\mathfrak{N}}_n^{\mathbb{H}}$ is an isomorphism.

Proof. This follows from Proposition 1.16. We only prove (ii)(a) and the rest is similar. From Proposition 1.16 we have $\varepsilon_{\mathbb{R}}^{\mathbb{C}} : \hat{\mathfrak{M}}_n^{\mathbb{H}} \xrightarrow{\cong} \hat{\mathfrak{M}}_n^{\mathbb{C}^2}$ for $n \equiv 3, 4, 5 \bmod 8$. Then from the commutative square

$$\begin{tikzcd}
\hat{\mathfrak{M}}_n^{\mathbb{H}} \arrow[r, "\varepsilon_{\mathbb{R}}^{\mathbb{C}}"] & \hat{\mathfrak{M}}_n^{\mathbb{C}^2} \\
\hat{\mathfrak{M}}_{n+1}^{\mathbb{H}} \arrow[u, "i^*"] \arrow[r, "\varepsilon_{\mathbb{R}}^{\mathbb{C}}"] & \hat{\mathfrak{M}}_{n+1}^{\mathbb{C}^2} \arrow[u, "i^*"]
\end{tikzcd}$$

we conclude $\varepsilon_{\mathbb{R}}^{\mathbb{C}} : \hat{\mathfrak{N}}_n^{\mathbb{H}} \xrightarrow{\cong} \hat{\mathfrak{N}}_n^{\mathbb{C}^2}$ for $n \equiv 4 \bmod 8$. The same holds for $n \equiv 3 \bmod 8$, but in that case both $\hat{\mathfrak{N}}_n^{\mathbb{H}}$ and $\hat{\mathfrak{N}}_n^{\mathbb{C}^2}$ are zero. ∎

So far we have determined the groups $\hat{\mathfrak{M}}_n^{\mathbb{K}}$ and described their generators. We now consider the graded group

$$\hat{\mathfrak{M}}_*^{\mathbb{K}} := \sum_{n \geq 0} \hat{\mathfrak{M}}_n^{\mathbb{K}}.$$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let V and W be $\mathbb{Z}_2$-graded $\mathbb{K}$-modules of Cl_m and Cl_n respectively. Then $V \hat{\otimes}_{\mathbb{K}} W$ is made into a $\mathbb{Z}_2$-graded $\mathbb{K}$-module of $\mathrm{Cl}_{m+n} \cong \mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_n$ by linearly extending the action given on simple elements via

$$(x \hat{\otimes} y) \cdot (v \hat{\otimes} w) = (-1)^{\deg y \deg v} (x \cdot v) \hat{\otimes} (y \cdot w),$$

where x, y, v, w are homogeneous elements of $\mathrm{Cl}_m, \mathrm{Cl}_n, V, W$ respectively. The $\mathbb{Z}_2$-graded tensor product then induces a natural pairing

$$\hat{\mathfrak{M}}_m^{\mathbb{K}} \otimes_{\mathbb{Z}} \hat{\mathfrak{M}}_n^{\mathbb{K}} \to \hat{\mathfrak{M}}_{m+n}^{\mathbb{K}}.$$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $(\hat{\mathfrak{M}}_*^{\mathbb{K}}, \oplus, \hat{\otimes}_{\mathbb{K}})$ is a commutative graded ring with unit. Moreover, $i^* \hat{\mathfrak{M}}_*^{\mathbb{K}}$ is a homogenenous ideal, and therefore $(\hat{\mathfrak{N}}_*^{\mathbb{K}}, \oplus, \hat{\otimes}_{\mathbb{K}})$ is a graded ring.

In contrast, since $\mathbb{H}$ is not commutative, there is no good notion of tensor product in the category of left $\mathbb{H}$-modules, so $\hat{\mathfrak{M}}_*^{\mathbb{H}}$ does not form a ring. Nevertheless we have:

Lemma 1.18. The natural pairing

$$\hat{\mathfrak{M}}_m^{\mathbb{R}} \otimes_{\mathbb{Z}} \hat{\mathfrak{M}}_n^{\mathbb{H}} \to \hat{\mathfrak{M}}_{m+n}^{\mathbb{H}}$$

induced by the $\mathbb{Z}_2$-graded tensor product makes $\hat{\mathfrak{M}}_*^{\mathbb{H}}$ into a graded $\hat{\mathfrak{M}}_*^{\mathbb{R}}$-module. Moreover this pairing descends to a pairing

$$\hat{\mathfrak{N}}_m^{\mathbb{R}} \otimes_{\mathbb{Z}} \hat{\mathfrak{N}}_n^{\mathbb{H}} \to \hat{\mathfrak{N}}_{m+n}^{\mathbb{H}}$$

making $\hat{\mathfrak{N}}_*^{\mathbb{H}}$ into a graded $\hat{\mathfrak{N}}_*^{\mathbb{R}}$-module.

Proof. The first assertion is obvious. That the paring descends follows from the commutative diagram

$$\begin{tikzcd}
\hat{\mathfrak{M}}_{m+1}^{\mathbb{R}} \otimes_{\mathbb{Z}} \hat{\mathfrak{M}}_n^{\mathbb{H}} \arrow[r, "i^* \otimes 1"] \arrow[d] & \hat{\mathfrak{M}}_m^{\mathbb{R}} \otimes_{\mathbb{Z}} \hat{\mathfrak{M}}_n^{\mathbb{H}} \arrow[d] & \hat{\mathfrak{M}}_m^{\mathbb{R}} \otimes_{\mathbb{Z}} \hat{\mathfrak{M}}_{n+1}^{\mathbb{H}} \arrow[l, "1 \otimes i^*"'] \arrow[d] \\
\hat{\mathfrak{M}}_{m+n+1}^{\mathbb{H}} \arrow[r, "i^*"] & \hat{\mathfrak{M}}_{m+n}^{\mathbb{H}} & \hat{\mathfrak{M}}_{m+n+1}^{\mathbb{H}} \arrow[l, "i^*"']
\end{tikzcd}$$

which is induced from the commutative square

$$\begin{tikzcd}
\mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_n \arrow[r, "\cong"] \arrow[d, "1 \hat{\otimes} i_*"] & \mathrm{Cl}_{m+n} \arrow[d, "i_*"] \\
\mathrm{Cl}_m \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_{n+1} \arrow[r, "\cong"] & \mathrm{Cl}_{m+n+1}
\end{tikzcd}$$

∎

Remark 1.19. One can also interpret elements in $\hat{\mathfrak{M}}_n^{\mathbb{H}}$ as $\mathbb{Z}_2$-graded $\mathbb{R}$-modules of $\mathrm{Cl}_{n,\mathbb{H}}$, and then use Proposition 1.7(i) to equip $\hat{\mathfrak{M}}_*^{\mathbb{H}}$ with a graded $\hat{\mathfrak{M}}_*^{\mathbb{R}}$-module structure. It is clear this is the same one as discussed in the lemma.

Lemma 1.20. For $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$, in $\hat{\mathfrak{M}}^{\mathbb{K}}_*$ we have

$$\begin{aligned}\Delta^+_{8,\mathbb{R}}\hat{\otimes}\Delta_{n,\mathbb{K}} &= \Delta_{n+8,\mathbb{K}} \quad \text{for } n\not\equiv 0 \mod 4\\ \Delta^+_{8,\mathbb{R}}\hat{\otimes}\Delta^\pm_{n,\mathbb{K}} &= \Delta^\pm_{n+8,\mathbb{K}} \quad \text{for } n\equiv 0 \mod 4.\end{aligned}$$

Proof. For $n\not\equiv 0 \mod 4$, $\Delta^+_{8,\mathbb{R}}\hat{\otimes}\Delta_{n,\mathbb{K}}$ must be irreducible by dimension counts, but $\Delta_{n+8,\mathbb{K}}$ is the unique equivalence class of irreducible $\mathbb{Z}_2$-graded $\mathbb{K}$-modules of Cl_{n+8}. The proof is similar for $n\equiv 0 \mod 4$ by observing $(\omega_8\hat{\otimes}1)\cdot(1\hat{\otimes}\omega_n)=\omega_{n+8}$ in $\mathrm{Cl}_8\mathbin{\hat{\otimes}}\mathrm{Cl}_n\cong\mathrm{Cl}_{n+8}$. ■

Proposition 1.21. For $\mathbb{K}=\mathbb{R}$ or $\mathbb{H}$, $\Delta^+_{8,\mathbb{R}}\hat{\otimes}_{\mathbb{R}}-$ induces "periodicity" isomorphisms of graded $\hat{\mathfrak{M}}^{\mathbb{R}}_*$- and $\hat{\mathfrak{N}}^{\mathbb{R}}_*$-modules

$$\hat{\mathfrak{M}}^{\mathbb{K}}_* \xrightarrow{\cong} \hat{\mathfrak{M}}^{\mathbb{K}}_{*+8}, \quad \hat{\mathfrak{N}}^{\mathbb{K}}_* \xrightarrow{\cong} \hat{\mathfrak{N}}^{\mathbb{K}}_{*+8}.$$

Proof. The first isomorphism follows from Lemma 1.20. The second isomorphism follows from the first one and the commutative square

$$\begin{tikzcd}
\hat{\mathfrak{M}}^{\mathbb{K}}_{n+1} \arrow[r, "\Delta^+_{8,\mathbb{R}}\hat{\otimes}-"] \arrow[d, "i^*"] & \hat{\mathfrak{M}}^{\mathbb{K}}_{n+9} \arrow[d, "i^*"] \\
\hat{\mathfrak{M}}^{\mathbb{K}}_{n} \arrow[r, "\Delta^+_{8,\mathbb{R}}\hat{\otimes}-"] & \hat{\mathfrak{M}}^{\mathbb{K}}_{n+8}
\end{tikzcd}$$

■

This 8-fold periodicity splits into two 4-fold periodicities, interchanging $\mathbb{R}$ and $\mathbb{H}$.

Lemma 1.22. (i) In $\hat{\mathfrak{M}}^{\mathbb{H}}_*$ we have

$$\begin{aligned}\Delta_{n,\mathbb{R}}\hat{\otimes}\Delta^+_{4,\mathbb{H}} &= \Delta_{n+4,\mathbb{H}} \quad \text{for } n\not\equiv 0 \mod 4\\ \Delta^\pm_{n,\mathbb{R}}\hat{\otimes}\Delta^+_{4,\mathbb{H}} &= \Delta^\pm_{n+4,\mathbb{H}} \quad \text{for } n\equiv 0 \mod 4\end{aligned}$$

(ii) As equivalence classes of $\mathbb{Z}_2$-graded $\mathbb{R}$-modules of $\mathrm{Cl}_{n,\mathbb{H}}\mathbin{\hat{\otimes}_{\mathbb{R}}}\mathrm{Cl}_{4,\mathbb{H}}\cong\mathrm{Cl}_{n+4}\otimes_{\mathbb{R}}\mathbb{R}(4)$, we have

$$\begin{aligned}\Delta_{n,\mathbb{H}}\hat{\otimes}\Delta^+_{4,\mathbb{H}} &= \Delta_{n+4,\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{R}^4 \quad \text{for } n\not\equiv 0 \mod 4\\ \Delta^\pm_{n,\mathbb{H}}\hat{\otimes}\Delta^+_{4,\mathbb{H}} &= \Delta^\pm_{n+4,\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{R}^4 \quad \text{for } n\equiv 0 \mod 4\end{aligned}$$

Proof. The proof is similar to that of Lemma 1.20. Note under the isomorphism $\mathrm{Cl}_{n,\mathbb{H}}\mathbin{\hat{\otimes}_{\mathbb{R}}}\mathrm{Cl}_{4,\mathbb{H}}\cong\mathrm{Cl}_{n+4}\otimes_{\mathbb{R}}\mathbb{R}(4)$, $(\omega_n\hat{\otimes}1)\cdot(1\hat{\otimes}\omega_4)$ is identified with $\omega_{n+4}\otimes 1$. ■

Proposition 1.23. (i) $-\hat{\otimes}_{\mathbb{R}}\Delta^+_{4,\mathbb{H}}$ induces isomorphisms of graded $\hat{\mathfrak{M}}^{\mathbb{R}}_*$- and $\hat{\mathfrak{N}}^{\mathbb{R}}_*$-modules

$$\hat{\mathfrak{M}}^{\mathbb{R}}_* \xrightarrow{\cong} \hat{\mathfrak{M}}^{\mathbb{H}}_{*+4}, \quad \hat{\mathfrak{N}}^{\mathbb{R}}_* \xrightarrow{\cong} \hat{\mathfrak{N}}^{\mathbb{H}}_{*+4}.$$

(ii) By identifying equivalence classes of $\mathbb{Z}_2$-graded $\mathbb{R}$-modules of $\mathrm{Cl}_{n+4}\otimes_{\mathbb{R}}\mathbb{R}(4)$ with those of Cl_{n+4} through Morita equivalence, $-\hat{\otimes}_{\mathbb{R}}\Delta^+_{4,\mathbb{H}}$ induces isomorphisms of graded $\hat{\mathfrak{M}}^{\mathbb{R}}_*$- and $\hat{\mathfrak{N}}^{\mathbb{R}}_*$-modules

$$\hat{\mathfrak{M}}^{\mathbb{H}}_* \xrightarrow{\cong} \hat{\mathfrak{M}}^{\mathbb{R}}_{*+4}, \quad \hat{\mathfrak{N}}^{\mathbb{H}}_* \xrightarrow{\cong} \hat{\mathfrak{N}}^{\mathbb{R}}_{*+4}.$$

Proof. This follows from Lemma 1.22. ■

Remark 1.24. Since $\Delta^+_{4,\mathbb{H}}\hat{\otimes}_{\mathbb{R}}\Delta^+_{4,\mathbb{H}} = \Delta^+_{8,\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{R}^4$, the composition of the two 4-fold periodicity isomorphisms in Lemma 1.22 recovers the full 8-fold periodicity.

1.3. **Atiyah-Bott-Shapiro isomorphism.** For any $\mathbb{Z}_2$-graded $\mathbb{R}$-module $V = V^0 \oplus V^1$ of Cl_n, we associate to it an element $\varphi(V) \in \mathrm{KO}(D^n, \partial D^n)$ by setting

$$\varphi(V) := [\mathcal{V}^0, \mathcal{V}^1; \mu]$$

where D^n is the unit disk in $\mathbb{R}^n$, $\mathcal{V}^\alpha = D^n \times V^\alpha$ for $\alpha = 0, 1$, and μ is the Clifford module multiplication. That is

$$\mu : \mathbb{R}^n \times V \to V, \quad (e, v) \mapsto \mu_e(v) = e \cdot v$$

We note the Clifford multiplication $\mu_e : V \to V$ interchanges V^0 and V^1, and satisfies $\mu_e^2 = -\|e\|^2 \cdot 1$ for $e \in \mathbb{R}^n$. In particular, when restricted to ∂D^n, μ is a skew-adjoint isomorphism.

It is clear $\varphi(V)$ only depends on the isomorphism class of Clifford modules, thus we have a homomorphism

$$\varphi : \hat{\mathfrak{M}}^{\mathbb{R}}_n \to \mathrm{KO}(D^n, \partial D^n) \cong \mathrm{KO}^{-n}(\mathrm{pt}).$$

If the $\mathbb{Z}_2$-graded module V arises by restriction of some $\mathbb{Z}_2$-graded module of Cl_{n+1}, then the isomorphism μ extends to D^n by identifying D^n with the upper hemisphere of $S^n = \partial D^{n+1} \subset \mathbb{R}^{n+1} \subset \mathrm{Cl}_{n+1}$. Therefore, φ descends to a (graded) homomorphism

$$\varphi : \hat{\mathfrak{N}}^{\mathbb{R}}_* \to \mathrm{KO}^{-*}(\mathrm{pt}).$$

Similar constructions apply to $\mathbb{C}$- and $\mathbb{H}$-modules as well, yielding graded homomorphisms:

$$\varphi^c : \hat{\mathfrak{N}}^{\mathbb{C}}_* \to \mathrm{KU}^{-*}(\mathrm{pt}),$$
$$\varphi^h : \hat{\mathfrak{N}}^{\mathbb{H}}_* \to \mathrm{KSp}^{-*}(\mathrm{pt}).$$

It is a celebrated theorem of Atiyah, Bott and Shapiro [ABS64] that φ and φ^c are isomorphisms of graded rings. In analogy, we will prove φ^h is an isomorphism. For this, we need

Lemma 1.25. The following square commutes.

$$\begin{tikzcd}
\hat{\mathfrak{N}}^{\mathbb{R}}_* \otimes \hat{\mathfrak{N}}^{\mathbb{H}}_* \arrow[r, "\hat{\otimes}"] \arrow[d, "\varphi\otimes\varphi^h"] & \hat{\mathfrak{N}}^{\mathbb{H}}_* \arrow[d, "\varphi^h"] \\
\mathrm{KO}^{-*}(\mathrm{pt}) \otimes \mathrm{KSp}^{-*}(\mathrm{pt}) \arrow[r, "\boxtimes"] & \mathrm{KSp}^{-*}(\mathrm{pt})
\end{tikzcd}$$

Here $\boxtimes$ is the module multiplication of KO on KSp.

Proof. The proof is the same as that of [ABS64, Proposition 11.1]. ∎

Theorem 1.26. $\varphi^h : \hat{\mathfrak{N}}^{\mathbb{H}}_* \to \mathrm{KSp}^{-*}(\mathrm{pt})$ is an isomorphism of graded $\mathrm{KO}^{-*}(\mathrm{pt})$-modules.

Proof. We identify $\hat{\mathfrak{N}}^{\mathbb{R}}_*$ with $\mathrm{KO}^{-*}(\mathrm{pt})$ through φ. From Lemma 1.25, φ^h is a morphism of $\mathrm{KO}^{-*}(\mathrm{pt})$-modules. Since both $\hat{\mathfrak{N}}^{\mathbb{H}}_*$ and $\mathrm{KSp}^{-*}(\mathrm{pt})$ are free $\mathrm{KO}^{-*}(\mathrm{pt})$-modules of rank one in degrees ≥ 4, it suffices to prove φ^h is an isomorphism in degree 0 and 4. In degree 0, $\mathrm{KSp}^0(\mathrm{pt})$ is generated by $\mathbb{H} \to \mathrm{pt}$, and $\varphi^h(\Delta^+_{0,\mathbb{H}})$ hits this generator. In degree 4, consider the commutative square

$$\begin{tikzcd}
\hat{\mathfrak{N}}^{\mathbb{H}}_4 \arrow[r, "\varphi^h"] \arrow[d, "\rho"] & \mathrm{KSp}^{-4}(\mathrm{pt}) \arrow[d, "\rho"] \\
\hat{\mathfrak{N}}^{\mathbb{R}}_4 \arrow[r, "\varphi"] & \mathrm{KO}^{-4}(\mathrm{pt})
\end{tikzcd}$$

By construction, $\rho(\Delta^+_{4,\mathbb{H}}) = \Delta^+_{4,\mathbb{R}}$, hence $\rho : \hat{\mathfrak{N}}^{\mathbb{H}}_4 \to \hat{\mathfrak{N}}^{\mathbb{R}}_4$ is an isomorphism. From [ABS64, Theorem 11.5], $\varphi : \hat{\mathfrak{N}}^{\mathbb{R}}_4 \to \mathrm{KO}^{-4}(\mathrm{pt})$ is an isomorphism. Finally thanks to [Bot59, 3.14], $\rho : \mathrm{KSp}^{-4}(\mathrm{pt}) \to \mathrm{KO}^{-4}(\mathrm{pt})$ is an isomorphism. We conclude φ^h is an isomorphism in degree 4. This completes the proof. ∎

From now on whenever it is clear in the context, we will identify $\hat{\mathfrak{N}}_*^{\mathbb{R}}, \hat{\mathfrak{N}}_*^{\mathbb{C}}$ and $\hat{\mathfrak{N}}_*^{\mathbb{H}}$ with the corresponding K groups of a point through φ, φ^c and φ^h respectively.

Further, we fix a set of (additive) generators for $\hat{\mathfrak{N}}_*^{\mathbb{K}}$. For $\mathbb{K} = \mathbb{C}$ or $\mathbb{C}^2$, let $\mathbf{\Delta}_{n,\mathbb{K}}$ denote the residue class of $\Delta_{n,\mathbb{K}}$ or $\Delta_{n,\mathbb{K}}^+$ in $\hat{\mathfrak{N}}_n^{\mathbb{K}}$. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, we choose generators more carefully. If $n \not\equiv 0 \mod 4$ we let $\mathbf{\Delta}_{n,\mathbb{K}}$ denote the residue class of $\Delta_{n,\mathbb{K}}$. If $n \equiv 0 \mod 8$, we let $\mathbf{\Delta}_{n,\mathbb{K}}$ denote the residue class of $\Delta_{n,\mathbb{K}}^+$ and for $n \equiv 4 \mod 8$, we let $\mathbf{\Delta}_{n,\mathbb{H}}$ denote the residue class of $\Delta_{n,\mathbb{K}}^-$. Our choice for $n \equiv 4 \mod 8$ is so made that $\varepsilon_{\mathbb{R}}^{\mathbb{C}}(\mathbf{\Delta}_{8k+4,\mathbb{H}}) = \mathbf{\Delta}_{8k+4,\mathbb{C}^2}$ and $\rho_{\mathbb{C}}^{\mathbb{R}}(\mathbf{\Delta}_{8k+4,\mathbb{C}}) = \mathbf{\Delta}_{8k+4,\mathbb{R}}$. When the degree n is clear in the context, we will simply write $\mathbf{\Delta}_{\mathbb{K}}$ for $\mathbf{\Delta}_{n,\mathbb{K}}$.

1.4. **Right modules and bimodules.** We now briefly discuss right and bi-modules of the quaternionic Clifford algebras. They will be used in computing Clifford index later on.

Let us begin by observing the category of right $\mathrm{Cl}_{n,\mathbb{H}}$-modules is naturally isomorphic to the category of left $\mathrm{Cl}_{n,\mathbb{H}}$-modules. Consider the transpose endomorphism $(-)^t : \mathrm{Cl}_n \to \mathrm{Cl}_n$ given on basis by

$$(e_{i_1}e_{i_2}\cdots e_{i_k})^t := e_{i_k}\cdots e_{i_2}e_{i_1} \quad (i_1 < \cdots < i_k).$$

It is easy to see that $(a^t)^t = a$ and $(ab)^t = b^t a^t$ for all $a, b \in \mathrm{Cl}_n$. Therefore the transpose $(-)^t$ is an isomorphism of algebras $(-)^t : \mathrm{Cl}_n \xrightarrow{\cong} \mathrm{Cl}_n^{op}$.

We extend the transpose to $\mathrm{Cl}_{n,\mathbb{H}}$ by setting on simple elements

$$(a \otimes z)^t := a^t \otimes \bar{z} \quad \text{for } a \in \mathrm{Cl}_n, z \in \mathbb{H}$$

and then linear extension. The extended transpose is an isomorphism of algebras $(-)^t : \mathrm{Cl}_{n,\mathbb{H}} \xrightarrow{\cong} \mathrm{Cl}_{n,\mathbb{H}}^{op}$, which in turn induces an isomorphism between the category of left and right modules of $\mathrm{Cl}_{n,\mathbb{H}}$ as follows.

Given a left module V of $\mathrm{Cl}_{n,\mathbb{H}}$, we can define a right module $\widetilde{V}$ whose underlying vector space is V, on which the right $\mathrm{Cl}_{n,\mathbb{H}}$-multiplication is given by

$$v \cdot a := (a^t) \cdot v$$

where $a \in \mathrm{Cl}_{n,\mathbb{H}}$ and $v \in V$. Then $V \mapsto \widetilde{V}$ gives an isomorphism between the category of left and right modules of $\mathrm{Cl}_{n,\mathbb{H}}$. Since we have classified all (finite-dimensional) left modules of $\mathrm{Cl}_{n,\mathbb{H}}$, we also obtain a classification of right modules.

Next we consider bimodules, i.e. left modules over

$$\mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,\mathbb{H}}^{op} \cong \mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathrm{Cl}_{n,\mathbb{H}}\,.$$

One can easily classify these algebras using Table 1 and consequently classify their left modules. Of particular interest is the canonical bimodule of $\mathrm{Cl}_{n,\mathbb{H}}$ which is $\mathrm{Cl}_{n,\mathbb{H}}$ itself via left and right multiplications.

For $n = 8k+4$, $\mathrm{Cl}_{8k+4,\mathbb{H}}$ is of form $\mathbb{R}(N)$. From standard representation theory, we have an isomorphism of real $\mathrm{Cl}_{8k+4,\mathbb{H}}$-bimodules

$$\mathrm{Cl}_{8k+4,\mathbb{H}} \cong \Delta_{8k+4,\mathbb{H}} \otimes_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}}.$$

Here $\Delta_{8k+4,\mathbb{H}}$ denotes the underlying ungraded left $\mathbb{R}$-module of $\mathrm{Cl}_{8k+4,\mathbb{H}}$ obtained from either $\Delta_{8k+4,\mathbb{H}}^+$ or $\Delta_{8k+4,\mathbb{H}}^-$. Since each one of the two inequivalent $\mathbb{Z}_2$-gradings on $\Delta_{8k+4,\mathbb{H}}$ is obtained from the other by interchanging the grading, either one gives the same $\mathbb{Z}_2$-grading on the tensor product $\Delta_{8k+4,\mathbb{H}} \otimes_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}}$, and the bimodule isomorphism above is now a $\mathbb{Z}_2$-graded one.

For $n = 8k+5$, $\mathrm{Cl}_{8k+5,\mathbb{H}}$ is of form $\mathbb{C}(N)$. Therefore all real modules of $\mathrm{Cl}_{8k+5,\mathbb{H}}$ are naturally $\mathbb{C}$-vector spaces, and similarly we have an isomorphism of real (and also complex) $\mathrm{Cl}_{8k+5,\mathbb{H}}$-bimodules

$$\mathrm{Cl}_{8k+5,\mathbb{H}} \cong \Delta_{8k+5,\mathbb{H}} \otimes_{\mathbb{C}} \widetilde{\Delta}_{8k+5,\mathbb{H}}.$$

For $n = 8k+6$, $\mathrm{Cl}_{8k+6,\mathbb{H}}$ is of form $\mathbb{H}(N)$, therefore every left (resp. right) real modules of $\mathrm{Cl}_{8k+6,\mathbb{H}}$ admits a right (resp. left) $\mathbb{H}$-action that commutes with the $\mathrm{Cl}_{8k+6,\mathbb{H}}$-action. In particular, $\Delta_{8k+6,\mathbb{H}}$ is a right $\mathbb{H}$-module and $\widetilde{\Delta}_{8k+6,\mathbb{H}}$ a left $\mathbb{H}$-module. Thus $\Delta_{8k+6,\mathbb{H}} \otimes_{\mathbb{H}} \widetilde{\Delta}_{8k+6,\mathbb{H}}$ makes sense and we have an isomorphism of real $\mathrm{Cl}_{8k+6,\mathbb{H}}$-bimodules

$$\mathrm{Cl}_{8k+6,\mathbb{H}} \cong \Delta_{8k+6,\mathbb{H}} \otimes_{\mathbb{H}} \widetilde{\Delta}_{8k+6,\mathbb{H}}.$$

For $n = 8k$, let us first consider $\mathbb{C}\mathrm{l}_{8k,\mathbb{H}}$. We have $\mathbb{C}\mathrm{l}_{8k,\mathbb{H}} \cong \Delta_{8k,\mathbb{C}^2} \otimes_{\mathbb{C}} \widetilde{\Delta}_{8k,\mathbb{C}^2}$ as $\mathbb{C}\mathrm{l}_{8k,\mathbb{H}}$-bimodules and therefore as $\mathrm{Cl}_{8k,\mathbb{H}}$-bimodules. Since $\mathbb{C}\mathrm{l}_{8k,\mathbb{H}} \cong \mathrm{Cl}_{8k,\mathbb{H}} \oplus_{\mathbb{R}} \mathrm{Cl}_{8k,\mathbb{H}}$ as real $\mathrm{Cl}_{8k,\mathbb{H}}$-bimodules, we conclude there is an isomorphism of real $\mathrm{Cl}_{8k,\mathbb{H}}$-bimodules

$$\mathrm{Cl}_{8k,\mathbb{H}} \oplus \mathrm{Cl}_{8k,\mathbb{H}} \cong \Delta_{8k,\mathbb{C}^2} \otimes_{\mathbb{C}} \widetilde{\Delta}_{8k,\mathbb{C}^2}.$$

We will write this as $\mathrm{Cl}_{8k,\mathbb{H}} \cong \frac{1}{2}\Delta_{8k,\mathbb{C}^2} \otimes_{\mathbb{C}} \widetilde{\Delta}_{8k,\mathbb{C}^2}$.

To summarize, we have proved:

Proposition 1.27. For $n \equiv 0, 4, 5, 6 \bmod 8$, there are isomorphisms of $\mathbb{Z}_2$-graded $\mathrm{Cl}_{n,\mathbb{H}}$-bimodules:

$$\begin{aligned}
\mathrm{Cl}_{8k,\mathbb{H}} &\cong \frac{1}{2}\Delta_{8k,\mathbb{C}^2} \otimes_{\mathbb{C}} \widetilde{\Delta}_{8k,\mathbb{C}^2} \\
\mathrm{Cl}_{8k+4,\mathbb{H}} &\cong \Delta_{8k+4,\mathbb{H}} \otimes_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}} \\
\mathrm{Cl}_{8k+5,\mathbb{H}} &\cong \Delta_{8k+5,\mathbb{H}} \otimes_{\mathbb{C}} \widetilde{\Delta}_{8k+5,\mathbb{H}} \\
\mathrm{Cl}_{8k+6,\mathbb{H}} &\cong \Delta_{8k+6,\mathbb{H}} \otimes_{\mathbb{H}} \widetilde{\Delta}_{8k+6,\mathbb{H}}
\end{aligned}$$

■

1.5. **KQ-theory and (1,1)-periodicity.** So far we have been restricting our attention to Clifford algebras associated to positive definite quadratic forms. In this subsection we study Clifford algebras associated to (non-degenerate) indefinite quadratic forms, as well as their modules. Since over $\mathbb{C}$ all non-degenerate quadratic forms look the same, we deal only with $\mathbb{R}$- and $\mathbb{H}$-modules here.

Let $\mathrm{Cl}_{r,s}$ be the Clifford algebra on $\mathbb{R}^{r+s} = \mathbb{R}^r \times \mathbb{R}^s$ with respect to the quadratic form $\|x\|^2 - \|y\|^2$ of signature (r, s) where $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. These algebras are also $\mathbb{Z}_2$-graded and there are $\mathbb{Z}_2$-graded algebra isomorphisms (see [LM89, Prop. 3.2])

$$\mathrm{Cl}_{r+r',s+s'} \cong \mathrm{Cl}_{r,s} \hat{\otimes}_{\mathbb{R}} \mathrm{Cl}_{r',s'} \tag{4}$$

for all $r, r', s, s' \geq 0$. Denote by $\mathrm{Cl}_{r,s,\mathbb{H}} := \mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{H}$ the quaternionification of $\mathrm{Cl}_{r,s}$.

Proposition 1.28. There are isomorphisms of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebras

$$\begin{aligned}
\mathrm{Cl}_{r+1,s+1} &\cong \mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{R}(2) \\
\mathrm{Cl}_{r+4,s,\mathbb{H}} &\cong \mathrm{Cl}_{r,s} \otimes_{\mathbb{R}} \mathbb{R}(8) \\
\mathrm{Cl}_{r+4,s} &\cong \mathrm{Cl}_{r,s,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{R}(2)
\end{aligned}$$

Proof. For the proof of the first isomorphism, see [LM89, Theorem 4.1]. We note in particular $\mathrm{Cl}_{1,1} = \mathbb{R}(2)$. The other two isomorphisms follow from (4) and Proposition 1.2. ■

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{H}$, let $\hat{\mathfrak{M}}^{\mathbb{K}}_{r,s}$ denote the Grothendieck group of (finite dimensional) $\mathbb{Z}_2$-graded $\mathbb{K}$-modules of $\mathrm{Cl}_{r,s}$, and set

$$\hat{\mathfrak{N}}^{\mathbb{K}}_{r,s} = \hat{\mathfrak{M}}^{\mathbb{K}}_{r,s} / i^* \hat{\mathfrak{M}}^{\mathbb{K}}_{r+1,s}$$

where i^* is induced by the inclusion $\mathbb{R}^r \times \mathbb{R}^s \hookrightarrow \mathbb{R}^{r+1} \times \mathbb{R}^s$, $(x, y) \mapsto (x, 0, y)$. Then naturally $(\hat{\mathfrak{M}}^{\mathbb{R}}_{*,*} = \sum_{r,s} \hat{\mathfrak{M}}^{\mathbb{R}}_{r,s}, \oplus, \hat{\otimes})$ is a bigraded ring over which $\hat{\mathfrak{M}}^{\mathbb{H}}_{*,*} = \sum_{r,s} \hat{\mathfrak{M}}^{\mathbb{H}}_{r,s}$ is a bigraded module. These structures descend to make $\hat{\mathfrak{N}}^{\mathbb{H}}_{*,*} = \sum_{r,s} \hat{\mathfrak{N}}^{\mathbb{H}}_{r,s}$ a bigraded module over the bigraded ring $\hat{\mathfrak{N}}^{\mathbb{R}}_{*,*} = \sum_{r,s} \hat{\mathfrak{N}}^{\mathbb{R}}_{r,s}$.

In [Ati66], Atiyah showed $\hat{\mathfrak{N}}^{\mathbb{R}}_{*,*}$ is naturally isomorphic to Real K-theory of a point:

$$\hat{\mathfrak{N}}^{\mathbb{R}}_{*,*} \cong \mathrm{KR}^{*,*}(\mathrm{pt}). \tag{5}$$

Recall the Real K-theory KR is a variant of K-theory defined on the category of real spaces. A real space is simply a space with involution, for example the set of complex points of a real algebraic variety with conjugation. Another important example is $\mathbb{R}^{r,s}$ whose underlying space is $\mathbb{R}^r \times \mathbb{R}^s$ with involution given by $(x, y) \mapsto (x, -y)$ for all $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. When $r = s$, we write $\mathbb{R}^{r,r} = \mathbb{C}^r$ where the involution on $\mathbb{C}^r$ is the complex conjugation. A Real bundle over a real space (X, f) is a complex vector bundle E over X together with a involution $j : E \to E$ covering the involution f on the base such that $j : E_x \to E_{fx}$ is $\mathbb{C}$-antilinear for all $x \in X$ and $j^2 \equiv 1$. For a real space X, $\mathrm{KR}(X)$ is the Grothendieck group of Real bundles over X. The reduced group $\widetilde{\mathrm{KR}}$, the compactly supported group KR_{cpt} and the relative group for a pair $\mathrm{KR}(-,-)$ are defined in the usual manner. If the involution on X is trivial, then $\mathrm{KR}(X) = \mathrm{KO}(X)$. Similar to the KO-theory, the KR-theory is a multiplicative theory with multiplication induced from tensor product: given Real bundles $(E, j) \to X$ and $(E', j') \to X'$, the bundle $\pi^* E \otimes_{\mathbb{C}} \pi'^* E'$ with $J = \pi^* j \otimes \pi'^* j'$ is a Real bundle over $X \times X'$, where π, π' are projections from $X \times X'$ onto X and X' respectively.

For any compact real pair (X, Y) we define higher KR groups by

$$\mathrm{KR}^{r,s}(X,Y)^1 = \mathrm{KR}(X \times D^{r,s}, X \times S^{r,s} \cup Y \times D^{r,s})$$

where $D^{r,s}$ and $S^{r,s}$ are the unit disk and unit sphere in $\mathbb{R}^{r,s}$ respectively with restricted involutions. The higher groups have their reduced, compactly supported counterparts as well.

With all these understood, we remark that $\mathrm{KR}^{r,s}_{cpt}(X) = \mathrm{KR}_{cpt}(X \times \mathbb{R}^{r,s})$, and that $\mathrm{KR}^{*,*}(\mathrm{pt})$ is a bigraded ring. In particular, $\mathrm{KR}^{r,0}(\mathrm{pt}) = \mathrm{KR}_{cpt}(\mathbb{R}^r) = \mathrm{KO}_{cpt}(\mathbb{R}^r) = \mathrm{KO}^{-r}(\mathrm{pt})$. We refer the reader to [Ati66] for more on KR-theory.

Now the isomorphism (5) is in fact a bigraded ring isomorphism. It is established in two steps. First, consider the algebra-with-involution $\mathrm{Cl}(\mathbb{R}^{r,s})$ whose underlying algebra is simply Cl_{r+s} on which the involution $c : \mathrm{Cl}(\mathbb{R}^{r,s}) \to \mathrm{Cl}(\mathbb{R}^{r,s})$ is extended from the involution on $\mathbb{R}^{r,s}$. A Real module of $\mathrm{Cl}(\mathbb{R}^{r,s})$ is a complex module V of $\mathrm{Cl}(\mathbb{R}^{r,s})$ together with a $\mathbb{C}$-antilinear involution $c : V \to V$ such that

$$c(a \cdot v) = c(a) \cdot c(v)$$

for all $a \in \mathrm{Cl}(\mathbb{R}^{r,s})$ and all $v \in V$. If $V = V^0 \oplus V^1$ is $\mathbb{Z}_2$-graded and $c(V^\alpha) = V^\alpha$ for $\alpha = 0, 1$, then V is a $\mathbb{Z}_2$-graded Real module for $\mathrm{Cl}(\mathbb{R}^{r,s})$. One can form the tensor product of Real modules as follows: let V, W be $\mathbb{Z}_2$-graded Real modules for $\mathrm{Cl}(\mathbb{R}^{r,s})$ and $\mathrm{Cl}(\mathbb{R}^{r',s'})$ respectively, then their tensor product $V \hat{\otimes}_{\mathbb{C}} W$ with the induced involution $c_V \hat{\otimes} c_W$ is a Real module for $\mathrm{Cl}(\mathbb{R}^{r,s}) \otimes \mathrm{Cl}(\mathbb{R}^{r',s'}) \cong \mathrm{Cl}(\mathbb{R}^{r+r',s+s'})$.

Now let $\hat{\mathfrak{M}}R_{r,s}$ denote the Grothendieck group of $\mathbb{Z}_2$-graded Real modules of $\mathrm{Cl}(\mathbb{R}^{r,s})$ and put $\hat{\mathfrak{N}}R_{r,s} = \hat{\mathfrak{M}}R_{r,s} / i^* \hat{\mathfrak{M}}R_{r+1,s}$. Then there are natural isomorphisms

$$\hat{\mathfrak{M}}^{\mathbb{R}}_{r,s} \xrightarrow{\cong} \hat{\mathfrak{M}}R_{r,s}$$
$$\hat{\mathfrak{N}}^{\mathbb{R}}_{r,s} \xrightarrow{\cong} \hat{\mathfrak{N}}R_{r,s}$$

by assigning to each $\mathbb{Z}_2$-graded $\mathbb{R}$-module V of $\mathrm{Cl}_{r,s}$, the $\mathbb{C}$-vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the involution given by complex conjugation and with the $\mathrm{Cl}(\mathbb{R}^{r,s})$-multiplication given by setting

$$(x, y) \cdot w := xw + iyw$$

for all $(x, y) \in \mathbb{R}^r \times \mathbb{R}^s = \mathbb{R}^{r,s}$.

Second, given a $\mathbb{Z}_2$-graded Real module $V = V^0 \oplus V^1$ for $\mathrm{Cl}(\mathbb{R}^{r,s})$, we construct an element $[\mathcal{V}^0, \mathcal{V}^1; \mu]$ in $\mathrm{KR}_{cpt}(\mathbb{R}^{r,s})$ by setting $\mathcal{V}^\alpha = \mathbb{R}^{r,s} \times V^\alpha$ for $\alpha = 0, 1$ and $\mu : \mathcal{V}^0 \to \mathcal{V}^1$ is again the Clifford module multiplication $\mu(z, v) = (z, z \cdot v)$ for $z = (x, y) \in \mathbb{R}^{r,s}$ and $v \in V^0$. Due to Atiyah, this turns out yields a bigraded ring isomorphism

$$\hat{\mathfrak{N}}R_{*,*} \xrightarrow{\cong} \mathrm{KR}^{*,*}(\mathrm{pt}).$$

We now analogously consider quaternionic modules of $\mathrm{Cl}_{r,s}$ and identify $\hat{\mathfrak{N}}^{\mathbb{H}}_{*,*}$ with the coefficient groups of certain K-theory. The appropriate K-theory is the Quaternionic K-theory KQ, defined also on the category of real spaces.

Definition 1.29. Let (X, f) be a real space. A Quaternionic bundle, or simply a Q-bundle, over X is a complex vector bundle E over X equipped with a map $j : E \to E$ covering f such that $j : E_x \to E_{fx}$ is $\mathbb{C}$-antilinear for all $x \in X$ and such that $j^2 \equiv -1$. For a real space X, $\mathrm{KQ}(X)$ is the Grothendieck group of Q-bundles over X.

Similar to the case of KR-theory, one can define KQ_{cpt} and $\mathrm{KQ}^{r,s}$ etc. Here we list some features of KQ.

[1] our notation $\mathrm{KR}^{r,s}$ agrees with [LM89], but differs from Atiyah's [Ati66] by switching r and s.

First, KQ is a module theory over KR, i.e. there is an external product

$$\mathrm{KR}^{r,s}(X) \otimes \mathrm{KQ}^{r',s'}(X') \to \mathrm{KQ}^{r+r',s+s'}(X \times X')$$

which is induced from the tensor product of Real and Quaternionic bundles: given $(E, j) \to X$ a Real bundle over X and $(E', j') \to Y$ a Quaternionic bundle over X', then $\pi^* E \otimes_{\mathbb{C}} \pi'^* E'$ with $\pi^* j \otimes \pi'^* j'$ is a Quaternionic bundle over $X \times X'$, where π, π' are projections from $X \times X'$ onto X and X' respectively.

Second, for any compact real space X, multiplication with the generator of $\mathrm{KQ}^{4,0}(\mathrm{pt}) \cong \mathrm{KSp}^{-4}(\mathrm{pt})$ yields an isomorphism

$$\mathrm{KR}^{*,*}(X) \xrightarrow{\cong} \mathrm{KQ}^{*+4,*}(X) \tag{6}$$

In particular $\mathrm{KQ}^{*+4,*}(\mathrm{pt})$ is a free module over $\mathrm{KR}^{*,*}(\mathrm{pt})$ generated by $\mathrm{KQ}^{4,0}(\mathrm{pt})$.

Finally, KQ-theory satisfies a $(1,1)$-periodicity: multiplication with the generator of $\mathrm{KR}^{1,1}(\mathrm{pt})$ yields an isomorphism

$$\mathrm{KQ}^{*,*}(X) \xrightarrow{\cong} \mathrm{KQ}^{*+1,*+1}(X) \tag{7}$$

We remark the same (1,1)-periodicity holds for KR-theory as well (see [Ati66]).

Both (6) and (7) should be known to Dupont (see [Dup69]), but he did not make them explicit and we fail to find complete proofs in the literature. Since the (1,1)-periodicity is central to our construction of topological index later on, we decide to supply justifications for (6) and (7) in the Appendix. The main point is to consider the $\mathbb{Z}_2$-graded theory $\mathrm{KM} = \mathrm{KR} \oplus \mathrm{KQ}$ where KR and KQ are put in degree 0 and 1 respectively. Then KM is a multiplicative theory whose multiplication respects its $\mathbb{Z}_2$-grading. Most of the results in [Ati66] can be worked out for KM as pointed out by Dupont. The corresponding results for KQ follow at once by exploiting the $\mathbb{Z}_2$-grading on KM.

We are now one construction away from stating and proving an Atiyah-Bott-Shapiro type isomorphism for KQ-theory.

Definition 1.30. By a Quaternionic module, or simply a Q-module, over the algebra-with-involution $\mathrm{Cl}(\mathbb{R}^{r,s})$ we mean a finite dimensional complex module V for $\mathrm{Cl}(\mathbb{R}^{r,s})$ together with a $\mathbb{C}$-antilinear map $j : V \to V$ so that $j^2 = -1$ and

$$j(a \cdot v) = c(a) \cdot j(v)$$

for all $a \in \mathrm{Cl}(\mathbb{R}^{r,s})$ and all $v \in V$. If in addition $V = V^0 \oplus V^1$ is $\mathbb{Z}_2$-graded with the property that $j(V^\alpha) = V^\alpha$ for $\alpha = 0, 1$, then V is called a $\mathbb{Z}_2$-graded Q-module for $\mathrm{Cl}(\mathbb{R}^{r,s})$.

Of course one can tensor $\mathbb{Z}_2$-graded Quaternionic modules by $\mathbb{Z}_2$-graded Real modules to obtain Quaternionic modules: simply form the $\mathbb{Z}_2$-graded tensor product over $\mathbb{C}$ and equip it with the tensor product of the two structures.

Let $\hat{\mathfrak{M}}Q_{r,s}$ denote the Grothendieck group of $\mathbb{Z}_2$-graded Q-modules of $\mathrm{Cl}(\mathbb{R}^{r,s})$ and define $\hat{\mathfrak{N}}Q_{r,s} = \hat{\mathfrak{M}}Q_{r,s}/i^*\hat{\mathfrak{M}}Q_{r+1,s}$.

Theorem 1.31. There are isomorphisms of bigraded $\mathrm{KR}^{*,*}(\mathrm{pt})$-modules

$$\hat{\mathfrak{N}}^{\mathbb{H}}_{*,*} \cong \hat{\mathfrak{N}}Q_{*,*} \cong \mathrm{KQ}^{*,*}(\mathrm{pt}).$$

Proof. Given any $\mathbb{H}$-module V of $\mathrm{Cl}_{r,s}$, let $V_{\mathbb{C}}$ be the underlying $\mathbb{C}$-module and $j : V_{\mathbb{C}} \to V_{\mathbb{C}}$ the multiplication by $j \in \mathbb{H}$. Then we may realize $(V_{\mathbb{C}}, j)$ as a Q-module for $\mathrm{Cl}(\mathbb{R}^{r,s})$ by setting

$$(x, y) \cdot v = x \cdot v + iy \cdot v$$

for all $(x, y) \in \mathbb{R}^r \times \mathbb{R}^s$ and then extend to an action of $\mathrm{Cl}(\mathbb{R}^{r,s})$ on $V_{\mathbb{C}}$. It is straightforward to verify $j(z \cdot v) = c(z) \cdot j(v)$ for all $z = (x, y)$ and all $v \in V_{\mathbb{C}}$ using $ij = -ji$. Hence $(V_{\mathbb{C}}, j)$ is indeed a well-defined Q-module for $\mathrm{Cl}(\mathbb{R}^{r,s})$. The functor $V \mapsto (V_{\mathbb{C}}, j)$ is clearly invertible, thus induces isomorphisms

$$\hat{\mathfrak{M}}^{\mathbb{H}}_{r,s} \xrightarrow{\cong} \hat{\mathfrak{M}}Q_{r,s}$$
$$\hat{\mathfrak{N}}^{\mathbb{H}}_{r,s} \xrightarrow{\cong} \hat{\mathfrak{N}}Q_{r,s}.$$

For the second isomorphism, given any $\mathbb{Z}_2$-graded Q-module (V, j) of $\mathrm{Cl}(\mathbb{R}^{r,s})$, we define an element $\varphi^Q(V) = [\mathcal{V}^0, \mathcal{V}^1; \mu]$ in $\mathrm{KQ}_{cpt}(\mathbb{R}^{r,s})$ by setting $\mathcal{V}^\alpha = \mathbb{R}^{r,s} \times V^\alpha$ and μ the Clifford multiplication. Given our previous discussions, it is now a routine to check $\varphi^Q(V)$ is well-defined and yields a homomorphism

$$\varphi^Q : \hat{\mathfrak{N}}Q_{r,s} \to \mathrm{KQ}^{r,s}(\mathrm{pt}).$$

The proof of Lemma 1.25 carries through to show φ^Q is a map of $\hat{\mathfrak{N}}R_{*,*} \cong \mathrm{KR}^{*,*}(\mathrm{pt})$-modules.

Finally to see φ^Q is an isomorphism, we use the Morita equivalences established in Proposition 1.28. From the first isomorphism in Proposition 1.28, we have (1,1)-periodicities

$$\hat{\mathfrak{N}}R_{*,*} \cong \hat{\mathfrak{N}}R_{*+1,*+1}$$
$$\hat{\mathfrak{N}}Q_{*,*} \cong \hat{\mathfrak{N}}Q_{*+1,*+1}.$$

Both are induced by multiplication with the generator of $\hat{\mathfrak{N}}R_{1,1} \cong \mathrm{KR}^{1,1}(\mathrm{pt}) = \mathbb{Z}$. These "periodicity" isomorphisms are compatible with the (1,1)-periodicities for KR- and KQ-theories: they are both induced by multiplication with the generator of $\mathrm{KR}^{1,1}(\mathrm{pt})$. Therefore, we are reduced to the case $r \geq 4$, i.e. we must show

$$\varphi^Q : \hat{\mathfrak{N}}Q_{*+4,*} \to \mathrm{KQ}^{*+4,*}(\mathrm{pt})$$

is an isomorphism. Now from the second isomorphism in Proposition 1.28, we can deduce $\hat{\mathfrak{N}}Q_{*+4,*}$ is a free $\hat{\mathfrak{N}}R_{*,*}$-module generated by $\hat{\mathfrak{N}}Q_{4,0} = \hat{\mathfrak{N}}^{\mathbb{H}}_{4,0} = \hat{\mathfrak{N}}^{\mathbb{H}}_4 = \mathbb{Z}$. Since $\mathrm{KQ}^{*+4,*}(\mathrm{pt})$ is a free module over $\mathrm{KR}^{*,*}(\mathrm{pt})$ generated by $\mathrm{KR}^{4,0}(\mathrm{pt}) = \mathrm{KSp}^{-4}(\mathrm{pt})$ and $\varphi^Q : \hat{\mathfrak{N}}Q_{4,0} \to \mathrm{KQ}^{4,0}(\mathrm{pt})$ coincides with $\varphi^h : \hat{\mathfrak{N}}^{\mathbb{H}}_4 \to \mathrm{KSp}^{-4}(\mathrm{pt})$ which we have proved to be an isomorphism, we conclude $\varphi^Q : \hat{\mathfrak{N}}Q_{*,*} \to \mathrm{KQ}^{*,*}(\mathrm{pt})$ is an isomorphism as desired. ∎

2. Spin^h vector bundles

2.1. **Spin^h structures on vector bundles.** Recall the group $\mathrm{Spin}(n)$ can be viewed as a subgroup of the multiplicative group $\mathrm{Cl}_n^\times$ of the real Clifford algebra Cl_n. Let $\mathrm{Sp}(1)$ be the group of unit quaternions, then we have a natural group homomorphism

$$\mathrm{Spin}(n) \times \mathrm{Sp}(1) \to \mathrm{Cl}_{n,\mathbb{H}}^\times = (\mathrm{Cl}_n \otimes_{\mathbb{R}} \mathbb{H})^\times,$$

whose kernel is the "diagonal" $\mathbb{Z}_2$ generated by $(-1,-1)$. By modding out the kernel, we obtain the group

$$\mathrm{Spin}^h(n) := \mathrm{Spin}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2 \subset \mathrm{Cl}_{n,\mathbb{H}}^\times.$$

Since $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^0$, we see $\mathrm{Spin}^h(n) \subset \mathrm{Cl}_{n,\mathbb{H}}^0$. From here, the representation theory of $\mathrm{Spin}^h(n)$ is closely related to that of $\mathrm{Cl}_{n,\mathbb{H}}$. For $V = V^0 \oplus V^1$ a $\mathbb{Z}_2$-graded $\mathbb{R}$-module (resp. $\mathbb{C}$-module) of $\mathrm{Cl}_{n,\mathbb{H}}$, we see V^0 is invariant under the $\mathrm{Cl}_{n,\mathbb{H}}^0$-action, hence by restricting the action of $\mathrm{Cl}_{n,\mathbb{H}}^0$ to $\mathrm{Spin}^h(n)$, V^0 becomes a real (resp. complex) representation of $\mathrm{Spin}^h(n)$.

Proposition 2.1. Let V be an irreducible $\mathbb{Z}_2$-graded real (resp. complex) module of $\mathrm{Cl}_{n,\mathbb{H}}$. Then V^0 is an irreducible real (resp. complex) representation of $\mathrm{Spin}^h(n)$.

Proof. Since V is an irreducible $\mathbb{Z}_2$-graded module of $\mathrm{Cl}_{n,\mathbb{H}}$, V^0 must be an irreducible module of $\mathrm{Cl}_{n,\mathbb{H}}^0$, otherwise V^0 contains a non-trivial proper submodule W^0 which then extends to a non-trivial $\mathbb{Z}_2$-graded proper submodule $W = W^0 \otimes_{\mathrm{Cl}_{n,\mathbb{H}}^0} \mathrm{Cl}_{n,\mathbb{H}} \subset V$. Now we note $\mathrm{Spin}(n)$ contains a set of generators of Cl_n^0, namely $e_{i_1} e_{i_2} \cdots e_{i_k}$ for k even, and meanwhile $\mathrm{Sp}(1)$ contains a set of generators of $\mathbb{H}$. Therefore $\mathrm{Spin}^h(n)$ contains a set of generators of $\mathrm{Cl}_{n,\mathbb{H}}^0$. This implies V^0, irreducible over $\mathrm{Cl}_{n,\mathbb{H}}^0$, is an irreducible representation of $\mathrm{Spin}^h(n)$. ∎

But $\mathrm{Spin}^h(n)$ owns more irreducible representations than $\mathrm{Cl}_{n,\mathbb{H}}$. For instance, through projections onto its two factors, $\mathrm{Spin}^h(n)$ admits two natural orthogonal representations

$$\begin{aligned} &\mathrm{Spin}^h(n) \to \mathrm{Spin}(n)/\mathbb{Z}_2 = \mathrm{SO}(n) \\ &\mathrm{Spin}^h(n) \to \mathrm{Sp}(1)/\mathbb{Z}_2 = \mathrm{SO}(3) \end{aligned} \tag{8}$$

Thus irreducible representations of $\mathrm{SO}(n)$ and $\mathrm{SO}(3)$ also become irreducible representations of $\mathrm{Spin}^h(n)$. By contrast, $\mathrm{Cl}_{n,\mathbb{H}}$ has only one or two irreducible representations from Table 1.

Now putting the two projections in (8) together, we obtain a short exact sequence of groups:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^h(n) \to \mathrm{SO}(n) \times \mathrm{SO}(3) \to 1 \tag{9}$$

The $\mathbb{Z}_2$ corresponds to ± 1 in $\mathrm{Cl}_{n,\mathbb{H}}$. So $\mathrm{Spin}^h(n)$ is a central extension of $\mathrm{SO}(n) \times \mathrm{SO}(3)$ by $\mathbb{Z}_2$. Group extensions of this type are classified by

$$H^2(BSO(n) \times BSO(3); \mathbb{Z}_2) = \{0, w_2, w_2', w_2 + w_2'\}$$

where $w_2 \in H^2(BSO(n); \mathbb{Z}_2)$ and $w_2' \in H^2(BSO(3); \mathbb{Z}_2)$ stand for the corresponding second Stiefel-Whitney classes. Clearly $\mathrm{Spin}^h(n)$ is the extension that corresponds to $w_2 + w_2'$; the other three elements $0, w_2, w_2'$ correspond to $\mathbb{Z}_2 \times \mathrm{SO}(n) \times \mathrm{SO}(3)$, $\mathrm{Spin}(n) \times \mathrm{SO}(3)$ and $\mathrm{SO}(n) \times \mathrm{Sp}(1)$ respectively. We note $w_2 = 0$ for $n < 2$, nevertheless the above assertion still holds. It is therefore convenient for us to make the following definition:

Definition 2.2. Let E be an oriented vector bundle of rank n furnished with a metric, and $P_{\mathrm{SO}}(E)$ is the oriented frame bundle of E. We say E admits a spin^h structure if one of the following equivalent conditions is satisfied:

(i) there is a rank 3 oriented vector-bundle-with-metric $\mathfrak{h}_E$ such that $w_2(\mathfrak{h}_E) = w_2(E)$.
(ii) there is a principal $\mathrm{Spin}^h(n)$-bundle $P_{\mathrm{Spin}^h}(E)$ and a map of principal bundles $P_{\mathrm{Spin}^h}(E) \to P_{\mathrm{SO}}(E)$ which is equivariant respect to $\mathrm{Spin}^h(n) \to \mathrm{SO}(n)$ in (8).

With a fixed choice of $\mathfrak{h}_E$ or $P_{\mathrm{Spin}^h}(E)$, we say E is a spin^h vector bundle. The bundles $\mathfrak{h}_E$ and $P_{\mathrm{Spin}^h}(E)$ are then called the canonical bundle and the structure bundle of the spin^h vector bundle E respectively.

Even though we used metrics in our definition, the existence of spin^h structures is really a topological (in fact homotopy-theoretical) question. The primary obstruction to the existence spin^h structures is the fifth integral Stiefel-Whiney class W_5 [AM21]; there are non-trivial secondary obstructions as well. We insist on including metrics in our discussion for it will be convenient for us later to construct Dirac operators.

Definition 2.3. We say a smooth manifold M a spin^h manifold if its tangent bundle is equipped with a spin^h structure. Spin^h manifolds with boundary and spin^h cobordism relations are defined in the usual way.

Example 2.4. Every closed oriented riemannian manifold of dimension ≤ 7 admits spin^h structures [AM21]. Every oriented riemannian 4-manifold (including non-compact ones) admits two natural spin^h structures whose canonical bundles are the bundle of self-dual two forms and the bundle of anti-self-dual two forms.

Example 2.5. Let F be a spin vector bundle of rank m on Y and E a spin^h vector bundle of rank n on X, then $F \times E$ is a spin^h vector bundle on $Y \times X$ with the canonical bundle $\mathfrak{h}_{F\times E} = \pi_X^* \mathfrak{h}_E$ where $\pi_X : Y \times X \to X$ is the projection onto X. Let $P_{\mathrm{Spin}}(F)$ denote the structural principal $\mathrm{Spin}(m)$-bundle associated to F, then the structure bundle $P_{\mathrm{Spin}^h}(F \times E)$ of $F \times E$ is derived from the principal bundle $P_{\mathrm{Spin}}(F) \times P_{\mathrm{Spin}^h}(E)$ through the natural homomorphism

$$\mathrm{Spin}(m) \times \mathrm{Spin}^h(n) \to \mathrm{Spin}^h(m+n)$$

induced from the isomorphism given in Proposition 1.7.

2.2. **Quaternionic Clifford and ^{h}spinor bundles.** Recall for a spin vector bundle $F \to Y$ of rank m, its Clifford bundle is defined to be the bundle of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra

$$\mathrm{Cl}(F) = P_{\mathrm{Spin}}(F) \times_{\mathrm{Ad}} \mathrm{Cl}_m$$

with the natural inherited $\mathbb{Z}_2$-grading, where $P_{\mathrm{Spin}}(F)$ is the principal $\mathrm{Spin}(m)$-bundle associated to F and $\mathrm{Spin}(m)$ acts on Cl_m through the adjoint representation

$$\mathrm{Ad} : \mathrm{Spin}(m) \to \mathrm{Aut}(\mathrm{Cl}_m), \quad g \mapsto \mathrm{Ad}_g(x) := gxg^{-1}, \text{ for } x \in \mathrm{Cl}_m .$$

Since $-1 \in \ker \mathrm{Ad}$, the adjoint representation descends to a representation $\mathrm{Ad} : \mathrm{SO}(m) \to \mathrm{Aut}(\mathrm{Cl}_n)$. As such, the Clifford bundle in fact only relies on the metric on F. Alternatively $\mathrm{Cl}(F)$ can be described as

$$\mathrm{Cl}(F) = \left(\sum_{r=0}^{\infty} \bigotimes^r F \right) \Big/ I(F),$$

where $I(F)$ is the bundle of ideals, whose fibre at $y \in Y$ is the two-sided ideal $I(F_y)$ in $\sum_{r=0}^{\infty} \bigotimes^r F_y$, generated by elements $e \otimes e + \|e\|^2$ for $e \in F_y$. In particular $\mathrm{Cl}(F_y)$ is the Clifford algebra generated by F_y with respect to the inner product on F_y.

Definition 2.6. The quaternionic Clifford bundle of a spin^h vector bundle $E \to X$ of rank n is the bundle of $\mathbb{Z}_2$-graded $\mathbb{R}$-algebra over X

$$\mathrm{Cl}_{\mathbb{H}}(E) = P_{\mathrm{Spin}^h}(E) \times_{\mathrm{Ad}^h} \mathrm{Cl}_{n,\mathbb{H}}$$

with the natural inherited $\mathbb{Z}_2$-grading, where $\mathrm{Spin}^h(n)$ acts on $\mathrm{Cl}_{n,\mathbb{H}}$ through the adjoint representation

$$\mathrm{Ad}^h : \mathrm{Spin}^h(n) \to \mathrm{Aut}(\mathrm{Cl}_{n,\mathbb{H}}), \quad g \mapsto \mathrm{Ad}^h_g(x) := gxg^{-1}, \text{ for } x \in \mathrm{Cl}_{n,\mathbb{H}}.$$

Since $(1,-1) \in \ker \mathrm{Ad}^h$, $\mathrm{Cl}_{\mathbb{H}}(E)$ depends only on the metrics on E and $\mathfrak{h}_E$. In fact

Lemma 2.7. $\mathrm{Cl}_{\mathbb{H}}(E) = \mathrm{Cl}(E) \otimes_{\mathbb{R}} \mathrm{Cl}^0(\mathfrak{h}_E)$.

Proof. Ad^h descends to a representation $\mathrm{SO}(n) \times \mathrm{SO}(3) \to \mathrm{Aut}(\mathrm{Cl}_{n,\mathbb{H}})$ which is clearly induced from tensoring the adjoint representation of $\mathrm{Spin}(n)$ on Cl_n and the adjoint representation of $\mathrm{Sp}(1) = \mathrm{Spin}(3)$ on $\mathbb{H} = \mathrm{Cl}^0_3$. ■

So the construction of the quaternionic Clifford bundle does not really require a spin^h structure. However, the presence of the spin^h structure will allow us to construct interesting bundles of modules over the quaternionic Clifford bundle.

Definition 2.8. Let $E \to X$ be a spin^h vector bundle of rank n. A real ^{h}spinor bundle of E is a bundle of the form

$$S_{\mathbb{H}}(E, V) := P_{\mathrm{Spin}^h}(E) \times_\mu V,$$

where V is a $\mathbb{R}$-module of $\mathrm{Cl}_{n,\mathbb{H}}$ and μ is the composition $\mathrm{Spin}^h(n) \subset \mathrm{Cl}^\times_{n,\mathbb{H}} \to GL_{\mathbb{R}}(V)$.

Similarly a complex ^{h}spinor bundle of E is a bundle of the form

$$S_{\mathbb{C}^2}(E, V_{\mathbb{C}}) := P_{\mathrm{Spin}^h}(E) \times_\mu V_{\mathbb{C}},$$

where $V_{\mathbb{C}}$ is a $\mathbb{C}$-module of $\mathrm{Cl}_{n,\mathbb{H}}$.

If the module V (or $V_{\mathbb{C}}$) is $\mathbb{Z}_2$-graded, the corresponding bundle is said to be $\mathbb{Z}_2$-graded.

Example 2.9 (fundamental $\mathbb{Z}_2$-graded ^{h}spinor bundle)**.** We denote the corresponding $\mathbb{Z}_2$-graded real ^{h}spinor bundle constructed from the $\mathbb{Z}_2$-graded modules $\Delta_{n,\mathbb{H}}$ (resp. $\Delta^{\pm}_{n,\mathbb{H}}$ if $n \equiv 0 \bmod 4$) by $\$_{\mathbb{H}}(E)$ (resp. $\$^{\pm}_{\mathbb{H}}(E)$). Similarly, $\$_{\mathbb{C}^2}(E)$ (resp. $\$^{\pm}_{\mathbb{C}^2}(E)$ if n even) denotes the $\mathbb{Z}_2$-graded complex ^{h}spinor bundle that corresponds to $\Delta_{n,\mathbb{C}^2}$ (resp. $\Delta^{\pm}_{n,\mathbb{C}^2}$). We call them the fundamental $\mathbb{Z}_2$-graded (real or complex) ^{h}spinor bundles of E.

Lemma 2.10. Let $S_{\mathbb{H}}(E)$ be a real ^{h}spinor bundle of a spin^h bundle E. Then $S_{\mathbb{H}}(E)$ is a bundle of modules over the bundle of algebras $\mathrm{Cl}_{\mathbb{H}}(E)$.

The corresponding facts hold in the complex and $\mathbb{Z}_2$-graded cases.

Proof. The diagram

$$\begin{tikzcd} P_{\mathrm{Spin}^h}(E) \times \mathrm{Cl}_{n,\mathbb{H}} \times V \arrow[r, "\mu"] \arrow[d, "\sigma_g"] & P_{\mathrm{Spin}^h}(E) \times V \arrow[d, "\sigma'_g"] \\ P_{\mathrm{Spin}^h}(E) \times \mathrm{Cl}_{n,\mathbb{H}} \times V \arrow[r, "\mu"] & P_{\mathrm{Spin}^h}(E) \times V \end{tikzcd}$$

given by

$$\begin{tikzcd}
(p,x,v) \arrow[r] \arrow[d] & (p,xv) \arrow[d] \\
(pg^{-1},gxg^{-1},gv) \arrow[r] & (pg^{-1},gxv)
\end{tikzcd}$$

clearly commutes. Therefore μ descends to a mapping $\mu : \mathrm{Cl}_{\mathbb{H}}(E)\otimes_{\mathbb{R}} S_{\mathbb{H}}(E) \to S_{\mathbb{H}}(E)$ which is easily seen to have the desired properties. The corresponding argument goes through in the complex and $\mathbb{Z}_2$-graded case. ■

We say that two (real or complex, graded or ungraded) ^{h}spinor bundles of E are equivalent if they are equivalent as bundles of $\mathrm{Cl}_{\mathbb{H}}(E)$-modules. A (real or complex, graded or ungraded) bundle of $\mathrm{Cl}_{\mathbb{H}}(E)$-module is called irreducible if at each $x \in X$ the fibre is irreducible as a module over $\mathrm{Cl}_{\mathbb{H}}(E_x)$.

It is clear every ^{h}spinor bundle of E can be decomposed into a direct sum of irreducible ones. With the assumption that X is connected, the number of equivalence classes of irreducible (real or complex, graded or ungraded) $\mathrm{Cl}_{\mathbb{H}}(E)$-modules is exactly the number of irreducible (real or complex, graded or ungraded) modules of $\mathrm{Cl}_{n,\mathbb{H}}$; further the irreducible ones are exactly the fundamental ones.

2.3. **Thom classes and Thom isomorphisms.** In [ABS64], φ is upgraded, for each spin vector bundle $F \to Y$ of rank m, to a homomorphism

$$\varphi_F : \hat{\mathfrak{N}}^{\mathbb{R}}_m \to \mathrm{KO}(D(F), \partial D(F)) = \widetilde{\mathrm{KO}}(Th(F))$$

where $Th(F) = D(F)/\partial D(F)$ is the Thom space of F. If $F' \to Y'$ is another spin vector bundle of rank m', then we have a commutative diagram

$$\begin{tikzcd}
\hat{\mathfrak{N}}^{\mathbb{R}}_m \otimes \hat{\mathfrak{N}}^{\mathbb{R}}_{m'} \arrow[r, "\hat{\otimes}"] \arrow[d, "\varphi_F\otimes\varphi_{F'}"] & \hat{\mathfrak{N}}^{\mathbb{R}}_{m+m'} \arrow[d, "\varphi_{F\times F'}"] \\
\widetilde{\mathrm{KO}}(Th(F)) \otimes \widetilde{\mathrm{KO}}(Th(F')) \arrow[r, "\boxtimes"] & \widetilde{\mathrm{KO}}(Th(F\times F'))
\end{tikzcd} \tag{10}$$

where $\boxtimes$ is the external product in KO.

Depending whether we treat representatives of elements in $\hat{\mathfrak{N}}^{\mathbb{H}}_*$ as $\mathbb{Z}_2$-graded $\mathbb{R}$-modules of the quternionic Clifford algebras or as $\mathbb{Z}_2$-graded $\mathbb{H}$-modules of the real Clifford algebras, we may analogously upgrade φ^h in two different directions:

- for each *spin*^{h} vector bundle, we obtain a homomorphism from $\hat{\mathfrak{N}}^{\mathbb{H}}_*$ to the reduced KO-group of its Thom space;
- for each *spin* vector bundle, we obtain a homomorphism from $\hat{\mathfrak{N}}^{\mathbb{H}}_*$ to the reduced KSp-group of its Thom space.

We now spell out our construction in the spin^h case. Let $E \to X$ be a spin^h vector bundle of rank n, let $D(E), \partial D(E)$ denote the (closed) unit disk and sphere bundle of E respectively. Let $\pi : D(E) \to X$ be the bundle projection.

For any $\mathbb{Z}_2$-graded $\mathbb{R}$-module V of $\mathrm{Cl}_{n,\mathbb{H}}$, we have the associated $\mathbb{Z}_2$-graded ^{h}spinor bundle $S_{\mathbb{H}}(E,V)$. Then the pull-backs of the degree 0 and degree 1 parts of $S_{\mathbb{H}}(E,V)$ are canonically isomorphic on $\partial D(E)$ via the map

$$\mu_e : \left(\pi^* S_{\mathbb{H}}\left(E, V^0\right)\right)_e \to \left(\pi^* S_{\mathbb{H}}\left(E, V^1\right)\right)_e$$

given at $e \in \partial D(E)$ by

$$\mu_e(\sigma) = e\cdot\sigma.$$

That is, Clifford multiplication by e itself. Since $e\cdot e = -\|e\|^2 = -1$, each map μ_e is an isomorphism. This defines a difference element

$$\varphi^h_E(V) := \left[\pi^* S_{\mathbb{H}}\left(E, V^0\right), \pi^* S_{\mathbb{H}}\left(E, V^1\right); \mu\right] \in \mathrm{KO}(D(E), \partial D(E)) \cong \widetilde{\mathrm{KO}}(Th(E)).$$

Clearly $\varphi^h_E(V)$ depends only on the equivalence class of V. If V is restricted from a $\mathbb{Z}_2$-graded module of $\mathrm{Cl}_{n+1,\mathbb{H}}$, i.e. $[V]$ belongs to $i^*\hat{\mathfrak{M}}^{\mathbb{H}}_{n+1}$, then we may embed E into $E \oplus \mathbb{R}$, where $\mathbb{R}$ is the trivialized bundle with a nowhere zero cross-section e_{n+1} and a metric so that e_{n+1} is of norm one.

This way the ^{h}spinor bundle $S_{\mathbb{H}}(E,V)$ is contained in $S_{\mathbb{H}}(E\oplus\mathbb{R},V)$ and $\pi^*\not{S}_{\mathbb{H}}(E,V)$ extends to a bundle over $D(E\oplus\mathbb{R})$. Then

$$\tilde{\mu}_e(\sigma)=(e+(\sqrt{1-\|e\|^2})e_{n+1})\cdot\sigma \quad \text{for } e\in D(E)\subset D(E\oplus\mathbb{R})$$

extends the isomorphism μ on $\partial D(E)$ to an isomorphism on $D(E)$. This means φ^h_E descends to a group homomorphism

$$\varphi^h_E:\hat{\mathfrak{N}}^{\mathbb{H}}_n\to\widetilde{\mathrm{KO}}(Th(E)).$$

Remark 2.11. φ^h_E is functorial with respect to pull-backs of spin^h vector bundles. That is, if $f: X'\to X$ is a continuous map, let $Th(f):Th(f^*E)\to Th(E)$ denote the map between Thom spaces induced from the bundle map $f^*E\to E$ covering f, then we have a commutative diagram:

$$\begin{tikzcd} & \hat{\mathfrak{N}}^{\mathbb{H}}_n \arrow[dl, "\varphi^h_E"'] \arrow[dr, "\varphi^h_{f^*E}"] & \\ \widetilde{\mathrm{KO}}(Th(E)) \arrow[rr, "Th(f)^*"] & & \widetilde{\mathrm{KO}}(Th(f^*E)) \end{tikzcd}$$

For $F\to Y$ a spin vector bundle of rank m, the same construction carries over to give two homomorphisms

$$\varphi_F:\hat{\mathfrak{N}}^{\mathbb{R}}_m\to\widetilde{\mathrm{KO}}(Th(F)),\quad \varphi^h_F:\hat{\mathfrak{N}}^{\mathbb{H}}_m\to\widetilde{\mathrm{KSp}}(Th(F)).$$

We note φ_F coincides with the one obtained in [ABS64]; moreover if F is the trivial bundle over a point, then φ_F and φ^h_F coincide with φ and φ^h respectively. Despite the similarity in notation, φ^h_E and φ^h_F are very different: they are defined for different types of vector bundles and they land in different types of K-groups.

The following proposition shows, through the morphisms we constructed above, the module structure of $\hat{\mathfrak{N}}^{\mathbb{H}}_*$ over $\hat{\mathfrak{N}}^{\mathbb{R}}_*$ is compatible with the external product in real K theory.

Proposition 2.12. Let $E\to X$ be a spin^h vector bundle of rank n and $F\to X$ a spin vector bundle of rank m. Suppose $F\times E$ is given the spin^h structure as in Example 2.5. Then the following diagram commutes:

$$\begin{tikzcd} \hat{\mathfrak{N}}^{\mathbb{R}}_m\otimes\hat{\mathfrak{N}}^{\mathbb{H}}_n \arrow[r, "\hat{\otimes}"] \arrow[d, "\varphi_F\otimes\varphi^h_E"] & \hat{\mathfrak{N}}^{\mathbb{H}}_{m+n} \arrow[d, "\varphi^h_{F\times E}"] \\ \widetilde{\mathrm{KO}}(Th(F))\otimes\widetilde{\mathrm{KO}}(Th(E)) \arrow[r, "\boxtimes"] & \widetilde{\mathrm{KO}}(Th(F\times E)) \end{tikzcd} \tag{11}$$

Proof. The proof is the same as that of [ABS64, Prop. 11.1]. ∎

As applications of these upgraded homomorphisms, we have:

Theorem 2.13. Let $F\to Y$ be a spin vector bundle of rank $8k+4$ over a finite CW-complex Y. Then multiplication with the class

$$\varphi^h_F(\mathbf{\Delta}_{\mathbb{H}})\in\widetilde{\mathrm{KSp}}(Th(F))$$

induces a Thom isomorphism

$$\mathrm{KO}^\sharp(Y)\xrightarrow{\cong}\widetilde{\mathrm{KSp}}^\sharp(Th(F))$$

where $\sharp$ means summing over all integers.

Proof. Restricted to the fibre over each $y\in Y$ we have $\varphi^h_F(\mathbf{\Delta}_{\mathbb{H}})|_y=\varphi^h_{F_y}(\mathbf{\Delta}_{\mathbb{H}})=\varphi^h(\mathbf{\Delta}_{\mathbb{H}})$ is the generator of $\mathrm{KSp}(D(F_y),\partial D(F_y))\cong\mathrm{KSp}^{-8k-4}(\mathrm{pt})$ by Theorem 1.26. Since $\mathrm{KSp}^{-8k-4}(\mathrm{pt})$ generates $\mathrm{KSp}^\sharp(\mathrm{pt})$ as a free $\mathrm{KO}^\sharp(\mathrm{pt})$-module, then a standard argument using Mayer-Vietoris sequence and five-lemma proves the desired Thom isomorphism. ∎

Theorem 2.14. Let $E\to X$ be a spin^h vector bundle of rank $8k+4$. Then

$$\varphi^h_E(\mathbf{\Delta}_{\mathbb{H}})\in\widetilde{\mathrm{KO}}(Th(E))$$

restricts to each fibre over $x \in X$ generates $\widetilde{\mathrm{KO}}(Th(E_x)) \cong \mathrm{KO}^{-8k-4}(\mathrm{pt}) \cong \mathbb{Z}$. Moreover, multiplication with the class $\varphi^h_E(\mathbf{\Delta}_{\mathbb{H}})$ induces a Thom isomorphism

$$\mathrm{KO}^\sharp(X)[\frac{1}{2}] \xrightarrow{\cong} \widetilde{\mathrm{KO}}^\sharp(Th(E))[\frac{1}{2}].$$

We emphasize it is necessary to invert 2 in order to obtain an isomorphism. Indeed, when X is a point the map $\mathrm{KO}^\sharp(\mathrm{pt}) \to \widetilde{\mathrm{KO}}^\sharp(S^{8k+4}) = \mathrm{KO}^{\sharp-8k-4}(\mathrm{pt}) \cong \mathrm{KO}^{\sharp-4}(\mathrm{pt})$ is never an isomorphism since the 2-torsions on both sides are placed in different degrees.

Proof. Restricted to the fibre at $x \in X$, $E_x \to \{x\}$ is a trivial bundle and $\varphi^h_{E_x}$ coincides with the composition

$$\rho\varphi^h : \hat{\mathfrak{N}}^{\mathbb{H}}_{8k+4} \xrightarrow{\varphi^h} \mathrm{KSp}^{-8k-4}(\mathrm{pt}) \xrightarrow{\rho} \mathrm{KO}^{-8k-4}(\mathrm{pt}).$$

Then the first assertion follows from $\rho(\mathbf{\Delta}_{8k+4,\mathbb{H}}) = \mathbf{\Delta}_{8k+4,\mathbb{R}}$ and $\rho\varphi^h = \varphi\rho$. The second assertion follows from that $\mathrm{KO}^{-8k-4}(\mathrm{pt})[\frac{1}{2}]$ generates $\mathrm{KO}^\sharp(\mathrm{pt})[\frac{1}{2}]$ as a free $\mathrm{KO}^\sharp(\mathrm{pt})[\frac{1}{2}]$-module. ■

Remark 2.15. For $E \to X$ spin^h vector bundle of rank $8k$, the KO-class $\varphi^h_E(\mathbf{\Delta}_{\mathbb{H}})$ in fact lifts to a KSp-class, using the intrinsic quaternionic structure of $\mathbf{\Delta}_{8k,\mathbb{H}}$. Then similarly $\varphi^h_E(\mathbf{\Delta}_{\mathbb{H}})$ induces an isomorphism $\mathrm{KO}^\sharp(X)[\frac{1}{2}] \xrightarrow{\cong} \widetilde{\mathrm{KSp}}^\sharp(Th(E))[\frac{1}{2}]$. Again 2 must be inverted for this homomorphism to be an isomorphism. It is for this Thom isomorphism that we mentioned in the introduction one can define Thom classes for spin^h vector bundles in symplectic K-theory, however we will prefer to work with the KO-class for rank $8k+4$ spin^h vector bundles in this paper.

One can apply the same construction to complex modules. In [ABS64] it is shown:

Theorem 2.16 ([ABS64])**.** Let $F \to Y$ be a spin vector bundle of rank $2m$, then there is a homomorphism

$$\varphi^c_F : \hat{\mathfrak{N}}^{\mathbb{C}}_{2m} \to \widetilde{\mathrm{KU}}(Th(F))$$

so that multiplication with the class $\varphi^c_F(\mathbf{\Delta}_{\mathbb{C}})$ induces a Thom isomorphism

$$\mathrm{KU}^\sharp(Y) \xrightarrow{\cong} \widetilde{\mathrm{KU}}^\sharp(Th(F)).$$

■

Analogously for spin^h vector bundles, we have:

Theorem 2.17. Let $E \to X$ be a spin^h vector bundle of rank $2n$, then there is a homomorphism

$$\varphi^c_E : \hat{\mathfrak{N}}^{\mathbb{C}^2}_{2n} \to \widetilde{\mathrm{KU}}(Th(E))$$

so that $\varphi^c_E(\mathbf{\Delta}_{\mathbb{C}^2}) \in \widetilde{\mathrm{KU}}(Th(E))$ restricted to each fibre over $x \in X$ is *twice* the generator of $\widetilde{\mathrm{KU}}(Th(E_x)) \cong \mathrm{KU}^{-2n}(\mathrm{pt})$. Moreover, multiplication with the class $\varphi^c_E(\mathbf{\Delta}_{\mathbb{C}^2})$ induces a Thom isomorphism

$$\mathrm{KU}^\sharp(X)[\frac{1}{2}] \xrightarrow{\cong} \widetilde{\mathrm{KU}}^\sharp(Th(E))[\frac{1}{2}].$$

Proof. It suffices to show $\varphi^c_E(\mathbf{\Delta}_{\mathbb{C}^2})$ restricted to each fibre is twice the generator. Indeed, restricted to the fibre over $x \in X$, $\varphi^c_{E_x}(\mathbf{\Delta}_{\mathbb{C}^2}) = \varphi^c(\mathbf{\Delta}_{\mathbb{C}^2}) = \varphi^c(2\mathbf{\Delta}_{\mathbb{C}}) = 2\varphi^c(\mathbf{\Delta}_{\mathbb{C}})$; and $\varphi^c(\mathbf{\Delta}_{\mathbb{C}})$ generates $\mathrm{KU}^{-2n}(\mathrm{pt})$ by [ABS64]. ■

These Thom isomorphisms motivate the following definition.

Definition 2.18. Let E be a spin^h vector bundle of rank n. If $n \equiv 4 \pmod 8$, then

$$\Theta_E := \varphi^h_E(\mathbf{\Delta}_{\mathbb{H}})$$

is called the weak KO-Thom class of E. If $n \equiv 0 \pmod 2$, then

$$\Lambda_E := \varphi^c_E(\mathbf{\Delta}_{\mathbb{C}^2})$$

is called the weak KU-Thom class of E.

We point out from Corollary 1.17 we have $\varepsilon^{\mathbb{C}}_{\mathbb{R}}(\Theta_E) = \Lambda_E$ for $n \equiv 4 \pmod 8$.

The weak KO-Thom class enjoys a nice multiplicative property.

Proposition 2.19. Let E be a spin^h vector bundle of rank $8k+4$ and F a spin vector bundle of rank $8l$. Denote by $\Xi_F = \varphi_F(\mathbf{\Delta}_{\mathbb{R}})$ the KO-Thom class of F. Then

$$\Theta_{F\times E} = \Xi_F \cdot \Theta_E$$

Proof. This follows from Proposition 2.12. ∎

The Chern character of the weak KU-Thom class is calculated below. Then using $\varepsilon(\Theta_E) = \Lambda_E$ for $n \equiv 4 \bmod 8$, the Pontryagin character of Θ_E is obviously given by $\mathrm{ph}(\Theta_E) = \mathrm{ch}(\Lambda_E)$.

Proposition 2.20. Let $E \to X$ be a spin^h vector bundle of rank $2n$. Then

$$\mathrm{ch}(\Lambda_E) = (-1)^n U_E \cdot \left(2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_E)}}{2}\right)\hat{\mathfrak{A}}(E)^{-1}\right)$$

where $U_E \in \widetilde{H}^{2n}(Th(E);\mathbb{Z})$ is the singular cohomology Thom class of E and $\hat{\mathfrak{A}}(E)$ is the total $\hat{A}$-class of E.

Proof. We prove this for the universal spin^h vector bundle $\mathbb{E}_{2n} \to BSpin^h(2n)$. Consider the pull-back diagram

$$\begin{tikzcd} \mathbb{F}_{2n} \arrow[r, "f^*"] \arrow[d] & \mathbb{E}_{2n} \arrow[d] \\ BSpin(2n)\times BSp(1) \arrow[r, "f"] & BSpin^h(2n) \end{tikzcd}$$

induced by the quotient map $\mathrm{Spin}(2n)\times \mathrm{Sp}(1) \to \mathrm{Spin}^h(2n)$, where $\mathbb{F}_{2n}$ is the universal spin vector bundle on $BSpin(2n)$. Let $\mathbb{F}_3$ be the universal 3-plane bundle on $BSpin(3) = BSp(1)$. Then we claim $f^*\,\mathrm{Cl}_{\mathbb{H}}(\mathbb{E}_{2n}) \cong \mathrm{Cl}(\mathbb{F}_{2n}) \otimes_{\mathbb{R}} \mathrm{Cl}^0(\mathbb{F}_3)$. Indeed, note that $f^*\mathbb{E}_{2n} = \mathbb{F}_{2n}$ and $f^*\mathfrak{h}_{\mathbb{E}_{2n}} = \mathbb{F}_3$. Since the action of $SU(2) = \mathrm{Sp}(1) \subset \mathbb{H} \subset \mathbb{H}\otimes_{\mathbb{R}}\mathbb{C} = \mathbb{C}(2)$ on $\mathbb{C}^2$ through the matrix multiplication of $\mathbb{C}(2)$ on $\mathbb{C}^2$ is the canonical representation $SU(2) \to U(2)$. It follows that

$$Th(f)^*\varphi^c_{\mathbb{E}_{2n}}(\mathbf{\Delta}_{\mathbb{C}^2}) = \varphi^c_{\mathbb{F}_{2n}}(\mathbf{\Delta}_{\mathbb{C}}) \cdot [\mathbb{U}]$$

where $\mathbb{U}$ is the tautological complex 2-plane bundle on $BSU(2) = BSp(1)$. From [Hir95] we have $\mathrm{ch}\,\varphi^c_{\mathbb{F}_{2n}}(\mathbf{\Delta}_{\mathbb{C}}) = (-1)^n U_{\mathbb{F}_{2n}} \cdot \hat{\mathfrak{A}}(\mathbb{F}_{2n})^{-1}$. We shall prove $\mathrm{ch}(\mathbb{U}) = 2\cosh\left(\frac{\sqrt{p_1(\mathbb{F}_3)}}{2}\right)$ in the lemma below. Then the proposition follows from applying Chern character and using that f, $Th(f)$ induce isomorphisms on rational cohomology. ∎

Lemma 2.21. Let $\mathbb{F}_3 \to BSU(2)$ and $\mathbb{U} \to BSU(2)$ be as in the proof of Theorem 2.23. Then $\mathrm{ch}(\mathbb{U}) = 2\cosh\left(\frac{\sqrt{p_1(\mathbb{F}_3)}}{2}\right)$.

Proof. The natural representation $SU(2) \to U(2)$ is irreducible with weights $1, -1$. By applying spitting principle, we may write $c(\mathbb{U}) = (1+x)(1-x) = 1-x^2$. Since $\mathbb{F}_3$ is induced from the rotation representaion $SU(2) \to SO(3)$, $\mathbb{F}_3\otimes_{\mathbb{R}}\mathbb{C}$ corresponds to the adjoint representation of $SU(2)$, which is irreducible with weights $2, 0, -2$. Therefore by splitting principle we can write $c(\mathbb{F}_3\otimes_{\mathbb{R}}\mathbb{C}) = (1+2x)(1-2x) = 1-4x^2$. Now $p_1(\mathbb{F}_3) = -c_2(\mathbb{F}_3\otimes_{\mathbb{R}}\mathbb{C}) = 4x^2$, hence symbolically $x = \frac{\sqrt{p_1(\mathbb{F}_3)}}{2}$. So $\mathrm{ch}(\mathbb{U}) = e^x + e^{-x} = 2\cosh(x) = 2\cosh\left(\frac{\sqrt{p_1(\mathbb{F}_3)}}{2}\right)$. This expression makes sense since cosh is an even function. ∎

2.4. **Riemann-Roch theorem for spin^h maps.** In this subsection, we prove a Riemann-Roch theorem for spin^h maps. Along the way, we pick out a special characteristic class for spin^h manifolds, whose role is analogous to the $\hat{A}$-class for spin manifolds.

Definition 2.22. Let X and Y be closed oriented smooth manifolds. A continuous map $f : X \to Y$ is called a spin^h map if there exists an oriented rank 3 real vector bundle $\mathfrak{h}_f$ on X so that

$$w_2(X) + f^*w_2(Y) = w_2(\mathfrak{h}_f).$$

The bundle $\mathfrak{h}_f$ is called the canonical bundle of the spin^h map f.

Theorem 2.23. (i) Let $\dim X \equiv \dim Y \bmod 2$. Then a spin^h map $f : X \to Y$ induces a group homomorphism $f_! : \mathrm{KU}(X) \to \mathrm{KU}(Y)$ such that

$$\operatorname{ch} f_!(\xi) \cdot \hat{\mathfrak{A}}(TY) = f_* \left(\operatorname{ch} \xi \cdot 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_f)}}{2} \right) \hat{\mathfrak{A}}(TX) \right)$$

where TX, TY are the tangent bundles of X, Y respectively, f_* is the umkehr homomorphism.
(ii) If moreover $\dim X - \dim Y \equiv 4 \bmod 8$, then there is a group homomorphism $\tilde{f}_! : \mathrm{KO}(X) \to \mathrm{KO}(Y)$ so that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{KO}(X) & \xrightarrow{\tilde{f}_!} & \mathrm{KO}(Y) \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ \mathrm{KU}(X) & \xrightarrow{f_!} & \mathrm{KU}(Y) \end{array}$$

Proof. Since only the homotopy class of f is relevant to the theorem, we may assume f is smooth. Let $g : X \to S^{2n}$ be a smooth embedding of X. Then $f : X \to Y$ can be factored into smooth embedding $\iota : f \times g : X \to Y \times S^{2n}$ followed by the projection $\pi : Y \times S^{2n} \to Y$. Since $w_2(S^{2n}) = 0$, ι is a spin^h map with $\mathfrak{h}_\iota = \mathfrak{h}_f$; meanwhile π is a spin map, that is $\pi^* w_2(Y) = w_2(Y \times S^{2n})$.

Suppose we have proved (i) for F. Then since π is spin, by Riemann-Roch theorem for spin maps (from [Hir95], enhanced in [ABS64]), π incudes a homomorphism $\pi_! : \mathrm{KU}(Y \times S^{2n}) \to \mathrm{KU}(Y)$ satisfying $\operatorname{ch} \pi_!(-) \cdot \hat{\mathfrak{A}}(TY) = \pi_*(\operatorname{ch}(-) \cdot \hat{\mathfrak{A}}(T(Y \times S^{2n}))$. Hence $f_! = \pi_! \iota_!$ is as desired. So we can assume f is an embedding. Let E be the normal bundle of X in Y whose rank is $2n = \dim Y - \dim X$, then $w_2(E) = w_2(Y) + f^* w_2(X) = w_2(\mathfrak{h}_f)$. So E is spin^h with canonical bundle $\mathfrak{h}_E = \mathfrak{h}_f$. Identify a closed tubular neighborhood of X with $D(E)$, then we have $\mathrm{KU}(D(E), \partial D(E)) \cong \mathrm{KU}(Y, Y - X)$ and $H^*(D(E), \partial D(E); \mathbb{Q}) \cong H^*(Y, Y - X; \mathbb{Q})$ by excision. Recall the umkehr homomorphism f_* is the composition

$$f_* : H^*(X; \mathbb{Q}) \xrightarrow{\times U_E} H^*(D(E), \partial D(E); \mathbb{Q}) \cong H^*(Y, Y - X; \mathbb{Q}) \xrightarrow{\text{restrict}} H^*(Y; \mathbb{Q}).$$

Define $f_!$ to be the composition

$$f_! : \mathrm{KU}(X) \xrightarrow{\times \Lambda_E} \mathrm{KU}(D(E), \partial D(E)) \cong \mathrm{KU}(Y, Y - X) \xrightarrow{\text{restrict}} \mathrm{KU}(Y).$$

Using Proposition 2.20 and the multiplicative property of the $\hat{A}$-class: $f^* \hat{\mathfrak{A}}(TY) = \hat{\mathfrak{A}}(TX \oplus E) = \hat{\mathfrak{A}}(TX) \cdot \hat{\mathfrak{A}}(E)$, we conclude $f_!$ satisfies (i).

For (ii), by using the same embedding trick as before so that π has relative dimension divisible by 8, and noticing from [Hir95] and [ABS64] $\pi_!$ in this case lifts to a homomorphism between KO-groups, we may assume f is an embedding. Now E is of rank $8k + 4$, as such $\Lambda_E = \varepsilon(\Theta_E)$. Define $\tilde{f}_!$ to be the composition

$$\tilde{f}_! : \mathrm{KO}(X) \xrightarrow{\times \Theta_E} \mathrm{KO}(D(E), \partial D(E)) \cong \mathrm{KO}(Y, Y - X) \xrightarrow{\text{restrict}} \mathrm{KO}(Y).$$

Then $\tilde{f}_!$ clearly is as required. ∎

Remark 2.24. Even though when defining $f_!$, we made a choice of embedding $X \hookrightarrow S^{2n}$, $f_!$ in fact does not depend on such a choice due to the multiplicative property Proposition 2.19 of the weak KO-Thom class. Indeed had we chosen two different embeddings, we may find a common larger embedding. So we can assume $X^d \subset \mathbb{R}^{d+8k+4} \subset \mathbb{R}^{d+8k+8l+4}$, then the normal bundle of $\mathbb{R}^{d+8k+4}$ in $\mathbb{R}^{d+8k+8l+4}$ is spin of rank $8l$. By Proposition 2.19 and that $\Delta_{8l,\mathbb{R}} \in \mathrm{KO}^{-8k}(\mathrm{pt})$ is the Bott generator, we conclude $f_!$ is independent of the choice of the embedding.

Corollary 2.25. (i) Let X be a closed spin^h manifold of dimension $n \equiv 0 \bmod 2$ with canonical bundle $\mathfrak{h}_X$. Suppose ξ is a complex vector bundle on X. Then the (rational) number

$$\langle \operatorname{ch} \xi \cdot 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2} \right) \hat{\mathfrak{A}}(TX), [X] \rangle$$

is an integer, where $\langle -, [X] \rangle$ means pairing with the fundamental class of X.

(ii) If further $n \equiv 0 \bmod 8$ and γ is a real vector bundle on X, then the (rational) number

$$\langle \mathrm{ph}\,\gamma \cdot 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2}\right)\hat{\mathfrak{A}}(TX), [X]\rangle$$

is an even integer, where $\mathrm{ph}\,\gamma = \mathrm{ch}(\gamma \otimes_{\mathbb{R}} \mathbb{C})$ is the Pontryagin character of γ.

Proof. For (i), apply Theorem 2.23(i) to the spin^h-map $f : X \to \mathrm{pt}$. Then we have

$$\langle \mathrm{ch}\,\xi \cdot 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2}\right)\hat{\mathfrak{A}}(TX), [X]\rangle = \langle \mathrm{ch}\, f_!\xi, [\mathrm{pt}]\rangle \in \mathbb{Z}.$$

For (ii), apply Theorem 2.23(ii) to the spin^h-map $f : X \to \mathrm{pt} \hookrightarrow S^4$. Then we have

$$\langle \mathrm{ph}\,\gamma \cdot 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2}\right)\hat{\mathfrak{A}}(TX), [X]\rangle = \langle \mathrm{ph}\,\tilde{f}_!\gamma, [S^4]\rangle \in 2\mathbb{Z}.$$

The asserted integralities follows from Bott's theory (see [Hir95]). ∎

Remark 2.26. These integrality results are first obtained by Mayer [May65] in studying immersions of manifolds into spin manifolds. They are also used to construct non-spin^h 8-manifolds [AM21].

Definition 2.27. Let X be a closed spin^h manifold of even dimension. We define its $\hat{A}^h$-genus twisted by a complex vector bundle ξ to be the integer

$$\hat{A}^h(X, \xi) := \langle \mathrm{ch}\,\xi \cdot 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2}\right)\hat{\mathfrak{A}}(TX), [X]\rangle$$

We define the $\hat{A}^h$-genus of X to be the integer

$$\hat{A}^h(X) := \langle 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2}\right)\hat{\mathfrak{A}}(TX), [X]\rangle.$$

Remark 2.28. It follows from Corollary 2.25 that $\hat{A}^h(X)$ is further an even integer when the dimension of X is divisible by 8. This is the spin^h counterpart of Rokhlin's theorem.

Example 2.29. Let M be a closed oriented riemannian 4-fold. We furnish M into a spin^h manifold by setting $\mathfrak{h}_M = \Lambda^+_M$ (resp. Λ^-_M) where Λ^+_M (resp. Λ^-_M) is the bundle of self-dual (resp. anti-self-dual) two forms, and denote the resulting spin^h manifold by M_+ (resp. M_-). Then since $p_1(\Lambda^{\pm}_M) = p_1(M) \pm 2e(M)$ (see e.g. [Wal04, pp. 195]) where $e(M)$ is the Euler class of M, we can compute

$$\begin{aligned}\hat{A}^h(M_\pm) &= \langle (2 + \frac{p_1(\Lambda^{\pm}_M)}{4})(1 - \frac{p_1(M)}{24}), [M]\rangle \\ &= \langle \frac{p_1(M)}{6} \pm \frac{e(M)}{2}, [M]\rangle \\ &= \frac{1}{2}(Sign(M) \pm \chi(M)).\end{aligned}$$

Here $Sign(M)$ and $\chi(M)$ are the signature and euler characteristic of M respectively. Therefore, as long as $\chi(M) \neq 0$, $M_\pm$ are different spin^h manifolds. For instance $\hat{A}^h(\mathbb{HP}^1_\pm) = \pm 1$ and $\hat{A}^h(\mathbb{CP}^2_+) = 2$ whence $\hat{A}^h(\mathbb{CP}^2_-) = -1$.

2.5. **Characteristic classes of spin^h bundles.** We calculate the cohomology of (the classifying space of) the stable spin^h group, which serves as an input for applying Adams spectral sequence to analyze the spin^h cobordism groups, especially at prime 2. The cohomology for unstable spin^h groups can be obtained using the beautiful method of [Qui71], however we do not persuit it here.

To begin with, recall that Spin^h is a central extension of $\mathrm{SO}\times\mathrm{SO}(3)$ by $\mathbb{Z}_2$, which is classified by $w_2 + w_2' \in H^2(BSO\times BSO(3);\mathbb{Z}_2)$. Therefore we have a pull-back diagram

$$\begin{array}{ccc} BSpin^h & \longrightarrow & PK(\mathbb{Z}_2,2) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ BSO\times BSO(3) & \xrightarrow{f} & K(\mathbb{Z}_2,2) \end{array}$$

where $PK(\mathbb{Z}_2,2)\to K(\mathbb{Z}_2,2)$ is the path space fiberation, and f is induced by w_2+w_2'. Let i_2 denote the generator of $H^2(K(\mathbb{Z}_2,2);\mathbb{Z}_2)\cong\mathbb{Z}_2$. It is well known the mod 2 cohomology of $K(\mathbb{Z}_2,2)$ is a polynomial algebra generated by i_2 and $Sq^I(i_2)$ where I runs over all multi-indices $(2^r,2^{r-1},\dots,1)$. In other words,

$$H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2)\cong\mathbb{Z}_2[i_2, Sq^1(i_2), Sq^2Sq^1(i_2),\dots,Sq^I(i_2),\dots]$$

where $I=(2^r,2^{r-1},\dots,2,1)$.

Lemma 2.30. $f^*: H^*(K(\mathbb{Z}_2,2);\mathbb{Z}_2)\to H^*(BSO\times BSO_3;\mathbb{Z}_2)=\mathbb{Z}_2[w_2,w_3,\dots]\otimes\mathbb{Z}_2[w_2',w_3']$ is monic.

Proof. For oriented bundles $Sq^1w_2=w_3$. Then inductively using $Sq^{n-1}w_n=w_{2n-1}$+decomposables (see [Sto68, pp. 291]), we get

$$\begin{aligned} Sq^0(w_2+w_2') &= w_2+w_2' \\ Sq^1(w_2+w_2') &= w_3+w_3' \\ Sq^I(w_2+w_2') &= w_{2^{r+1}+1}+\text{decomposables}\quad (r\geq 1). \end{aligned}$$

It is clear these are algebraically independent, thus proving f^* is monic. ∎

Proposition 2.31. $\pi^*: H^*(BSO\times BSO(3);\mathbb{Z}_2)\to H^*(BSpin^h;\mathbb{Z}_2)$ maps the subalgebra

$$\mathbb{Z}_2[w_i | i\geq 2, i\neq 2^{r+1}+1, r\geq 1]$$

isomorphically onto $H^*(BSpin^h;\mathbb{Z}_2)$.

Proof. Let E^* denote the Serre spectral sequence for $\pi: BSpin^h\to BSO\times BSO(3)$ and E'^* that of $PK(\mathbb{Z}_2,2)\to K(\mathbb{Z}_2,2)$. The map f induces a map $f^*: E'^*\to E^*$ between spectral sequences. Since E^* is an $H^*(BSO\times BSO(3);\mathbb{Z}_2)$-module, one has an induced spectral sequence map

$$\mathbb{Z}_2[w_i | i\geq 2, i\neq 2^{r+1}+1, r\geq 1]\otimes E'^*\to E^*$$

by means of f^* and module multiplication. This is an isomorphism on the second page by the calculations done in the proof of Lemma 2.30. Therefore by Zeeman's comparison theorem, this map is an isomorphism on ∞-page. Therefore the proposition follows from that the path space $PK(\mathbb{Z}_2,2)$ is contractible. ∎

Remark 2.32. The classes $w_{2^{r+1}+1}$ are not identically zero, but decomposable in $H^*(BSpin^h;\mathbb{Z}_2)$. For instance, using Theorem 2.34 below one can prove $w_9=w_2w_7+w_3w_6$.

Our next step is to apply the Bockstein spectral sequence to recover the 2-local cohomology of $BSpin^h$, so first of all we must understand the action of Sq^1. Recall for oriented bundles $Sq^1(w_{2i})=w_{2i+1}$, so $\mathbb{Z}_2[w_{2i},w_{2i+1}]$ is a subalgebra invariant under Sq^1. However, in the mod 2-cohomology of $BSpin^h$, the class $w_{2^{r+1}+1}$ is not an algebraic generator, so we would like to replace $w_{2^{r+1}}$ by another indecomposable class of the same degree, i.e. a class of the form $(w_{2^{r+1}}+\text{decomposables})$, on which Sq^1 vanishes.

The Wu class $\nu_{2^{r+1}}$ is known to be indecomposable (see [Sto68, pp. 315]), we shall verify $Sq^1\nu_{2^{r+1}}=0$ in $H^*(BSpin^h;\mathbb{Z}_2)$, therefore $\nu_{2^{r+1}}$ is exactly the class we are looking for.

Proposition 2.33. In $H^*(BSpin^h;\mathbb{Z}_2)$ we have $Sq^1\nu_{2^{r+1}}=0$ for $r\geq 1$.

Proof. Let w, ν denote the total Stiefel-Whitney class, the total Wu class respectively, and let Sq denote the total Steenrod square. They are related by Wu's relation $Sq(\nu) = w$. Suppose $\overline{U}$ is the Thom class of the stable normal bundle to the bundle in question, then $Sq(\overline{U}) = \overline{w} \cdot \overline{U}$ where $\overline{w}$ is the total Stiefel-Whitney class of the stable normal bundle, satisfying $w \cdot \overline{w} = 1$.

Applying Sq to $\nu \cdot \overline{U}$ and using Cartan's formula we get

$$Sq(\nu \cdot \overline{U}) = Sq(\nu) \cdot Sq(\overline{U}) = w \cdot \overline{w} \cdot \overline{U} = \overline{U}.$$

Then since $\chi(Sq)$ is the inverse to Sq, where χ is the canonical involution of the Steenrod algebra, we get $\nu \cdot \overline{U} = \chi(Sq)\overline{U}$.

Now from Adem's relation $Sq^2 Sq^{4k-1} = Sq^{4k} Sq^1$ we obtain

$$\begin{aligned}(Sq^1 \nu_{4k}) \cdot \overline{U} &= Sq^1(\nu_{4k}\overline{U}) = Sq^1 \chi(Sq^{4k})\overline{U} = \chi(Sq^1)\chi(Sq^{4k})\overline{U} \\ &= \chi(Sq^{4k} Sq^1)\overline{U} = \chi(Sq^2 Sq^{4k-1})\overline{U} = \chi(Sq^{4k-1})\chi(Sq^2)\overline{U} \\ &= \chi(Sq^{4k-1}) Sq^2 \overline{U} = \chi(Sq^{4k-1})(w_2 \overline{U}).\end{aligned}$$

Here we used $Sq^1 \overline{U} = 0$ and $w_2 = \overline{w}_2$ since the bundles in question are orientable. Next we note from [Dav74]

$$\chi(Sq^{2^{r+1}-1}) = Sq^{2^r} Sq^{2^{r-1}} \cdots Sq^2 Sq^1,$$

therefore

$$\begin{aligned}Sq^1 \nu_{2^{r+1}} \cdot \overline{U} &= \chi(Sq^{2^{r+1}-1})(w_2 \overline{U}) \\ &= Sq^{2^r} Sq^{2^{r-1}} \cdots Sq^2 Sq^1 (w_2 \overline{U}) \\ &= Sq^{2^r} Sq^{2^{r-1}} \cdots Sq^4 (Sq^1 \nu_4 \cdot \overline{U}).\end{aligned}$$

By Thom isomorphism, we are reduced to proving $Sq^1 \nu_4 = 0$. For oriented bundles, $\nu_4 = w_4 + w_2^2$ and thus $Sq^1 \nu_4 = Sq^1 w_4 = w_5$. But the integral fifth Stiefel-Whitney class vanishes for spin^h bundles [AM21, Corollary 2.5], so its mod 2 reduction w_5 must also vanish for spin^h bundles. This completes the proof. ■

We now obtain a better description of the mod 2 cohomology of $BSpin^h$.

Theorem 2.34. *$\pi^* : H^*(BSO \times BSO(3); \mathbb{Z}_2) \to H^*(BSpin^h; \mathbb{Z}_2)$ is epic, with kernel generated by $w_2 + w_2'$, $w_3 + w_3'$, and $Sq^1 \nu_{2^{r+1}}$ for all $r \geq 1$. In particular, π^* induces an isomorphism*

$$H^*(BSpin^h; \mathbb{Z}_2) \cong H^*(BSO; \mathbb{Z}_2)/(Sq^1 \nu_{2^{r+1}}, r \geq 1).$$

Corollary 2.35. *$H(H^*(BSpin^h; \mathbb{Z}_2), Sq^1) \cong \mathbb{Z}_2[w_2^2, w_{2k}^2, \nu_{2^{r+1}} | k \neq 2^j, r \geq 1]$.*

Proof. From the above theorem, the mod 2 cohomology of $BSpin^h$ is isomorphic to

$$\mathbb{Z}_2[w_2, Sq^1 w_2; w_{2k}, Sq^1 w_{2k}; \nu_{2^{r+1}} | k \neq 2^j, r \geq 1]$$

whose cohomology with respect to Sq^1 can now be easily obtained by applying Kunneth theorem. The result clearly is as claimed. ■

Corollary 2.36. *All torsion in $H^*(BSpin^h; \mathbb{Z})$ has order 2.*

Proof. Since $H(H^*(BSpin^h; \mathbb{Z}_2), Sq^1)$ is concentrated in even degrees, all higher Bocksteins vanish, hence by Bockstein spectral sequence all 2-primary torsion of $H^*(BSpin^h; \mathbb{Z})$ has order 2. At odd primes, namely with 2 inverted, $H^*(BSpin^h; \mathbb{Z}[\frac{1}{2}]) \cong H^*(BSO \times BSO(3); \mathbb{Z}[\frac{1}{2}])$ is torsion-free. The statement thus follows. ■

At this point, we have a rather complete description of the characteristic classes for spin^h vector bundles. Putting torsion aside, the integral characteristic classes are the Pontryagin classes for the bundle in question together with the first Pontryagin class for the canonical bundle associated to the spin^h structure. The mod 2 characteristic classes are the Stiefel-Whitney classes for the bundle subject to universal relations generated by $Sq^1 \nu_{2^{r+1}} = 0$ for $r \geq 1$. Certain mod 2 classes admit integral lifts. The square of the even Stiefel-Whitney classes are lifted to the Pontryagin classes. The odd Stiefel-Whitney classes are lifted to their integral counterpart. Finally the Wu classes in degrees power of two (except for ν_2) all have integral lifts.

3. Spin^h Dirac index

3.1. **Dirac operator.** Let X be a closed spin^h manifold of dimension n with canonical bundle $\mathfrak{h}_X$. We choose, once and for all, a riemannian connection on $P_{\mathrm{SO}}(\mathfrak{h}_X)$. Then $P_{\mathrm{Spin}^h}(TX)$ admits a natural connection inherited from the Levi-Civita connection on $P_{\mathrm{SO}}(TX)$ and the riemannian connection on $P_{\mathrm{SO}}(\mathfrak{h}_X)$. Suppose S is a ^{h}spinor bundle of TX, i.e. S is a bundle of the form $P_{\mathrm{Spin}^h}(TX)\times_\mu V$ for some $\mathrm{Cl}_{n,\mathbb{H}}$-module V. Then S is a bundle of $\mathrm{Cl}_{\mathbb{H}}(X)$-module, and consequently a bundle of $\mathrm{Cl}(X)$-module. Moreover, S is equipped with a connection ∇^S induced from $P_{\mathrm{Spin}^h}(TX)$. As usual we define the Dirac operator $D:\Gamma(S)\to\Gamma(S)$ to be the first order elliptic differential operator

$$D := \sum_{i=1}^n e_i\cdot\nabla^S_{e_i}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame of X, and $\cdot$ means Clifford multiplication.

If $S=S^0\oplus S^1$ is a $\mathbb{Z}_2$-graded one, then D clearly interchanges the two factors. Written in matrix form

$$D=\begin{pmatrix}0 & D^1\\ D^0 & 0\end{pmatrix}$$

where $D^0:\Gamma(S^0)\to\Gamma(S^1)$ and $D^1:\Gamma(S^1)\to\Gamma(S^0)$. Of course as usual the Dirac operator is formally self-adjoint, namely $(D^0)^*=D^1$ and $(D^1)^*=D^0$. In particular $\ker D^1=\operatorname{coker} D^0$.

Recall all ^{h}spinor bundles are direct sums of the fundamental ones.

Definition 3.1. Let X be a closed spin^h manifold of dimension n. We define its fundamental $\mathbb{Z}_2$-graded real ^{h}spinor bundle to be

$$\$_{\mathbb{H}}(X) := \begin{cases}\$_{\mathbb{H}}(TX) & \text{if } n\neq 0 \bmod 4\\ \$^+_{\mathbb{H}}(TX) & \text{if } n\equiv 0 \bmod 8\\ \$^-_{\mathbb{H}}(TX) & \text{if } n\equiv 4 \bmod 8\end{cases}$$

and denote the corresponding Dirac operator to be $\not{D}_{\mathbb{H},X}$. Similarly we define its fundamental $\mathbb{Z}_2$-graded complex ^{h}spinor bundle to be

$$\$_{\mathbb{C}^2}(X) := \begin{cases}\$_{\mathbb{C}^2}(TX) & \text{if } n \text{ odd}\\ \$^+_{\mathbb{C}^2}(TX) & \text{if } n \text{ even}\end{cases}$$

and denote the corresponding Dirac operator to be $\not{D}_{\mathbb{C}^2,X}$.

Theorem 3.2. *Let X be a closed spin^h manifold of dimension $2n$ and ξ a complex vector bundle over X. Then*

$$\operatorname{ind}(\not{D}^0_{X,\xi}) = \hat{A}^h(X,\xi)$$

where $\not{D}_{X,\xi}$ is the Dirac operator on $\$_{\mathbb{C}^2}(X)\otimes_{\mathbb{C}}\xi$. In particular $\operatorname{ind}(\not{D}^0_{\mathbb{C}^2,X})=\hat{A}^h(X)$.

Proof. This follows from Atiyah-Singer index theorem. Let $x_1,\dots,x_n$ be virtual Chern roots of X then from Atiyah-Singer index theorem we have

$$\operatorname{ind}(\not{D}^0_{X,\xi}) = \left\langle \left(\operatorname{ch}\left(\$^0_{\mathbb{C}^2}(X)\right)-\operatorname{ch}\left(\$^1_{\mathbb{C}^2}(X)\right)\right)\cdot\operatorname{ch}\xi\cdot\prod_{i=1}^n\frac{x_i}{1-e^{-x_i}}\cdot\frac{1}{1-e^{x_i}},[X]\right\rangle.$$

Meanwhile from Proposition 2.20 we have

$$\operatorname{ch}\left(\$^0_{\mathbb{C}^2}(X)\right)-\operatorname{ch}\left(\$^1_{\mathbb{C}^2}(X)\right) = (-1)^n 2\cosh\left(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2}\right)\prod_{i=1}^n x_i\cdot\frac{\sinh(x_i/2)}{x_i/2}.$$

The theorem now follows from a straightforward computation. ∎

3.2. $\mathrm{Cl}_{k,\mathbb{H}}$**-linear operator.**

Definition 3.3. By a $\mathrm{Cl}_{k,\mathbb{H}}$-Dirac bundle over a riemannian manifold X we mean a real Dirac bundle $\mathfrak{S}$ over X, together with a left action $\mathrm{Cl}_{k,\mathbb{H}} \to \mathrm{Aut}(\mathfrak{S})$ which is parallel and commutes with the multiplication by elements of $\mathrm{Cl}(X)$.

Definition 3.4. A $\mathrm{Cl}_{k,\mathbb{H}}$-Dirac bundle is $\mathfrak{S}$ is said to be $\mathbb{Z}_2$-graded if it carries a $\mathbb{Z}_2$-grading $\mathfrak{S} = \mathfrak{S}^0 \oplus \mathfrak{S}^1$ as a Dirac bundle, which is simultaneously a $\mathbb{Z}_2$-grading for the $\mathrm{Cl}_{k,\mathbb{H}}$-action, that is

$$\mathrm{Cl}_{k,\mathbb{H}}^{\alpha} \cdot \mathfrak{S}^{\beta} \subseteq \mathfrak{S}^{\alpha+\beta}$$

for all $\alpha, \beta \in \mathbb{Z}_2$.

Any $\mathrm{Cl}_{k,\mathbb{H}}$-Dirac bundle $\mathfrak{S}$ has a canonically associated Dirac operator $\mathfrak{D}$, which commutes with the $\mathrm{Cl}_{k,\mathbb{H}}$-action. If $\mathfrak{S}$ is $\mathbb{Z}_2$-graded, then $\mathfrak{D}$ is decomposed as

$$\mathfrak{D} = \begin{pmatrix} 0 & \mathfrak{D}^1 \\ \mathfrak{D}^0 & 0 \end{pmatrix}$$

where $\mathfrak{D}^0 : \Gamma(\mathfrak{S}^0) \to \Gamma(\mathfrak{S}^1)$ and $\mathfrak{D}^1 : \Gamma(\mathfrak{S}^1) \to \Gamma(\mathfrak{S}^0)$. Then

$$\ker \mathfrak{D} = \ker \mathfrak{D}^0 \oplus \ker \mathfrak{D}^1$$

is a $\mathbb{Z}_2$-graded $\mathrm{Cl}_{k,\mathbb{H}}$-module.

Definition 3.5. Let $\mathfrak{S}$ be a $\mathbb{Z}_2$-graded $\mathrm{Cl}_{k,\mathbb{H}}$-Dirac bundle over a closed manifold. The analytic index of the Dirac operator $\mathfrak{D}$ of $\mathfrak{S}$ is the residue class

$$\mathrm{ind}^h(\mathfrak{D}) = [\ker \mathfrak{D}] \in \hat{\mathfrak{N}}_k^{\mathbb{H}} \cong \mathrm{KSp}^{-k}(\mathrm{pt}).$$

Example 3.6 ($\mathrm{Cl}_{k,\mathbb{H}}$-ification)**.** Let S be any ordinary real $\mathbb{Z}_2$-graded Dirac bundle over a closed manifold X, and let D be its Dirac operator. We now consider an irreducible $\mathbb{Z}_2$-graded module V over $\mathrm{Cl}_{k,\mathbb{H}}$, and take the tensor product

$$\mathfrak{S} = S \hat{\otimes}_{\mathbb{R}} V$$

where V is considered as the trivialized bundle $V \times X \to X$. This bundle is naturally a $\mathbb{Z}_2$-graded $\mathrm{Cl}_{k,\mathbb{H}}$-Dirac bundle. The associated Dirac operator $\mathfrak{D}$ on $\mathfrak{S}$ is simply $D \hat{\otimes} \mathrm{Id}_V$. Consequently we have that

$$\ker \mathfrak{D} = (\ker D) \hat{\otimes} V$$

and in particular $\ker \mathfrak{D}^0 = (\ker D^0 \otimes V^0) \oplus (\ker D^1 \otimes V^1)$. To determine the residue class $[\ker \mathfrak{D}]$ in $\hat{\mathfrak{N}}_{k,\mathbb{H}} = \hat{\mathfrak{M}}_{k,\mathbb{H}} / i^* \hat{\mathfrak{M}}_{k+1,\mathbb{H}}$ we recall the isomorphism $\hat{\mathfrak{M}}_{k,\mathbb{H}} \xrightarrow{\cong} \mathfrak{M}_{k-1,\mathbb{H}}$ by taking the degree zero part. Since $V^0 \oplus V^1$ is a $\mathrm{Cl}_{k,\mathbb{H}}$-module, we have $[V^0] + [V^1] = 0$ in $\mathfrak{M}_{k-1,\mathbb{H}} / i^* \mathfrak{M}_{k,\mathbb{H}}$. So in $\mathfrak{M}_{k-1,\mathbb{H}} / i^* \mathfrak{M}_{k,\mathbb{H}}$

$$[\ker \mathfrak{D}^0] = (\dim_{\mathbb{R}} \ker D^0 - \dim_{\mathbb{R}} \ker D^1)[V] = (\mathrm{ind}\, D^0) \cdot [V].$$

Now that $[V]$ generates $\hat{\mathfrak{N}}_k^{\mathbb{H}}$, we conclude

$$[\ker \mathfrak{D}] = \begin{cases} \mathrm{ind}\, D^0 & \text{if } k \equiv 0 \bmod 4 \\ \mathrm{ind}\, D^0 \mod 2 & \text{if } k \equiv 5, 6 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

Example 3.7 (The fundamental case)**.** Let X be a closed spin^h manifold of dimension n. Consider the ^{h}spinor bundle

$$\mathfrak{S}\!\!\!/(X) := P_{\mathrm{Spin}^h}(X) \times_l \mathrm{Cl}_{n,\mathbb{H}}$$

whose Dirac operator is denoted by $\mathfrak{D}\!\!\!/$, where $\mathrm{Spin}^h(n) \subset \mathrm{Cl}_{n,\mathbb{H}}^{\times}$ acts on $\mathrm{Cl}_{n,\mathbb{H}}$ through the left multiplication. We remark the principle symbol of $\mathfrak{D}\!\!\!/$ is $\sigma_\xi(\mathfrak{D}\!\!\!/) = i\xi$ where tangent vectors ξ act by left Clifford multiplication. Clearly $\mathfrak{S}\!\!\!/(X)$ admits a right $\mathrm{Cl}_{n,\mathbb{H}}$-action that commutes with $\mathfrak{D}\!\!\!/$, we can turn this into a left one by means of the transpose. This way, $\mathfrak{S}(X)$ is a $\mathrm{Cl}_{n,\mathbb{H}}$-Dirac bundle, and it follows from Proposition 1.27 that

$$\mathfrak{S}(X) \cong \begin{cases} \$_{\mathbb{H}}(X) \otimes_{\mathbb{R}} \Delta_{n,\mathbb{H}} & \text{if } n \equiv 4 \bmod 8 \\ \$_{\mathbb{H}}(X) \otimes_{\mathbb{C}} \Delta_{n,\mathbb{H}} & \text{if } n \equiv 5 \bmod 8 \\ \$_{\mathbb{H}}(X) \otimes_{\mathbb{H}} \Delta_{n,\mathbb{H}} & \text{if } n \equiv 6 \bmod 8 \\ \frac{1}{2}\$_{\mathbb{C}^2}(X) \otimes_{\mathbb{C}} \Delta_{n,\mathbb{C}^2} & \text{if } n \equiv 0 \bmod 8 \end{cases}$$

Note that the tilde's are removed for we have turned right $\mathrm{Cl}_{n,\mathbb{H}}$-actions into left ones. Also we remark that for $n \equiv 6 \bmod 8$, the tensor $\otimes_{\mathbb{H}}$ is equating the right $\mathbb{H}$-multiplication on $\$_{\mathbb{H}}(X)$ with the right $\mathbb{H}$-multiplication on $\Delta_{n,\mathbb{H}}$.

From here we can extract the analytic index of $\mathfrak{D}$ as follows.

For $n = 8k+4$, this is exactly the case of Example 3.6 hence $\mathrm{ind}^h_{8k+4}(\mathfrak{D}) = \mathrm{ind}(\not{D}^0_{\mathbb{H},X})$. Now note that $\varepsilon^{\mathbb{C}}_{\mathbb{R}} : \hat{\mathfrak{N}}^{\mathbb{H}}_{8k+4} \to \hat{\mathfrak{N}}^{\mathbb{C}^2}_{8k+4}$ is an isomorphism and $\varepsilon(\boldsymbol{\Delta}_{8k+4,\mathbb{H}}) = \boldsymbol{\Delta}_{8k+4,\mathbb{C}^2}$, we see $\mathrm{ind}(\not{D}^0_{\mathbb{H},X}) = \mathrm{ind}(\not{D}^0_{\mathbb{C}^2,X}) = \hat{A}^h(X)$. So we have

$$\mathrm{ind}^h_{8k+4}(\mathfrak{D}) = \hat{A}^h(X).$$

For $n = 8k+5$, the situation is similar to Example 3.6, we analogously have

$$\ker \mathfrak{D} = \ker \not{D}_{\mathbb{H},X} \hat{\otimes}_{\mathbb{C}} \Delta_{8k+5,\mathbb{H}}.$$

Note that the volume element $\omega_{8k+5} \in \mathrm{Cl}_{8k+5,\mathbb{H}}$ is central satisfying $\omega^2_{8k+5,\mathbb{H}} = -1$. Thus ω_{8k+5} generates a subalgebra isomorphic to $\mathbb{C}$, and consequently $\mathrm{Cl}_{8k+5,\mathbb{H}} = \mathrm{Cl}^0_{8k+5,\mathbb{H}} \oplus \omega_{8k+5} \mathrm{Cl}_{8k+5,\mathbb{H}} \cong \mathrm{Cl}^0_{8k+5,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{C}$. Similarly for any $\mathbb{Z}_2$-graded module V of $\mathrm{Cl}_{8k+5,\mathbb{H}}$, we have $V \cong V^0 \oplus \omega_{8k+5} V^0 \cong V^0 \otimes_{\mathbb{R}} \mathbb{C}$. It follows that $\ker \not{D}_{\mathbb{H},X} \cong \ker \not{D}^0_{\mathbb{H},X} \otimes_{\mathbb{R}} \mathbb{C}$ and $\Delta_{8k+5,\mathbb{H}} \cong \Delta^0_{8k+5,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{C}$. So we have

$$\begin{aligned} \ker \mathfrak{D} &= \ker \not{D}_{\mathbb{H},X} \hat{\otimes}_{\mathbb{C}} \Delta_{8k+5,\mathbb{H}} \\ &\cong (\ker \not{D}^0_{\mathbb{H},X} \otimes_{\mathbb{R}} \mathbb{C}) \hat{\otimes}_{\mathbb{C}} (\Delta^0_{8k+5,\mathbb{H}} \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong (\ker \not{D}^0_{\mathbb{H},X} \otimes_{\mathbb{R}} \Delta^0_{8k+5,\mathbb{H}}) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \ker \not{D}^0_{\mathbb{H},X} \otimes_{\mathbb{R}} \Delta_{8k+5,\mathbb{H}}. \end{aligned}$$

One easily checks the $\mathbb{Z}_2$-grading on $\ker \mathfrak{D}$ coincides with the one inherited from $\Delta_{8k+5,\mathbb{H}}$. Therefore we conclude

$$\mathrm{ind}^h_{8k+5}(\mathfrak{D}) = \dim_{\mathbb{R}} \ker \not{D}^0_{\mathbb{H},X} = \dim_{\mathbb{C}} \ker \not{D}_{\mathbb{H},X} \pmod 2.$$

There is a strong analogy in dimensions $n = 8k+6$. The volume element ω_{8k+6} generates a subalgebra $\mathbb{C}^{\omega}$ of $\mathrm{Cl}^0_{8k+6,\mathbb{H}}$ that is isomorphic to $\mathbb{C}$; moreover ω together with $e = e_{8k+6}$ generate a subalgebra $\mathbb{H}^{\omega,e}$ of $\mathrm{Cl}_{8k+6,\mathbb{H}}$ that is isomorphic to $\mathbb{H}$. Then we have $\mathrm{Cl}_{8k+6,\mathbb{H}} \cong \mathrm{Cl}^0_{8k+6,\mathbb{H}} \otimes_{\mathbb{C}^{\omega}} \mathbb{H}^{\omega,e}$. Such structure is carried on by its modules as well. Then the same analysis as in the $8k+5$ case proves

$$\ker \mathfrak{D} \cong \ker \not{D}^0_{\mathbb{H},X} \otimes_{\mathbb{C}} \Delta_{8k+6,\mathbb{H}}$$

and consequently

$$\mathrm{ind}^h_{8k+6}(\mathfrak{D}) = \dim_{\mathbb{C}} \ker \not{D}^0_{\mathbb{H},X} = \dim_{\mathbb{H}} \ker \not{D}_{\mathbb{H},X} \pmod 2.$$

Finally for $n = 8k$, recall the forgetful morphism $\varepsilon : \hat{\mathfrak{N}}^{\mathbb{C}^2}_{8k} \to \hat{\mathfrak{N}}^{\mathbb{H}}_{8k}$ is an isomorphism, and $\Delta_{8k,\mathbb{C}^2}$ generates $\hat{\mathfrak{N}}^{\mathbb{C}^2}_{8k}$. The argument of Example 3.6 extended to the complex case yields

$$\mathrm{ind}^h_{8k}(\mathfrak{D}) = \frac{1}{2} \mathrm{ind}(\not{D}^0_{\mathbb{C}^2,X}) = \frac{1}{2} \hat{A}^h(X).$$

In particular $\hat{A}^h(X)$ is an even integer in the case $n = 8k$. Of course this also follows from our Riemann-Roch theorem.

Definition 3.8. We define the Clifford index of a closed spin^h manifold X of dimension n twisted by a real vector bundle γ, denoted by $\hat{\mathcal{A}}^h(X,\gamma)$, to be the index of the $\mathrm{Cl}_{n,\mathbb{H}}$-Dirac bundle $\mathfrak{S}(X) \otimes_{\mathbb{R}} \gamma$. If γ is the trivial line bundle $\mathbb{R}$, then we simply call $\hat{\mathcal{A}}^h(X,\mathbb{R}) =: \hat{\mathcal{A}}^h(X)$ the Clifford index of X.

We have proved the following.

Theorem 3.9. Let X be a closed spin^h manifold of dimension n. When $n \equiv 5$ or $6 \bmod 8$, let $\mathcal{H} = \ker \not{D}_{\mathbb{H},X}$ denote the space of harmonic ^{h}spinors, that is, the kernel of the Dirac operator on the $\mathbb{Z}_2$-graded fundamental real ^{h}spinor bundle of X. Then

$$\hat{\mathcal{A}}^h(X) = \begin{cases} \hat{A}^h(X)/2 & \text{if } n \equiv 0 \bmod 8 \\ \hat{A}^h(X) & \text{if } n \equiv 4 \bmod 8 \\ \dim_{\mathbb{C}} \mathcal{H} \pmod 2 & \text{if } n \equiv 5 \bmod 8 \\ \dim_{\mathbb{H}} \mathcal{H} \pmod 2 & \text{if } n \equiv 6 \bmod 8 \end{cases} \tag{12}$$

The same argument goes through with $\not{\mathfrak{S}}(X)$ replaced by $\not{\mathfrak{S}}(X) \otimes_{\mathbb{R}} \gamma$, so we have:

Theorem 3.10. Let X be a closed spin^h manifold of dimension n and γ a real vector bundle over X whose complexification is denoted by $\gamma_{\mathbb{C}}$. When $n \equiv 5$ or $6 \bmod 8$, let $\mathcal{H}_\gamma = \ker(\not{D}_{\mathbb{H},X} \otimes \mathrm{Id}_\gamma)$ denote the space of harmonic ^{h}spinors in γ, that is, the kernel of the Dirac operator on the $\mathbb{Z}_2$-graded fundamental real ^{h}spinor bundle of X twisted by γ. Then

$$\hat{\mathcal{A}}^h(X,\gamma) = \begin{cases} \hat{A}^h(X,\gamma_{\mathbb{C}})/2 & \text{if } n \equiv 0 \bmod 8 \\ \hat{A}^h(X,\gamma_{\mathbb{C}}) & \text{if } n \equiv 4 \bmod 8 \\ \dim_{\mathbb{C}} \mathcal{H}_\gamma \pmod 2 & \text{if } n \equiv 5 \bmod 8 \\ \dim_{\mathbb{H}} \mathcal{H}_\gamma \pmod 2 & \text{if } n \equiv 6 \bmod 8 \end{cases} \tag{13}$$

■

It is easy to see $\hat{A}^h(X)$ is a spin^h cobordism invariant using Chern-Weil theory and Stokes theorem. As for the $\mathbb{Z}_2$-valued invariants, it appears they rely on a prior choice of the connection on the canonical bundle, however we will show the Clifford index $\hat{\mathcal{A}}^h$ in all dimensions, including the more refined 2-torsion part, is a spin^h cobordism invariant. This will be achieved by identifying $\hat{\mathcal{A}}^h$ with the analytic index of certain family of quaternionic elliptic operators.

3.3. **Index of a family of quaternionic operators.** Recall that a complex vector bundle E is said to be quaternionic if E is equipped with a real vector bundle automorphism $j : E \to E$ which is $\mathbb{C}$-antilinear in each fiber and $j^2 = -1$. The space of sections $\Gamma(E)$ is equipped with a quaternionic structure given by j^*.

Suppose now E, F are quaternionic vector bundles over a closed manifold X and $P : \Gamma(E) \to \Gamma(F)$ is an elliptic differential operator. We say P is quaternionic if $Pj_E^* = j_F^* P$. In local terms, $P = \sum A^\alpha(x)\partial^{|\alpha|}/\partial x^\alpha$ plus lower order terms, where the A^α's are complex-matrix-valued functions with $A^\alpha j_E = j_F A^\alpha$. The principal symbol $\sigma_\xi(P) = \sum A^\alpha(x)(\sqrt{-1}\xi)^\alpha$ of P thus satisfies

$$\sigma_\xi(P) j_E = j_F \sigma_{-\xi}(P) \tag{14}$$

for any tangent vector ξ of X. The symbol class of a quaternionic elliptic differential operator therefore lands in KQ-theory.

Definition 3.11. Given a closed manifold X, consider the tangent bundle $\pi : TX \to X$ to be equipped with the canonical involution $f : TX \to TX$ defined by $f(e) = -e$, i.e. the fiberwise antipodal map. Given any quaternionic vector bundle (E, j) over X, π^*E is in a natural way a Quaternionic bundle over the real space (TX, f) by setting $J : \pi^*E \to \pi^*E$ to be

$$J(x, \xi, e) = (x, -\xi, j(e)).$$

Suppose now E, F are quaternionic vector bundles over X, then for any quaternionic elliptic operator $P : \Gamma(E) \to \Gamma(F)$, the Quaternionic symbol class of P is defined to be the element

$$[\pi^*E, \pi^*F; \sigma(P)] \in \mathrm{KQ}_{cpt}(TX).$$

Note (14) says $\sigma(P)$ is an isomorphism of Quaternionic bundles outside the zero section of TX.

To define the topological index of a quaternionic elliptic operator, we need a version of Thom isomorphism for KQ-theory which is explained in the Appendix.

Theorem 3.12 (Atiyah, Dupont)**.** Let E be a Real bundle over the locally compact real space X. Then multiplication by λ_E induces isomorphisms

$$\mathrm{KR}_{cpt}(X) \xrightarrow{\cong} \mathrm{KR}_{cpt}(E)$$
$$\mathrm{KQ}_{cpt}(X) \xrightarrow{\cong} \mathrm{KQ}_{cpt}(E)$$

where $\lambda_E \in \mathrm{KR}_{cpt}(E)$ is defined by the exterior algebra of E.

Remark 3.13. Locally these Thom isomorphisms are compositions of (1,1)-periodicities.

Now we can define the topological index of a quaternionic elliptic operator P as follows. We first choose an embedding $f : X \hookrightarrow \mathbb{R}^m$. The associated embedding $TX \hookrightarrow T\mathbb{R}^m$ is compatible with involutions, i.e. is a mapping of real spaces. If N is the normal bundle to X in $\mathbb{R}^m$, then $\pi^*N \oplus \pi^*N \cong \pi^*N \otimes \mathbb{C}$ is the normal bundle to TX in $T\mathbb{R}^m$. We consider this to be a Real bundle over TX (with complex conjugation as its involution). Then similar to the construction in our Riemann-Roch theorem, we can define a map

$$f_! : \mathrm{KQ}_{cpt}(TX) \to \mathrm{KQ}_{cpt}(T\mathbb{R}^m) \tag{15}$$

by composing the Thom isomorphism with the map induced by the inclusion of the normal bundle as a tubular neighborhood of TX in $T\mathbb{R}^m$. This inclusion can be easily chosen to be compatible with involutions. We now identify $T\mathbb{R}^m = \mathbb{R}^{m,m} = \mathbb{C}^m$, then $\mathrm{KQ}_{cpt}(T\mathbb{R}^m) \cong \mathrm{KQ}^{n,n}(pt) \cong \mathrm{KQ}^{0,0}(\mathrm{pt}) \cong \mathbb{Z}$. Therefore we can define the topological index of P to be the integer $f_!(\sigma(P))$.

As usual, the fact that the topological index is independent of our choice of the embedding follows from the multiplicative property of the KR-Thom class for Real bundles.

The discussion of symbol class and topological index naturally extends to families of quaternionic operators.

Definition 3.14. Let P be a family of quaternionic elliptic operators on a closed manifold X parametrized by a compact Hausdorff space A. Let $\mathfrak{X} \to A$ denote the underlying family of manifolds, and let $\boldsymbol{\sigma}(P) \in \mathrm{KQ}_{cpt}(T\mathfrak{X})$ be the symbol class of the family. The topological index of the family P is defined to be the element

$$\text{t-ind}(P) = q_! f_! \boldsymbol{\sigma}(P) \in \mathrm{KQ}_{cpt}(A) \cong \mathrm{KSp}_{cpt}(A)$$

where $f_! : \mathrm{KQ}_{cpt}(T\mathfrak{X}) \to \mathrm{KQ}_{cpt}(A \times T\mathbb{R}^m)$ is constructed similar to (15) and $q_! : \mathrm{KQ}_{cpt}(A \times \mathbb{C}^n) \to \mathrm{KQ}_{cpt}(A)$ is the natural isomorphism given by the Thom isomorphism.

The forgetful morphism $\mathrm{KSp}_{cpt}(A) \to \mathrm{KU}_{cpt}(A)$ is not always injective, so the index we just defined is more refined than the usual index of P as a family of complex operators.

One can of course define the analytic index for such a family P of quaternionic elliptic operators by setting

$$\text{a-ind}(P) = [\ker P] - [\operatorname{coker} P] \in \mathrm{KSp}(A).$$

To be more precise, if the dimensions of $\ker P_a$ and $\operatorname{coker} P_a$ are constant for $a \in A$, then $\ker P$ and $\operatorname{coker} P$ define two quaternionic bundles over A. In this case, a-ind(P) is defined to be the difference class $[\ker P] - [\operatorname{coker} P]$. In general, $\ker P_a$ and $\operatorname{coker} P_a$ are not constant dimensional, then we must first "stabilize" the situation as Atiyah and Singer did in the complex case in [AS71a]; we suffice to note the treatment in [AS71a, sec.2] can be easily made to respect the quaternionic structures.

The analytic index, of course, coincides with the topological index.

Theorem 3.15. Let P be a family of quaternionic elliptic operators on a closed manifold parametrized by a compact Hausdorff space A. Then

$$\text{a-ind}(P) = \text{t-ind}(P).$$

The proof of this quaternionic version of the index theorem proceeds just as in the case of real and complex families [AS71a, AS71b]. Given that the proof for real and complex cases has become a common knowledge, we will not attempt to duplicate the proof for the quaternionic case here. Instead, we will point out the key places where changes must be made to adapt the argument for real and complex families to quaternionic families.

Recall that such index theorem for families essentially relies on checking the following three axioms.

Lemma 3.16. The analytic index

$$\text{a-ind} : \mathrm{KQ}_{cpt}(T\mathfrak{X}) \to \mathrm{KQ}(A)$$

is a homomorphism of $\mathrm{KR}(A)$-modules, which in the special case $\mathfrak{X} = A = \mathrm{pt}$ is the identity map.

Lemma 3.17 (Excision)**.** Let $\mathfrak{X} \to A$ and $\mathfrak{X}' \to A$ be two families over A with compact fibers X, X' respectively and let $f : \mathcal{O} \hookrightarrow \mathfrak{X}$, $f' : \mathcal{O}' \hookrightarrow \mathfrak{X}'$ be inclusions of open sets, with a smooth equivalence $\mathcal{O} \cong \mathcal{O}'$ compatible with the maps to A. Then, identifying $\mathcal{O}'$ with $\mathcal{O}$, the following diagram commutes:

$$\begin{tikzcd} & \mathrm{KQ}_{cpt}(T\mathfrak{X}) \arrow[dr, "\text{a-ind}"] & \\ \mathrm{KQ}_{cpt}(T\mathcal{O}) \arrow[ur, "f_!"] \arrow[dr, "f'_!"'] & & \mathrm{KQ}(A) \\ & \mathrm{KQ}_{cpt}(T\mathfrak{X}') \arrow[ur, "\text{a-ind}"'] & \end{tikzcd}$$

Lemma 3.18 (Multiplicativity)**.** Let $\mathcal{E} \to \mathfrak{X}$ be a family of oriented smooth vector bundles of rank n, and let $\mathcal{S} = S(\mathcal{E} \oplus \mathbb{R})$ be the family of n-sphere bundle compactified from $\mathcal{E}$. Then the following diagram commutes:

$$\begin{tikzcd} \mathrm{KQ}_{cpt}(T\mathfrak{X}) \arrow[rr, "i_!"] \arrow[dr, "\text{a-ind}"'] & & \mathrm{KQ}_{cpt}(T\mathcal{S}) \arrow[dl, "\text{a-ind}"] \\ & \mathrm{KQ}(A) & \end{tikzcd}$$

where $i_!$ is multiplication by the fundamental equivariant symbol $b \in \mathrm{KR}_{\mathrm{SO}_n}(TS^n)_{cpt}$ (cf. [AS71b]).

The argument of [AS71a, AS71b] for the excision and multiplicative properties goes through easily in the quaternionic case, only Lemma 3.16 requires special attention. There are several implicit facts hidden in our statement of Lemma 3.16. First the analytic index depends only on the homotopy class of the symbol class, which is a consequence of [Mat71, Main Theorem III]. Second, every element in $\mathrm{KQ}_{cpt}(T\mathfrak{X})$ can be represented by some symbol class. And finally the homomorphism a-ind is well-defined, i.e it does not depend on the choice of symbol-class-representatives. The second and the last points can be proved no differently from the real and complex cases.

3.4. **Topological formula of $\mathrm{Cl}_{k,\mathbb{H}}$-index.** Assume E is a $\mathbb{Z}_2$-graded $\mathrm{Cl}_{k,\mathbb{H}}$-bundle over a closed riemannian manifold X. Further assume E carries a bundle metric for which the Clifford multiplication by unit vectors in $\mathbb{R}^k$ is orthogonal and the multiplication by unit quaternions is orthogonal. Let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint operator and assume P is $\mathrm{Cl}_{k,\mathbb{H}}$-linear and $\mathbb{Z}_2$-graded. Recall we defined the index $\mathrm{ind}^h(P) \in \mathrm{KSp}^{-k}(\mathrm{pt})$ in terms of the $\mathrm{Cl}_{k,\mathbb{H}}$-module $\ker P$. We shall now give a topological formula for this index.

Since P and $(1 + P^*P)^{-1/2}P$ have the same kernel, we may assume P has degree zero. With respect to the splitting $E = E^0 \oplus E^1$, P can be written as

$$P = \begin{pmatrix} 0 & P^1 \\ P^0 & 0 \end{pmatrix}$$

where $P^1 = (P^0)^*$. Now we construct a family $\mathscr{P}$ of quaternionic elliptic operators parametrized by $\mathbb{R}^k$ by assigning to each $v \in \mathbb{R}^k$ the operator

$$\mathscr{P}_v^0 : \Gamma(E^0) \to \Gamma(E^1)$$

defined by the restriction to E^0 of the operator

$$\mathscr{P}_v := v + P$$

where "v" denotes Clifford multiplication by v. Since both Clifford multiplication and P are $\mathbb{H}$-linear, so is $\mathscr{P}$. Also since P commutes with Clifford multiplication, there is a "conjugate" family $\overline{\mathscr{P}}_v = v - P$ satisfying

$$\overline{\mathscr{P}}_v \mathscr{P}_v = \mathscr{P}_v \overline{\mathscr{P}}_v = -(\|v\|^2 + P^2).$$

Therefore $\mathscr{P}^0_v$ is invertible for all $v \neq 0$. Since the invertible $\mathbb{H}$-linear operators on quaternionic Hilbert spaces form a contractible set (see [Seg69], [Mat71]), we could pass to a family parametrized by S^k, however the calculation will be more illustrating if we treat $\mathscr{P}^0$ as a family with "compact support", whose index lies in $\mathrm{KSp}_{cpt}(\mathbb{R}^k) \cong \mathrm{KSp}^{-k}(\mathrm{pt})$.

Theorem 3.19. Let P be an elliptic self-adjoint $\mathbb{Z}_2$-graded $\mathrm{Cl}_{k,\mathbb{H}}$-operator on a closed manifold X. Then

$$\mathrm{ind}^h(P) = \text{a-ind}(\mathscr{P}^0).$$

Proof. Set $K^0 = \ker P^0 \subset \Gamma(E^0)$ and $K^1 = \ker P^1 \cong \mathrm{coker}(P^0) \subset \Gamma(E^1)$. Then K^0, K^1 are finite dimensional $\mathbb{H}$-subspaces of $\Gamma(E^0)$ and $\Gamma(E^1)$ respectively. By assumption the quaternionic structure on E is compatible with the its bundle metric, so there are L^2-orthogonal compliments V^0, V^1 to K^0, K^1 respectively. Then the family $\mathscr{P}^0_v$ decomposes as a direct sum of two operators: the first summand $V^0 \xrightarrow{\mathscr{P}^0_v} V^1$ is an $\mathbb{H}$-isomorphism for all $v \in \mathbb{R}^k$, thus can be ignored for the purpose of computing the index; whilst the second summand is just $K^0 \xrightarrow{\mathscr{P}^0_v = v} K^1$ which is independent of variables on X. Therefore the analytic index of $\mathscr{P}^0$ is

$$\text{a-ind}(\mathscr{P}) = [K^0, K^1; v] \in \mathrm{KSp}_{cpt}(\mathbb{R}^k) \cong \mathrm{KSp}^{-k}(\mathrm{pt}).$$

Under the isomorphism $\mathrm{KSp}^{-k}(\mathrm{pt}) \cong \hat{\mathfrak{N}}^{\mathbb{H}}_k$, this corresponds exactly to the element represented by $\ker P = K^0 \oplus K^1$, i.e. it corresponds exactly to $\mathrm{ind}^h(P)$. ■

In view of Theorem 3.15, $\text{a-ind}(\mathscr{P}^0) = \text{t-ind}(\mathscr{P}^0)$. So Theorem 3.19 can be applied to give a topological formula for the Clifford index of spin^h manifolds.

3.5. **Cobordism invariance of Clifford index.** Let X be a closed spin^h manifold of dimension n. Recall X carries a canonical $\mathrm{Cl}_{n,\mathbb{H}}$-Dirac bundle $\mathfrak{S}(X) := P_{\mathrm{Spin}^h}(X) \times_l \mathrm{Cl}_{n,\mathbb{H}}$ with Dirac operator $\mathfrak{D}$. Then by Theorem 3.19, the Clifford index of X, i.e. $\mathrm{ind}^h(\mathfrak{D})$, coincides with the index of the family $\mathscr{D}^0$ defined by setting

$$\mathscr{D}^0_{v,x} = v + \mathfrak{D}^0_x.$$

To compute the topological index of this family, we must understand its symbol class $\boldsymbol{\sigma}(\mathscr{D}^0) \in \mathrm{KQ}_{cpt}(\mathbb{R}^n \times TX)$. For this we note $\mathbb{R}^n \times TX \to X$ is a Real bundle over X whose fibre at $x \in X$ is $\mathbb{R}^n \times T_xX$ with involution $(v, \xi) \mapsto (v, -\xi)$. The fibre of the bundle $\mathfrak{S}(X)$ at $x \in X$ is the Clifford algebra $\mathrm{Cl}_{\mathbb{H}}(T_xX) \cong \mathrm{Cl}_{n,\mathbb{H}}$. Tangent vectors $\xi \in T_xX$ act by right Clifford multiplication and vectors $v \in \mathbb{R}^n$ act by left multiplication. The principle symbol of $\mathscr{D}^0$ is the map $\boldsymbol{\sigma}(\mathscr{D}^0) : \Gamma(\pi^*\mathfrak{S}^0) \to \Gamma(\pi^*\mathfrak{S}^1)$ defined by

$$\boldsymbol{\sigma}_{v,\xi}(\mathscr{D}^0) = v + i\xi = R_v + iL_\xi$$

where L, R stand for left and right Clifford multiplications respectively. When restricted to any fibre $\mathbb{R}^n \times T_xX \cong \mathbb{R}^{n,n} \cong \mathbb{C}^n$ the symbol class becomes

$$[\mathrm{Cl}^0_{n,\mathbb{H}}, \mathrm{Cl}^1_{n,\mathbb{H}}; R_v + iL_\xi] \in \mathrm{KQ}_{cpt}(\mathbb{C}^n).$$

We claim this is a generator for $\mathrm{KQ}_{cpt}(\mathbb{C}^n) \cong \mathbb{Z}$. Indeed when $n = 0$, $\mathrm{Cl}_{0,\mathbb{H}} = \mathrm{Cl}^0_{0,\mathbb{H}} = \mathbb{H}$ and our claim trivially holds. For $n \geq 1$, since $[\mathrm{Cl}^0_n, \mathrm{Cl}^1_n; R_v + iL_\xi] \in \mathrm{KR}_{cpt}(\mathbb{C}^n)$ generates $\mathrm{KR}_{cpt}(\mathbb{C}^n)$ (see [LM89, Prop. 10.2]), by (1,1)-periodicity

$$[\mathrm{Cl}^0_{n,\mathbb{H}}, \mathrm{Cl}^1_{n,\mathbb{H}}; R_v + iL_\xi] = [\mathrm{Cl}^0_n \otimes_{\mathbb{C}} \mathbb{H}, \mathrm{Cl}^1_n \otimes_{\mathbb{C}} \mathbb{H}; R_v + iL_\xi]$$

generates $\mathrm{KQ}_{cpt}(\mathbb{C}^n)$ as claimed.

Before we proceed, let us make an easy but useful observation. Consider the bundle $P_{\mathrm{Spin}^h(X)} \times_r \mathrm{Cl}_{n,\mathbb{H}}$ where $r : \mathrm{Spin}^h(X) \to \mathrm{Aut}(\mathrm{Cl}_{n,\mathbb{H}})$ is the transpose of right Clifford multiplication. Through the transpose isomorphism $\mathrm{Cl}_{n,\mathbb{H}} \cong \mathrm{Cl}^{op}_{n,\mathbb{H}}$, l and r are *equivalent* real representations of $\mathrm{Spin}^h(n)$.

Therefore we have a bundle isomorphism $\not{\mathfrak{S}}(X) \cong P_{\mathrm{Spin}^h}(X) \times_r \mathrm{Cl}_{n,\mathbb{H}}$ under which the symbol class becomes

$$\boldsymbol{\sigma}(\mathscr{D}^0) = [\pi^*\not{\mathfrak{S}}^0, \pi^*\not{\mathfrak{S}}^1; L_v + iR_\xi].$$

Now choose a smooth embedding $f : X \hookrightarrow \mathbb{R}^{n+8k+4}$ and let N denote its normal bundle, which we identify with the tubular neighborhood of X. This induces an embedding $TX \hookrightarrow T\mathbb{R}^{n+8k+4} \cong \mathbb{C}^{n+8k+4}$ of real spaces with normal bundle $TN \cong \pi_X^*N \oplus \pi_X^*N \cong \pi_X^*N \otimes_{\mathbb{R}} \mathbb{C}$ where $\pi_X : TX \to X$ is the projection. Then we get the following commutative diagram of bundle maps:

$$\begin{array}{ccc} TX & \xleftarrow{\bar{p}} & \pi_X^*N \otimes \mathbb{C} \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_N} \\ X & \xleftarrow{p} & N \end{array}$$

The vertical maps are Real bundles. By taking zero sections, we obtain a diagram of embeddings

$$\begin{array}{ccccc} X & \xrightarrow{j} & N & \xrightarrow{k} & \mathbb{R}^{n+8k+4} \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle i_{\mathbb{R}}} \\ TX & \xrightarrow{\bar{j}} & \pi_X^*N \otimes \mathbb{C} & \xrightarrow{\bar{k}} & \mathbb{C}^{n+8k+4} \end{array}$$

We would like to show the following induced diagram commutes:

$$\begin{array}{ccccc} \mathrm{KR}(X) & \xrightarrow{j_!} & \mathrm{KR}_{cpt}(N) & \xrightarrow{k_!} & \mathrm{KR}_{cpt}(\mathbb{R}^{n+8k+4}) \\ \downarrow{\scriptstyle (i_X)_!} & & \downarrow{\scriptstyle (i_N)_!} & & \downarrow{\scriptstyle (i_{\mathbb{R}})_!} \\ \mathrm{KQ}_{cpt}(\mathbb{R}^n \times TX) & \xrightarrow{\bar{j}_!} & \mathrm{KQ}_{cpt}(\mathbb{R}^n \times (\pi_X^*N \otimes \mathbb{C})) & \xrightarrow{\bar{k}_!} & \mathrm{KQ}_{cpt}(\mathbb{R}^n \times \mathbb{C}^{n+8k+4}) \end{array}$$

We must explain the definition of each morphism. Staring from the first row, since $X, N, \mathbb{R}^{n+8k+4}$ carry trivial involutions, KR-groups of these spaces coincide with their KO-groups. With this understood, $j_!$ is the Thom homomorphism induced by the weak KO-Thom class of N, and $k_!$ is the restriction map. The second row is similar, $\bar{j}_!$ is the Thom isomorphism induced by the Real bundle $\pi^*N \otimes \mathbb{C}$ and $\bar{k}_!$ is the restriction map.

The vertical maps are more complicated. $(i_X)_!$ is the map induced by multiplying the symbol class $\boldsymbol{\sigma}(\mathscr{D}^0)$. To define $(i_N)_!$, we note since $\pi_X^*N \otimes \mathbb{C} \cong TN \cong p^*TX \oplus p^*N$ and $w_2(TX) = w_2(N)$, the bundle $TN \to N$ has structure group

$$G = \mathrm{Spin}(n) \times \mathrm{Spin}(8k+4)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ is the diagonal $\{(1,1), (-1,-1)\}$. The group G acts on $\mathrm{Cl}_n \otimes_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}}$ via $\rho = (r, r)$, that is $\mathrm{Spin}(n)$ (resp. $\mathrm{Spin}(8k+4)$) acts on Cl_n (resp. $\widetilde{\Delta}_{8k+4,\mathbb{H}}$) through the transpose of right Clifford multiplication. It is easy to see ρ is a well-defined representation of G: the two factorwise defined actions commute and descends to a G-representation. Therefore, we obtain a $\mathbb{Z}_2$-graded vector bundle over N:

$$E = P_G(TN) \times_\rho (\mathrm{Cl}_n \otimes_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}})$$

This bundle carries a natural right quaternionic structure inherited from the right $\mathbb{H}$-module structure on $\widetilde{\Delta}_{8k+4,\mathbb{H}}$, which we turn into a left one by conjugation. So E is a $\mathbb{Z}_2$-graded Q-bundle over N, and thus pulls back by π_N to a Q-bundle on TN. Define $(i_N)_!$ to be the homomorphism induced by multiplication with

$$[\pi_N^*E^0, \pi_N^*E^1; L_v + iR_\xi] \in \mathrm{KQ}_{cpt}(\mathbb{R}^n \times TN)$$

where L_v and R_ξ stands for the left Clifford multiplication by $\mathbb{R}^n$ and the right Clifford multiplication by $\xi \in TN$. We similarly define $(i_{\mathbb{R}})_!$ using that the structure group of the bundle $\mathbb{C}^{n+8k+4} \cong T\mathbb{R}^{n+8k+4} \to \mathbb{R}^{n+8k+4}$ can be reduced to $\mathrm{Spin}(n) \times \mathrm{Spin}(8k+4)/\mathbb{Z}_2$. It clearly follows that $\bar{k}_!(i_N)_! = (i_{\mathbb{R}})_! k_!$.

We now show $\bar{j}_!(i_X)_! = (i_N)_! j_!$. To elaborate, we replace the name of each morphism by the $\mathbb{Z}_2$-graded bundle inducing it:

$$\begin{array}{ccc} \mathrm{KR}(X) & \xrightarrow{\$_{\mathbb{H}}(N)\otimes\mathbb{C}} & \mathrm{KR}_{cpt}(N) \\ \downarrow{\scriptstyle \mathfrak{S}(X)} & & \downarrow{\scriptstyle E} \\ \mathrm{KQ}_{cpt}(\mathbb{R}^n \times TX) & \xrightarrow{\mathbb{C}\mathrm{l}(N)} & \mathrm{KQ}_{cpt}(\mathbb{R}^n \times (\pi^* N \otimes \mathbb{C})) \end{array}$$

The commutativity of this diagram follows from two facts. First, there is a $\mathbb{Z}_2$-graded Q-bundle isomorphism over N:

$$p^*\mathfrak{S}(X) \otimes_{\mathbb{R}} p^* \mathrm{Cl}(N) \cong E \otimes_{\mathbb{R}} p^* \$_{\mathbb{H}}(N).$$

Indeed both bundles correspond to the $\mathbb{Z}_2$-graded representation (r, Ad, r)

$$\mathrm{Cl}_{n,\mathbb{H}} \otimes_{\mathbb{R}} \mathrm{Cl}_{8k+4} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \mathrm{Cl}_{8k+4,\mathbb{H}} \cong \mathrm{Cl}_n \otimes_{\mathbb{R}} \Delta_{8k+4,\mathbb{H}} \otimes_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}}$$

of the group

$$\mathrm{Spin}(n) \times \mathrm{Spin}(8k+4) \times \mathrm{Sp}(1)/\{(1,1,1),(-1,-1,-1)\}$$

where $\mathrm{Spin}(n)$ acts through the transpose of right Clifford multiplication on Cl_n, $\mathrm{Spin}(8k+4)$ acts through adjoint representation on Cl_{8k+4} and $\mathrm{Sp}(1)$ acts by conjugate of right multiplication. Second, keeping track of the isomorphisms (away from zero section) between even and odd parts of these bundles, the "free" $\mathbb{R}^n$ always acts from the left and the tangent vectors acts from the right.

Finally, we assert:

Proposition 3.20. $(i_{\mathbb{R}})_! : \mathrm{KR}_{cpt}(\mathbb{R}^{n+8k+4}) \to \mathrm{KQ}_{cpt}(\mathbb{R}^n \times \mathbb{C}^{n+8k+4})$ is an isomorphism.

Proof. Under the isomorphisms

$$\mathrm{KR}_{cpt}(\mathbb{R}^{n+8k+4}) \cong \hat{\mathfrak{N}}^{\mathbb{R}}_{n+8k+4,0}$$

and

$$\mathrm{KQ}_{cpt}(\mathbb{R}^n \times \mathbb{C}^{n+8k+4}) \cong \hat{\mathfrak{N}}^{\mathbb{H}}_{2n+8k+4,n+8k+4},$$

$(i_{\mathbb{R}})_!$ is identified with multiplication by

$$\mathrm{Cl}_n \hat{\otimes}_{\mathbb{R}} \widetilde{\Delta}_{8k+4,\mathbb{H}} \in \hat{\mathfrak{N}}^{\mathbb{H}}_{n,n+8k+4}$$

where $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{8k+4}$ acts by $L_x + (-1)^\alpha R_y + (-1)^\alpha R_z$. By (1,1)-periodicity, this element simply corresponds to

$$\widetilde{\Delta}_{8k+4,\mathbb{H}} \in \hat{\mathfrak{N}}^{\mathbb{H}}_{0,8k+4}.$$

The proposition then follows from the algebraic lemma below. ■

Lemma 3.21. $\widetilde{\Delta}_{8k+4,\mathbb{H}}$ generates $\hat{\mathfrak{N}}^{\mathbb{H}}_{0,8k+4}$ and $\hat{\mathfrak{N}}^{\mathbb{R}}_{*,0} \otimes \hat{\mathfrak{N}}^{\mathbb{H}}_{0,8k+4} \xrightarrow{\hat{\otimes}} \hat{\mathfrak{N}}^{\mathbb{H}}_{*,8k+4}$ is an isomorphism.

Proof. We first observe that there is a $\mathbb{Z}_2$-graded isomorphism of real algebras $\mathrm{Cl}_{0,8k+4} \cong \mathrm{Cl}_{8k+4,0}$. Consider the linear map $f : \mathbb{R}^{8k+4} \to \mathrm{Cl}_{8k+4,0}$ defined on the standard orthonormal basis $\{e_i : 1 \leq i \leq 8k+4\}$ by

$$f(e_i) = e_i \omega_{8k+4}.$$

It is easy to verify $f(e_i)^2 = 1$ and $f(e_i)f(e_j) + f(e_j)f(e_i) = 0$ for all i, j. Therefore f extends to an algebra map $f : \mathrm{Cl}_{0,8k+4} \to \mathrm{Cl}_{8k+4,0}$. We note f preserves the $\mathbb{Z}_2$-grading and it maps onto a set of generator. Now since the two algebras in question have the same dimension, we conclude f is an isomorphism of $\mathbb{Z}_2$-graded algebras. By $\mathbb{Z}_2$-graded tensoring with $\mathrm{Cl}_{*,0}$, f induces isomorphisms $\mathrm{Cl}_{*+8k+4,0} \cong \mathrm{Cl}_{*,8k+4}$ for all $* \geq 0$. It follows that $\hat{\mathfrak{N}}^{\mathbb{H}}_{0,8k+4} \cong \hat{\mathfrak{N}}^{\mathbb{H}}_{8k+4,0} \cong \mathbb{Z}$.

We claim $\widetilde{\Delta}_{8k+4,\mathbb{H}}$ generates $\hat{\mathfrak{N}}^{\mathbb{H}}_{0,8k+4}$. Indeed since $\mathrm{Cl}_{0,8k+4} \cong \mathrm{Cl}_{8k+4,0}$, the dimensions of irreducible $\mathbb{H}$-modules of these two algebras must be the same, so by a dimension count $\widetilde{\Delta}_{8k+4,\mathbb{H}}$ is the unique (up to equivalence) irreducible $\mathbb{H}$-module for $\mathrm{Cl}_{0,8k+4}$. Finally the fact that $\hat{\mathfrak{N}}^{\mathbb{R}}_{*,0} \otimes \hat{\mathfrak{N}}^{\mathbb{H}}_{*,8k+4} \xrightarrow{\hat{\otimes}} \hat{\mathfrak{N}}^{\mathbb{H}}_{*,8k+4}$ is an isomorphism now follows from the isomorphism

$$\hat{\mathfrak{N}}^{\mathbb{R}}_{*,0} \otimes \hat{\mathfrak{N}}^{\mathbb{H}}_{8k+4,0} \xrightarrow{\hat{\otimes}} \hat{\mathfrak{N}}^{\mathbb{H}}_{*+8k+4,0}$$

which is a consequence of 8-fold periodicity and Proposition 1.23. ■

To summarize, we have identified the index of $\not{\mathscr{D}}^0$ with $q_!k_!j_!(1)$ where $q_! : \mathrm{KO}_{cpt}(\mathbb{R}^{n+8k+4}) \cong \mathrm{KSp}^{-n}(\mathrm{pt})$ is the periodicity isomorphism.

Theorem 3.22. Let X be a spin^h manifold of dimension n. Let $f : X \hookrightarrow \mathbb{R}^{n+8k+4}$ be a smooth embedding. Denote by $f_! : \mathrm{KO}(X) \to \mathrm{KO}_{cpt}(\mathbb{R}^{n+8k+4})$ the composition of the Thom homomorphism and the restriction map, and denote by $q_! : \mathrm{KO}_{cpt}(\mathbb{R}^{n+8k+4}) \cong \mathrm{KSp}^{-n}(\mathrm{pt})$ the periodicity isomorphism. Then

$$\hat{A}^h(X) = q_!f_!(1) \in \mathrm{KSp}^{-n}(\mathrm{pt}).$$

In particular, $\hat{\mathcal{A}}^h$ is a spin^h-cobordism invariant.

Proof. Since $\hat{A}^h$ does not depend on the choice of the embedding f, we may choose k to be large enough. The classifying map of the normal bundle to X in $\mathbb{R}^{n+8k+4}$, by Pontryagin-Thom construction, induces a map $S^{n+8k+4} \to MSpin^h(8k+4)$. The universal weak KO-Thom class $\Theta_{8k+4} \in \widetilde{KO}(MSpin(8k+4))$ corresponds to a map $MSpin^h(8k+4) \to BO$. From definition $q_!f_!(1)$ is exactly the homotopy class of the composition

$$S^{n+8k+4} \to MSpin^h(8k+4) \to BO.$$

Now if X bounds a spin^h manifold (with the restricted spin^h structure), then the homotopy class of $S^{n+8k+4} \to MSpin^h(8k+4)$ is trivial by a standard Pontryagin-Thom argument. This proves the cobordism invariance. ∎

In fact the spin^h-cobordism invariance of $\hat{\mathcal{A}}^h$ follows quickly from that when X bounds a spin^h manifold (in a spin^h fashion), the symbol class of $\not{\mathscr{D}}^0$ is trivial since the Clifford multiplications extend over to the zero section. The proof we present here is more complicated, but has its own benefit. To elaborate, we assume for the moment the reader is familiar with generalized homology theories and the language of spectra (see e.g. [Whi62]).

The map induced by the weak-KO-Thom class

$$\Theta_{8k+4} : MSpin^h(8k+4) \to BO \subset BO \times \mathbb{Z}$$

assembles into a spectrum map from the Thom spectrum of spin^h cobordism to the Ω-spectrum of the symplectic K-theory. This is a consequence of the following commutative square

$$\begin{array}{ccc} S^8 \wedge MSpin^h(8k+4) & \longrightarrow & MSpin^h(8k+12) \\ \Big\downarrow {\scriptstyle id\wedge\Theta_{8k+4}} & & \Big\downarrow {\scriptstyle \Theta_{8k+12}} \\ S^8 \wedge (BO \times \mathbb{Z}) & \xrightarrow{\quad Bott \quad} & BO \times \mathbb{Z} \end{array}$$

where the top map is induced by the bundle $\mathbb{R}^8 \oplus \mathbb{E}_{8k+4}$ and the bottom map is the Bott periodicity map. The commutativity follows from that $\Xi_{\mathbb{R}^8} \in \widetilde{\mathrm{KO}}(S^8)$ is the Bott generator and the multiplicative property Proposition 2.19. Thus we obtain a natural transformation from spin^h cobordism theory to symplectic K-theory. From this point of view, $\hat{\mathcal{A}}^h$ is simply the evaluation of this natural transformation at a point. Further, using the multiplicative property of the weak-KO-Thom class, one can show $\hat{\mathcal{A}}^h : MSpin^h \to KSp$ is a module map over the ring homomorphism $\hat{\mathcal{A}} : MSpin \to KO$ where $\hat{\mathcal{A}}$ is the well-known spin-orientation of KO, defined using the KO-Thom class for spin vector bundles. Indeed, the following square commutes:

$$\begin{array}{ccc} MSpin(8l) \wedge MSpin^h(8k+4) & \longrightarrow & MSpin^h(8l+8k+4) \\ \Big\downarrow {\scriptstyle \Xi_{8l}\wedge\Theta_{8k+4}} & & \Big\downarrow {\scriptstyle \Theta_{8l+8k+4}} \\ (BO \times \mathbb{Z}) \wedge (BO \times \mathbb{Z}) & \xrightarrow{\quad \boxtimes \quad} & (BO \times \mathbb{Z}) \end{array}$$

where $\Xi_{8l} \in \widetilde{\mathrm{KO}}(MSpin(8l))$ is the universal KO-Thom class of Spin$(8l)$-bundles and the top map is induced by the bundle $\mathbb{F}_{8l} \oplus \mathbb{E}_{8k+4}$.

A non-trivial consequence is the following:

Theorem 3.23. $\hat{A}^h : \Omega_*^{\mathrm{spin}^h}(\mathrm{pt}) \to \mathrm{KSp}^{-*}(\mathrm{pt})$ is epic. In particular $\Omega_n^{\mathrm{spin}^h} \neq 0$ for $n \equiv 5, 6 \bmod 8$.

Proof. Since $\hat{\mathcal{A}}^h$ is equivariant with respect to the surjective ring homomorphism $\hat{\mathcal{A}} : \Omega_*^{\mathrm{spin}}(\mathrm{pt}) \to \mathrm{KO}^{-*}(\mathrm{pt})$, and since $\mathrm{KSp}^{-*-4}(\mathrm{pt})$ is a free $\mathrm{KO}^{-*}(\mathrm{pt})$-module generated by $\mathrm{KSp}^{-4}(\mathrm{pt})$, it suffices to show $\hat{\mathcal{A}}^h$ is onto in degrees 0 and 4. But clearly $\hat{\mathcal{A}}^h(\mathrm{pt}) = \hat{A}^h(\mathrm{pt})/2 = 1$ and $\hat{\mathcal{A}}^h(\mathbb{HP}^1_+) = \hat{A}^h(\mathbb{HP}^1_+) = 1$. ■

Remark 3.24. With 2 inverted, $\mathrm{Spin}^h \simeq \mathrm{Spin} \times \mathrm{Sp}(1)$ and consequently

$$\Omega_*^{\mathrm{spin}^h}(\mathrm{pt})[\frac{1}{2}] \cong \Omega_*^{\mathrm{spin}}(\mathbb{HP}^\infty)[\frac{1}{2}] \cong \Omega_*^{\mathrm{spin}}(\mathrm{pt}) \otimes_{\mathbb{Z}} H_*(\mathbb{HP}^\infty; \mathbb{Z}[\frac{1}{2}]).$$

This implies $\Omega_n^{\mathrm{spin}^h}$ is a 2-primary torsion group for $n \equiv 5, 6 \bmod 8$.

Also, this natural transformation helps determine the generators of the spin^h cobordism groups in low dimensions. The determination of all spin^h cobordism groups seems to be considerably hard.

Proposition 3.25. Let $F : \Omega_*^{\mathrm{spin}^h}(\mathrm{pt}) \to \Omega_*^{\mathrm{SO}}(\mathrm{pt})$ be the forgetful homomorphism. Then

$$(F, \hat{\mathcal{A}}^h) : \Omega_n^{\mathrm{spin}^h}(\mathrm{pt}) \to \Omega_n^{\mathrm{SO}}(\mathrm{pt}) \oplus \mathrm{KSp}^{-n}(\mathrm{pt})$$

is an isomorphism for $n \leq 5$.

Sketch of proof. The surjectivity is clear since $\hat{\mathcal{A}}^h$ is surjective by the previous theorem and F is also surjective: one can enrich oriented manifolds of dimensions ≤ 5 with spin^h structures (see Example 2.4). Meanwhile a formidable computation of the spin^h cobordism groups in low dimensions shows in dimensions ≤ 5 the spin^h cobordism groups are abstractly isomorphic to

$$\mathbb{Z}, 0, 0, 0, \mathbb{Z} + \mathbb{Z}, \mathbb{Z}_2 + \mathbb{Z}_2.$$

Details will not be given. These groups are also abstractly isomorphic to $\Omega_*^{\mathrm{SO}}(\mathrm{pt}) \oplus \mathrm{KSp}^{-*}(\mathrm{pt})$ in dimensions ≤ 5. Consequently surjectivity forces isomorphism. ■

It is now easy to see $\Omega_4^{\mathrm{spin}^h}(\mathrm{pt})$ is generated by $\mathbb{HP}^1_+$ and $\mathbb{CP}^2_+$, since $\mathbb{CP}^2$ generates Ω_4^{SO} and $\mathbb{HP}^1$ is zero in Ω_4^{SO} but $\hat{\mathcal{A}}^h(\mathbb{HP}^1_+) = 1$. Similarly $\Omega_5^{\mathrm{spin}^h}(\mathrm{pt})$ is generated by $\mathbb{RP}^1 \times \mathbb{HP}^1_+$ and $\mathrm{SU}(3)/\mathrm{SO}(3)$. Here $\mathbb{RP}^1$ is viewed as a spin manifold with its non-trivial spin structure and $\mathrm{SU}(3)/\mathrm{SO}(3)$ carries a natural spin^h structure whose canonical bundle is the natural principal $\mathrm{SO}(3)$-bundle $\mathrm{SO}(3) \to \mathrm{SU}(3) \to \mathrm{SU}(3)/\mathrm{SO}(3)$.

Remark 3.26. In fact, using standard notations for homotopy theorists, with the knowledge of the cohomology of $BSpin^h$ calculated in Section 2.5, one can show the spectrum map

$$MSpin^h \to ksp \vee \Sigma^4 H\mathbb{Z} \vee \Sigma^5 H\mathbb{Z}_2$$

labeled by $\hat{\mathcal{A}}^h$, p_1U and w_2w_3U induces an isomorphism on 2-local cohomology up to degree 6. In degree 7, the induced map on mod 2 cohomology is epic with a one-dimensional kernel reflecting the relation $Sq^3(w_2^2U) = Sq^2(w_2w_3U) = w_2^2w_3U$. It follows that the above spectrum map lifts to $MSpin^h \to ksp \vee F$ where $\Sigma^{-4}F$ is the fiber of $H\mathbb{Z} \vee \Sigma H\mathbb{Z}_2 \to \Sigma^3 H\mathbb{Z}_2$ labeled by Sq^3, Sq^2. This lifted map is an isomorphism on 2-local cohomology up to degree 7, hence $\Omega_6^{\mathrm{spin}^h} \cong \mathbb{Z}_2 + \mathbb{Z}_2$.

3.6. **Boundary defect and invariants of real vector bundles.** Let X be a spin^h manifold with boundary ∂X of dimension $2n$, so that ∂X has dimension $2n-1$. Assume the riemannian metric on X coincides with a product metric on $\partial X \times [0,1]$ in a neighborhood of the boundary. Recall X carries a fundamental $\mathbb{Z}_2$-graded complex ^{h}spinor bundle $\$_{\mathbb{C}^2}(X) = \$^0_{\mathbb{C}^2}(X) \oplus \$^1_{\mathbb{C}^2}(X)$ that admits a Dirac operator

$$\not{D}_{\mathbb{C}^2,X} = \begin{pmatrix} & \not{D}^1_{\mathbb{C}^2,X} \\ \not{D}^0_{\mathbb{C}^2,X} & \end{pmatrix}$$

where $\not{D}^0_{\mathbb{C}^2,X} : \Gamma(X, \$^0_{\mathbb{C}^2}(X)) \to \Gamma(X, \$^1_{\mathbb{C}^2}(X))$ is a first order elliptic operator. The restriction of the bundle $\$^0_{\mathbb{C}^2}(X)$ to ∂X can be identified with the complex ^{h}spinor bundle over ∂X

$$S_{\partial X} := P_{\mathrm{Spin}^h}(\partial X) \times_\mu \Delta^0_{2n,\mathbb{C}^2} \tag{16}$$

where $\Delta^0_{2n,\mathbb{C}^2}$ is viewed as a $\mathrm{Cl}_{2n-1,\mathbb{H}}$-module through the isomorphism $\mathrm{Cl}_{2n-1,\mathbb{H}} \cong \mathrm{Cl}^0_{2n,\mathbb{H}}$. Choose, in a neighborhood of the boundary, a local framing $e_1, \dots, e_{2n}$ for X so that e_{2n} is the inward normal direction. In local terms

$$\not{D}^0_{\mathbb{C}^2,X} = e_{2n} \cdot (\nabla_{e_{2n}} + \sum_{i=1}^{2n-1} e_i e_{2n} \cdot \nabla_{e_i}) = e_{2n} \cdot (\nabla_{e_{2n}} + D)$$

where D, through the identification $\not{S}^0_{\mathbb{C}^2}(X)|_{\partial X} = S_{\partial X}$ is the Dirac operator on $S_{\partial X}$. In particular D is a first order self-adjoint elliptic operator. As such, A has a discrete spectrum with real eigenvalues.

Two invariants are attached to $SpecD$, the spectrum of the operator D: the multiplicity of the eigenvalue 0

$$h = \dim_{\mathbb{C}} \ker D$$

and the eta-invariant $\eta(0)$ where η is the analytic continuation of

$$\eta(s) = \sum_{\lambda \in SpecD - 0} (sign\lambda)|\lambda|^{-s}.$$

If we impose the following global boundary condition for $\not{D}^0_{\mathbb{C}^2,X}$

$$P(f|_{\partial X}) = 0, \quad f \in \Gamma(X, \not{S}^0_{\mathbb{C}^2,X}) \tag{17}$$

where P is the spectral projection of D corresponding to eigenvalues ≥ 0, then the Atiyah-Patodi-Singer index theorem [APS75] asserts:

$$\mathrm{ind}(\not{D}^0_{\mathbb{C}^2,X}) = \int_X \alpha_0(x) - \frac{h + \eta(0)}{2}.$$

where $\alpha_0(x)$ is certain locally defined differential form on X. To determine $\alpha_0(x)$, one suffices to do a local computation, so we can assume X is a spin manifold and the canonical bundle $\mathfrak{h}_X$ is reduced from a $\mathrm{Sp}(1)$-bundle through the covering map $\mathrm{Sp}(1) \to \mathrm{SO}(3)$. Now that $\Delta_{2n,\mathbb{C}^2}$, when viewed as a representation of $\mathrm{Spin}(2n) \times \mathrm{Sp}(1)$, is the tensor product $\Delta_{2n,\mathbb{C}} \otimes \mathbb{C}^2$ where $\mathbb{C}^2$ is considered the 2-dimensional irreducible representation for $\mathrm{Sp}(1) = \mathrm{SU}(2)$, the $\mathbb{Z}_2$-graded complex ^{h}spinor bundle $\not{S}_{\mathbb{C}^2}(X)$ can be written as $\not{S}_{\mathbb{C}}(X) \otimes \xi$ where $\not{S}_{\mathbb{C}}(X)$ is the usual $\mathbb{Z}_2$-graded complex spinor bundle for spin manifolds that corresponds to the complex Clifford module $\Delta_{2n,\mathbb{C}}$, and where ξ is the rank 2 complex vector bundle associated to the 2-dimensional irreducible representation of $Sp(1)$. This is exactly the twisted situation considered in [APS75, 4.3], therefore α_0 is the Chern-Weil form representative of $\mathrm{ch}(\xi)\hat{\mathfrak{A}}(X)$. By Lemma 2.21 this form is identical to $2\cosh(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2})\hat{\mathfrak{A}}(X)$. Therefore we have proved:

Theorem 3.27. Let X be a $2n$-dimensional spin^h manifold with boundary ∂X. Let $\not{D}_{\mathbb{C}^2,X}$ be the Dirac operator on the fundamental $\mathbb{Z}_2$-graded complex ^{h}spinor bundle. Then the index of $\not{D}^0_{\mathbb{C}^2,X}$ with the global boundary condition (17) is given by

$$\mathrm{ind}(\not{D}^0_{\mathbb{C}^2,X}) = \int_X 2\cosh(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2})\hat{\mathfrak{A}}(X) - \frac{h + \eta(0)}{2}$$

where h is the dimension of the null-space of the Dirac operator D on the complex ^{h}spinor bundle $S_{\partial X}$ (defined by (16)) over ∂X, and $\eta(0)$ is the eta-invariant of D.

The corresponding statement of course holds for Dirac operators with coefficients in a hermitian vector bundle. Suppose ξ be a hermitian vector bundle with a unitary connection and that, near the boundary, the metric and connection are constant in the normal direction. Then the above theorem generalizes to

$$\mathrm{ind}(\not{D}^0_{X,\xi}) = \int_X \mathrm{ch}(\xi) \cdot 2\cosh(\frac{\sqrt{p_1(\mathfrak{h}_X)}}{2})\hat{\mathfrak{A}}(X) - \frac{h_\xi + \eta_\xi(0)}{2} \tag{18}$$

where $\not{D}^0_{X,\xi}$ is as defined in Theorem 3.2 and h_ξ, $\eta_\xi(0)$ relate to the Dirac operator D_ξ on $S_{\partial X} \otimes_{\mathbb{C}} \xi$.

Using the indices discussed above, we can associate a family of *numerical* invariants to real vector bundles over an arbitrary compact manifold (with corners). To begin with, let $\gamma \to Y$ be a real vector bundle over a compact manifold Y (with corners). Let $f : X \to Y$ be a smooth mapping from a closed n-dimensional spin^h manifold X into Y. We define a pairing

$$\langle X \xrightarrow{f} Y, \gamma \to Y \rangle = \begin{cases} \hat{\mathcal{A}}^h(X, f^*\gamma) \in \mathbb{Z} & \text{if } n \equiv 0, 4 \bmod 8 \\ \hat{\mathcal{A}}^h(X, f^*\gamma) \in \mathbb{Z}_2 & \text{if } n \equiv 5, 6 \bmod 8 \\ \frac{1}{2}\left(h_{f^*\gamma_{\mathbb{C}}} + \eta_{f^*\gamma_{\mathbb{C}}}(0)\right) \pmod 1 \in \mathbb{R}/\mathbb{Z} & \text{if } n \equiv 3, 7 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

We shall state properties of this pairing whose proof will appear elsewhere.

This paring relies only on the spin^h cobordism class of the map $X \xrightarrow{f} Y$ and the stable class of the real vector bundle γ. Moreover this pairing is additive with respect to both disjoint union of mappings and Whitney sum of bundles. Therefore this pairing yields, for each n, a group homomorphism

$$\mathrm{KO}(Y) \to Hom(\Omega_n^{\mathrm{spin}^h}(Y), R_n)$$

where $R_n = \mathbb{Z}, \mathbb{Z}_2, \mathbb{R}/\mathbb{Z}$ or 0 depending on the dimension n. It is natural to ask whether or not the homomorphism

$$\mathrm{KO}(Y) \to \bigoplus_n Hom(\Omega_n^{\mathrm{spin}^h}(Y), R_n)$$

is injective, or equivalently, do the above *numerical* invariants completely determine the bundle γ up to stable equivalence? Further, due to the differential-geometric nature of these invariants, one can enhance them to be invariants of real vector bundles with connections. Then the same question can be asked.

We emphasize that the answer can never be "yes" if one discards the $\mathbb{R}/\mathbb{Z}$-valued invariants, since the $\mathbb{Z}$- and $\mathbb{Z}_2$-valued invariants cannot detect odd primary torsions, thus considering manifolds-with-boundary is necessary. We would like to show, in a separate article that, with appropriate modifications to these invariants, the answers are affirmative.

Appendix A. KM-theory

Definition A.1. Let (X, f) be a real space. An M-bundle over X is a pair (E, j) consisting of a complex vector bundle E over X together with a real bundle map $j : E \to E$ covering f so that $j : E_x \to E_{fx}$ is $\mathbb{C}$-antilinear and $j^4 \equiv 1$. We say j is the M-structure on E. In the special case X is a point with trivial involution, we say (E, j) is an M-vector space.

It is clear both Real bundles and Quaternionic bundles are M-bundles. It may be helpful to think of the Real theory is associated to the group $\mathbb{Z}_2$ while the M-theory is associated to the group $\mathbb{Z}_4$. The group $\mathbb{Z}_4$ admits a natural even-odd filtration where the even subgroup is isomorphic to $\mathbb{Z}_2$. Even though the sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ does not split, our KM-theory does. Indeed with the assumption that X is connected, every M-bundle is a direct sum of a Real one and a Quaternionic one.

Proposition A.2. Let (E, j) be an M-bundle over the *connected* real space (X, f). Then there is a natural M-bundle isomorphism

$$E \cong (1 + j^2)E \oplus (1 - j^2)E$$

where $(1 + j^2)E$, endowed with j, is a Real bundle and $(1 - j^2)E$ Quaternionic.

Proof. Notice that $j^2 : E \to E$ is a complex linear automorphism of E. Since $j^4 \equiv 1$, at $x \in X$, j^2 decomposes E_x into a direct sum of eigenspaces

$$\ker(1 - j_x^2) \oplus \ker(1 + j_x^2).$$

The continuity of j_x^2 with respect to x implies the dimensions of $\ker(1 \mp j_x^2)$ are upper semi-continuous with respect to x, whence the sum of the dimensions of $\ker(1 \mp j_x^2)$ is a constant. As such both the dimensions of $\ker(1 \mp j_x^2)$ are locally constant in x. Since X is now assumed to be connected, we conclude $\ker(1 \mp j^2) = (1 \pm j^2)E$ define complex vector bundles over X. It is easy to see when equipped with j these two bundles are Real and Quaternionic respectively. The asserted M-bundle isomorphism follows at once. ∎

So KM $=$ KR$\oplus$KQ can be viewed as the Grothendieck group of M-bundles. When dealing with Quaternionic bundles, it is better to think of them as M-bundles, since the theory KM is multiplicative while the theory KQ is not. The multiplication on KM is of course induced by tensor product of complex vector bundles. A special feature for this product is that the product of two Quaternionic bundles is Real. That said, we see the multiplication in KM-theory respects its $\mathbb{Z}_2$-grading; in particular KR is a subring of KM and KQ is a KR-module

Most of the results in [Ati66] for Real bundles and KR-theory now hold for M-bundles and KM-theory, one suffices to replace the Real structures therein by the M-structures. In particular, adopting the notation of [Ati66], we have the following projective bundle formula:

Proposition A.3. Let L be a Real line-bundle (i.e. of complex rank one) over the real compact space X, H is the standard Real line-bundle over the projective bundle $\mathbb{P}(L\oplus 1)$ where 1 is understood to be the trivialized Real bundle over X. Then as a $\mathrm{KM}(X)$-algebra, $\mathrm{KM}(\mathbb{P}(L\oplus 1))$ is generated by H subject to the single relation

$$([H]-[1])([L][H]-1)=0.$$

The Thom isomorphism for Real bundles and (1,1)-periodicity follow in a quite formal way.

Theorem A.4. Let E be a Real vector bundle over the real compact space X. Then

$$\phi : \mathrm{KM}(X) \to \mathrm{KM}_{cpt}(E)$$

is an isomorphism where $\phi(x) = \lambda_E \cdot x$ and λ_E is the element of $\mathrm{KR}_{cpt}(E)$ defined by the exterior algebra of E.

Theorem A.5. Let $b = [H] - 1 \in \mathrm{KR}^{1,1}(\mathrm{pt}) = \mathrm{KR}(\mathbb{CP}^1)$. Then the homomorphism

$$\beta : \mathrm{KM}^{r,s}(X, Y) \to \mathrm{KM}^{r+1,s+1}(X, Y)$$

given by $x \mapsto bx$ is an isomorphism.

Since the homomorphisms ϕ and β are both induced by multiplication with Real bundles, they preserve the $\mathbb{Z}_2$-grading KM $=$ KR$\oplus$KQ, i.e. they send KR to KR and KQ to KQ. So the corresponding theorems hold for KQ-theory as well. This explains (7) and Theorem 3.12.

Recall we have defined $\mathrm{KM}^{r,s}$ for $r, s \geq 0$ using

$$\mathrm{KM}^{r,s}(X, Y) = \mathrm{KM}(X \times D^{r,s}, X \times S^{r,s} \cup Y \times D^{r,s}),$$

which in the special case $s = 0$ coincides with the usual suspension groups KM^{-r}. Now thanks to the (1,1)-periodicity, we can define KM-groups with positive indices by putting $\mathrm{KM}^r = \mathrm{KM}^{0,r}$. Then we have a natural isomorphism $\mathrm{KM}^{r,s} \cong \mathrm{KM}^{s-r}$. This justifies the use of the group KM^4 in [Dup69], in fact this is the main reason why we did not directly quote Dupont's results.

Now we can quote [Dup69] to prove (6).

Proposition A.6. For $r, s \geq 0$, multiplication with the generator of $\mathrm{KQ}^{4,0}(\mathrm{pt})$ yields an isomorphism

$$\mathrm{KR}^{r,s}(\mathrm{pt}) \xrightarrow{\cong} \mathrm{KQ}^{r+4,s}(\mathrm{pt}).$$

Proof. From [Dup69, (6)], we know multiplication with the generator of $\mathrm{KQ}^4(\mathrm{pt}) \cong \mathrm{KQ}^{0,4}(\mathrm{pt})$ gives an isomorphism

$$\mathrm{KQ}^{r+4,s}(\mathrm{pt}) \cong \mathrm{KR}^{r+4,s+4}(\mathrm{pt}).$$

On the other hand, the (1,1)-periodicity gives

$$\mathrm{KR}^{r+4,s+4}(\mathrm{pt}) \cong \mathrm{KR}^{r,s}(\mathrm{pt}).$$

Combining the two isomorphisms and summing over all $r, s \geq 0$, we obtain isomorphisms of bigraded-groups

$$\mathrm{KR}^{*,*}(\mathrm{pt}) \cong \mathrm{KR}^{*+4,*+4}(\mathrm{pt}) \cong \mathrm{KQ}^{*+4,*}(\mathrm{pt}).$$

Now observe the above isomorphisms are homomorphisms of $\mathrm{KR}^{*,*}(\mathrm{pt})$-modules, the proposition thus follows. ∎

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