Generalized Frölicher spectral sequence

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Let X be an almost complex manifold and A be its de Rham algebra of complex valued differential forms. Cirici and Wilson defined a filtration on (A, d) by

$$F^pA^n = \ker\overline{\mu} \cap A^{p,n-p} \bigoplus \oplus_{i>p} A^{i,n-i}$$

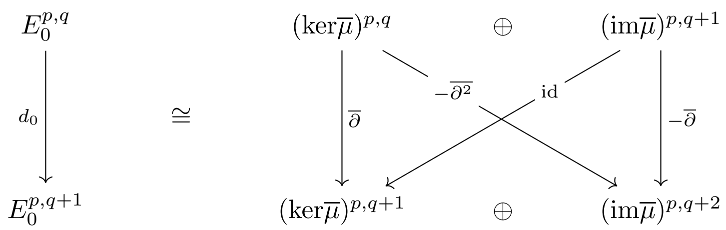
This filtration then yields a spectral sequence converging to the (complex) de Rham cohomology of X. The purpose of this note is to compute its E_0- and E_1-page.

To begin with, let us describe E_0-page. By definition

$$E_0^{p,q} = F^p(A^{p+q})/F^{p+1}(A^{p+q}) = \ker\overline{\mu} \cap A^{p,q} \oplus \frac{A^{p+1,q-1}}{\ker\overline{\mu}} \cong (\ker\overline{\mu})^{p,q} \oplus (\operatorname{im}\overline{\mu})^{p,q+1}.$$

The differential $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is induced from d. Notice for degree reasons E_0-page does not detect ∂, μ, so d_0 is made from $\overline{\mu}$ and $\overline{\partial}$ only. (Compare this to the complex case where E_0-page of the Frölicher spectral sequence only detects $\overline{\partial}$.)

So we must compute the effect of $\overline{\mu}$ and $\overline{\partial}$ on $\ker\overline{\mu} \cap A^{p,q}$ and $\frac{A^{p+1,q-1}}{\ker\overline{\mu}} \cong (\operatorname{im}\overline{\mu})^{p,q+1}$. Using the relation $\partial\overline{\mu} + \overline{\mu}\partial + \overline{\partial}^2 = 0$ and the isomorphism above, one sees d_0 can be described as



(1)

Written in matrix

$$d_0 \cong \begin{pmatrix} \overline{\partial} & \mathrm{id} \\ -\overline{\partial}^2 & -\overline{\partial} \end{pmatrix}$$

Now consider the short exact sequence of cochain complexes

$$0 \to (\operatorname{im}\overline{\mu})^{p,q} \oplus (\operatorname{im}\overline{\mu})^{p,q+1} \to E_0^{p,q} \to H_{\overline{\mu}}^{p,q} = \frac{(\ker\overline{\mu})^{p,q}}{(\operatorname{im}\overline{\mu})^{p,q}} \to 0$$

We claim that $((\operatorname{im}\overline{\mu})^{p,q} \oplus (\operatorname{im}\overline{\mu})^{p,q+1}, d_0)$ is acyclic. Then note it follows from (1) that d_0 descends to $\overline{\partial}$ on $H_{\overline{\mu}}^{p,q}$, so we have

Proposition. $E_1^{p,q} \cong H^q(H_{\overline{\mu}}^{p,*}, \overline{\partial})$.

It remains to prove the acyclicity of $((\operatorname{im}\overline{\mu})^{p,q} \oplus (\operatorname{im}\overline{\mu})^{p,q+1}, d_0)$. For this, we apply the following lemma.

Lemma. Let $C^\bullet$ be a graded vector space and $\delta : C^\bullet \to C^\bullet$ an endomorphism of degree 1. Then $C^\bullet \oplus C^\bullet[1]$ with $d = \begin{pmatrix} \delta & \mathrm{id} \\ -\delta^2 & -\delta \end{pmatrix}$ is an acyclic cochain complex.

Proof. If $(x, y) \in C^k \oplus C^{k+1}$ satisfies $d(x, y) = 0$, then it is easy to show $(x, y) = d(0, x)$. $\square$